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Compact generalized polygons and Moore graphs as stable graphs

Nils Rosehr

Abstract

We introduce stable graphs as a common generalization of compact generalized polygons with closed adjacency, stable planes and other types of graphs with continuous geometric operations; non-bipartite structures like Moore graphs are also included. Topological and graph-theoretical properties of stable graphs are established, and generalized polygons are characterized among all stable graphs by means of topological properties. Some results about Moore graphs, which might help to find infinite examples, are included.

Keywords: topological generalized polygons, stable graphs, stable planes

MSC 2000: 51E12, 51H10

1 Introduction

Let $\mathcal{G} = (V, E)$ be a graph with *vertex set* V and *edge set* E; our graphs will always be without loops, so every element of E is a subset of V with precisely two elements. Two vertices $v, w \in V$ with $\{v, w\} \in E$ are called *adjacent* or *neighbours*; the set of all neighbours of a vertex is called a *panel* (to distinguish it from neighbourhoods in the topological sense). A *path of length k from v_0 to v_k* of $\mathcal{G}$ is a finite sequence $(v_0, v_1, \dots, v_k)$ of vertices such that v_{i-1} and v_i are adjacent for $i = 1, \dots, k$. The *distance* $d(v, w)$ of vertices v and w is the shortest length of a path from v to w or ∞, if there is none. The *diameter* of a graph is the supremum of all these distances. A path $(v_0, \dots, v_k)$ is called *non-stammering*, if $v_i \neq v_{i+2}$ for $i = 0, \dots k-2$. The *girth* of a graph is the shortest positive length of a non-stammering path from some vertex to itself or infinity, if there is none. Note that a graph with girth g has at least diameter d with $2d \leq g$. We will frequently use that $(v_0, \dots, v_k)$ is the unique non-stammering path from v_0 to v_k and that $d(v_0, v_k) = k$, if the girth is greater than $2k$. A *Moore graph* is a graph with finite diameter $d \geq 2$ and girth $2d + 1$. A graph is called *bipartite*, if its vertex set is the disjoint union of *classes* P and L such that all edges have one vertex in P and one in L. A *generalized polygon* or *d-gon* is a bipartite graph with finite diameter $d \geq 3$ and girth $2d$.

It is easy to see that Moore graphs are *regular* graphs, i.e. all panels have the same cardinality (local projectivities as in Lemma 2.1 are defined on panels minus one vertex). Damerell and Bannai–Ito have shown that apart from the cycles of odd length all finite Moore graphs have diameter 2; see [5] or [1]. The only known non-trivial finite Moore graphs are the Petersen graph with 10 vertices and the Hoffman–Singleton graph with 50 vertices. They have also shown that any other finite Moore graph has valency 57 and $3250 = 57^2 + 1$ vertices, but it is not known whether such a graph exists. For infinite Moore graphs the situation is different. It is possible to give free constructions of Moore graphs for all diameters and with an infinite vertex set of an arbitrary cardinality; also there is considerable freedom in this construction so that it is not possible to classify all infinite Moore graphs without any further assumptions. In this paper we will investigate topological assumptions.

For generalized polygons the situation is quite different. There is a wealth of examples: Moufang polygons related to algebraic structures (see [23]), compact projective planes or generalized 3-gons (see [19]), finite generalized quadrangles (see [16]), generalized quadrangles related to circle geometries (see [20]) or related to isoparametric hypersurfaces (see [22, 14, 12]). But there are also similarities: Finite generalized polygons whose panels have at least three elements have diameter 3, 4, 6 or 8 (see [7]), and for a wide class of compact generalized polygons the diameter is restricted to 3, 4 or 6. For compact generalized hexagons only the two split Cayley hexagons over the fields $\mathbb{R}$ and $\mathbb{C}$ are known; see [25, Section 9.3.7].

We will denote the set of pairs of vertices at distance $n \in \mathbb{N}_0$ with $D_n := d^{-1}(n)$ in this paper and regard D_n as a relation on V. For a vertex v the set $D_n(v) = \{w \in V : (v, w) \in D_n\}$ is defined as usual for relations; so $D_1(v)$ is a panel. In extension of the symbol D_n we will also write $D_{n,m} := D_n \cup D_m$ and $D_{\leq n} := D_0 \cup D_1 \cup \dots \cup D_n$ and so on, as well as $D_A := d^{-1}(A)$ for $A \subseteq \mathbb{N}_0$.

In order to be able to say anything about Moore graphs we might assume, as suggested by generalized polygons, that the vertex set is equipped with a compact topology, but for this assumption to actually be a restriction the topology has to be related to the graph. In the area of topological incidence geometry, that is for topological bipartite graphs, it is common to assume that the adjacency relation is closed. This is not an option here, as the following lemma shows. (Note for readers which are not familiar with the notion of nets: we will show later that most spaces we are dealing with are second countable so that working with sequences instead of nets would be sufficient in most situations.)

Lemma 1.1. *A compact Moore graph with a closed adjacency relation is finite.*

Proof. Assume that (V, E) is a Moore graph with diameter k such that V is a compact topological space and D_1 is closed in V^2. Let $D_1(v_1)$ be an infinite panel and (v_2^σ) be an injective sequence in it. Note that then all panels are infinite. By passing to subnets we can assume that (v_2^σ) converges to some $v_0 \in D_1(v_1)$ and that (v_2^σ) is a net in $D_1(v_1) \setminus \{v_0\}$, because V is compact and D_1 is closed. For every σ we can choose v_3^σ, v_4^σ,…,v_{k+1}^σ such that $(v_0, v_1, v_2^\sigma, v_3^\sigma, \dots, v_{k+1}^\sigma)$ is a non-stammering path. Because (V, E) has diameter k and girth $2k+1$, the vertices v_0 and v_{k+1}^σ have distance k and so there are further vertices v_{k+2}^σ,…,v_{2k}^σ such that $(v_0, v_1, v_2^\sigma, v_3^\sigma, \dots, v_{2k}^\sigma, v_0)$ is a closed non-stammering path of length $2k+1$. Because V is compact, we can assume that this net of paths has a limit $(v_0, v_1, v_0, v_3, v_4, \dots, v_{2k}, v_0)$, which is a path, because the adjacency D_1 is closed. Thus there is a closed path of odd length $2k-1 < 2k+1$, a contradiction. So we have shown that all panels are finite, which implies that V is finite. □

The following observation is fundamental for all that follows.

Observation 1.2. *Let (V, E) be a Moore graph or a generalized polygon with a compact topology on V such that D_1 is closed in $V^2 \setminus D_0$, and let k be the largest integer such that $2k$ is smaller than the girth of (V, E); for $k = 2$ assume also that V is a Hausdorff space. Then the well-defined map*

$$f\colon D_k \to V^{k+1}, (v, w) \mapsto p, \text{ where } p \text{ is the path from } v \text{ to } w,$$

is continuous, and its domain D_k is open in V^2.

Proof. For the topological closure of D_1 in V^2 we have $\overline{D_1} \subseteq D_{0,1}$ by assumption. Note that we have the representations $D_{\leq 2} = \overline{D_1}\,\overline{D_1} \cup \overline{D_1}$ and $D_{\leq l+1} = D_{\leq l}\overline{D_1}$ as composite relations for $l \geq 2$. Since V is compact, products of closed relations on V are closed (which can be seen by an easy argument with nets), so by induction $D_{\leq l}$ is closed for all $l \geq 2$. In the case $k = 2$ we have by assumption that $D_{\leq 1} = D_0 \cup \overline{D_1}$ is closed. Thus in the case of a Moore graph $D_k = D_{\geq k}$ is open.

If (V, E) is a generalized polygon, then $D_{k+1} = D_{\geq k+1}$ is open. Thus $D_{k+1}(v)$ is an open neighbourhood of w for $(v, w) \in D_{k+1}$; it is contained in one of the classes P or L of the bipartite graph (V, E). Therefore P and L are open. Since also $U := D_{k,k+1} = D_{\geq k}$ is open, it follows that the sets $(P^2 \cup L^2) \cap U$ and $(P \times L \cup L \times P) \cap U$ are open, one of which is D_k.

In both cases the map $f\colon D_k \to V^{k+1}$ is continuous, because its codomain V^{k+1} is compact and because its graph

$$\{((x,y),(v_0,\ldots,v_k)) : (x,y) = (v_0,v_k) \in D_k, (v_{i-1},v_i) \in D_1 \text{ for } i = 1,\ldots,k\}$$

is closed in $D_k \times V^{k+1}$, as we can replace D_1 with $\overline{D_1}$ in this description. □

Contrary to Moore graphs, the above proof shows that in the case of generalized polygons the adjacency relation is in fact closed in the larger set V^2 as P and L are closed in V. For a graph (V,E) with a topology on V, we call the adjacency relation D_1 *semi-closed* if it is closed in $V^2 \setminus D_0$, or equivalently, if $\overline{D_1} \subseteq D_{0,1}$.

Definition 1.3. A graph $\mathcal{G} = (V,E)$ is called *stable*, or more precisely *k-stable* for $k \in \mathbb{N}$ with $k \geq 2$, if the girth of $\mathcal{G}$ is greater than $2k$, if all panels contain at least three vertices and if V caries a topology such that D_k is open in V^2 and the *geometric map*

$$f\colon D_k \to V^{k+1}, (v,w) \mapsto p, \text{ where } p \text{ is the path from } v \text{ to } w,$$

is continuous. We will always denote the geometric map of a k-stable graph by f and its coordinate functions by f_i for $i = 0,\ldots,k$. Furthermore for $0 \leq i \leq l \leq k$ and $(x,y) \in D_l$ we denote the i-th vertex in a non-stammering path from x to y by $f_{l,i}(x,y)$.

We give an example to explain the idea of stability. Let P be an arbitrary non-empty open subset of $\mathbb{R}^2$ and let L be the set of ordinary affine lines in $\mathbb{R}^2$ meeting P in at least one point. Then (V,E) with $V := P \dot\cup L$ and $E := \big\{\{v,w\} : v \in P, v \in w \in L\big\}$ is a 2-stable graph. As an aside, note that for $P = \mathbb{R}^2$ we get the incidence graph of the real affine plane and for P an open disc we get the Klein model for the hyperbolic plane. The vertices in P are usually called points and the ones in L lines. Note that $P^2 \cap D_2 = P^2 \setminus \mathrm{id}_P$, i.e. any two distinct points can be joined by a line. For lines the dual property that any two lines meet in a point is false. But the fact that D_2 is open implies that two meeting lines can be "moved" slightly and they will still meet; this property is responsible for the name *stability*. Note that it is false in $\mathbb{R}^3$, and for this reason the stability axiom rules out geometries of higher rank. For a simple non-bipartite 2-stable example see the graphs $\mathcal{G}_r$ below.

Concerning the definition of stable graphs it can be shown that it is enough to ask for the continuity of the coordinate function f_1 only, which is important, for example, for the application to (semi)-biplanes; see [17, Proposition 5.2].

In [17] the notion of a k-stable graph is more general; it entails graphs with a topology where vertices at distance k have unique joining paths only locally, that is, no assumption about the girth is made; the graphs according to Definition 1.3 are called *monocursal k-stable graphs* there. We treat only monocursal stable graphs here, because we are mainly interested in Moore graphs and generalized polygons, and we drop the qualifier monocursal for ease of language. However, most of the results we will obtain hold in a much more general context; see [17].

We will call a graph *compact*, *locally compact*, *locally connected*, or *discrete*, if the vertex space carries a topology which has this property. We will see later that this usually means that all panels share the respective property.

There are numerous examples of stable graphs: Given a non-discrete k-stable graph and an open subset of the vertex space such that the induced subgraph has no vertices without neighbours the induced subgraph defines a k-stable graph. So starting with a stable generalized polygon we obtain many examples. There are also examples which cannot be embedded in generalized polygons as an open subgraph, for example $\mathcal{G}_r = (V_r, \{\{v,w\} \subseteq V_r : |v-w| = 1\})$ for $V_r := \{x \in \mathbb{R}^2 : 1-r < |x| < r\}$ and $1/2 < r \leq 1/\sqrt{3}$ is a 2-stable graph which is not bipartite; see also [21], [9] and [17, Section 3]. In [18] it is shown that every graph-connected non-discrete stable graph can be embedded in a maximal graph-connected stable graph as an open subgraph, which means that all stable graphs are obtained as substructures of these maximal objects; furthermore a large class of Cayley graphs on $\mathrm{C}_2 \ltimes \mathbb{R}^2$, which is maximal with respect to this embedding is exhibited. For many more examples see [17].

Of course Observation 1.2 says that every compact Moore graph or compact generalized polygon with a semi-closed adjacency relation and large enough panels is a stable graph. One of our main theorems says that there is a certain converse: a locally connected non-discrete stable graph with compact panels is a generalized polygon. Since we will also show that every locally compact stable graph is metrizable as well as either locally connected or totally disconnected, this result implies that every stable Moore graph with compact panels and a compact vertex space is defined on the Cantor set.

2 Properties of stable graphs

2.1 Local projectivities and local coordinates

Let v_0 and v_{k+1} be two vertices at maximal distance $k+1$ in a generalized $(k+1)$-gon. The geometric map $f_1(v_{k+1}, \cdot)\colon D_1(v_0) \to D_1(v_{k+1})$ is a bijection — a so called *projectivity*. The following basic lemma provides local versions of

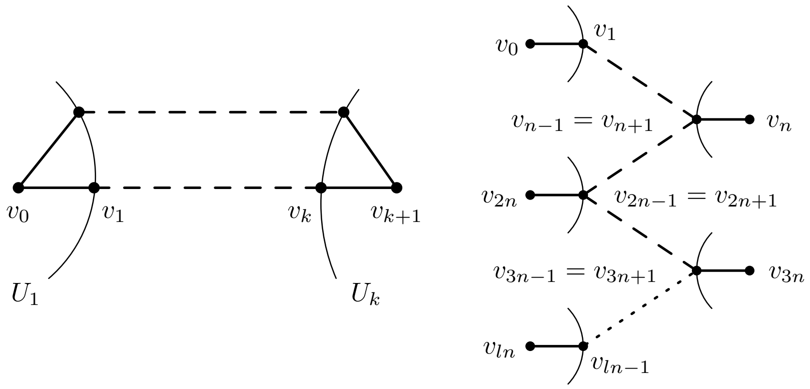

Figure 1: A local projectivity and a staircase path with its perspectivity

these projectivities for stable graphs; see Figure 1.

Lemma 2.1. *Let (V, E) be a k-stable graph, and let $(v_0, \ldots, v_{k+1})$ be a non-stammering path. Then there is an open neighbourhood $U_1 \times U_k$ of (v_1, v_k) in $D_1(v_0) \times D_1(v_{k+1})$ such that $f_1(v_{k+1}, \cdot)\colon U_1 \to U_k$ is a homeomorphism mapping v_1 to v_k.*

Proof. Set $U_1 := D_1(v_0) \cap f_1(v_{k+1}, \,\cdot\,)^{-1}(D_k(v_0))$ and $U_k := D_1(v_{k+1}) \cap f_1(v_0, \,\cdot\,)^{-1}(D_k(v_{k+1}))$. Then $f_1(v_{k+1}, \,\cdot\,)\colon U_1 \to U_k$ and $f_1(v_0, \,\cdot\,)\colon U_k \to U_1$ are inverse to each other and continuous. □

The maps from the previous lemma are called *local projectivities*. If we apply this lemma l-times we get *local perspectivities* as the following lemma states. It owes its name to the shape of the path used in its formulation; see Figure 1.

Staircase lemma 2.2. *Let (V, E) be a k-stable graph, and set $n := k + 1$. Let $l \in \mathbb{N}$, and let $v_0, \ldots, v_{ln} \in V$ such that $v_{in-1} = v_{in+1}$ for $i = 1, \ldots, l-1$ and $(v_{(i-1)n}, \ldots, v_{in})$ is a non-stammering path for $i = 1, \ldots, l$. Then there is an open neighbourhood $U_1 \times U_{ln-1}$ of (v_1, v_{ln-1}) in $D_1(v_0) \times D_1(v_{ln})$ such that*

$$f_1(v_{ln}, \,\cdot\,) \circ \cdots \circ f_1(v_{2n}, \,\cdot\,) \circ f_1(v_n, \,\cdot\,)\colon U_1 \to U_{ln-1}$$

is a homeomorphism mapping v_1 to v_{ln-1}.

Recall that two topological spaces X and Y are *locally homeomorphic* if any two elements x and y of X and Y respectively have homeomorphic neighbourhoods, and that X is called *locally homogeneous* if it is locally homeomorphic with itself. Of course, a space that is locally homeomorphic to some space is locally homogeneous.

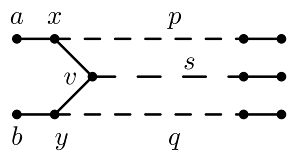

Figure 2: The staircase path making panels locally homogeneous

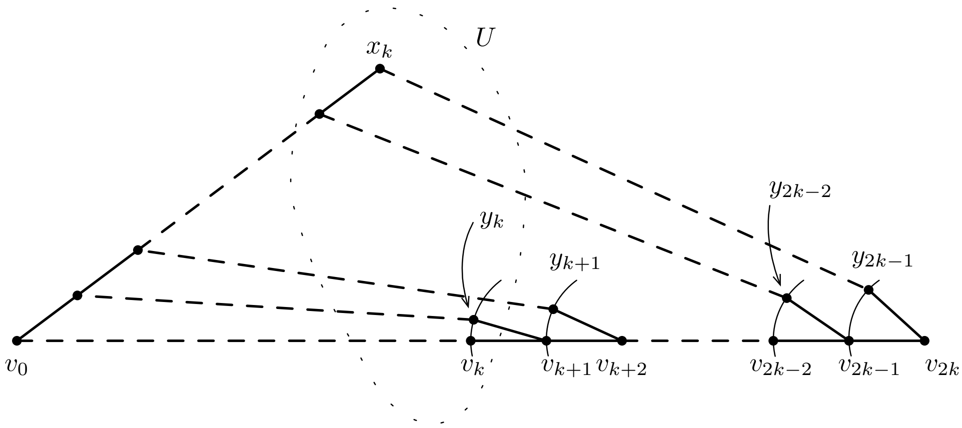

Figure 3: Local coordinates given by the map g

Proposition 2.3. *Let $(v_0, \dots, v_{2l})$ be a path of even length $2l \in \mathbb{N}_0$ of a stable graph. Then the panels $D_1(v_0)$ and $D_1(v_{2l})$ are locally homeomorphic. In particular panels are locally homogeneous.*

Proof. Let (V, E) be a k-stable graph. In order to show the case $l = 0$ let $v \in V$ and $x, y \in D_1(v)$; see Figure 2. Because all panels contain at least three elements, we can choose $a, b \in V$, $p, q \in V^k$ and $s \in V^{k-1}$ such that (a, x, p), (a, x, v, s), (v, x, p), (b, y, q), (b, y, v, s) and (v, y, q) are non-stammering paths. For $p = (x_1, \dots, x_k)$ set $p^{\leftrightharpoons} := (x_1, \dots, x_{k-1}, x_k, x_{k-1}, \dots, x_1)$ and define $q^{\leftrightharpoons}$ and $s^{\leftrightharpoons}$ analogously. Then

$$(v, x, p^{\leftrightharpoons}, x, a, x, v, s^{\leftrightharpoons}, v, y, b, y, q^{\leftrightharpoons}, y, v)$$

is a path as in the staircase Lemma 2.2. Thus $D_1(v)$ is locally homogeneous. Since $(v, x, p^{\leftrightharpoons}, x, a)$ is also such a path, $D_1(v)$ and $D_1(a)$ are locally homeomorphic, and induction completes the proof. □

Next we will show that there are local coordinates of the vertex set in terms of the panels.

Theorem 2.4. *Let (V, E) be a k-stable graph, and let $(v_0, \dots, v_k)$ be a non-stammering path. Then there are open neighbourhoods U of v_k in V and W of $(v_1, \dots, v_k)$ in $D_1(v_0) \times \cdots \times D_1(v_{k-1})$ and a homeomorphism $h\colon U \to W$ such that $h(D_{k-l}(v_l)) = \big(\{(v_1, \dots, v_l)\} \times V^{k-l}\big) \cap W$ for any $l = 0, \dots k$.*

Proof. We extend the given path to a non-stammering path $(v_0, \dots, v_{2k})$ and show that there are open neighbourhoods U of v_k in V and W in $D_1(v_{k+1}) \times \cdots \times D_1(v_{2k})$ such that

$$g\colon U \to W, x_k \mapsto \big(f_1(v_{k+i}, f_i(v_0, x_k))\big)_{i=1,\dots,k}$$

is a homeomorphism; see Figure 3. We have

$$g(D_{k-l}(v_l)) = \big(\{(v_k, \dots, v_{k+l-1})\} \times V^{k-l}\big) \cap W,$$

because the girth is greater than $2k$, and the theorem follows if we compose g with the local projectivities $D_1(v_{k+i}) \to D_1(v_{i-1})$ for $i = 1, \dots, k$ according to Lemma 2.1.

We may set $U := f(v_0, \cdot\,)^{-1}(D_k(v_k) \times \cdots \times D_k(v_{2k}))$. Because U is open, the set of all $(y_k, \dots, y_{2k-1}) \in V^k$ such that

$$\pi_{y_{2k-1}} \circ \pi_{y_{2k-2}} \circ \cdots \circ \pi_{y_{k-1}} \circ \pi_{y_k}(v_0) \in U, \quad \text{where } \pi_y := f_1(\,\cdot\,, y) \text{ for } y \in V,$$

is also open by stability. Let W be its intersection with $D_1(v_{k+1}) \times \cdots \times D_1(v_{2k})$. Then it is easy to see that g and the map defined on W by the above expression are inverse to each other. □

In a non-bipartite k-stable graph there is a cycle of odd length; so there are two adjacent vertices v and w such that $D_1(v)$ and $D_1(w)$ are locally homeomorphic by Proposition 2.3. The same is true if k is even: local projectivities can be used to see that $D_1(v_0)$ and $D_1(v_{k+1})$ are locally homeomorphic for a non-stammering path $(v_0, \dots, v_{k+1})$, and Proposition 2.3 yields that $D_1(v_0)$ and $D_1(v_1)$ are locally homeomorphic. Thus Theorem 2.4 yields the following result.

Corollary 2.5. *Let $\mathcal{G}$ be a graph-connected k-stable graph. Then all panels and classes (in the bipartite case) of $\mathcal{G}$ are locally homogeneous. If $\mathcal{G}$ is not bipartite or k is even, then the vertex space of $\mathcal{G}$ is locally homogeneous and any two panels are locally homeomorphic.*

2.2 Separation properties

The vertex space of a stable graph is in general not a Hausdorff space. However, this only happens if the adjacency relation is not closed, and vertices which are close together in the graph theoretical sense can always be separated. More precisely we have the following result.

Theorem 2.6. *Let $\mathcal{G}$ be a k-stable graph. Then all panels are Hausdorff spaces, and the vertex set is a locally Hausdorff space and in particular a T_1-space. Vertices at distance at most $2k$ or greater than $4k$ can be separated. Every panel has an open neighbourhood which is a Hausdorff space.*

The relations D_l for $l = 0, 1, \ldots, k$ and in particular the adjacency relation are locally closed, and if D_1 is closed, then V is a Hausdorff space.

Proof. Let $\mathcal{G} = (V, E)$ and $(v_0, v_l) \in D_l$. If $0 < l \leq k$, choose v_i such that $(v_0, \ldots, v_{l+k})$ is a non-stammering path; the domain of $g := f_l(\,\cdot\,, f_{k-l}(\,\cdot\,, v_{l+k}))$ is an open neighbourhood of (v_0, v_l) which is disjoint from the diagonal $\mathrm{id}_V \subseteq V^2$, because the girth of $\mathcal{G}$ is greater than $2k$. If $k \leq l < 2k$, choose a non-stammering path $(v_0, \ldots, v_l)$ and consider $f(\,\cdot\,, f_{2k-l}(v_{l-k}, \,\cdot\,))$. For $l = 2k$ we can choose (using the case $l = 2$) separating neighbourhoods U_{k-1} and U_{k+1} of v_{k-1} and v_{k+1} respectively. Then $f_1(v_k, \,\cdot\,)^{-1}(U_{k-1})$ and $f_1(v_k, \,\cdot\,)^{-1}(U_{k+1})$ are separating neighbourhoods of v_0 and v_{2k}. If $4k < l$, then $D_k(x)$ and $D_k(y)$ are open disjoint neighbourhoods of v_0 and v_l for $x \in D_k(v_0)$ and $y \in D_k(v_l)$ respectively.

Panels are Hausdorff spaces as they have diameter 2; thus V is a locally Hausdorff space by Theorem 2.4. Let $v \in V$, and choose $w_1, w_2, w_3 \in D_{k-1}(v)$ such that the elements $f_{k-1,1}(v, w_i)$ for $i = 1, 2, 3$ are distinct. Then the open subsets $U_i := D_k(w_i)$ for $i = 1, 2, 3$ are Hausdorff spaces by what we have shown already, and we have $D_1(v) \cap U_i = D_1(v) \setminus \{f_{k-1,1}(v, w_i)\}$. Thus the open subset $\bigcup_{i,j \in \{1,2,3\}, i \neq j} U_i \cap U_j$ contains $D_1(v)$, and it is easy to see that it is a Hausdorff space.

The local closedness of D_l follows as V is a locally Hausdorff space and $(x, y) \in D_l \cap \operatorname{dom} g$ if and only if $g(x, y) = y$ with the map g from above.

If v_0 and v_1' are distinct vertices, then there is a $v_1 \in D_1(v_0) \setminus D_1(v_1')$, because otherwise there would be a circle of length 4. Choose $v_2, \ldots, v_k \in V$ such that $(v_0, \ldots, v_k)$ is a non-stammering path. Since D_1 is closed and the map $(x, y) \mapsto (f_1(x, v_k), y)$ is continuous, there is an open neighbourhood $U_0 \times U_1'$ of (v_0, v_1') such that $f_1(U_0, v_k) \times U_1'$ is disjoint from D_1. But then U_0 and U_1' are disjoint, since otherwise there would be a vertex $x \in U_0$ such that $(f_1(x, v_k), x) \in D_1 \cap (f_1(U_0, v_k) \times U_1')$. $\square$

Let (V, E) be a generalized $(k+1)$-gon with a vertex v satisfying $D_1(v) \cong \mathbb{S}_1$. If we remove two edges $\{v, w_1\}$ and $\{v, w_2\}$ from E and replace v by new vertices v_1 and v_2 such that the respective panels of these vertices are precisely the connected components of $D_1(v) \setminus \{w_1, w_2\}$ and such that the open neighbourhoods of v_i for $i = 1, 2$ are precisely the open neighbourhoods of v with v replaced by v_i, then the resulting graph is again a k-stable graph and the vertices v_1 and v_2 cannot be separated. Their distance is $2k + 2$.

The above example seems a little pathological and in related situations non-Hausdorff graphs are often excluded from the treatment. We would like to stress that this would be unnatural in the realm of stable graphs, because there are interesting maximal examples with a large automorphism group whose vertex space is not a Hausdorff space; see [24]. These graphs are related to shift planes.

Consequently we never assume any separation properties implicitly: our compact spaces are not assumed to be Hausdorff spaces, and our locally compact spaces are spaces with a neighbourhood basis of compact spaces.

Sometimes the property of being a locally Hausdorff space allows to conclude stronger separation properties. We note the following lemma for later use in the proof of Theorem 3.6; it can be applied to compact vertex sets of graph-theoretical diameter at most $2k$.

Lemma 2.7. *Let X be a locally Hausdorff space and $C \subseteq X$ a compact subspace such that any two distinct elements in C can be separated in X. Then there is a neighbourhood of C which is a Hausdorff space.*

Proof. Let $x \in C$ and H_x be an open neighbourhood of x which is a Hausdorff space. For any $y \in C \setminus H_x$ let $\tilde{U}_y$ and $\tilde{V}_y$ be disjoint open neighbourhoods of x and y respectively. Because $C \setminus H_x$ is compact, there are $y_1, \dots, y_n \in C \setminus H_x$ such that $C \setminus H_x \subseteq \tilde{V}_{y_1} \cup \dots \cup \tilde{V}_{y_n}$. Then any element of $U_x := H_x \cap \tilde{U}_{y_1} \cap \dots \cap \tilde{U}_{y_n}$ can be separated from any distinct element of $V_x := H_x \cup \tilde{V}_{y_1} \cup \dots \cup \tilde{V}_{y_n} \supseteq C$. There are $x_1, \dots, x_m \in C$ such that $C \subseteq U_{x_1} \cup \dots \cup U_{x_m}$, because the U_x for $x \in C$ cover C. As all V_x contain C, the subspace $(U_{x_1} \cup \dots \cup U_{x_m}) \cap V_{x_1} \cap \dots \cap V_{x_m}$ is a neighbourhood of C which is a Hausdorff space. □

2.3 Openness and continuity

Proposition 2.8. *Let (V, E) be a k-stable graph and $0 \le i \le l \le k$ integers.*

(a) *The geometric maps $f_{l,i}$ are continuous and open.*

(b) *The maps $f_{l,i}(v_0, \cdot\,)\colon D_l(v_0) \to D_i(v_0)$ are open.*

(c) *If U is an open subset of V, then $D_l(U)$ is open in V.*

Proof. (a) Let $(v_0, v_l) \in D_l$. Choose v_i such that $(v_0, \dots, v_{l+k})$ is a non-stammering path. The map $f_i(\,\cdot\,, f_{k-l}(\,\cdot\,, v_{l+k}))$ is continuous and defined on an open neighbourhood W of (v_0, v_l) by stability. It equals the restriction $f_{l,i}|_W$ on D_l, which is therefore continuous.

To show openness let now W be an arbitrary open neighbourhood of (v_0, v_l); see Figure 4. Choose further $v_j \in V$ such that $(v_{i-k}, \dots, v_{i+k})$ is a non-

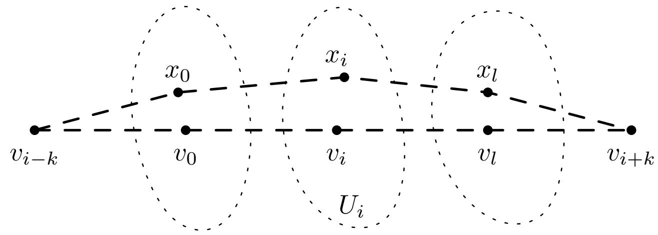

Figure 4: Openness of the geometric maps

stammering path. Then v_{i-1} and v_{i+1} can be separated by open disjoint neighbourhoods by Theorem 2.6. Thus there is an open neighbourhood U_i of v_i such that for all $x_i \in U_i$ the path $(f(v_{i-k}, x_i), f(x_i, v_{i+k}))$ is a non-stammering path from v_{i-k} via x_i to v_{i+k}; furthermore by continuity U_i can be chosen so small that $(f_{k-i}(v_{i-k}, x_i), f_{l-i}(x_i, v_{i+k}))$ is contained in W. For $x_i \in U_i$ we have

$$f_{l,i}\big(f_{k-i}(v_{i-k}, x_i), f_{l-i}(x_i, v_{i+k})\big) = x_i,$$

and therefore $U_i \subseteq f_{l,i}(W)$.

(b) This follows if we specialize to $x_i \in D_i(v_0) \cap U_i$.

(c) We have $D_l(U) = f_l(U \times V)$, and this set is open by (a). □

The open subsets $D_k(v)$ for a vertex v cover the vertex space of a k-stable graph, and any of them is contained in a graph component and a class in the bipartite case. Thus the graph-components and the classes of a stable graph are open. Any collection of graph-components of a k-stable graph forms a k-stable graph.

By the continuity and openness of $f_{2,1}$ we have the following result.

Corollary 2.9. *If P is a class of a bipartite stable graph (V, E), then the topology of $V \setminus P$ is determined by the topology of P.*

2.4 The addition

There is a local addition and a local subtraction on panels as the following result shows. We will use this addition to prove the existence of local isotopies on panels which is a strong form of local homogeneity.

Lemma 2.10. *Let (V, E) be a stable graph, and let $v \in V$ and $o \in D_1(v)$. Then there are an open neighbourhood U of o in $D_1(v)$ and continuous operations*

$$\pm \colon U \times U \to D_1(v)$$

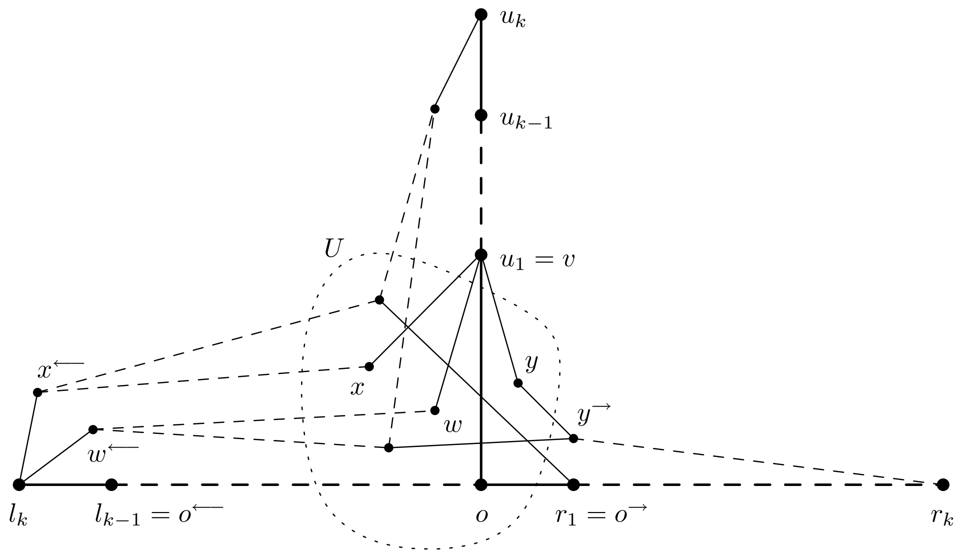

Figure 5: The addition

satisfying $o+x = x = x+o$ *and* $(x+y)-y = x = (x-y)+y$ *for vertices* $x, y \in U$ *for which all terms are defined.*

Proof. Let (V, E) be a k-stable graph. Set $l_0 := r_0 := u_0 := o$ and $u_1 := v$, and choose distinct vertices $l_i, r_i, u_i \in V$ such that $l_0, \ldots, l_k$, $r_0, \ldots, r_k$ and $u_0, \ldots, u_k$ are paths and $l_k, r_k, u_k \in D_k(o)$; see Figure 5. If defined for $x, y, z \in V$ set

$$x^{\longleftarrow} := f_1(l_k, x)\,, \quad y^{\rightarrow} := f_1(y, r_k) \quad \text{and}$$
$$\pi(x, y, z) := f_1\big(u_1, f_1(z^{\rightarrow}, f_1(u_k, f_1(y^{\rightarrow}, x^{\longleftarrow})))^{\longleftarrow}\big)\,.$$

By the choice of the l_i, r_i, and u_i we have $\pi(o, o, o) = o$, because $o^{\rightarrow} = r_1$, $o^{\longleftarrow} = l_{k-1}$, $f_1(r_1, l_{k-1}) = o$, $f_1(u_k, o) = u_{k-1}$, $f_1(r_1, u_{k-1}) = o$ and $f_1(u_1, l_{k-1}) = o$. Thus there is an open neighbourhood U of o in $D_1(v)$ such that $U^3 \subseteq \operatorname{dom} \pi$, because D_k is open in V^2 and f is continuous. The maps

$$\pm \colon U \times U \to D_1(v) \quad \text{are defined by}$$
$$x + y := \pi(x, o, y) \quad \text{and} \quad x - y := \pi(x, y, o).$$

Let $x, y, w \in U$. It is easy to show that $(x+y)-y = x$ if $x+y \in U$ and that $(w-y)+y = w$ if $w-y \in U$; see Figure 5. Also it is straightforward to see that o is a left and a right neutral element. $\square$

The following application constitutes a strong homogeneity property which will be crucial in the proof of the fact that panels are cohomology manifolds. We prove that two points which are connected by a path in the topological sense not only have homeomorphic neighbourhoods, but that these neighbourhoods can be shifted along the continuous path in the following sense.

Lemma 2.11. *Let (V, E) be a stable graph. Every element o of any panel has arbitrary small neighbourhoods W such that for every continuous path $\lambda\colon [0,1] \to W$ there is a neighbourhood $W' \subseteq W$ of $\lambda(0)$ and an isotopy $h\colon [0,1] \times W' \to W$, i.e., $h(\,\cdot\,, \lambda(0)) = \lambda$ and $h(t, \,\cdot\,)\colon W' \to h(t, W')$ is a homeomorphism for all $t \in [0,1]$.*

Proof. Let $\pm\colon U \times U \to V$ be the local operations with zero $o \in U$ as in Lemma 2.10. Let W be an open neighbourhood of o in U such that $\pm\colon W \times W \to U$, and let $\lambda\colon [0,1] \to W$ be a continuous map. Define

$$h\colon [0,1] \times W \to V, (t, v) \mapsto \big(v - \lambda(0)\big) + \lambda(t).$$

Then $h(t, \lambda(0)) = \big(\lambda(0) - \lambda(0)\big) + \lambda(t) = \big((o + \lambda(0)) - \lambda(0)\big) + \lambda(t) = o + \lambda(t) = \lambda(t)$ and in particular $h([0,1], \lambda(0)) \subseteq W$. So by compactness of $[0,1]$ there is a neighbourhood W' of $\lambda(0)$ such that $h\colon [0,1] \times W' \to W$. We have

$$(h(t,v) - \lambda(t)) + \lambda(0) = (v - \lambda(0)) + \lambda(0) = v$$

for all $v \in W'$. Thus $h(t, \,\cdot\,)\colon W' \to h(t, W')$ is a homeomorphism. □

2.5 The multiplication

As we will see later the following result has far reaching applications and poses strong restrictions on the topology of the vertex space and the panels of a stable graph.

Proposition 2.12. *Let (V, E) be a k-stable graph, let $(v_0, \dots, v_{2k})$ be a non-stammering path, and set $(o_1, o_2, o_3) := (v_1, v_{k+1}, v_{2k-1})$. Then there is an open neighbourhood $U_1 \times U_2$ of (o_1, o_2) in $D_1(v_0) \times D_1(v_k)$ and a continuous multiplication*

$$\cdot\, : U_1 \times U_2 \to D_1(v_{2k})$$

satisfying $o_1 \cdot y = o_3 = x \cdot o_2$ for all $(x, y) \in U_1 \times U_2$; furthermore the right-multiplication $U_1 \to D_1(v_{2k}), x \mapsto x \cdot y$ is an open and injective map for every $y \in U_2 \setminus \{o_2\}$.

Proof. If defined for $x, y \in V$ set

$$g(x, y) := f_1(v_{2k}, f_1(y, x));$$

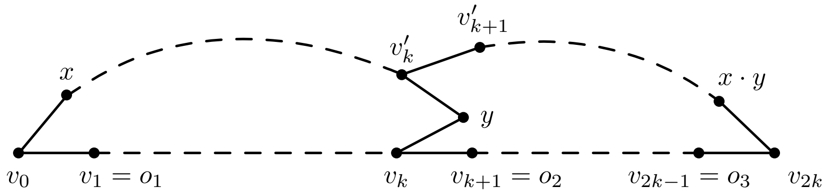

Figure 6: The multiplication

see Figure 6. Let $(x, y) \in \operatorname{dom} g$. If $x = o_1$ and $y \in D_1(v_k)$, then $f_1(y, x) = v_k$ and $f_1(v_{2k}, v_k) = v_{2k-1}$. If $y = o_2 = v_{k+1}$, then $v'_k := f_1(y, x) \in D_1(v_{k+1})$ and therefore $f_1(v_{2k}, v'_k) = v_{2k-1}$. Thus in either case $g(x, y) = o_3$. In particular $g(o_1, o_2) = o_3$, so $\operatorname{dom} g$ is an open neighbourhood of (o_1, o_2), and we can choose the neighbourhood $U_1 \times U_2$ as required and define the multiplication to be the restriction of g to this neighbourhood.

It is easy to see that the right-multiplications are injective, because (V, E) has girth greater than $2k$, and as local projectivities they are open maps. □

The following result says that the multiplication of the previous proposition can be chosen such that it contains a given compact proper subset of a panel in its domain of definition. In view of Proposition 2.14 it holds for non-discrete stable graphs.

Lemma 2.13. *Let (V, E) be a k-stable graph. For $(v, w) \in D_1$ let C be a proper compact subset of $D_1(v)$ and assume that $D_1(w)$ is not discrete. Then for any $o_1 \in C$ there is a multiplication $\cdot : U_1 \times U_2 \to U_3$ with zeros o_1, o_2 and o_3 as in Proposition* 2.12 *containing $C \times \{o_2\}$ in its domain of definition.*

Proof. Choose a path $(w_2, \ldots, w_{2k+1})$ of distinct vertices such that C is contained in $D_1(w_2)$. If defined for $x, y, z \in V$ set

$$h(x, y, z) := f_1\big(f_1(w_{2k+1}, z), f_1(y, x)\big);$$

see Figure 7. Since $h(c, w_{k+1}, w_{k+1}) = f_1(w_{2k}, w_k) = w_{2k-1}$ for all $c \in C$, we have that $C \times \{(w_{k+1}, w_{k+1})\} \subseteq \operatorname{dom} h$. Since C is compact and $\operatorname{dom} h$ is open there is an open neighbourhood $W_1 \times W_2$ of $C \times \{w_{k+1}\}$ such that $W_1 \times {W_2}^2 \subseteq \operatorname{dom} h$. Let $v_1 = o_1 \in C$. The set $f_1(v_1, W_2)$ is an open neighbourhood of w_2 (Proposition 2.8) in the non-discrete set $D_1(v_1)$ (Proposition 2.3); so there is a $v_{k+1} \in W_2$ such that $(v_1, \ldots, v_{k+1}) := f(v_1, v_{k+1})$ satisfies $v_2 \neq w_2$. Set $v_0 := w_2$ and $(v_{k+1}, \ldots, v_{2k+1}) := f(v_{k+1}, w_{2k+1})$. As panels are Hausdorff spaces W_2 can be chosen so small that $v_{k-1} \neq v_{k+1}$. Then $(v_0, \ldots, v_{2k})$ is a non-stammering path. Note that with the notation from the proof of Proposition 2.12 we have $g = h(\,\cdot\,, \,\cdot\,, v_{k+1})$. So again by the compactness of C there is an open

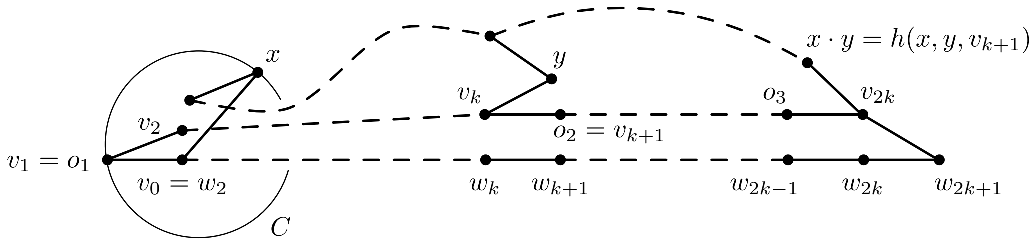

Figure 7: The map h

neighbourhood $U_1 \times U_2$ of $C \times \{o_2\}$ contained in the domain of the multiplication defined from g as in the proof of Proposition 2.12. □

2.6 Discreteness

It is not excluded that the vertex space of a stable graph is a discrete topological space. However a graph (V, E) with the discrete topology on V satisfies the topological assumptions for k-stable graphs for any k. If (V, E) is a k-stable graph and V is not discrete, then k is unique, and the discreteness can be seen in the panels as the following proposition shows.

Proposition 2.14. *The vertex space of a graph-connected k-stable graph is discrete if and only if some panel has an isolated element.*

If this is not the case, then the girth of the graph is at most $2k + 2$, and in particular k is uniquely determined.

Proof. Let (V, E) be a k-stable graph. Of course, if V is discrete, then every vertex is an isolated element of any panel it is contained in. If a panel $D_1(v)$ has an isolated element, then the panel is discrete by local homogeneity (Proposition 2.3); furthermore every panel $D_1(w)$ such that v and w can be connected by a path of even length is also discrete. Thus, if k is even, then all panels are discrete by Lemma 2.1, and therefore V is discrete by Corollary 2.4.

Now let k be odd, and let $p = (v_0, \dots, v_{2k})$ be a non-stammering path such that the panel $D_1(v_0)$ is discrete. We will use the multiplication $\cdot : U_1 \times U_2 \to D_1(v_{2k})$ from Proposition 2.12 for open subsets $U_1 \subseteq D_1(v_0)$ and $U_2 \subseteq D_1(v_k)$ to show that $D_1(v_k)$ is discrete; this implies that V is discrete as in the even case.

We assume that $D_1(v_k)$ is not discrete, which implies that $D_1(v_1)$ is not discrete, as $k - 1$ is even. So by Lemma 2.13 we can assume that U_1 contains an element $x \neq o_1$. Now the map $h \colon U_2 \to D_1(v_{2k}), y \to x \cdot y$ is continuous and

$h^{-1}(o_3) = \{o_2\}$ by Proposition 2.12, because $x \neq o_1$ and the right multiplication with $y \neq o_2$ is injective. So the discreteness of $D_1(v_{2k})$ implies that $\{o_2\}$ is open in the panel $D_1(v_k)$, which is therefore discrete in contradiction to our assumption that it is not.

The statement about the girth follows from the existence of local projectivities (Lemma 2.1): if the domain of definition of such a bijection contains more than one element, then the geometric maps defining this map provide cycles of length $2k + 2$; see Figure 1 on page 162. □

The above proposition implies that a stable graph is discrete if and only if all its panels are discrete.

2.7 The dichotomy of connectedness

In this subsection we show as a second application of the local multiplication that the graph components of the vertex space are either locally connected or totally disconnected.

Theorem 2.15. *Let (V, E) be a graph-connected locally compact k-stable graph. Then V and all panels are locally connected or every open Hausdorff subspace of V and in particular all panels are totally disconnected.*

Proof. Note that V and all panels have a basis consisting of locally compact Hausdorff spaces. Assume that there is a connected subset Z of an open Hausdorff subspace of V with $|Z| \geq 2$. By Theorem 2.4 about local coordinates we need to show that all panels are locally connected.

(1) Let X be a locally compact Hausdorff space, $x \in X$ and U a neighbourhood of x. If the connected component of x in U is $\{x\}$, then $\{x\}$ is also the component relative to X: It suffices to show that x has a neighbourhood basis of open and closed sets in X. Let W be an open neighbourhood of x such that $\overline{W} \subseteq U$ is compact. In a compact space the connected component of a point is the intersection of all open and closed neighbourhoods; see [6, 6.1.23]. Thus there are finitely many open and closed subsets of $\overline{W}$ whose intersection is contained in W, since $\overline{W} \setminus W$ is compact. This intersection is open and closed in X, because it is open in W and closed in $\overline{W}$.

(2) If $(u_0, u_k) \in D_k$ and if $D_1(u_0)$ is not totally disconnected, then $D_1(u_k)$ is locally connected: There are open subsets U_i of panels and a multiplication $\cdot : U_1 \times U_2 \to U_3$ with zeros $o_i \in U_i$ for $i = 1, 2, 3$, $U_2 \subseteq D_1(u_0)$ and $U_3 \subseteq D_1(u_k)$ as in Proposition 2.12. Let C be a compact neighbourhood of o_2 in U_2. By local homogeneity (Proposition 2.3) and (1) the connected component Z of o_2 in C contains at least two elements. On the one

hand, $U_1 \cdot Z = \bigcup_{z \in Z \setminus \{o_2\}} U_1 \cdot z$ is a neighbourhood of o_3, because right-multiplication is an open map. On the other hand $U_1 \cdot Z = \bigcup_{u \in U_1} u \cdot Z$ is connected, because $o_3 = u \cdot o_2 \in u \cdot Z$ for all $u \in U_1$. Thus o_3 has a neighbourhood basis of connected sets, because U_3 can be chosen arbitrarily small. By local homogeneity $D_1(u_k)$ is locally connected.

(3) All panels are locally connected: Choose $(v_0, v_k) \in D_k$ with $v_0 \in Z$. By (1) we can assume that $Z \times \{v_k\} \subseteq \operatorname{dom} f$. For the smallest $l \in \{1, \ldots, k\}$ such that there is a $u_0 \in V$ with $f_l(Z, v_k) = \{u_0\}$ we have that $f_{l-1}(Z, u_0)$ is a connected subset of the panel $D_1(u_0)$ with at least two elements. Now let $u_i \in V$ such that $(u_0, \ldots, u_{k+1})$ is a non-stammering path. Then $D_1(u_k)$ is locally connected by (2) and $D_1(u_{k+1})$ is not totally disconnected, which can be seen with local projectivities (Lemma 2.1). So by (2) again $D_1(u_0)$ and $D_1(u_1)$ are locally connected. Now all panels are locally connected by Proposition 2.3, because every vertex v is connected to u_0 or to u_1 by a path of even length. □

2.8 Metrizability

The vertex space of a compact generalized polygon is metrizable; see [8, 1.5]. The following generalization for stable graphs guarantees metrizability in the case of Hausdorff spaces.

Theorem 2.16. *Let (V, E) be a locally compact stable graph. Then every non-discrete graph-component is second countable, and every regular subspace of V is metrizable. In particular every panel and every open subset of V which is a Hausdorff space is metrizable.*

Proof. By the remark before Corollary 2.9 we may assume that (V, E) is graph-connected and non-discrete, because the topological sum of metric spaces is metrizable.

(1) Every panel is first-countable: Let $\cdot : U_1 \times U_2 \to U_3$ be a multiplication with open subsets U_i of panels and $o_i \in U_i$ for $i = 1, 2, 3$ as in Proposition 2.12. Because every panel is non-discrete by Proposition 2.14, there is an injective sequence in U_2 with a cluster point in U_2. Panels are locally homogeneous by Proposition 2.3, so we have a sequence (v_n) in $U_2 \setminus \{o_2\}$ converging to o_2. If C is a compact neighbourhood of o_1 such that $C \times \{o_2\} \subseteq \operatorname{dom} \cdot$ and U is a neighbourhood of o_3 in U_3, then there is a neighbourhood W of o_2 such that $\cdot$ is defined on $C \times W$ and $C \cdot W \subseteq U$, because C is compact. Thus $C \cdot v_n \subseteq C \cdot W \subseteq U$ for large $n \in \mathbb{N}$ and consequently $\{C \cdot v_n : n \in \mathbb{N}\}$ is a neighbourhood basis at o_3, since these right-multiplications are open maps. Now panels are first-countable, since panels are locally homogeneous.

(2) Every vertex of a panel has a neighbourhood in the panel with a countable basis: Let the operations $\pm\colon U \times U \to D_1(v)$ be as in Lemma 2.10 with zero o. Let C and U' be neighbourhoods of o in U such that $U' + U' \subseteq U$, $C \subseteq U'$ is compact and U' is open. Let the sequence (U_n) of open subsets of U' be a neighbourhood basis at o. Note that $U_n + u$ for $u \in U'$ is a neighbourhood of u, since $(U_n + u) - u = U_n$ and $U \to X, x \mapsto x - u$ is continuous. Thus there are finite sets $C_n \subseteq C$ such that $C \subseteq U_n + C_n$ for all $n \in \mathbb{N}$, because C is compact. We show that $\{C \cap (U_n + c) : n \in \mathbb{N}, c \in C_n\}$ is a basis of C. Let W be an open neighbourhood of $w \in C$ in U'. Choose $c_n \in C_n$ such that $w \in U_n + c_n$, and assume that there is $u_n \in U_n$ such that $u_n + c_n \notin W$ for all $n \in \mathbb{N}$. Since $w - c_n, u_n \in U_n$, the sequences $(w - c_n)$ and (u_n) converge to o, and since C is compact, we can assume that (c_n) converges, namely to $w = (w - c_n) + c_n$. Thus the sequence $(u_n + c_n)$ in $U \setminus W$ converges to $w \in W$, a contradiction.

(3) The vertex space is second-countable: Using local coordinates (Corollary 2.4) we see that every vertex has a neighbourhood in V with a countable basis. If U is an open subset of V and $\mathcal{B}$ be a countable basis of U, then $\{f_{2,1}(X,Y) : X, Y \in \mathcal{B}\}$ is a countable basis of $D_1(U)$, because $f_{2,1}$ is a continuous and open map by Proposition 2.8(a). Thus by induction $D_{\leq l}(U)$ has a countable basis for all $l \in \mathbb{N}$, and consequently $V = \bigcup_{l \in \mathbb{N}} D_{\leq l}(U)$ has a countable basis.

(4) Finally panels as well as open subsets of V which are Hausdorff spaces are regular, and such spaces with a countable basis are metrizable; see [6, 4.2.9]. □

2.9 Generalized manifolds

In this subsection we prove that a large class of stable graphs has a locally contractible vertex space. Together with the local addition this will allow us to show that finite- and positive-dimensional locally compact stable graphs are cohomology manifolds. Let us recall the *small inductive dimension*, or simply *dimension*, $\operatorname{ind} X$ of a topological space X: it is defined by $\operatorname{ind} \emptyset = -1$ and $\operatorname{ind} X \leq n$ if X has a base consisting of sets U with $\operatorname{ind} \partial U \leq n - 1$ for $n \in \mathbb{N}_0$; see [6, 7.1]. For example a non-empty space is zero-dimensional if and only if every point has a neighbourhood base of open and closed sets. For locally compact Hausdorff space this is equivalent to the space being totally disconnected; see [6, 6.2.9]. The Euclidean spaces $\mathbb{R}^n$ satisfy $\operatorname{ind} \mathbb{R}^n = n$; see [6, 7.3.19]. Every subset A of a regular space X satisfies $\operatorname{ind} A \leq \operatorname{ind} X$; see [6, 7.1.1]. Note that a locally compact stable graph is positive-dimensional if and only if all its panels are non-discrete and locally connected.

An *ANR* or *absolute neighbourhood retract* X is a metric space with the following property: for any metric space Y of which X is a closed subspace there is a neighbourhood U of X in Y such that X is a retract of U. A locally contractible second-countable metric space of finite small inductive dimension is an ANR; see [10, V.7.1] (by [6, 4.1.16, 7.3.3] the small inductive dimension and the covering dimension, which is used in [10], agree).

Easier to understand are *ENRs* or *Euclidean neighbourhood retracts*: an ENR is a subset of some $\mathbb{R}^n$ which is a retract of an open subset of $\mathbb{R}^n$. Because a retract of a space is a closed subset of that space, every ENR is locally compact. This means it can also be embedded as a closed subset in some $\mathbb{R}^n$; see [3, E.2]. Since an ENR is a retract and $\mathbb{R}^n$ is locally contractible, ENRs are locally contractible. Thus an ENR is an ANR, because its small inductive dimension is finite.

Theorem 2.17. *In a positive-dimensional locally compact stable graph every panel is locally arcwise connected and locally contractible.*

In a finite- and positive-dimensional locally compact stable graph all open Hausdorff subspaces of the vertex space are ANRs and all open Hausdorff subspaces of graph-components as well as panels are ENRs.

Proof. Let (V, E) be a stable graph as above and $v \in V$. By Proposition 2.16 the panel $D_1(v)$ is metrizable. It is locally connected, and locally compact by assumption, so the panel is locally arcwise connected by a theorem related to the Hahn–Mazurkiewicz theorem; see [4, 9.B.1].

We show that $D_1(v)$ is locally contractible. Let $\cdot : U_1 \times U_2 \to D_1(v)$ be a multiplication with zeros $o_1 \in U_1$, $o_2 \in U_2$ and $o_3 \in D_1(v)$ as in Proposition 2.12. Let U be an open neighbourhood of o_3, and choose an open neighbourhood $U_1' \times U_2'$ of (o_1, o_2) such that $\cdot : U_1' \times U_2' \to U$. Let $\lambda\colon [0,1] \to U_2'$ be an arc from o_2 to some $y \in U_2' \setminus \{o_2\}$. The right-multiplication $g\colon U_1' \to U, x \mapsto x \cdot y$ is injective and open. Set $W := \operatorname{im} g$ and define

$$\Lambda\colon [0,1] \times W \to U, (t, x) \mapsto g^{-1}(x) \cdot \lambda(t).$$

Then $\Lambda(0, x) = g^{-1}(x) \cdot o_2 = o_1$ and $\Lambda(1, x) = g^{-1}(x) \cdot y = g(g^{-1}(x)) = x$ for all $x \in W$, so Λ is a homotopy from the constant map $o_2\colon W \to U$ to the inclusion $W \to U$. By local homogeneity panels are locally contractible, and using local coordinates (Proposition 2.3 and Corollary 2.4) we see that the vertex space is locally contractible.

Let X be a panel or an open subset of V which is a Hausdorff space. We show that X is an ANR. This is a local property by [10, III.8.1]; so we can assume that X is a second-countable metric space by Proposition 2.16. Then X has finite small inductive dimension by assumption, and we obtain that X is an ANR as a locally contractible space.

If X is contained in a graph component, then X is second-countable by 2.16 and therefore embeddable into some Euclidean space; see [11, V 3]. As an ANR it is also an ENR. □

Theorem 2.18. *Every panel and open Hausdorff subspace of the vertex space of a finite- and positive-dimensional locally compact stable graph is a cohomology manifold over any countable principal ideal domain with a unit.*

Proof. Let U be an open subset of a panel such that U is locally arcwise connected and locally contractible (Theorem 2.17), second-countable and metrizable (Proposition 2.16) and connected. Let L be a countable principal ideal domain with a unit. Since U is a locally compact Hausdorff space of finite inductive dimension, it also has finite cohomology dimension over L; see [2, II.16.38]. Because U is locally contractible, it is semi-locally 1-connected; i.e., every element has a neighbourhood whose embedding into U induces the trivial homomorphism between the respective fundamental groups. The local addition allows to construct local isotopies (Lemma 2.11); so we can assume that U is locally isotopic in the sense of [2, V.17.2]. Now a theorem of Bredon (see [2, V.17.6]) says that U is a cohomology manifold over L, because U is cohomology locally connected of any degree as U is locally contractible. This proves the theorem, since being a cohomology manifold is a local property. □

A topological space X is said to have the *domain invariance property* if for any homeomorphic subsets U and V of X the subset U is open if and only if V is open. Note that any cohomology manifold and in particular any Euclidean space $\mathbb{R}^n$ has the domain invariance property; see [2, V.16.9].

Let X be a locally homogeneous space with a basis consisting of cohomology manifolds (and therefore locally compact Hausdorff spaces). Note that we do not assume X to be a Hausdorff space. Let U be an open subset of X and $f\colon U \to V \subseteq X$ a homeomorphism. For any $x \in U$ there is an open neighbourhood W which is a locally compact Hausdorff space such that $f(U \cap W)$ is contained in a cohomology manifold and hence in a Hausdorff space with domain invariance. Then the open mapping theorem in [19, 51.19] implies that $f(U \cap W)$ is open. Thus $V = f(U)$ is open. We have shown that X has the domain invariance property and obtain the following corollary to Theorem 2.18.

Corollary 2.19. *The vertex space and all panels of a finite- and positive-dimensional locally compact stable graph have the domain invariance property.*

By a result of Bing–Borsuk second-countable cohomology manifolds with small inductive dimension $n \leq 2$ are locally homeomorphic to $\mathbb{R}^n$; for a proof see [2, II.16.38, V.16.32]. Thus we have the following result.

Corollary 2.20. *In a positive-dimensional locally compact stable graph panels of small inductive dimension $n \leq 2$ are locally homeomorphic to $\mathbb{R}^n$.*

2.10 Compact panels

In this subsection we prove properties of compact panels and derive properties of the vertex space.

Lemma 2.21. *Panels of locally connected non-discrete stable graphs have no proper non-empty open compact subsets.*

Proof. Let (V, E) be a locally connected non-discrete k-stable graph, and let $C \neq \emptyset$ be a compact and open proper subset of $D_1(v_1)$. By Proposition 2.12 and Lemma 2.13 there is a continuous multiplication with zeros o_i defined on a neighbourhood of $C \times \{o_2\}$ such that $C \cdot o_2 = \{o_3\}$ and $C \cdot y$ is open for $y \neq o_2$. Because C is compact, the sets $C \cdot y$ form a neighbourhood basis of o_3; see the proof of Theorem 2.16, part (1). By continuity the sets $C \cdot y$ are compact; so we obtain a neighbourhood basis of open and closed sets, since panels are Hausdorff spaces. By local connectedness panels are discrete, which is excluded. □

Since panels are Hausdorff spaces, we have that for compact panels their closed subsets are precisely their compact subsets, which proves the following corollary.

Corollary 2.22. *Compact panels of locally connected non-discrete stable graphs are connected.*

The conclusion of Lemma 2.21 also has implications on panels in the vicinity of a compact panel, as shown in the following lemma, which we state in a purely topological fashion. Part (b) says, roughly speaking, that a 'compact union' of compact connected sets in a locally compact space is compact; the compactness of the union is 'measured' with a relation satisfying a *weak continuity* assumption.

Lemma 2.23. *Let X and Y be locally compact spaces, and let $R \subseteq X \times Y$ be a locally closed relation such that for every $x \in X$ the subspace $R(x)$ of Y has no proper non-empty open and compact subspaces. Assume furthermore that for every $x \in X$ and every neighbourhood U of $R(x)$ in Y the set $R^{-1}(U)$ is a neighbourhood of x. Then the following holds.*

(a) *The relation R is continuous at any $x \in X$ for which $R(x)$ is compact, i.e. for any neighbourhood U of $R(x)$ there is a neighbourhood T of x such that*

$R(T) \subseteq U$. Furthermore T can be chosen such that $R(t)$ is compact for all $t \in T$.

(b) *If X is compact and all $R(x)$ are compact for $x \in X$, then R and the projection $R(X)$ are compact.*

Proof. We have assumed that R is a locally closed relation; so let W be an open subset of $X \times Y$ such that R is a closed subset of W.

(a) Let x and U be as above. Because $X \times Y$ is locally compact, the compact set $\{x\} \times R(x) \subseteq W$ has a compact neighbourhood of the form $T' \times U' \subseteq W$ satisfying $U' \subseteq U$. Thus $R \cap (T' \times U')$ is closed in $T' \times U'$, and $R(t) \cap U'$ is closed in U' and therefore compact for all $t \in T'$. Assume that there is no neighbourhood $T \subseteq T'$ of x satisfying $R(T) \subseteq U'$. Then there is a net (x_σ) in T' converging to x such that $R(x_\sigma) \nsubseteq U'$ and $R(x_\sigma)$ contains points of the interior U'° of U', because $R^{-1}(U'^\circ)$ is a neighbourhood of x by assumption. We have that $R(x_\sigma) \cap (U' \setminus U'^\circ) \neq \emptyset$, because otherwise $R(x_\sigma) \cap U'^\circ = R(x_\sigma) \cap U'$ would be a proper non-empty open and compact subset of $R(x_\sigma)$, which is not possible by assumption. Choose $u_\sigma \in R(x_\sigma) \cap (U' \setminus U'^\circ)$. The net (u_σ) has an accumulation point u in the compact set $U' \setminus U'^\circ$. Since $R \cap (T' \times U')$ is closed in $T' \times U'$, we have on the other hand that $u \in R(x) \subseteq U'^\circ$, which is a contradiction. Thus there is a neighbourhood $T \subseteq T'$ such that $R(T) \subseteq U' \subseteq U$, and we also have that $R(t) = R(t) \cap U'$ is compact for all $t \in T$.

(b) We need to show that any net (x_σ, y_σ) in R has a cluster point in R. By the compactness of X we can assume that (x_σ) converges to some $x \in X$. As X is locally compact, there is a compact neighbourhood U of $R(x)$ contained in the open subset $W(x)$ of Y. Then (a) implies that $R(x_\sigma)$ is finally contained in U and the net (y_σ) has therefore a cluster point $y \in U \subseteq W(x)$. By the closedness of R in W we have $(x, y) \in R$. Thus R and the image $R(X)$ are compact. □

Note that the assumption of *not containing any proper non-empty open and compact subspaces* cannot be replaced by the stronger assumption of connectedness, because it is not satisfied in the application Proposition 2.24(a).

To get a feeling for the condition on $R(x)$ and to see why the assumption of compactness of $R(x)$ in (a) is necessary consider the following example: let X be the set of ordinary affine lines of $Y = \mathbb{R}^2$, and let R be the reversed element relation; then this defines a 2-stable graph, and $U = \mathbb{R} \times]-1, 1[$ and $x = \mathbb{R} \times \{0\}$ show that the assumption of compactness in (a) is necessary.

By Lemma 2.21, Theorem 2.6 and Proposition 2.8(c) the assumptions of the previous lemma are satisfied for $R = D_1$, and we have the following corollary.

Proposition 2.24. *Let (V, E) be a locally compact locally connected non-discrete stable graph.*

(a) *The set of vertices v for which $D_1(v)$ is compact is open.*

(b) *If $C \subseteq V$ and all panels $D_1(v)$ for $v \in C$ are compact, then $D_1(C)$ is compact.*

(c) *The relation D_1 is continuous at any v for which $D_1(v)$ is compact, i.e. for any neighbourhood U of $D_1(v)$ there is a neighbourhood T of v such that $D_1(T) \subseteq U$.*

Note that using induction Proposition 2.24(b) implies that $D_{\leq l}(v)$ is compact for all $l \in \mathbb{N}_0$, if all panels are compact. We will see in the next section that there is a much stronger result.

3 Characterization of generalized polygons

In this section we characterize generalized $(k+1)$-gons among k-stable graphs by topological properties. The following lemma is basic and says that the set of vertices at distance at most $k-2$ of a given vertex is topologically a 'thin' set.

Lemma 3.1. *Let (V, E) be a locally connected k-stable graph, and let U be an open and connected subset of V. Fix $v \in V$ and $k_2 \leq k-2$, and assume that $D_{\leq l}(v)$ is closed for all $l \leq k_2$. Then $U \setminus D_{\leq l}(v)$ is connected for all $l \leq k_2$.*

Proof. In a first step we prove the existence of certain geometric neighbourhoods of elements in $D_{\leq k_2}(v)$. The rest of the proof is then purely topological.

(1) For $l \in \{0, \ldots, k_2\}$ and $w \in D_l(v)$ there are arbitrarily small neighbourhoods U_w of w such that $U_w \setminus D_l(v)$ is connected and dense in U_w: Let $(v_0, \ldots, v_k)$ be a non-stammering path with $(v, w) = (v_{k-l}, v_k)$. By Theorem 2.4 about local coordinates there are homeomorphic connected neighbourhoods U_w of w in V and $U_1 \times \cdots \times U_k$ of $(v_1, \ldots, v_k)$ in $D_1(v_0) \times \cdots \times D_1(v_{k-1})$ such that $U_w \setminus D_l(v)$ is homeomorphic to $\big(U_1 \times \cdots \times U_{k-l} \setminus \{(v_1, \ldots, v_{k-l})\}\big) \times U_{k-l+1} \times \cdots \times U_k$. This set is connected, since $k - l \geq 2$. It is dense, because panels have no isolated points by Proposition 2.14.

(2) We will now prove the lemma by induction over l. Let $l \in \{0, \ldots, k_2\}$ and assume that $U' := U \setminus D_{<l}(v)$ is connected. By assumption U' is open and $A := D_l(v) \cap U'$ is closed in U'. Let Z be a connected component of $U' \setminus A$. We show that the closure $\overline{Z}$ taken in U' is open in U'. Let $w \in \overline{Z}$. If $w \notin A$, then w has a connected neighbourhood disjoint from A. This neighbourhood meets the component Z and is therefore contained in it. If $w \in A$ there is a neighbourhood U_w as in (1). Since it meets

the component Z, the connected subset $U_w \setminus A$ also meets this component and is therefore contained in it. By the density property from (1) we have $U_w \subseteq \overline{U_w \setminus A} \subseteq \overline{Z}$. This proves that $\overline{Z}$ is open and therefore equal to U'. Now because of $Z \subseteq \overline{Z} \setminus A \subseteq \overline{Z}$ the set $\overline{Z} \setminus A$ is connected and therefore contained in the component Z. Thus $U' \setminus A = \overline{Z} \setminus A = Z$ is connected. □

The following proposition is the basis for the characterization theorem for generalized polygons.

Proposition 3.2. *Let (V, E) be a locally connected k-stable bipartite graph, and let $v \in V$ be a vertex such that $D_{\leq l}(v)$ is closed for all $l \leq k$. Then any connected subset of V meeting $D_k(v)$ is contained in $D_{\leq k}(v)$.*

Proof. Let U be a connected subset of V meeting $D_k(v)$. We can assume that U is a connected component. It is closed and also open, because V is locally connected. The bipartiteness of (V, E) implies that $D_{k+2\mathbb{Z}}(v)$ is an open and closed subset of V. Thus U is contained in this subset. Consequently $U \cap D_k(v)$ is closed in $U \setminus D_{\leq k-2}(v)$, because $D_{\leq k}(v)$ is closed; the intersection is also closed by stability. But $U \setminus D_{\leq k-2}(v)$ is connected by Lemma 3.1, so $U \cap D_k(v) = U \setminus D_{\leq k-2}(v)$, which completes the proof. □

Assume that the adjacency relation D_1 is closed and that $D_{<k}(v)$ is compact. It can be seen, for example, with nets that $D_{\leq l}(v)$ is closed for all $l \leq k$ (cf. the proof of Observation 1.2); so we have the following corollary.

Corollary 3.3. *Let (V, E) be a locally connected k-stable bipartite graph with a closed adjacency relation D_1. If $v \in V$ is a vertex such that $D_{<k}(v)$ is compact, then $D_{\leq k}(v)$ contains the connected component of every vertex in $D_k(v)$.*

This is a generalization of a result of Löwen (see [15, 1.15]) for stable planes which says that a compact line meets every line; indeed a compact line meets some other line; thus it meets all lines, because the line space is connected. The above corollary also implies that a stable triangle with compact line pencils is a linear space (any two points can be joined by a line) if the point space is connected. We give another version of the above corollary which does not assume that the adjacency relation is closed.

Corollary 3.4. *Let $\mathcal{G} = (V, E)$ be a locally connected k-stable polygon such that V is a Hausdorff space. If $v \in V$ is a vertex such that all panels $D_1(w)$ are compact for $w \in D_{<k}(v)$, then $D_{\leq k}(v)$ contains the connected component of every vertex in $D_k(v)$.*

Proof. Because $D_1(v)$ and $D_1(w)$ for some $w \in D_1(v)$ are compact, all panels in the graph component of v are locally compact by local homogeneity; so with local coordinates we see that also the component is locally compact. By induction Proposition 2.24(b) implies that $D_{\leq l}(v)$ is compact for all $l \leq k$. Thus we can apply Proposition 3.2, because V is a Hausdorff space □

For the next theorem we need the following lemma.

Lemma 3.5. *The vertex space of a locally connected stable graph with only compact panels is a Hausdorff space.*

Proof. Let (V, E) be a locally connected stable graph such that all panels are compact. It is enough to show the lemma for non-discrete graph-connected stable graphs, as graph-components are open, and discrete spaces are Hausdorff spaces. We consider the two cases of odd and even k separately, and in each case we will assume that there are two vertices which cannot be separated and derive the contradiction that they can be separated after all.

Let k be even and $w \in V$. Because vertices at distance at most $2k$ can be separated by Theorem 2.6 and because $D_{\leq k}(w)$ is compact by Corollary 2.24(b), there is a neighbourhood U of $D_{\leq k}(w)$ which is a Hausdorff space by Lemma 2.7. Choose a connected open neighbourhood $W \subseteq U$ of w. Consider the k-stable graph induced on U now. Since this stable graph is not discrete, panels that meet W have infinitely many points with W in common. Thus, because k is even, W contains a vertex at distance k from w. By Corollary 3.4 we have $W \subseteq D_{\leq k}(w)$. It has been shown that every vertex w has a neighbourhood contained in $D_{\leq k}(w)$. Thus, if there were two vertices w and w' which cannot be separated, neighbourhoods of w and w' as above would meet, and therefore the two vertices would have distance at most $2k$. This yields that w and w' can be separated.

Let k be odd and $(v, w) \in D_1$. We leave out a few details which are similar to the even case. There is a neighbourhood U of $D_{\leq k}(v)$ which is a Hausdorff space. Choose a connected neighbourhood $W \subseteq U$ of w. Because k is odd, we can argue as in the even case that there is a vertex in W at distance k from v. Thus W is contained in $D_{\leq k}(v)$. Now assume that w and w' are two vertices that cannot be separated. Then for any $v \in D_1(w)$ and $v' \in D_1(w')$ and neighbourhoods W and W' of w and w' as above there is an $x \in W \cap W'$. Thus we have $d(v, v') \leq 2k$, and therefore v and v' can be separated. This holds for all $v \in D_1(w)$ and $v' \in D_1(w')$. Thus, since $D_1(w')$ is compact, there are disjoint neighbourhoods X and Y of v and $D_1(w')$ respectively. By Proposition 2.24(c) there is a neighbourhood Y' of w' such that $D_1(Y') \subseteq Y$. Now $D_1(X)$ and Y' are separating neighbourhoods for w and w'. □

Here is the main theorem of this section. It says that the k-stable graphs which are generalized polygons can be characterized by the compactness of panels. It is not necessary to assume that the graph is bipartite, that the girth is $2k+2$, or that the vertex space is a Hausdorff space.

Theorem 3.6. *Let a graph-connected locally connected non-discrete stable graph be given. Then the following statements are equivalent.*

(a) *The graph is a generalized polygon, and the vertex set is locally compact.*

(b) *The vertex set is compact, and the adjacency relation is closed.*

(c) *All panels are compact.*

Proof. The implication (a)⇒(b) follows from [13, 2.5.5, 2.5.2], and (b)⇒(c) follows, because closed subsets of compact spaces are compact.

Assume that $\mathcal{G} = (V, E)$ is a k-stable graph as above such that all panels are compact. Then V is a Hausdorff space by Lemma 3.5. Let us consider the bipartite case first. By Corollary 2.22 all panels are connected. Thus any two vertices at distance 2 (and therefore at even distance) are contained in a connected set. Thus V consists of precisely two connected components, the two classes of the bipartite graph $\mathcal{G}$. Now Corollary 3.4 implies that the diameter of $\mathcal{G}$ is at most $k+1$. It cannot be k, because $\mathcal{G}$ is bipartite and the girth is at least $2k+1$. Thus the girth is $2k+2$, and we have shown that $\mathcal{G}$ is a generalized $(k+1)$-gon.

We still need to exclude the non-bipartite case. Consider the bipartite graph $\mathcal{G}\times \mathrm{K}_2$ with vertex set $V\times\{1,2\}$, where two vertices (v,i) and (w,j) are adjacent if and only if $\{v,w\} \in E$ and $i \neq j$. It has compact panels, and it is graph-connected, because $\mathcal{G}$ is graph-connected and not bipartite. Thus $\mathcal{G} \times \mathrm{K}_2$ is a generalized $(k+1)$-gon. Let v and w be two adjacent vertices of $\mathcal{G}$. In $\mathcal{G} \times \mathrm{K}_2$ the vertices $(v,1)$ and $(w,1)$ have an even distance less than or equal to $k+1$, and there is also a path of this even length from v to w in $\mathcal{G}$. So there is a path with equal end-points of odd length $l \leq k+2$ in $\mathcal{G}$. Because l is odd this path contains a cycle, and we can conclude that the girth of $\mathcal{G}$ is bounded by l, so $2k < l \leq k+2$. It follows that $k=1$, which is a contradiction, because we have excluded this case from our definition of stable graphs. □

Corollary 3.7. *Every locally connected non-discrete stable graph with compact panels is bipartite.*

Every totally disconnected compact metric space without isolated points is homeomorphic to the Cantor set; see [4, 6.C.11]. Thus by 2.16 we have the following result.

Corollary 3.8. *If all panels of an infinite stable Moore graph are compact, then each of them is homeomorphic to the Cantor set. If furthermore the vertex set is compact, then it is also homeomorphic to the Cantor set.*

If infinite stable Moore graphs exist at all, the analogy to generalized polygons does not work, because the adjacency relation is not closed in the case of stable Moore graphs by Lemma 1.1. Therefore it seems unlikely that there is a Moore graph with compact panels. On the other hand panels should be compact, because the diameter is the smallest possible. If there are infinite stable Moore graphs at all, I would expect that the vertex space is compact and that the *extended panels* $D(v) \cup \{v\}$ are compact for all vertices $v \in V$ and not all or maybe no panels are compact.

Problems 3.9. (a) Is there an infinite stable Moore graph with compact (extended) panels?

(b) Is there an infinite Moore graph with a compact vertex space and a semi-closed adjacency relation?

(c) Is there a stable Moore graph on a locally Euclidean space?

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Nils Rosehr

MATHEMATISCHES INSTITUT, UNIV. WÜRZBURG, AM HUBLAND, 97074 WÜRZBURG, GERMANY

e-mail: `rosehr@mathematik.uni-wuerzburg.de`

website: `http://www.mathematik.uni-wuerzburg.de/~rosehr`