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Deletions, extensions, and reductions of elliptic semiplanes

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Abstract

We present three constructions which transform some symmetric configuration $\mathcal{K}$ of type n_k into new symmetric configurations of types $(n+1)_k$, or n_{k-1}, or $((\lambda-1)\mu)_{k-1}$ if $n = \lambda\mu$. Applying them to Desarguesian elliptic semiplanes, an infinite family of new configurations comes into being, whose types fill large gaps in the parameter spectrum of symmetric configurations.

Keywords: configurations, elliptic semiplanes, 1-factors, Martinetti extensions

MSC 2000: 05B30

1 The parameter spectrum of configurations of type n_k

For notions from incidence geometry and graph theory, we refer to [10] and [7], respectively.

A *(tactical) configuration of type* (n_r, b_k) is a finite incidence structure consisting of a set of n points and a set of b lines such that (i) each line is incident with exactly k points and each point is incident with exactly r lines, (ii) two distinct points are incident with at most one line. If $n = b$ (or equivalently $r = k$), the configuration is called *symmetric* and its type is indicated by the symbol n_k.

The *deficiency* of a symmetric configuration $\mathcal{C}$ is $d := n - k^2 + k - 1$. The deficiency is zero if and only if $\mathcal{C}$ is a finite projective plane.

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Symmetric configurations of a given type n_k may or may not exist, and we call the type n_k *realizable* or *unrealizable*, accordingly.

Let Σ be the set of realizable types n_k. We refer to Σ as the *parameter spectrum of symmetric configurations*. The parameter spectrum is often displayed by means of the parameters d and k, see Table 1, which gives some more information [14, 20]: in row k, the entries n, *(n)* and **(n)** indicate types n_k for which the answer to the existence problem of a configuration is positive, undecided and negative, respectively (cf. [14, 18, 19, 3, 13, 22]).

$k\backslash d$	0	1	2	3	4	5	6	7	8	9
3 :	7	8	9	10	11	12	13	14	15	16
4 :	13	14	15	16	17	18	19	20	21	22
5 :	21	**(22)**	23	24	25	26	27	28	29	30
6 :	31	**(32)**	**(33)**	34	35	36	37	38	39	40
7 :	**(43)**	**(44)**	45	*(46)*	*(47)*	48	49	50	51	52
8 :	57	**(58)**	*(59)*	*(60)*	*(61)*	*(62)*	63	64	65	66
9 :	73	**(74)**	*(75)*	*(76)*	*(77)*	78	*(79)*	80	81	82
10 :	91	**(92)**	*(93)*	*(94)*	*(95)*	*(96)*	*(97)*	98	*(99)*	*(100)*
11 :	**(111)**	*(112)*	*(113)*	*(114)*	*(115)*	*(116)*	*(117)*	*(118)*	*(119)*	120
12 :	133	**(134)**	135	*(136)*	*(137)*	*(138)*	*(139)*	*(140)*	*(141)*	*(142)*

Table 1: The parameter spectrum of symmetric configurations

In the lower left triangle of Σ, the existence of instances is highly in doubt. As far as they exist, *elliptic semiplanes* dominate the region. Recall that an *elliptic semiplane of order* ν is a configuration of type $n_{\nu+1}$ satisfying the following axiom of parallels: given a non-incident point-line pair (p,l), there exists at most one line l' through p parallel to l (i.e. l and l' are not concurrent) and at most one point p' on l parallel to p (i.e. p and p' are not collinear). Dembowski [10] provided a classification of elliptic semiplanes in types called O, C, L, D and B, which we will use in the sequel.

Consider any finite projective plane of order n. An *anti-flag* is a non-incident point-line pair (p,l). The *pencil (of lines) through a point* is the set of lines that are incident with that point. By removing from a projective plane $\mathcal{P}$ an anti-flag (p,l) as well as the pencil through p and all the points on l, we obtain an elliptic semiplane $\mathcal{L}$ of type L [10] which is a configuration of type $(q^2-1)_q$ and deficiency $q-2$. Since projective planes of order q exist for each prime-power q, this construction furnishes an infinite family of configurations of type $(q^2-1)_q$. If $\mathcal{P} = \mathsf{PG}(2,q)$ is Desarguesian, the corresponding Desarguesian semiplane of type L will be denoted by $\mathcal{L}_q$. We call them the *anti-flag* examples. They lie in the second upper diagonal of Σ (called *anti-flag diagonal*).

A *flag* of a projective plane of order n is a point-line pair (p,l) with $p \mid l$. By removing from a projective plane $\mathcal{P}$ a flag (p,l) as well as the pencil through p

and all the points on l, we obtain an elliptic semiplane $\mathcal{C}$ of type C [10] which is a configuration of type $(q^2)_q$ with deficiency $q-1$. This construction furnishes an infinite family of configurations of type $(q^2)_q$. If $\mathcal{P} = \mathsf{PG}(2,q)$ is Desarguesian, the corresponding Desarguesian semiplane of type C will be denoted by $\mathcal{C}_q$. We call them the *flag* examples. They lie in the third upper diagonal of Σ (called *flag diagonal*).

There is a third series of elliptic semiplanes furnishing an instance for every $n = q^4 - q$, namely those of type D (cf. [10]), denoted by $\mathcal{D}_{q^2}$ and obtained as complements of Baer subplanes in $\mathsf{PG}(2,q^2)$, the first four being configurations of types 14_4, 78_9, 252_{16}, and 620_{25}.

For the region above the flag diagonal existence results are known for many types (cf. e.g. [14, 20, 23]), due to the following construction: a *Golomb ruler* of order k is a set of k positive integers $(\alpha_1, \ldots, \alpha_k)$ such that all the differences $|\alpha_i - \alpha_j|$ are pairwise distinct for $i,j = 1,\ldots,k$ with $i \neq j$. Its *length* is the largest integer α_i. A Golomb ruler is *optimal* if it has the smallest length among Golomb rulers of order k. Let l_k be the length of an optimal Golomb ruler of order k. In [14] Gropp pointed out that for each $k \geq 3$ there exists an integer $n_0(k)$ such that there is a configuration n_k for all $n \geq n_0(k)$, namely $n_0(k) := 2l_k + 1$ where l_k is the length of an optimal Golomb ruler of order k. By a *Golomb configuration* we mean a configuration of type $(2l_k + 1)_k$ coming from Gropp's construction. So far, values for the lengths of optimal Golomb rules have been computed for $4 \leq k \leq 25$, cf. e.g. [6, 24] and they give rise to Golomb configurations 7_3, 13_4, 23_5, 35_6, 51_7, 69_8, 89_9, 111_{10}, 145_{11}, 171_{12}, 213_{13}, 255_{14}, 303_{15}, 355_{16}, 399_{17}, 433_{18}, 493_{19}, 567_{20}, 667_{21}, 713_{22}, 745_{23}, 851_{24}, and 961_{25}. Denote by $d_G(k)$ the deficiency of a Golomb configuration of type $(2l_k + 1)_k$. Hence, for each $d(k) \geq d_G(k)$, there exists a configuration with parameters $(k, d(k))$.

In Figure 1, page 142, we exhibit the region Δ of Σ bounded by the anti-flag diagonal below and the Golomb configurations above, for which the existence of symmetric configurations is unknown.

In this paper, we introduce three operations, namely 1-*factor deletions* (Section 2), *Martinetti extensions* (Section 3), and *reductions of polysymmetric configurations* (Section 4), that allow to construct new configurations. In particular, as our main result, we prove the existence of three infinite classes of symmetric configurations

$$
\begin{array}{lll}
\mathcal{C}_q^{(\alpha R)(\beta M)(\gamma F)} & \text{of type} & (q^2 - \alpha q + \beta)_{q-\alpha-\gamma}\,, \\
\mathcal{L}_q^{(\alpha R)(\beta M)(\gamma F)} & \text{of type} & ((q+1-\alpha)(q-1) + \beta)_{q-\alpha-\gamma}\,, \\
\mathcal{D}_q^{(\alpha R)(\gamma F)} & \text{of type} & (q^4 - \alpha(q^2+q+1))_{q^2+1-\alpha-\gamma}\,,
\end{array}
$$

for feasible values of α, β, and γ (cf. Theorems 6.2, 6.3, 6.4).

As a consequence, we prove that at least 1752 (out of a total number of 2176)

types n_k with $(k^2)_k \leq n_k < (2l_k + 1)_k$ and $7 \leq k \leq 25$, whose deficiencies lie in the region Δ indicated in Table 2, are realizable (Section 7).

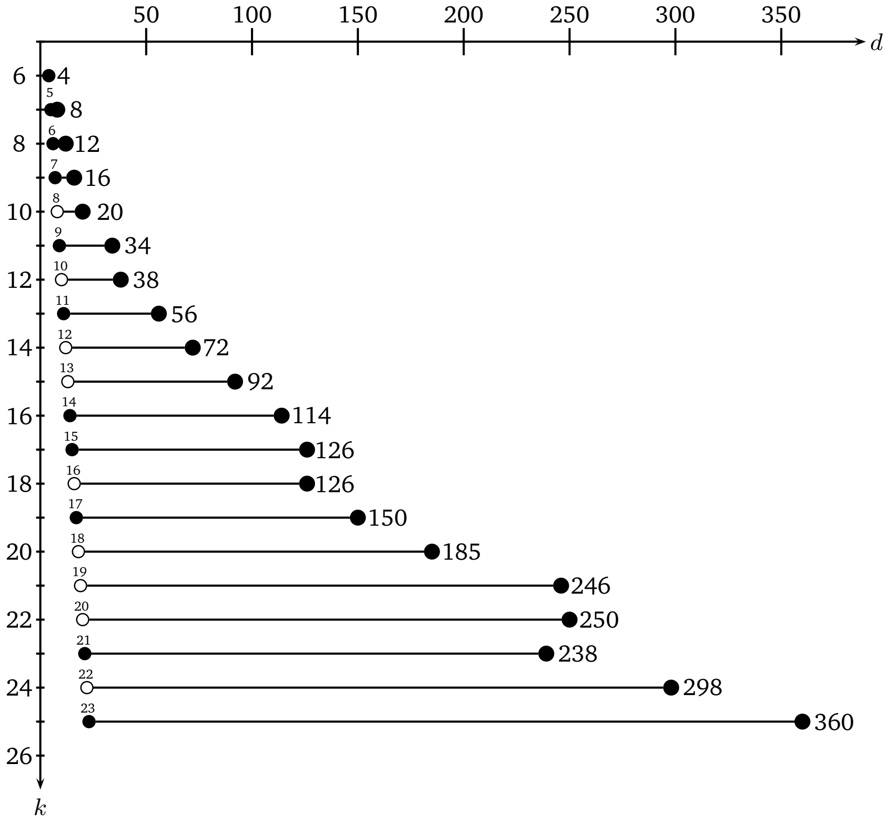

Figure 1: Small numbers indicate the deficiencies of configurations in the flag, diagonal and white dots the non-existence of such configurations. Big numbers indicate the deficiencies of Golomb configurations.

2 1-Factor deletions in Levi graphs

Let $\mathcal{K} = (P, L, |)$ be a configuration of type n_k. The *Levi graph* (or *incidence graph*) $\Lambda(\mathcal{K})$ of $\mathcal{K}$ has vertex set $V(\Lambda(\mathcal{K})) = P \cup L$ such that two vertices $p \in P$ and $l \in L$ are adjacent if and only if $p \mid l$ (cf. [8, 15]). It is well known that $\Lambda(\mathcal{K})$ is a bipartite k-regular graph of girth ≥ 6 on $2n$ vertices. Vice versa, each such graph determines either a self-dual configuration of type n_k or a pair of non-isomorphic configurations, dual to each other.

A corollary to the famous Marriage Theorem by Ph. Hall [16] states: *every k-regular bipartite graph Λ is 1-factorable* (cf. e.g. [17, Theorem 3.2]). This implies that the edge set $E(\Lambda)$ can be partitioned into a union of k pairwise disjoint 1-factors F_i, $i = 1, \ldots, k$.

Let Λ be the Levi graph of some configuration $\mathcal{K}$ of type n_k and choose a 1-factor F_i of Λ, for some $i \in \{1, \ldots, k\}$. Let $\Lambda^{(1F)}$ be the subgraph of Λ with vertex set $V(\Lambda^{(1F)}) = V(\Lambda)$ and edge set $E(\Lambda^{(1F)}) = E(\Lambda) \backslash E(F_i)$. Obviously, $\Lambda^{(1F)}$ is a $(k-1)$-regular bipartite graph on $2n$ vertices, which can be seen as the Levi graph of some configuration of type n_{k-1}. Since we are only interested in its type n_{k-1} being realizable, any such configuration will be denoted by $\mathcal{K}^{(1F)}$ and referred to as a configuration *obtained from $\mathcal{K}$ by a 1-factor deletion*.

This construction can be reiterated ν times for some $\nu \in \{1, \ldots, k-3\}$, for pairwise distinct 1-factors belonging to a fixed 1-factorisation of Λ. We denote the resulting configuration by $\mathcal{K}^{(\nu F)}$.

If we embed the parameter spectrum of symmetric configurations Σ into $\mathbb{R}^2$, the realizable types n_k, n_{k-1}, …, n_3 lie on a parabola since, for fixed n and k, the deficiency of the type $n_{k-\nu}$ seen as a function of $\nu = 0, \ldots, k-3$ reads

$$d(k-\nu) \ = \ -\nu^2 + (2k-1)\nu + d(k)$$

where $d(k) = n - k^2 + k - 1$ is the deficiency of $\mathcal{K}$ and does not depend on ν. The vertex of the parabola is the point $(\frac{1}{2}, (k-\frac{1}{2})^2 + d(k))$, which lies outside of Σ. Hence distinct types out of $\{n_k, n_{k-1}, \ldots, n_3\}$ have distinct deficiencies.

3 Parallel flags in configurations and Martinetti extensions

Two distinct points (lines) of a configuration $\mathcal{K} = (P, L, |)$ are said to be *parallel* if there is no line (point) incident with both of them. We extend this concept and call two flags (p_1, l_1) and (p_2, l_2), such that $p_1 \neq p_2$ and $l_1 \neq l_2$, *parallel* if both $\{p_1, p_2\}$ and $\{l_1, l_2\}$ make up pairs of parallel elements. A family of pairwise parallel flags in a configuration of type n_k is said to be a *hyperpencil* if it has cardinality $k-1$.

Definition 3.1. Let $\mathcal{K} = (P, L, |)$ be a configuration of type n_k and

$$\mathcal{H} = \{(p_i, l_i) : p_i \mid l_i \text{ for } i = 1, \ldots, k-1\}$$

a hyperpencil of parallel flags in $\mathcal{K}$. Then the *Martinetti extension* $\mathcal{K}_{\mathcal{H}}$ of $\mathcal{K}$ is the incidence structure obtained from $\mathcal{K}$ by

(i) deleting the incidences $p_i \mid l_i$, for $i = 1, \ldots, k-1$,

(ii) adding a new flag, say $(p_{\mathcal{H}}, l_{\mathcal{H}})$,

(iii) adding the new incidences $p_i \mid l_{\mathcal{H}}$ and $p_{\mathcal{H}} \mid l_i$ for $i = 1, \ldots, k-1$.

Remark 3.2. The case $k = 3$ has already been pointed out by Martinetti [21].

The following is a special case of [11, Proposition 2.5].

Proposition 3.3. *If $\mathcal{K}$ is a configuration of type n_k, then $\mathcal{K}_{\mathcal{H}}$ is a configurations of type $(n+1)_k$.* □

Given a configuration $\mathcal{K}$ of type n_k with a suitable hyperpencil of parallel flags, we are only interested in the existence of Martinetti extensions of $\mathcal{K}$ as configurations having realizable type $(n+1)_k$. Therefore any such configuration will be denoted by $\mathcal{K}^{(1M)}$.

Next we investigate the possibilities to iterate this construction.

Definition 3.4. Let $\mathcal{K}$ be a configuration of type n_k. Two hyperpencils

$$\mathcal{F} = \{(r_i,\, l_i) \,:\, r_i \mid l_i \text{ for } i = 1, \ldots, k-1\} \text{ and}$$
$$\mathcal{G} = \{(s_i,\, m_i) \,:\, s_i \mid m_i \text{ for } i = 1, \ldots, k-1\}$$

of parallel flags are *disjoint* if all involved elements r_i, s_i and l_i, m_i are distinct in pairs.

Corollary 3.5. *Let $\mathcal{K}$ be a configuration of type n_k and $\mathcal{F}, \mathcal{G}$ be two disjoint hyperpencils of parallel flags. Then $(\mathcal{K}_{\mathcal{F}})_{\mathcal{G}}$ is isomorphic to $(\mathcal{K}_{\mathcal{G}})_{\mathcal{F}}$ and is of type $(n+2)_k$.*

Proof. It is enough to apply [11, Proposition 2.5]. □

Accordingly, any configuration obtained from a configuration $\mathcal{K}$ of type n_k by ν Martinetti extensions will be denoted by $\mathcal{K}^{(\nu M)}$.

4 Reducing polysymmetric configurations

Let A be a square $(0,1)$-matrix. We call A *doubly k-stochastic* if there are k entries 1 in each row and column. Recall that, with each permutation π in the symmetric group $\mathcal{S}_\mu$, we can associate its *permutation matrix* $P_\pi = (p_{ij})_{1 \le i,j \le \mu}$ which is defined by $p_{ij} = 1$ if $\pi(i) = j$, and $p_{ij} = 0$ otherwise. Distinct permutations $\pi, \rho \in S_\mu$ (as well as the corresponding permutation matrices P_π and

P_ρ) are *disjoint* if $\pi(i) \neq \rho(i)$, for all $i = 1, \ldots, \mu$. A doubly k-stochastic $(0,1)$-matrix is called (λ, μ)-*polysymmetric* if it admits a block matrix structure with λ square blocks in which each block is either zero or a sum of pairwise disjoint permutation matrices from $\mathcal{S}_\mu$.

Let $\mathcal{K}$ be a configuration. Fix a labelling for the points and lines of $\mathcal{K}$ and consider the *incidence matrix* $H_\mathcal{K}$ of $\mathcal{K}$ (cf. e.g. [10, pp. 17–20]): there is an entry 1 or 0 in position (i,j) of $H_\mathcal{K}$ if and only if the point p_i and the line l_j are incident or non-incident, respectively. A configuration $\mathcal{K}$ of type $(\lambda\mu)_k$ is said to be *polysymmetric* if it admits an incidence matrix $H_\mathcal{K}$ which is (λ, μ)-*polysymmetric*. Obviously, $H_\mathcal{K}$ is doubly k-stochastic.

A concise representation for the incidence matrices of polysymmetric configurations can be obtained by the following Definition 4.1 and Proposition 4.2 which are generalizations of notions presented in [13]:

Definition 4.1. (i) A subset $S \subseteq \mathcal{S}_\mu$ is *admissible* if its elements are pairwise disjoint. For $1 \leq i, j \leq \lambda$, let $S_{i,j}$ be a collection of admissible subsets of $\mathcal{S}_\mu$ such that

$$\sum_{i=1}^{\lambda} |S_{i,j}| = k = \sum_{j=1}^{\lambda} |S_{i,j}|$$

for some k. Then the array $\mathcal{S} = (S_{i,j})$ is called $\mathcal{S}_\mu$-*scheme of rank k and order λ*. An $\mathcal{S}_\mu$-scheme is called *quasi-simple of excess* ϵ if for each $1 \leq i \leq \lambda$ there is exactly one $j_i \in \{1, \ldots, \lambda\}$ such that $|S_{i,j_i}| = \epsilon = k - \lambda + 1$, and $|S_{i,j}| = 1$ for all $j \in \{1, \ldots, \lambda\} \setminus \{j_i\}$.

(ii) For $S \subseteq \mathcal{S}_\mu$, we define $\mathcal{P}(S) = \sum_{\pi \in S} P_\pi$. If $S = \emptyset$ then $\mathcal{P}(S)$ is the zero matrix of order μ. If $\mathcal{S} = (S_{i,j})$ is an $\mathcal{S}_\mu$-scheme, then the *blow up* of $\mathcal{S}$ is the block matrix $A(\mathcal{S}) = (\mathcal{P}(S_{i,j}))$.

Proposition 4.2. *Each doubly k-stochastic (λ, μ)-polysymmetric $(0,1)$-matrix A can be represented by an $\mathcal{S}_\mu$-scheme $\mathcal{S} = (S_{i,j})$ of rank k and order λ and, conversely, each $\mathcal{S}_\mu$-scheme $\mathcal{S}$ of rank k and order λ induces a doubly k-stochastic (λ, μ)-polysymmetric $(0,1)$-matrix $A(\mathcal{S})$.* □

Consider the cyclic subgroup of $\mathcal{S}_\mu$ generated by the permutation $(1\,2\,\ldots\,\mu)$. We can identify this subgroup with the group $\mathbb{Z}_\mu$ of integers modulo μ, using the monomorphism

$$i \in \mathbb{Z}_\mu \quad \mapsto \quad (1 \quad 2 \quad \ldots \quad \mu)^i \ \in \ \mathcal{S}_\mu.$$

Thus, an $\mathcal{S}_\mu$-scheme all of whose entries belong to the subgroup generated by the permutation $(1\,2\,\ldots\,\mu)$ can be rewritten with entries in $\mathbb{Z}_\mu$ and will be called a $\mathbb{Z}_\mu$-*scheme*. This definition of a $\mathbb{Z}_\mu$-*scheme* is equivalent to the one given in [13].

Definition 4.3. Let $\mathcal{S} = (S_{i,j})$ be a quasi-simple $\mathcal{S}_\mu$-scheme of order λ, rank k, and excess ϵ. If $\epsilon = 1$ choose any (i,j) with $1 \leq i,j \leq \lambda$. If $\epsilon \neq 1$ choose (i,j) such that either $S_{i,j} = \emptyset$ or $|S_{i,j}| > 1$. A *reduced $\mathcal{S}_\mu$-scheme* $\mathcal{S}^{(i,j)}$ is an S_μ-scheme of order $\lambda - 1$, rank $k-1$, and excess ϵ obtained from $\mathcal{S}$ by deleting the i^{th} row and the j^{th} column.

Proposition 4.4. *Let $\mathcal{S}$ be a quasi-simple $\mathcal{S}_\mu$-scheme of order λ, rank k, and excess ϵ such that its blow up represents a polysymmetric configuration. Then the blow-up of the reduced $\mathcal{S}_\mu$-scheme $\mathcal{S}^{(i,j)}$ is a polysymmetric configuration of type $((\lambda-1)\mu)_{k-1}$.*

Proof. This follows from Proposition 4.2 and Definition 4.3. □

Hence, by Proposition 4.4, the process of reducing quasi-simple $\mathcal{S}_\mu$-schemes can be iterated. In particular, if $\mathcal{S}$ represents a polysymmetric configuration $\mathcal{K}$ of type $(\lambda\mu)_k$, iterated applications of Proposition 4.4 gives rise to a series of configurations of realizable types $((\lambda-\nu)\mu)_{k-\nu}$ for $\nu = 1, \dots, \lambda-1$. We denote any such configuration by $\mathcal{K}^{(\nu R)}$, since we are only interested in the reduced configurations as instances having realizable types $((\lambda-\nu)\mu)_{k-\nu}$.

If we embed the parameter spectrum of symmetric configurations Σ in $\mathbb{R}^2$, the reduced polysymmetric configurations lie on a parabola. In fact, for fixed λ, μ, and k, the deficiency of the type $((\lambda-\nu)\mu)_{k-\nu}$ as a function of $\nu = 0, \dots, \lambda-1$ reads

$$d(k-\nu) \;=\; -\nu^2 + (2k-\mu-1)\nu + d(k)$$

where $d(k) = \lambda\mu - k^2 + k - 1$ is the deficiency of $\mathcal{K}$ and does not depend on ν. The vertex of this parabola is the point $\left(\frac{\mu+1}{2}, \frac{(2k-\mu-1)^2}{2} + d(k)\right)$ that lies inside Σ. Hence configurations $\mathcal{K}^{(\nu R)}$ with distinct types may have one and the same deficiency.

5 Desarguesian elliptic semiplanes

In [1] and [2] we have found concise representations for incidence matrices of elliptic semiplanes of types C, L and D, for which in this section we describe how such representations can be read as $\mathcal{S}_q$, $\mathcal{S}_{q-1}$ and $\mathbb{Z}_{q^2+q+1}$-schemes, respectively.

Notation 5.1. For elliptic semiplanes of types C and L we need modified multiplication and addition tables for $\mathsf{GF}(q)$.

Let q be a fixed prime power and label the elements $g_1, \dots, g_q$ of $\mathsf{GF}(q)$ in such a way that $g_1 = 1$ and $g_q = 0$. Let M'_q be the matrix of order $q-1$

which represents the multiplication table of the multiplicative group $\mathsf{GF}(q)^* = \mathsf{GF}(q) \setminus \{0\}$:

$$M'_q := (m_{i,j}) \quad \text{with} \quad m_{i,j} := g_i g_j \quad \text{for} \quad i,j = 1,\dots,q-1.$$

Similarly, let A'_q be the matrix of order q which represents the difference table of the additive group $\mathsf{GF}(q)^+$:

$$A'_q := (a_{i,j}) \quad \text{with} \quad a_{i,j} := -g_i + g_j \quad \text{for} \quad i,j = 1,\dots,q.$$

Finally, define the matrices

$$M_q := \left(\begin{array}{c|c} & 0 \\ M'_q & \vdots \\ & 0 \\ \hline 0\dots0 & 0 \end{array}\right) \quad \text{and} \quad A_q := \left(\begin{array}{c|c} & 1 \\ A'_q & \vdots \\ & 1 \\ \hline 1\dots1 & 0 \end{array}\right)$$

of orders q and $q+1$, respectively.

With each element g of $\mathsf{GF}(q)$, we associate an element $\pi_g \in \mathcal{S}_q$: let

$$(P_g^+)_{i,j} := \begin{cases} 1 & \text{if } a_{i,j} = g \text{ in } A'_q \\ 0 & \text{otherwise} \end{cases}$$

be the *position matrix of the element g in A'_q*. Since P_g^+ is a permutation matrix of order q, there exists $\pi_g \in \mathcal{S}_q$ such that $P_g^+ = P_{\pi_g}$.

Similarly, with each element g of $\mathsf{GF}(q) \setminus \{0\}$, we associate an element $\rho_g \in \mathcal{S}_{q-1}$ as follows: let

$$(P_g^*)_{i,j} := \begin{cases} 1 & \text{if } m_{i,j} = g \text{ in } M'_q \\ 0 & \text{otherwise} \end{cases}$$

be the *position matrix of the element g in M'_q*. Again, P_g^* is a permutation matrix of order $q-1$, and hence there exists $\rho_g \in \mathcal{S}_{q-1}$ such that $P_g^* = P_{\rho_g}$.

Substituting each entry g by $\{\pi_g\}$, the matrix M_q over $\mathsf{GF}(q)$ becomes a quasi-simple $\mathcal{S}_q$-scheme $\mathcal{M}_q^+$, of rank q, order q, and excess 1. Similarly, substituting each entry $g \neq 0$ by $\{\rho_g\}$, and each 0 by $\emptyset$, the matrix A_q over $\mathsf{GF}(q)$ becomes a quasi-simple $\mathcal{S}_{q-1}$-scheme $\mathcal{A}_q^*$, of rank q, order $q+1$ and excess 0.

The following two propositions have been proved, with a slightly different notation, in [1] and [2].

Proposition 5.2. *The blow up of $\mathcal{M}_q^+$ is a polysymmetric incidence matrix for the Desarguesian elliptic semiplane $\mathcal{C}_q$ of type C, and $\mathcal{M}_q^+$ is a quasi-simple $\mathcal{S}_q$-scheme of rank q, order q, and excess 1, representing $\mathcal{C}_q$.* □

Proposition 5.3. *The blow up of $\mathcal{A}_q^*$ is a polysymmetric incidence matrix for the Desarguesian elliptic semiplane $\mathcal{L}_q$ of type L, and $\mathcal{A}_q^*$ is a quasi-simple $\mathcal{S}_{q-1}$-scheme of rank q, order $q+1$ and excess 0, representing $\mathcal{L}_q$.* □

Notation 5.4. We need a representation for Desarguesian projective planes $\mathsf{PG}(2, q^2)$ in terms of a $\mathbb{Z}_{q^2+q+1}$-scheme. To this purpose recall the following:

(i) each finite Desarguesian projective plane $\mathsf{PG}(2, q^2)$ admits a tactical decomposition into q^2-q+1 copies of a Baer subplane isomorphic to $\mathsf{PG}(2, q)$;

(ii) each finite Desarguesian projective plane of order q is cyclic and can be represented by a perfect difference set $D_q = \{s_0, \ldots, s_q\}$ modulo q^2+q+1 [5], which gives rise to a $\mathbb{Z}_{q^2+q+1}$-scheme of rank $q+1$, order 1 and excess $q+1$, namely the scheme consisting of the unique entry $\{s_0, \ldots, s_q\}$ of cardinality $q+1$.

Recall also that a *circulant matrix* $\mathrm{Circ}(c_0, c_1, \ldots, c_{q-1})$ is the matrix $C = (c_{i,j})$, of order q, where $c_{i,j} = c_{j-i}$ (indices taken modulo q) [9].

For $q = 2, \ldots, 5$ consider the following perfect difference sets:

$$D_2 = \{0, 1, 3\}; \quad D_3 = \{0, 1, 4, 6\}; \quad D_4 = \{0, 1, 4, 14, 16\};$$
$$D_5 = \{0, 1, 6, 18, 22, 29\}.$$

In these four cases, by a computer search we have found that the incidence matrices of $\mathsf{PG}(2, q^2)$ admit a concise representation as a circulant quasi-simple $\mathbb{Z}_{q^2+q+1}$-scheme of order q^2-q+1, rank q^2+1 and excess $q+1$:

$$C_2 = \mathrm{Circ}(D_2, 6, 6); \quad C_3 = \mathrm{Circ}(D_3, 12, 8, 11, 11, 8, 12);$$
$$C_4 = \mathrm{Circ}(D_4, 3, 20, 6, 12, 17, 5, 5, 17, 12, 6, 20, 3);$$
$$C_5 = \mathrm{Circ}(D_5, 4, 5, 24, 13, 21, 28, 23, 7, 17, 26, 26, 17, 7, 23, 28, 21, 13, 24, 5, 4).$$

Remark 5.5. The perfect difference sets in the main diagonal of these $\mathbb{Z}_{q^2+q+1}$-schemes highlight a decomposition of $\mathsf{PG}(2, q^2)$ into Baer subplanes.

6 Families of configurations obtained from elliptic semiplanes

In this section, we obtain new symmetric configurations by applying reductions of polysymmetric configurations, Martinetti extensions, and 1-factor deletions to Desarguesian elliptic semiplanes.

Reductions of schemes and 1-factor deletions can always be performed (within the obvious arithmetic bounds), while Martinetti extensions depend on the existence of parallel flags. The next lemma shows when a symmetric configuration, represented by a quasi-simple scheme, does have a set of parallel flags and how to choose such a set.

Lemma 6.1. *Let $\mathcal{C}$ be an $(mq)_k$ configuration whose incidence matrix is the blow-up $A(\mathcal{M})$ of a quasi-simple $\mathcal{S}_q$-scheme $\mathcal{M} = (M_{i,j})$, of order q, rank q and excess $\epsilon \le 1$. Label the points and lines of $\mathcal{C}$, $p_1, \ldots, p_{mq}$ and $l_1, \ldots, l_{mq}$, with respect to the rows and columns of $A(\mathcal{M})$. Let $M_{i,j} = \sigma \in \mathcal{S}_q$, for some $i, j \in \{1, \ldots, m\}$, and consider the set*

$$\mathcal{F}_\sigma = \big\{(p_{(i-1)q+r}\,,\, l_{(j-1)q+\sigma(r)}) \,:\, r = 1, \ldots, q\big\}.$$

Then the set $\mathcal{F}_\sigma$ is a set of q pairwise parallel flags in $\mathcal{C}$.

Proof. Let $M_{i,j} = \sigma \in \mathcal{S}_q$ be the entry (i,j) of $\mathcal{M}$. By definition of $A(\mathcal{M})$, the entry $A(\mathcal{M})_{((i-1)q+r,(j-1)q+\sigma(r))} = 1$ for each $r = 1, \ldots, q$. Therefore $\mathcal{F}_\sigma$ is indeed a set of q flags. Now we show that they are pairwise parallel. Suppose that for some $s, t \in \{1, \ldots, q\}$ with $s \ne t$, the points $p_{(i-1)q+s}$ and $p_{(i-1)q+t}$ were joined by some line, say l_u, for some $u \in \{1, \ldots, mq\}$. Then there would be an entry 1 in positions $((i-1)q+s, u)$ and $((i-1)q+t, u)$ of $A(\mathcal{M})$; by the Euclidean algorithm $u = xq + u'$ with $u' < q$; put $y := x$ and $v := q$ if $u' = 0$, as well as $y := x - 1$ and $v := u'$ otherwise; then the blow-up of $M_{i,y}$ would have two entries 1 in its v^{th} column and no longer be only just one permutation matrix, a contradiction, since $\mathcal{M}$ is quasi-simple of excess $\epsilon \le 1$. Analogously it can be shown that any two distinct lines $l_{(j-1)q+1}, \ldots, l_{(j-1)q+q}$ never meet. □

Theorem 6.2. *Let $\mathcal{C}_q$ be a Desarguesian elliptic semiplane of type C. Then, for each $\alpha \in \{0, \ldots, q-3\}$, $\beta \in \{0, \ldots, q-\alpha\}$, and $\gamma \in \{0, \ldots, q-\alpha-3\}$, there exists a configuration $\mathcal{C}_q^{(\alpha R)(\beta M)(\gamma F)}$ of type $(q^2 - \alpha q + \beta)_{q-\alpha-\gamma}$.*

Proof. By Proposition 5.2, $\mathcal{M} := \mathcal{M}_q^+$ is a quasi-simple $\mathcal{S}_q$-scheme of order q, rank q, and excess 1, representing an incidence matrix of $\mathcal{C}_q$. Let $\mathcal{M}_\alpha$ be the quasi-simple $\mathcal{S}_q$-scheme of excess 1 obtained by deleting α rows and columns of $\mathcal{M}$. Then, by Proposition 4.4, the configuration $\mathcal{C}_q^{(\alpha R)}$ whose incidence matrix is the blow-up of $\mathcal{M}_\alpha$ has type $((q-\alpha)q)_{q-\alpha}$. Since we deal only with configurations of type n_k with $k \ge 3$, the range of α is bounded by $q - 3$.

Next, we show that Martinetti extensions can be performed on the configuration $\mathcal{C}_q^{(\alpha R)}$. We choose β entries $\sigma_1, \ldots, \sigma_\beta$ in the quasi-simple $\mathcal{S}_q$-scheme $\mathcal{M}_\alpha$ of excess 1, no two of them in the same row or column. By Lemma 6.1, each set $\mathcal{F}_{\sigma_i}$ is a set of q pairwise parallel flags in $\mathcal{C}_q^{(\alpha R)}$. Thus choosing, say the first $q - \alpha - 1$ flags

$$\big\{(p_{(i-1)q+m}\,,\, l_{(j-1)q+\sigma(m)}) \,:\, m = 1, \ldots, q-\alpha-1\big\}$$

of $\mathcal{F}_{\sigma_i}$ we get a hyperpencil of parallel flags in $\mathcal{C}_q^{(\alpha R)}$, and by Definition 3.1 we may perform the Martinetti extension on $\mathcal{C}_q^{(\alpha R)}$. The way in which we have chosen the β entries in $\mathcal{M}_\alpha$ guarantees, by Definition 3.4 and Corollary 3.5 that we can simultaneously perform $\beta \leq q-\alpha$ such Martinetti extensions. Clearly, the resulting configuration $\mathcal{C}_q^{(\alpha R)(\beta M)}$ has type $((q-\alpha)q+\beta)_{q-\alpha}$. Finally, we apply a finite number γ of 1-factor deletions on $\mathcal{C}_q^{(\alpha R)(\beta M)}$, for $\gamma \in \{0,\dots,q-\alpha-3\}$. The resulting configuration $\mathcal{C}_q^{(\alpha R)(\beta M)(\gamma F)}$ has type $(q^2-\alpha q+\beta)_{q-\alpha-\gamma}$. □

Theorem 6.3. *Let $\mathcal{L}_q$ be a Desarguesian elliptic semiplane of type L. Then, for each $\alpha \in \{0,\dots,q-3\}$, $\beta \in \{0,\dots,q-\alpha\}$, and $\gamma \in \{0,\dots,q-\alpha-3\}$, there exists a configuration $\mathcal{L}_q^{(\alpha R)(\beta M)(\gamma F)}$ of type $((q+1-\alpha)(q-1)+\beta)_{q-\alpha-\gamma}$.*

Proof. Proposition 5.3 states that $\mathcal{N} := \mathcal{A}_q^*$ is a quasi-simple $\mathcal{S}_{q-1}$-scheme of order $q+1$, rank q, and excess 0, representing an incidence matrix of $\mathcal{L}_q$. Reordering rows and columns, if necessary, we may suppose that the zero entries lie in the main diagonal of $\mathcal{N}$. Let $\mathcal{N}_\alpha$ be the quasi-simple $\mathcal{S}_q$-scheme of excess 0, obtained by deleting, say the last α rows and columns of $\mathcal{N}$. Then, by Proposition 4.4, the configuration $\mathcal{L}_q^{(\alpha R)}$ whose incidence matrix is the blow-up of $\mathcal{N}_\alpha$ has type $((q+1-\alpha)q)_{q-\alpha}$. Since $k \geq 3$ the range of α is bounded by $q-3$.

Next, we apply Martinetti extensions and 1-factor deletions as in the proof of Theorem 6.2. □

Theorem 6.4. *Let $\mathcal{P}_{q^2} := \mathsf{PG}(2,q^2)$, let D_q be a perfect difference set modulo q^2+q+1 and suppose that $\mathcal{B}_{q^2}$ is a circulant quasi-simple $\mathbb{Z}_{q^2+q+1}$-scheme of order q^2-q+1, rank q^2+1 and excess $q+1$ which represents an incidence matrix for $\mathcal{P}_{q^2}$. Then for each $\alpha \in \{0,\dots,q^2-q\}$ and $\gamma \in \{0,\dots,q^2-\alpha-2\}$, there exists a configuration $\mathcal{D}_q^{(\alpha R)(\gamma F)}$ of type $(q^4-\alpha(q^2+q+1))_{q^2+1-\alpha-\gamma}$.*

Proof. By hypothesis, $\mathcal{B}_{q^2}$ is a circulant quasi-simple $\mathbb{Z}_{q^2+q+1}$-scheme of order q^2-q+1, rank q^2+1 and excess $q+1$ which represents an incidence matrix for $\mathcal{P}_{q^2}$. Let $\mathcal{B}_\alpha$ be the quasi-simple $\mathbb{Z}_{q^2+q+1}$-scheme of excess $q+1$, obtained by deleting, say the last α rows and columns of $\mathcal{B}_{q^2}$. Then, by Proposition 4.4, the configuration $\mathcal{P}_{q^2}^{(\alpha R)}$ whose incidence matrix is the blow-up of $\mathcal{B}_\alpha$ has type $(q^4-\alpha(q^2+q+1))_{q^2+1-\alpha}$. Since $k \geq 3$ the range of α is bounded by q^2-q.

Next, we apply 1-factor deletions as in the proof of Theorem 6.2. □

Remark 6.5. Reductions, Martinetti extensions, and 1-factor deletions of elliptic semiplanes give rise to configurations which, in general, are no longer elliptic semiplanes, the only exception being $\mathcal{D}_{q^2} := \mathcal{P}_{q^2}^{(1R)}$.

7 Applications and open problems

Applying Theorems 6.2, 6.3, and 6.4, we compute all the new realizable configuration types obtained from elliptic semiplanes within region Δ of Figure 1. For each $\alpha \in \{0, \ldots, q-3\}$, $\beta \in \{0, \ldots, q-\alpha\}$, and $\gamma \in \{0, \ldots, q-\alpha-3\}$, Theorems 6.2 and 6.3 imply that the configurations types $n_k = (q^2 - \alpha q + \beta)_{q-\alpha-\gamma}$ and $n_k = ((q+1-\alpha)(q-1)+\beta)_{q-\alpha-\gamma}$ are realizable. The types $133_{11}, 183_{13}, 307_{17}, 381_{19}, 553_{23}$ are realizable as a 1-factor deletion $\mathcal{P}_q^{(1F)}$ of the finite Desarguesian projective plane $\mathcal{P}_q$ with $q = 11, 13, 17, 19, 23$. Theorem 6.4 and the explicit representation of $\mathcal{P}_{q^2}$ (see Section 5) support the following types:

$$
\begin{array}{lcl}
231_{15}, 210_{14}, 189_{13} & : & \mathcal{P}_{16}^{((\nu+1)R)} \quad \text{for } \nu = 1,2,3 \\
589_{24}, 558_{23}, 434_{19}, 403_{18} & : & \mathcal{P}_{25}^{((\nu+1)R)} \quad \text{for } \nu = 1,2,6,7
\end{array}
$$

For $7 \le k \le 25$, the types lying in Δ that become realizable through our methods are listed in the following table:

k	k^2-1	intervals of realizable types n_k	$(2l_k+1)_k$
7	48	$48_7 \ldots 50_7$	51_7
8	63	$63_8 \ldots 68_8$	69_8
9	80	$80_9 \ldots 88_9$	89_9
10	99	110_{10}	111_{10}
11	120	$120_{11} \ldots 133_{11}$	145_{11}
12	143	$156_{12} \ldots 170_{12}$	171_{12}
13	168	$168_{13} \ldots 183_{13};\ 189_{13};\ 208_{13} \ldots 212_{13}$	213_{13}
14	195	$210_{14};\ 224_{14} \ldots 254_{14}$	255_{14}
15	224	$231_{15};\ 240_{15} \ldots 302_{15}$	303_{15}
16	255	$255_{16} \ldots 354_{16}$	355_{16}
17	288	$288_{17} \ldots 307_{17};\ 323_{17} \ldots 380_{17};\ 391_{17} \ldots 398_{17}$	399_{17}
18	323	$342_{18} \ldots 380_{18};\ 403_{18};\ 414_{18} \ldots 432_{18}$	433_{18}
19	360	$360_{19} \ldots 381_{19};\ 434_{19};\ 437_{19} \ldots 492_{19}$	493_{19}
20	399	$460_{20} \ldots 566_{20}$	567_{20}
21	440	$483_{21} \ldots 666_{21}$	667_{21}
22	483	$506_{22} \ldots 712_{22}$	713_{22}
23	528	$528_{23} \ldots 553_{23};\ 558_{23};\ 575_{23} \ldots 744_{23}$	745_{23}
24	575	$589_{24};\ 600_{24} \ldots 850_{24}$	851_{24}
25	624	$624_{25} \ldots 650_{25};\ 675_{25} \ldots 960_{25}$	961_{25}

Table 2: Realizable types for $7 \le k \le 25$ obtained through our methods

Funk has found configurations of types $107_{10}, 108_{10}, 109_{10}, 110_{10}$ through a computer search using *cyclic difference sets* [12]. Performing further computer searches on cyclic difference sets we have found the following configurations:

$$
\begin{array}{llll}
135_{11}: & \{0,1,3,7,23,35,49,73,78,117,125\}^{(135)} & 140_{11}: & \{0,1,3,7,12,27,44,58,80,93,122\}^{(140)} \\
136_{11}: & \{0,1,3,7,26,35,43,55,65,76,92\}^{(136)} & 141_{11}: & \{0,1,3,7,15,20,52,61,79,108,118\}^{(141)} \\
137_{11}: & \{0,1,3,7,12,43,60,73,93,112,122\}^{(137)} & 142_{11}: & \{0,1,3,7,12,27,45,67,92,113,126\}^{(142)} \\
138_{11}: & \{0,1,3,7,19,65,86,91,106,114,128\}^{(138)} & 143_{11}: & \{0,1,3,7,12,20,55,70,84,106,116\}^{(143)} \\
139_{11}: & \{0,1,3,7,12,29,39,62,86,105,126\}^{(139)} & 144_{11}: & \{0,1,3,7,12,22,40,69,96,113,121\}^{(144)}
\end{array}
$$

Balbuena [4] constructed configurations of types 207_{13}, 223_{14}, 238_{15}, 239_{15}, 574_{23}, 598_{24}, 599_{24}, and the authors in [1] exhibited the existence of a configuration of type 231_{15}.

Taking into account all these existence results there remain the following 402 configuration types lying in region Δ, for which realizability is an open problem:

k	k^2-1	no configuration known of type n_k	$(2l_k+1)_k$
10	99	$99_{10}\dots 106_{10}$	111_{10}
11	120	134_{11}	145_{11}
12	143	$143_{12}\dots 155_{12}$	171_{12}
13	168	$184_{13}\dots 188_{13}$; $190_{13}\dots 206_{13}$	213_{13}
14	195	$195_{14}\dots 209_{14}$; $211_{14}\dots 222_{14}$	255_{14}
15	224	$224_{15}\dots 230_{15}$; $232_{15}\dots 237_{15}$	303_{15}
16	255	–	355_{16}
17	288	$308_{17}\dots 322_{17}$; $381_{17}\dots 390_{17}$	399_{17}
18	323	$323_{18}\dots 341_{18}$; $381_{18}\dots 402_{18}$; $404_{18}\dots 413_{18}$	433_{18}
19	360	$382_{19}\dots 433_{19}$; 435_{19}; 436_{19};	493_{19}
20	399	$399_{20}\dots 459_{20}$	567_{20}
21	440	$440_{21}\dots 482_{21}$	667_{21}
22	483	$483_{22}\dots 505_{22}$	713_{22}
23	528	$554_{23}\dots 557_{23}$; $559_{23}\dots 573_{23}$	745_{23}
24	575	$575_{24}\dots 588_{24}$; $590_{24}\dots 597_{24}$	851_{24}
25	624	$651_{25}\dots 674_{25}$	961_{25}

Table 3: Configurations for which realizability remains unknown in Δ

Figure 2 on page 153 illustrates how the gaps are bounded parabolically and that they are closely related to the distribution of prime powers.

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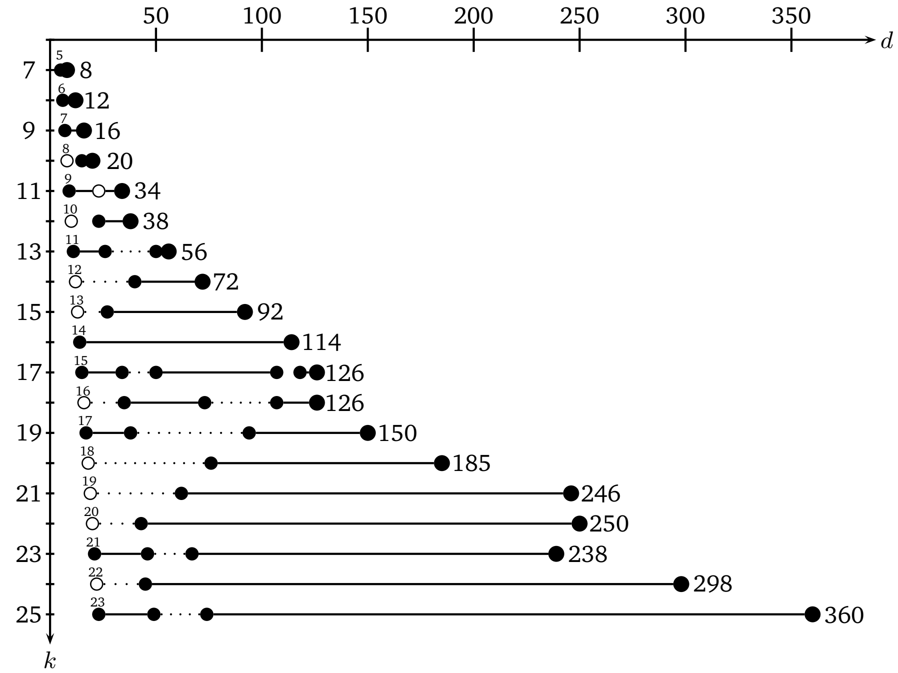

Figure 2: Region Δ including our new results

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