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Elation switching in real parallelisms

Esteban Diaz   Norman L. Johnson
Alessandro Montinaro*

Abstract

Switching techniques are developed that produce a variety of new parallelisms in $\mathsf{PG}(3, K)$, where K is a infinite field. When K is the field of real numbers, 2^{χ_0} mutually non-isomorphic parallelisms are constructed.

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1 Introduction

A *parallelism* of a projective space $\mathsf{PG}(3, F)$, where F is a skew field, is an equivalence relation on the set of lines satisfying the Euclidean parallel postulate. This concept originated with Clifford's work [3] in 1873, where there are two parallelisms. Indeed, there are characterizations of parallelisms admitting what are called *left and right parallelisms* and having certain other properties (see, e.g. the work of Karzel [11] and Karzel and Kroll [12]). More recently, there are a variety of parallelisms constructed by Betten and Riesinger over $\mathsf{PG}(3, \mathcal{R})$, where $\mathcal{R}$ is the field of real numbers (see [1]). Indeed, there are also a variety of *real parallelisms* constructed by the second author and R. Pomareda in [10]. It might be mentioned that the equivalence classes of a parallelism, called *spreads*, define affine translation planes in the associated four dimensional vector space over F and analysis of affine geometry then provides a strong technical device for the study of parallelisms.

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So, the concept of a parallelism is quite fundamental to the study of projective geometry and especially when the parallelisms are considered over infinite fields F.

In this article, the main focus is on new constructions of parallelisms in $\mathsf{PG}(3, K)$, where K is a subfield of the field of real numbers, although the field K actually need only be an ordered field. The construction involves a replacement procedure that we term *switching of spreads*. The authors [5] have developed this construction process previously for the finite case. Here, we consider a completely general procedure.

Previously, one of the authors (Johnson [7]) constructed a class of parallelisms in $\mathsf{PG}(3, K)$, where K is an arbitrary field which admits a quadratic extension. This particular construction uses a central collineation group G of a Pappian spread Σ lying in the parallelism so that G also acts as a collineation group of the parallelism and where G contains the full elation group E of Σ (or rather of the associated Pappian translation plane π_Σ) that fixes a given line ℓ pointwise, and where G acts transitively on the remaining spreads of the parallelism. There is a classification theorem of sorts that we might mention.

Theorem 1.1 (see Johnson and Pomareda [9]). *Let K be a skew field, Σ a spread in $\mathsf{PG}(3, K)$ and $\mathcal{P}$ a partial parallelism of $\mathsf{PG}(3, K)$ containing Σ.*

If $\mathcal{P}$ admits as a collineation group the full central collineation group G of Σ with a given axis ℓ that acts two-transitive on the remaining spread lines then

(1) *Σ is Pappian,*

(2) *$\mathcal{P}$ is a parallelism,*

(3) *the spreads of $\mathcal{P} - \{\Sigma\}$ are Hall, and*

(4) *G acts transitively on the spreads of $\mathcal{P} - \{\Sigma\}$.*

(5) *Moreover, $\mathcal{P}$ is one of the parallelisms of the construction of Johnson.*

Although we are mainly interested in infinite fields here, in the finite case, any such *transitive deficiency one group G* must contain the full elation group E that fixes a given line ℓ. The following result of the authors improves a similar theorem of Biliotti, Jha and Johnson [2], and whose work is required in the proof of the improvement.

Theorem 1.2 (Diaz, Johnson, Montinaro [4]; see also Biliotti, Jha and Johnson [2]). *Let $\mathcal{P}^-$ be a deficiency one partial parallelism in $\mathsf{PG}(3, q)$ that admits a collineation group in $P\Gamma L(4, q)$ acting transitively on the spreads of the partial parallelism. Let $\mathcal{P}$ denote the unique extension of $\mathcal{P}^-$ to a parallelism. Let the fixed spread be denoted by Σ_0 (the* socle*) and let the remaining $q^2 + q$ spreads of $\mathcal{P}^-$ be denoted by Σ_i, for $i = 1, 2, ..., q^2 + q$.*

(1) *Then* Σ_0 *is Desarguesian and* Σ_i *is a derived conical flock spread for* $i = 1, 2, \ldots, q^2 + q$.

(2) *Furthermore, the associated group* G *in* $\Gamma L(4, q)$ *acting on the associated Desarguesian affine plane* π_{Σ_0} *fixes a line* ℓ *of* Σ_0 *and contains the full elation group* E *with axis* ℓ *as a normal subgroup.*

Returning to the more general case for an arbitrary field K, there is something of a classical construction that we now mention.

Let Σ be any Pappian spread in $\mathsf{PG}(3, K)$ and let Σ' any spread which shares exactly a regulus R with Σ such that Σ' is derivable with respect to R. Assume that there exists a subgroup G^- of the central collineation group G with fixed axis L with the following properties:

(i) Every line skew to L and not in Σ is in $\Sigma' G^-$,

(ii) G^- is transitive on the reguli that share L, and

(iii) if g is a collineation of G^- such that for each $L' \in \Sigma'$ also $L'g \in \Sigma'$, then g is a collineation of Σ'.

Let $(Rg)^*$ denote the regulus opposite to Rg.

Theorem 1.3 (see Johnson [7]). *Under the above assumptions,*

$$\Sigma \cup \left\{ (\Sigma' g - Rg) \cup (Rg)^*; g \in G^- \right\}$$

is a parallelism of $\mathsf{PG}(3, K)$ *consisting of one Pappian spread* Σ *and the remaining spreads derived* Σ'*-spreads.*

Moreover, there are some related parallelisms, called the *derived parallelisms*.

Theorem 1.4 (see Johnson [6]). *Assume that*

$$\Sigma \cup \{ (\Sigma' g - Rg) \cup (Rg)^*; g \in G^- \}$$

is a parallelism. Then

$$\{\Sigma - R\} \cup R^* \cup \Sigma' \cup \left\{ (\Sigma' g - Rg) \cup (Rg)^*; g \in G^- - \{1\} \right\}$$

is a parallelism. In this case, the spreads are Hall, Σ' *(undetermined) and derived* Σ' *type spreads.*

In this article, we develop a construction method that uses the full elation subgroup E of G^- as follows: Suppose that Σ_2 is a spread in $\mathsf{PG}(3, K)$ such that $\Sigma_2 E$ is a set of mutually line disjoint spreads. If Σ_3 is a spread in $\mathsf{PG}(3, K)$

such that as a set of lines $\Sigma_3 E = \Sigma_2 E$ then it will turn out that $\Sigma_3 E$ is a set of mutually line disjoint spreads. What this means for parallelisms (or partial parallelisms) containing $\Sigma_2 E$, is that we may switch the sets of spreads $\Sigma_2 E$ with sets of spreads $\Sigma_3 E$. This process, called *elation switching*, produces a tremendous number of new parallelisms. If Σ_2 is Desarguesian and K is finite isomorphic to $\mathsf{GF}(q)$ then it is possible to completely determine all spreads Σ_3 such that $\Sigma_3 E$ switches with $\Sigma_2 E$. The authors show in [5] that Σ_3 must be a Kantor–Knuth or Desarguesian spread. However, when the construction is applied to an arbitrary infinite field, there are very few restrictions on the type of spreads Σ_3 that can be used. Our arguments center on conical flock spreads in the infinite case and the constructions given show that there are an enormous number of new parallelisms that may be constructed by this process. In particular, when K is the field of real numbers, there are uncountably many new parallelisms constructed.

2 Elation switching

As suggested previously, the application of this construction technique mentioned in the introduction has been applied most successfully when the spreads other than the Pappian spread are derived conical flock spreads and when the group contains a large normal subgroup that is a central collineation group. (By *conical flock spreads*, we intend to mean those spreads that correspond to flocks of quadratic cones.) The reader is directed to the Handbook [8] for the precise definitions and additional background.

Actually, there is a classification by procedure of such parallelisms.

Theorem 2.1 (see Johnson [6]). *Let $\mathcal{P}$ be a parallelism in $\mathsf{PG}(3, K)$, for K a field, that admits a Pappian spread Σ and a collineation group G^- fixing a line ℓ of Σ that acts transitively on the remaining spreads of $\mathcal{P}$.*

(1) *If K is finite and if G^- contains the full elation group with axis ℓ then the spreads of $\mathcal{P} - \{\Sigma\}$ are derived conical flock spreads.*

(2) *If G^- contains the full elation group with axis ℓ and, for ρ a spread of $\mathcal{P} - \{\Sigma\}$, G^-_ρ contains a non-trivial homology (i.e. homology in Σ) then the spreads of $\mathcal{P} - \{\Sigma\}$ are derived conical flock spreads.*

In a previous article on the above constructed over infinite fields, the second author constructed a variety of parallelisms over the reals (when K is the field of real numbers, $K = \mathcal{R}$.

The main idea is as follows. Let a Pappian spread Σ_1 defined as follows:

$$x = 0, y = x \begin{bmatrix} u & -t \\ t & u \end{bmatrix} \ \forall u, t \in \mathcal{R}.$$

We let Σ_2 be a spread in $\mathsf{PG}(3, \mathcal{R})$, defined by a function f:

$$x = 0, y = x \begin{bmatrix} u & -f(t) \\ t & u \end{bmatrix} \ \forall u, t \in \mathcal{R}$$

where f is a function such that $f(t) = t$ implies that $t = 0$ and $f(0) = 0$.

Thus, if a spread exists then the two spreads Σ_1 and Σ_2 share exactly the regulus $\mathcal{D}$ with partial spread:

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \ \forall u \in \mathcal{R}.$$

The following lemma of Johnson and Pomareda connects at least one of the groups G that we use for our construction.

Lemma 2.2 (Johnson and Pomareda [10]). *Let f be any continuous strictly increasing function on the field of real numbers such that $\lim_{x \to \pm\infty} f(t) = \pm\infty$.*

(1) *Then Σ_2 is a spread.*

(2) *Let $G^- = EH^-$ where H^- denotes the homology group of Σ_1 (or rather the associated affine plane) whose elements are given by*

$$\left\langle \begin{bmatrix} u & -t & 0 & 0 \\ t & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; u^2 + t^2 = 1 \right\rangle ,$$

and where E denotes the full elation group with axis $x = 0$.

(3) *Then G^- is transitive on the set of reguli of Σ_1 that share $x = 0$.*

Then for conditions that such spreads produce parallelisms along the same lines as when the second chosen spread is Pappian, we mention the following result:

Theorem 2.3 (Johnson and Pomareda [10]). *Under the above assumptions, assume also that f is symmetric with respect to the origin in the real Euclidean 2-space and $f(t_o + r) = f(t_o) + r$ for some t_o and r in the reals implies that $r = 0$.*

(1) *Then*

$$\Sigma_1 \cup \Sigma_2^* g; g \in G^-$$

is a partial parallelism $\mathcal{P}_f$ in $\mathsf{PG}(3, \mathcal{R})$, where Σ_2^* denotes the derived spread of Σ_2 by derivation of $\mathcal{D}$.

(2) *The above construction produces a parallelism if and only if* $f(t) - t$ *is surjective.*

(3) *When the function* f *produces a partial parallelism* $\mathcal{P}$ *and* $f(t) - t$ *is not an onto function then* $\mathcal{P}$ *is a proper maximal partial parallelism.*

(4) *If* $\mathcal{P}$ *is a proper maximal partial parallelism then so is any derived partial parallelism* $\mathcal{P}^*$.

In this article, we shall replace the conditions on f, with much more general conditions that are valid over essentially any field K and completely generalize the parallelisms constructed in the previous theorem.

With this background, we may now define the main concept of this article.

Definition 2.4. Let Σ_0 denote a Pappian spread in $\mathsf{PG}(3,K)$, where K is a field:

$$\left\{x = 0, y = x \begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix} \ \forall u, t \in K\right\}, \text{ where } \gamma_1 \text{ is a non-square in } K.$$

Let E denote the full elation group of Σ_0 with axis $x = 0$:

$$E = \left\langle \begin{bmatrix} 1 & 0 & u & \gamma_1 t \\ 0 & 1 & t & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; u, t \in K \right\rangle.$$

Let Σ_2 and Σ_3 be distinct spreads of $\mathsf{PG}(3,K)$ that share exactly the regulus

$$R = \left\{x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \ \forall u \in K\right\}.$$

Assume the following two conditions:

(i) $\Sigma_2 E = \Sigma_3 E$,

(ii) a line ℓ of $\Sigma_2 E$ is in a unique spread of $\Sigma_2 E$ if and only if ℓ is in a unique spread of $\Sigma_3 E$.

If the spreads Σ_2 and Σ_3 have properties (i) and (ii),we shall say that $\Sigma_2 E$ and $\Sigma_3 E$ are *E-switches* of each other (or that $\Sigma_2 E$ has been *switched* by $\Sigma_3 E$).

In the finite case, the authors have proved that if Σ_2 is Desarguesian (Pappian in the infinite case) and Σ_3 is a spread such that $\Sigma_3 E$ switches with $\Sigma_2 E$ then Σ_3 is either Kantor–Knuth or Desarguesian. In this setting a matrix spread set may be chosen so that the spread Σ_3 has the form

$$\left\{x = 0, y = x \begin{bmatrix} u + \alpha t + \beta f(t) & f(t) \\ t & u \end{bmatrix} \ \forall u, t \in K\right\},$$

where f, g are functions on $K \simeq \mathsf{GF}(q)$, α, β constants in K.

Then it turns out that f is completely determined as $f(t) = \gamma t^{\sigma}$, for $\gamma \in K$ and σ an automorphism of K. Furthermore, when K has even order $\sigma = 1$ and the spread is Desarguesian. In the odd order case, the plane is said to be Kantor–Knuth, if σ is not 1. Furthermore, in the odd order case, a change in basis can be made to further represent the spread as

$$\left\{ x = 0, y = x \begin{bmatrix} u & F(t) \\ t & u \end{bmatrix} \ \forall u, t \in K \right\} \tag{$*$}$$

where F is bijective on $K \simeq \mathsf{GF}(q)$, and $F(t) = \rho t^{\sigma}$, where ρ is a non-square and σ an automorphism of K.

But now in the infinite case, there are a wide variety of spreads that have the form of $(*)$ and these are the spreads that we now study in this article. We begin with a necessary and sufficient condition on the associated functions to have spreads of this form.

Theorem 2.5. *Let K be any field. Then*

$$\Sigma_f = \left\{ x = 0, y = x \begin{bmatrix} u & f(t) \\ t & u \end{bmatrix} \ \forall u, t \in K \right\},$$

where f is a function $K \to K$ such that $f(0) = 0$ is a spread if and only if for each $z \in K$, ρ_z is bijective where,

$$\rho_z(t) = f(t) - z^2 t.$$

Proof. Let (x_1, x_2, y_1, y_2), for $x_i, y_i \in K$, $i = 1, 2$. Σ_f is a spread if and only it defines an exact cover of the vectors. If $y_1 = y_2 = 0$ or $x_1 = x_2 = 0$ then such points belong to $x = 0$ and $y = 0$ respectively. Suppose that $x \begin{bmatrix} u & f(t) \\ t & u \end{bmatrix} = (0, 0)$, for $x = (x_1, x_2)$ then

$$\begin{aligned} x_1 u + x_2 t &= 0, \\ x_1 f(t) + x_2 u &= 0. \end{aligned}$$

If $x_1 = 0$ then $x_2 \neq 0$ and both $t = u = 0$. If $x_2 = 0$ then $x_1 \neq 0$ and $u = 0 = f(t)$. Hence, we must have $f(t) = 0$ if and only if $t = 0$.

In general,

$$\begin{aligned} x_1 u + x_2 t &= y_1, \\ x_2 u + x_1 f(t) &= y_2. \end{aligned}$$

for all $x_1, x_2, y, y_2 \in K$. If $x_1 = 0$, clearly, there is a unique solution for (u, t) and hence a unique $y = x \begin{bmatrix} u & f(t) \\ t & u \end{bmatrix}$ that covers the given point.

If $x_2 = 0$ then $x_1 f(t) = y_2$ provided f is bijective and again there is a unique solution for (u, t).

So, assume that $x_1 x_2 \neq 0$. Then,

$$\begin{aligned} u + zt &= y_1^*, \\ u + z^{-1} f(t) &= y_2^*, \end{aligned}$$

where $z = x_2/x_1$, $y_1^* = y_1/x_1$ and $y_2^* = y_2/x_2$. Note that y_1^* and y_2^* and z are then completely independent. Thus,

$$zt - z^{-1} f(t) = \frac{z^2 t - f(t)}{z}.$$

Since $f(t) - z^2 t$ is bijective for all elements z^2, we have a unique solution for t, which then produces a unique solution (u, t). This completes the proof of the theorem. □

Now we shall be interested in spreads of the above form that share precisely a regulus with a Pappian spread

$$\left\{ x = 0, y = x \begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix} \ \forall u, t \in K \right\}, \text{ where } \gamma_1 \text{ is a non-square in } K,$$

and such that the full elation subgroup of E that acts as a collineation group of the spread in question is

(i) $E^- = \left\langle \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; u \in K \right\rangle$, where

(ii) the full elation of Σ_1 with axis $x = 0$ is

$$E = \left\langle \begin{bmatrix} 1 & 0 & u & \gamma_1 t \\ 0 & 1 & t & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; u, t \in K \right\rangle, \text{ and}$$

(iii) such that for any elation $e \in E - E^-$ then $\Sigma_f e \cap \Sigma_f$ is empty.

The following proposition is essentially immediate and is left to the reader to verify.

Proposition 2.6. *A spread*

$$\Sigma_f = \left\{ x = 0, y = x \begin{bmatrix} u & f(t) \\ t & u \end{bmatrix} \ \forall u, t \in K \right\},$$

shares exactly the regulus

$$R_0 = \left\{ x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \ \forall u \in K \right\}$$

with the Pappian spread Σ_1 *if and only if*

$$f(t) - \gamma_1 t = 0$$

implies $t = 0$.

Note that in the finite case, f is forced to be additive and so the above condition implies that the function $f - \gamma_1$ is bijective (injective will suffice in the finite setting). This injective condition turns out to be an important condition for elation switching.

Definition 2.7. If the function g defined by $g(t) = f(t) - \gamma_1 t$ is injective, we shall say that the function f has the *regulus property*.

If the full elation group of a spread Σ_f of E is E^-, and for $e \in E - E^-$ then $\Sigma_f e \cap \Sigma_f$ is empty, we shall say that the spread has the *regulus-inducing property*.

For example, if $f(t) = \gamma_2 t$ and $\gamma_2 \neq \gamma_1$ then f will turn out to have the regulus-inducing property. For the automorphism type function $f(t) = \gamma_2 t^\sigma$ we need $\gamma_2 t_0^\sigma = \gamma_1 t_0$ for some t_0 if and only if $t = 0$.

More generally, we have the following description of spreads that have the regulus-inducing property.

Proposition 2.8. *A spread*

$$\Sigma_f = \left\{ x = 0, y = x \begin{bmatrix} u & f(t) \\ t & u \end{bmatrix} \ \forall u, t \in K \right\},$$

has the regulus-inducing property if and only if for $t_0, r \in K$

$$f(t_0 + r) = f(t_0) + \gamma_1 r$$

implies $r = 0$.

Proof. Let $e = \begin{bmatrix} 1 & 0 & u_0 & \gamma_1 r \\ 0 & 1 & r & u_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ map $y = x \begin{bmatrix} u & f(t_0) \\ t_0 & u \end{bmatrix}$ to

$$y = x \left(\begin{bmatrix} u & f(t_0) \\ t_0 & u \end{bmatrix} + \begin{bmatrix} u_0 & \gamma_1 r \\ r & u_0 \end{bmatrix} \right).$$

This line is back in the spread if and only if

$$f(t_0 + r) = f(t_0) + \gamma_1 r.$$

Hence, we wish this never to hold for non-zero values r, so we require that $r = 0$ in this case. □

Proposition 2.9. *The regulus property implies the regulus-inducing property.*

Proof. Now let $f(t) = \gamma_1 t + g(t)$, so we have, by assumption, that g is injective. Then consider the equation

$$f(t_0 + r) = \gamma_1(t_0 + r) + g(t_0 + r) = f(t_0) + \gamma_1 r = \gamma_1 t_0 + g(t_0) + \gamma_1 r. \quad (***)$$

So, for equation $(***)$ to imply that $r = 0$ is equivalent to the condition that

$$g(t_0 + r) = g(t_0), \text{ implies } r = 0.$$

Since g is assumed to be injective, this condition is automatically satisfied. So, the regulus property implies the regulus-inducing property. □

Corollary 2.10. *The set of lines*

$$\Sigma_f = \left\{ x = 0, y = x \begin{bmatrix} u & f(t) \\ t & u \end{bmatrix} \ \forall u, t \in K \right\},$$

is a spread admitting the regulus property, and regulus-inducing property if and only if

(i) $z \in K$, ρ_z *is bijective where*

$$\rho_z(t) = f(t) - z^2 t;$$

(ii) *the function* g *such that* $g(t) = f(t) - \gamma_1 t$ *is injective.*

These are the spreads that we shall use in the switching procedure, except that we shall further require that g is bijective.

Definition 2.11. Any spread admitting the properties (i), (ii) of the previous corollary with the extra condition that the function g is bijective shall be said to admit the *switching property*.

3 Main theorem on elation switching

Theorem 3.1. *Let Σ_0 denote a Pappian spread in* $\mathsf{PG}(3,K)$*, where K is a field:*

$$\left\{x=0, y=x\begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix} \ \forall u,t\in K\right\}, \text{ where } \gamma_1 \text{ is a non-square in } K.$$

Let E denote the full elation group of Σ_0 with axis $x=0$:

$$E=\left\langle \begin{bmatrix} 1 & 0 & u & \gamma_1 t \\ 0 & 1 & t & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; u,t\in K \right\rangle.$$

Assume that Σ_i is a spread in $\mathsf{PG}(3,K)$ *of the following form:*

$$\Sigma_i=\left\{x=0, y=x\begin{bmatrix} u & f_i(t) \\ t & u \end{bmatrix} \ \forall u,t\in K\right\}, \text{ for } i=2,3,$$

where f_i is a function $K\to K$ and such that both spreads admit the switching property.

Then $\Sigma_2 E$ switches with $\Sigma_3 E$.

Proof. Note that $\Sigma_2 E=\Sigma_3 E$ if and only if Σ_2 is in $\Sigma_3 E$ and Σ_3 is in $\Sigma_2 E$. Note that

$$y=x\begin{bmatrix} u & f_i(t) \\ t & u \end{bmatrix}$$

maps to

$$y=x\left\{\begin{bmatrix} u & f_i(t) \\ t & u \end{bmatrix}+\begin{bmatrix} u^* & \gamma_1 t^* \\ t^* & u^* \end{bmatrix}\right\}, \forall u^*,t^*\in K$$

by E and note that E fixes $x=0$ pointwise.

So consider, for $j\neq i$,

$$\begin{bmatrix} u & f_i(t) \\ t & u \end{bmatrix}=\begin{bmatrix} u & f_j(s) \\ s & u \end{bmatrix}+\begin{bmatrix} 0 & \gamma_1(t-s) \\ (t-s) & 0 \end{bmatrix}. \qquad (*)$$

Then

$$f_i(t)=f_j(s)+\gamma_1(t-s)$$

if and only if

$$f_j(t)-\gamma_1 t=f_j(s)-\gamma_1 s.$$

Therefore, given t in K, then there exists a unique s in K such that

$$f_j(t)-\gamma_1 t=f_i(s)-\gamma_1 s,$$

since ϕ_j and ϕ_i are both bijective. Hence, given u and t, there is a solution to $(*)$. Note that the argument is symmetric. Now suppose for u and t, there is another solution

$$\begin{bmatrix} u & f_i(t) \\ t & u \end{bmatrix} = \begin{bmatrix} u^* & f_j(s) \\ k & u^* \end{bmatrix} + \begin{bmatrix} w & \gamma_1 d \\ d & w \end{bmatrix}, \qquad (**)$$

it now follows easily that there is a unique solution to $(**)$, namely the unique solution to $(*)$.

Since the argument is symmetric, we have $\Sigma_2 E = \Sigma_3 E$. This establishes condition (i) of Definition 2.4.

Now take an element

$$y = x \begin{bmatrix} u_0 & f_i(t_0) \\ t_0 & u_0 \end{bmatrix}$$

and assume that there is an element e in E such that the image of this element is back in Σ_i. Then

$$\begin{bmatrix} u_0 & f_i(t_0) \\ t_0 & u_0 \end{bmatrix} + \begin{bmatrix} w & \gamma_1 r \\ r & w \end{bmatrix} = \begin{bmatrix} u_0 + w & f_i(t_0) + \gamma_1 r \\ t_0 + r & u_0 \end{bmatrix}.$$

But, this means that $f_i(t_0 + r) = f_i(t_0) + \gamma_1 r$, so that $r = 0$. Then this element e leaves Σ_{f_i} invariant. Again, since the argument is symmetric, we have that $\Sigma_2 E$ and $\Sigma_3 E$ are unions of disjoint spreads. Therefore, we have that $\Sigma_2 E$ switches with $\Sigma_3 E$. □

4 Deficiency one transitive groups

Let K be an ordered field such that all positive elements have square roots in K.

For example, if L is a subfield of the field of real numbers then the numbers L^C constructible from L by straight-edge and compass is such an ordered field.

So, let K be such an ordered field, and let Σ_1 denote the Pappian spread

$$\Sigma_1 = \left\{ x = 0, y = x \begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix} \ \forall u, t \in K \right\}$$

where γ_1 is a fixed negative element in K. Let

$$H = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}; w = \begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix}; u^2 - \gamma_1 t^2 = 1 \right\rangle$$

and let

$$\Sigma_2 = \left\{ x = 0, y = x \begin{bmatrix} u & \gamma_2 t \\ t & u \end{bmatrix} \ \forall u, t \in K \right\}$$

where γ_2 is a negative element in K, $\gamma_2 \neq \gamma_1$. Let R_0 denote the common regulus $\left\{ x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \; \forall u \in K \right\}$.

Theorem 4.1. $\Sigma_1 \cup \Sigma_2^* EH$ *is a parallelism in* $\mathsf{PG}(3, K)$. *Furthermore,*

$$E^- \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\rangle$$

is the subgroup of EH *that leaves* Σ_2 *invariant, where*

$$E^- = \left\langle \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; u \in K \right\rangle .$$

Proof. First of all we claim that EH is transitive on the set of reguli of Σ_1 that share $x = 0$. We note that E is transitive on the components of $\Sigma_1 - \{x = 0\}$. So the question then is H transitive on the reguli that share $x = 0$ and $y = 0$. Any such regulus has the following form:

$$R_t = \left\{ x = 0, y = 0, \; y = x \begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix} ; u \in K - \{0\} \right\} .$$

So, the question is whether R_0 can be mapped into R_t by an element of H. Hence, given $y = x \begin{bmatrix} 0 & \gamma_1 t \\ t & 0 \end{bmatrix}$, we need to find an element $y = x \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$, v not zero, such that

$$\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} w = \begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix} ,$$

for some $u \in K$.

Note that since clearly the reguli sharing $x = 0, y = 0$ are permuted by H, it just takes one appropriate image to establish that R_0 is mapped onto R_t, as any three distinct components generate a unique regulus of Σ_1. Since w commutes with $\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$, we only need to show that for some element $\begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix}$, for t not zero, there exist elements $\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$ and $\begin{bmatrix} u^* & \gamma_1 t^* \\ t^* & u^* \end{bmatrix}$, where $u^{*2} - \gamma_1 t^{*2} = 1$ such that

$$\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} u^* & \gamma_1 t^* \\ t^* & u^* \end{bmatrix} = \begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix} .$$

Let $u^2 - \gamma_1 t^2 = z$. Since $z > 0$, let $v = \sqrt{z}$, which exists by assumption. Now let $u^* = u/v$ and $t^* = t/v$. Then $(vu^*, vt^*) = (u, t)$ and $u^* - \gamma_1 t^* = (u/v)^2 - \gamma_1 (t/v)^2 = 1$. This establishes the transitivity.

This means that if Σ_2^* denotes the derived spread then $\Sigma_2^* EH$ will contain all Baer subplanes of Σ_1 that non-trivially intersect $x = 0$. Furthermore, if a Baer subplane of R_0 maps back into a Baer subplane of R_0 under an element g of EH then g leaves R_0^* invariant and hence leaves R_0 invariant. The subgroup of EH that leaves R_0 invariant is $E^- H^-$, where

$$E^- = \left\langle \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; u \in K \right\rangle,$$

and

$$H^- = \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\rangle.$$

These elements are collineations of Σ_2 and of Σ_2^*. Hence, it follows that there can be no line that non-trivially intersects $x = 0$ that is in two distinct spreads.

The remaining 'lines' of $\mathsf{PG}(3, K)$, apart from the components of Σ_1 are the Baer subplanes of Σ_1 that do not intersect $x = 0$. These have the basic form

$$y = x \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; a, b, c, d \in K,$$

where if $a = d$ then $b \neq \gamma_1 c$. That is, either $a \neq d$ or $b \neq \gamma_1 c$. Such a line will lie in a unique spread of $\Sigma_2^* EH$ if and only if it lies in a unique spread of $\Sigma_2 EH$. Therefore, there must be an element of Σ_2, $y = x \begin{bmatrix} u & \gamma_2 t \\ t & u \end{bmatrix}$ and an element $\rho \in EH$ such that

$$\left(y = x \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \rho = \left(y = x \begin{bmatrix} u & \gamma_2 t \\ t & u \end{bmatrix} \right).$$

Notice that we may apply an elation that adds $\begin{bmatrix} -u & -\gamma_1 t \\ -t & -u \end{bmatrix}$, as

$$\begin{bmatrix} u & \gamma_2 t \\ t & u \end{bmatrix} + \begin{bmatrix} -u & -\gamma_1 t \\ -t & -u \end{bmatrix} = \begin{bmatrix} 0 & (\gamma_2 - \gamma_1) t \\ 0 & 0 \end{bmatrix}.$$

This means that $y = x\begin{bmatrix}0 & b\\ 0 & 0\end{bmatrix}$, for any non-zero b does lie in a unique spread of $\Sigma_2 EH$. Suppose we have

$$\begin{aligned} y &= x\begin{bmatrix}0 & b\\ 0 & 0\end{bmatrix}\begin{bmatrix}w & \gamma_1 s\\ s & w\end{bmatrix}; w^2-\gamma_1 s^2 = 1\\ &= x\begin{bmatrix}bs & bw\\ 0 & 0\end{bmatrix}.\end{aligned}$$

Choose any z and e in K, at least one non-zero. If $z^2 - \gamma_1 e^2 = m = p^2$, then $(z/p)^2 - \gamma_1(e/p)^2 = 1$. Then letting $b = p$, $w = z/p$ and $s = e/p$, we see that we obtain

$$y = x\begin{bmatrix}e & z\\ 0 & 0\end{bmatrix},$$

for any e, z in K, not both zero, is in a unique spread of $\Sigma_2 EH_0$. Then adding $\begin{bmatrix}d & \gamma_1 c\\ c & d\end{bmatrix}$ (applying an elation), we obtain

$$y = x\begin{bmatrix}e+d & z+\gamma_1 c\\ c & d\end{bmatrix}.$$

Let $e + d = a$ and $z + \gamma_1 c = b$. Note that a, b, c, d are arbitrary except that if $a = d$, then $e = 0$ so that z is not zero and hence $b \neq \gamma_1 c$. This shows that every line of $\mathrm{PG}(3, K)$ is contained in a spread of $\Sigma_2 EH$. To ensure that no line is in two spreads of $\Sigma_2 EH$, we need only check that no line of $\Sigma_2 - R$ is in two spreads or equivalently that if

$$\left(y = x\begin{bmatrix}u & \gamma_2 t\\ t & u\end{bmatrix}\right) = \left(y = x\begin{bmatrix}u^* & \gamma_2 t^*\\ t^* & u^*\end{bmatrix}\right)\rho$$

for t nonzero and $\rho \in EH$ then $\Sigma_2\rho = \Sigma_2$. Therefore, the question is whether there exist matrices $\begin{bmatrix}r & \gamma_1 s\\ s & r\end{bmatrix}$ and $\begin{bmatrix}w & \gamma_1 k\\ k & w\end{bmatrix}$, such that $w^2 - \gamma_1 k^2 = 1$ and

$$\begin{bmatrix}u & \gamma_2 t\\ t & u\end{bmatrix}\begin{bmatrix}w & \gamma_1 k\\ k & w\end{bmatrix} = \left(\begin{bmatrix}u^* & \gamma_2 t^*\\ t^* & u^*\end{bmatrix} + \begin{bmatrix}r & \gamma_1 s\\ s & r\end{bmatrix}\right).$$

The elements that we are considering are now elements of Σ_1. The left hand side is

$$\begin{bmatrix}uw + \gamma_2 tk & \gamma_1 uk + \gamma_2 tw\\ tw + uk & \gamma_1 tk + uw\end{bmatrix}$$

and note that the matrix on the right hand side has equal $(1, 1)$ and $(2, 2)$-entries. Since $\gamma_1 \neq \gamma_2$ it then follows that $tk = 0$. Therefore, $k = 0$ then

$w = \pm 1$. Now the only possible elements of E that map one element of the regulus R_0 back into an element of R_0 requires that $s = 0$. But, now the element in question leaves Σ_2-invariant. This completes the proof of the theorem. □

Theorem 4.2. *Given any spread Σ_f, which is switchable (satisfies the switching property $f(t) - \gamma_1 t$ bijective and $f(t) - z^2 t$ bijective for all z).*

If Σ_2 is Pappian, then $\Sigma_f E$ switches with $\Sigma_2 E$.

Proof. We need only check that $\gamma_2 t - \gamma_1 t$ and $\gamma_2 t - z^2 t = -(-\gamma_2 + z^2)t$ define bijective functions, which is clear since $\gamma_2 \neq \gamma_1$ and $(-\gamma_2 + z^2) > 0$. □

5 The main theorem

We recall that our previous parallelism construction used the group EH and two Pappian spreads Σ_1 and Σ_2, where

$$E^{-} \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\rangle$$

is the subgroup that leaves Σ_2 invariant. We note that EH is transitive on the reguli of Σ_1 that share $x = 0$. Hence, H is transitive on the reguli of Σ_1 that share $x = 0, y = 0$. Let $\{h_i; i \in \lambda\}$ be a coset representation set for

$$\left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\rangle$$

in H. Using this coset representation set, we may now give our main theorem on elation switching.

Theorem 5.1. *Let K be any ordered field such the positive elements all have square roots. Let H denote the homology group with axis $y = 0$ and coaxis $x = 0$ of determinant 1 and let $\{h_i; i \in \lambda\}$ be a coset representation set for*

$$\left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\rangle$$

in H. For each $i \in \lambda$, choose any function f_i such that the functions $\rho_{i,z}$ and ϕ_i are bijective for each $z \in K$, where $f_i(0) = 0$ and

$$\rho_{i,z}(t) = f_i(t) - z^2 t \text{ and}$$
$$\phi_i(t) = f_i(t) - \gamma_1 t.$$

Let Σ_{f_i} denote the following spread:

$$\Sigma_f = \left\{ x = 0, y = x \begin{bmatrix} u & f_i(t) \\ t & u \end{bmatrix} \ \forall u, t \in K \right\}.$$

Then

$$\Sigma_1 \cup_{i \in \lambda} \Sigma^*_{f_i} E h_i$$

is a parallelism in $\mathsf{PG}(3, K)$.

Proof. We know that

$$\Sigma_1 \cup_{i \in \lambda} \Sigma^*_2 E \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\rangle h_i = \Sigma_1 \cup \Sigma^*_2 E H$$

is a parallelism by Theorem 4.1. But then we also have that

$$\Sigma_1 \cup_{i \in \lambda} \Sigma^*_2 E h_i$$

is a parallelism. By Theorem 4.2, we know that $\Sigma_f E$ switches with $\Sigma_2 E$, where f is any of the functions f_i. Choose any line ℓ of $\mathsf{PG}(3, K)$, then ℓ is in a unique spread of

$$\Sigma_1 \cup \Sigma^*_2 E H = \Sigma_1 \cup_{i \in \lambda} \Sigma^*_2 E h_i.$$

Either ℓ is a line of Σ_1 or there exists a unique h_j such that

$$\ell \in \Sigma^*_2 E h_i.$$

Assume that ℓ non-trivially intersects $x = 0$. Then, ℓ is a Baer subplane of a Rg, for $g \in EH$. We note that $(Rg)^*$ for $g \in EH$ is also in

$$\Sigma_1 \cup_{i \in \lambda} \Sigma^*_{f_i} E h_i.$$

Hence, we may assume that

$$\ell \in (\Sigma^*_2 - R) E h_i.$$

Assume that there is a line $m \in \Sigma_f E$ that does not intersect $x = 0$ that is in two spreads Σ_f and $\Sigma_f g$, for $g \in E$, then since $\Sigma_f E$ switches with $\Sigma_2 E$, we have a contradiction. Hence, every line of $\mathsf{PG}(3, K)$ not in Σ_1 is in a unique spread of $\Sigma^*_2 E h_i$ and hence is in a unique spread of $\Sigma^*_{f_i} E h_i$. This completes the proof of the theorem. □

6 Examples

Let F be any subfield of the field of real numbers and let $F^C = K$ denote the field of constructible numbers from F. If we let $\gamma_1 = -1$, choose any function f_i where $f_i(t) = \gamma_i t$, for $\gamma_i \neq -1$. Then it is clear that the following define bijective functions:

$$\begin{aligned}\rho_{i,z}(t) &= f_i(t) - z^2 t \text{ and} \\ \phi_i(t) &= f_i(t) - \gamma_1 t.\end{aligned}$$

Hence, we have the following result.

Theorem 6.1. *Let $K = F^C$, a field of constructible numbers from a subfield F of the field of real numbers. Let Σ_1 denote the Pappian spread*

$$\Sigma_1 = \left\{ x = 0, y = x \begin{bmatrix} u & -t \\ t & u \end{bmatrix} \ \forall u, t \in K \right\},$$

and let

$$E = \left\langle \begin{bmatrix} 1 & 0 & u & \gamma_1 t \\ 0 & 1 & t & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; u, t \in K \right\rangle,$$

$$H = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}; w = \begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix}; u^2 - \gamma_1 t^2 = 1 \right\rangle.$$

Let $\{h_i; i \in \lambda\}$ be a coset representation for

$$H^- = \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\rangle$$

in H. For each $i \in \lambda$, choose a negative number γ_i in K such that $\gamma_i \neq -1$ and finally let

$$\Sigma_i = \left\{ x = 0, y = x \begin{bmatrix} u & \gamma_i t \\ t & u \end{bmatrix} \ \forall u, t \in K \right\}.$$

Then

$$\Sigma_1 \cup_{i \in \lambda} \Sigma_i^* E h_i$$

is a parallelism in $\mathsf{PG}(3, K)$.

Remark 6.2. Let σ_i be an automorphism of K. We consider functions f_i such that $f_i(t) = \gamma_i t^{\sigma_i}$, where γ_i is a negative number. In order to obtain parallelisms in a manner similar to that of the previous theorem, we need to check that the following define bijective functions:

$$\begin{aligned} \rho_{i,z}(t) &= \gamma_i t^{\sigma} - z^2 t \text{ and} \\ \phi_i(t) &= \gamma_i t^{\sigma} - \gamma_1 t. \end{aligned}$$

Note that first set of functions $\rho_{i,z}$ are always

$$f_i(t) - z^2 t \ = \gamma_i t^{\sigma_i} - z^2 t = 0$$

if and only if $t = 0$ since $\gamma_i < 0$. Since the function is additive, we see $\rho_{i,z}$ is injective.

In general, the surjectivity of $\rho_{i,z}$ is not always guaranteed.

6.1 Examples over the reals

Let

$$f(t) = \left\{ \begin{array}{ll} \gamma_1 t - a^t + 1, & t \geq 0 \\ \gamma_1 t + b^{-t} - 1, & t < 0 \end{array} \right\},\ a, b \text{ both} > 1.$$

We see that $f(0) = 0$, f is continuous at all elements t of the reals and consider $f(t) - z^2 t$.

$$f(t) - z^2 t = \left\{ \begin{array}{ll} \gamma_1 t - a^t + 1 - z^2 t, & t \geq 0 \\ \gamma_1 t + b^{-t} - 1 - z^2 t, & t < 0 \end{array} \right\},\ a, b \text{ both} > 1.$$

Note that

$$\lim_{t \to \pm\infty} -f(t) = \pm\infty,$$

so that $f\,(t) - z^2 t$ is continuous and hence surjective. We note that the $\lim_{t \to 0} f(t) = 0$ and $f'(t)$ for t non-zero is

$$f'(t) - z^2 = \left\{ \begin{array}{ll} \gamma_1 - a^t \ln a - z^2, & t \geq 0 \\ -1 - b^{-t} \ln b - z^2, & t < 0 \end{array} \right\},\ a, b \text{ both} > 1,$$

which is never 0. Hence, $f(t) - z^2 t$ is bijective for each z. Now $f(t) - \gamma_1 t$

$$f(t) - \gamma_1 t = \left\{ \begin{array}{ll} -a^t + 1, & t \geq 0 \\ b^{-t} - 1, & t < 0 \end{array} \right\},\ a, b \text{ both} > 1,$$

and clearly this function is bijective. For example, assume that the function is not injective. Then the only questionable case is where $b^{-t}-1=1-a^s$, for $t<0$ and $s\geq 0$. But, then $a^s+b^{-t}=2$ and both a and $b>1$, then $a^s\geq 1$ and $b^{-t}>1$, a contradiction.

The more general version of the above set of examples is given in the following.

Theorem 6.3. *Let r be a strictly increasing continuous real function of the positive real numbers and let h be a strictly decreasing continuous real function on the negative real numbers. Choose any two real numbers a and $b>1$ (possibly equal). Then a function f defined as follows is switchable.*

$$f(t)=\left\{\begin{array}{ll}\gamma_1 t-a^{r(t)}+1, & t\geq 0\\ \gamma_1 t+b^{h(t)}-1, & t<0\end{array}\right\},\ a,b \textit{ both} >1,$$

$$\lim_{t\to 0^+} a^{r(t)}=1=\lim_{t\to 0^-} b^{h(t)}.$$

Proof. The function f is continuous on the field of real numbers, and we have $\lim_{t\to\infty}(1-a^{r(t)})=-\infty$ and $\lim_{t\to-\infty}(-1+b^{h(t)})=\infty$. This guarantees that $g(t)=f(t)-\gamma_1 t$ is a surjective function. For $s>0$, note that $g(s)>0$, since r is strictly increasing. Similarly for $t<0$, $h(t)<0$ since $\lim_{t\to 0^-}(b^{h(t)}-1)=0$ and h is strictly decreasing on the negative real numbers. Then our above argument shows that it is not possible that $b^{h(t)}+a^{r(s)}=2$, so that g is injective.

For z fixed, $f(t)-z^2t$ defines a continuous function.

$$f(t)-z^2t=\left\{\begin{array}{ll}\gamma_1 t-a^{r(t)}+1-z^2t, & t\geq 0\\ \gamma_1 t+b^{h(t)}-1-z^2t, & t<0\end{array}\right\},\ a,b \text{ both} >1.$$

If t is non-zero, we may take the derivative

$$f'(t)-z^2=\left\{\begin{array}{ll}\gamma_1-a^{r(t)}\ln a\ r'(t)-z^2, & t\geq 0\\ \gamma_1+b^{h(t)}\ln a\ h't)-z^2, & t<0\end{array}\right\},\ a,b \text{ both} >1.$$

It now follows exactly as in the previous example that we obtain bijective functions as required. □

Theorem 6.4. *Under the above assumptions, any such function f may be used to construct E-switchable spreads $\Sigma_f E=\Sigma_2 E$ and thus*

$$\Sigma_1\cup_{i\in\lambda}\Sigma^*_{f_i}Eh_i$$

is a parallelism for any set of choices of functions f_i.

7 The derive-underive parallelisms

We may now construct parallelisms from any parallelism of the type

$$\Sigma_1 \cup_{i\in\lambda} \Sigma^*_{f_i} Eh_i.$$

as follows: Choose an element eh_j of Eh_j, for some $j \in \lambda$. There is a regulus R_{eh_j} of Σ_1 that is derived in $\Sigma_{f_j} eh_j$ to construct $\Sigma^*_{f_j} eh_j$. Derive R_{eh_j} in Σ_1 to construct the Hall plane $\Sigma_1^{R^*_{eh_j}}$ and underive $R^*_{eh_j}$ in $\Sigma^*_{f_j} eh_j$ to construct $\Sigma_{f_j} eh_j$.

Theorem 7.1.

$$\Sigma_1^{R^*_{eh_j}} \cup_{i\in\lambda-\{j\}} \Sigma^*_{f_i} Eh_i \cup_{g\in E-\{e\}} \Sigma^*_{f_j} Eh_j \cup \Sigma_{f_j} eh_j$$

is a parallelism.

8 The variety of parallelisms

We note that although our original parallelism admits the group EH, the collineation group of certain of the constructed parallelisms can be made so that only E is a collineation group. Furthermore, certain of the derive-underive parallelisms can be found that do not admit a non-trivial collineation.

We note that to construct parallelisms over the reals of the type here considered, it is sufficient to constructs functions f_i with the conditions given in Theorem 5.1. We have also constructed 2^{χ_0} different functions f_i and therefore, we have also constructed 2^{χ_0} distinct parallelisms of the type

$$\Sigma_1 \cup_{i\in\lambda} \Sigma^*_{f_i} Eh_i.$$

Any isomorphism between two parallelisms

$$\Sigma_1 \cup_{i\in\lambda} \Sigma^*_{f_i} Eh_i$$

and

$$\Sigma_1 \cup_{i\in\lambda} \Sigma^*_{g_i} Eh_i$$

of this type necessarily is a collineation group of Σ_1, the Pappian plane over the field of complex numbers (assuming that none of the derived conical flock spreads are Pappian). Since our parallelisms admit E, it follows that any isomorphism must be a collineation of Σ_1 that leaves $x = 0, y = 0$ invariant and must permute the set of reguli of Σ_1 sharing $x = 0$, $y = 0$ and so must permute the $\Sigma^*_{f_i} h_i$. Hence, there is a collineation of Σ_1 that would map a function f_i

to a function g_j. But, this would mean that f_i and g_j are obtained as real linear combinations together with the automorphism of order 2. Clearly, it is easy to choose functions f_i and g_j that do not have this property and still produce parallelisms.

Theorem 8.1. *When K is the field of real numbers, there are 2^{χ_0} mutually non-isomorphic parallelisms of type*

$$\Sigma_1 \cup_{i\in\lambda} \Sigma^*_{f_i} Eh_i.$$

Remark 8.2. Similarly, if any of the derive-underive parallelisms are isomorphic, and if no derived conical flock spread can be a flock spread, any collineation would necessarily leave $\Sigma_{f_j} eh_j$ invariant and a conjugate would leave Σ_{f_j} invariant. Similar arguments then would show that there are 2^{χ_0} mutually non-isomorphic derive-underive parallelisms, none of which are isomorphic to any of the original parallelisms.

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Esteban Diaz
MATHEMATICS DEPT., UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242, USA

Norman L. Johnson
MATHEMATICS DEPT., UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242, USA
e-mail: njohnson@math.uiowa.edu

Alessandro Montinaro
DIPARTIMENTO DI MATEMATICA UNIVERSITÀ DI LECCE VIA ARNESANO, 73100 LECCE, ITALY
e-mail: alessandro.montinaro@unile.it