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Collineation groups with one or two orbits on the set of points not on an oval and its nucleus

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Abstract

Projective planes of even order admitting a collineation group fixing an oval and having one or two orbits on the set of points not on the oval and its nucleus are investigated.

Keywords: projective plane; collineation group; oval

MSC 2000: 51E23, 51A40

1 Introduction

Let Π be a projective plane of order n and $\mathcal{O}$ an *oval*: a set of $n+1$ points no three collinear. In this paper we only consider the case where n is even. In this case all the tangent lines to $\mathcal{O}$ are on the same point K, called the *nucleus* of $\mathcal{O}$. Denote by $\mathcal{E}$ the set of points not in $\mathcal{O} \cup \{K\}$. The aim of this paper is to investigate the following situation:

> *Let G be a collineation group fixing $\mathcal{O}$. Assume that G has one or two orbits on $\mathcal{E}$. Then determine the abstract structure of G, the plane and the oval.*

The case of one orbit is treated in [10], where we proved the following theorem:

Theorem 1.1 ([10, Theorem 2.4]). *Let Π be a projective plane of even order n. Then G is transitive on $\mathcal{E}$ if and only if the plane is desarguesian, $\mathcal{O}$ is the set of points of an irreducible conic and $G \trianglerighteq \mathsf{PSL}(2,n)$.*

The proof uses the classification of line-transitive maximal arcs [5] which depends on the classification of finite simple groups. We raised the question of finding a proof avoiding such classification.

In this paper we give a different proof that does not use that classification. We also provide a classification of projective planes and groups having two orbits on the set $\mathcal{E}$. We prove that there are precisely three possibilities: either

(1) the plane is the dual Lüneburg plane of order $n = 2^{2(2e+1)}$, some $e \geq 1$ and G contains a subgroup isomorphic to the Suzuki simple group $\mathsf{Sz}(\sqrt{n})$; or

(2) $\mathcal{O}$ is a two-transitive parabolic oval; or

(3) $n = 4$, $\Pi = \mathsf{PG}(2,4)$, $\mathcal{O}$ is the set of points of an irreducible conic and $G \trianglerighteq \mathsf{PSL}(2,4)$.

An oval $\mathcal{O}$ is called a *two-transitive parabolic oval* if there is a tangent line ℓ and a collineation group G fixing ℓ and acting 2-transitively on the points of $\mathcal{O}$ not on the tangent line. Two-transitive parabolic ovals are classified in [2]. For the sake of completeness we report on such a classification.

Theorem 1.2 ([2, Theorem 2.1]). *Let $\mathcal{O}$ be a two-transitive parabolic oval with group G. Then $\mathcal{O}$ is a translation oval, $n = 2^d$, G is isomorphic to a subgroup of $\mathsf{A\Gamma L}(1, 2^d)$ and contains the translation group of $\mathsf{A\Gamma L}(1, 2^d)$.*

2 One orbit

Let Π be a projective plane of even order n, $\mathcal{O}$ an oval with nucleus K and G a collineation group fixing $\mathcal{O}$. We say that G is *strongly irreducible* on $\mathcal{O}$ if G does not fix any point, secant line or suboval of $\mathcal{O}$. The main result of [1] states that if G is strongly irreducible on $\mathcal{O}$ and has *even* order, then G has involutory elations. Let $\langle\Delta\rangle$ be the subgroup generated by the set Δ of all elations. Then both the structure of $\langle\Delta\rangle$ and its action on $\mathcal{O}$ are determined (see [1] and [9], and note that both papers do not use the classification of finite simple groups).

Theorem 2.1. *Let G be strongly irreducible on $\mathcal{O}$ and assume $|G|$ even. Then one of the following holds.*

(1) *G fixes a line, and $\langle\Delta\rangle \cong O(\langle\Delta\rangle) \rtimes \langle\gamma\rangle$, where γ is a nontrivial elation. Moreover, G does not contain Baer involutions.*

(2) *G does not fix any line, and either*

   (a) *$n = q = 2^d \geq 4$, $\langle\Delta\rangle \cong \mathsf{PSL}(2,q)$, $\Pi = \mathsf{PG}(2,q)$ and $\mathcal{O}$ is the set of points of an irreducible conic; or*

   (b) *$n = 2^{2e+1}$, with $e \geq 1$, Π is the dual Lüneburg plane of order n and $G \trianglerighteq \mathsf{Sz}(\sqrt{n})$; or*

(c) $\langle\Delta\rangle \cong \mathsf{PSU}(3, q^2)$, *where* $n = q \geq 4$ *is a suitable power of* 2.

If G *is transitive on* $\mathcal{O}$, *then the case* $\langle\Delta\rangle \cong \mathsf{PSU}(3, q^2)$ *does not hold.*

In cases (2)(a) and (2)(b) the group G is 2-transitive on $\mathcal{O}$ and has one and two further point orbits on the set $\mathcal{E}$, respectively.

The case where G has precisely one orbit is the desarguesian case.

Theorem 2.2. *Let* G *be transitive on* $\mathcal{E}$. *Then* $n = q = 2^d$, *where* $d \geq 1$, $\Pi = \mathsf{PG}(2, q)$, $\mathcal{O}$ *is the set of points of an irreducible conic and, if* $q \geq 4$, $G \trianglerighteq \mathsf{PSL}(2, q)$.

Proof. First of all we note that G is transitive on $\mathcal{O}$. For, given $A, B \in \mathcal{O}$, choose $P \in KA$ and $Q \in KB$, with $P, Q \in \mathcal{E}$; hence there is $g \in G$ mapping P onto Q, and so A onto B.

Let N be a minimal normal subgroup of G. Then by [7, Theorem 4.3A and Corollary 4.3A] either

(A) N is a direct product of normal nonabelian simple subgroups of N which are conjugate under G; or

(B) N is an elementary abelian p-group for some prime p.

In case (A), G contains a nonabelian simple group, so that its order is even. By Theorem 2.1 either case (2)(a) or (2)(b) holds. The latter cannot hold, since in the dual Lüneburg plane the group fixing $\mathcal{O}$ has two further orbits on $\mathcal{E}$. Therefore only the case where $\Pi = \mathsf{PG}(2, q)$, $G \trianglerighteq \mathsf{PSL}(2, q)$ and $\mathcal{O}$ is a conic holds.

Now we examine case (B). Since G is transitive on $\mathcal{O}$ and $N \trianglelefteq G$, then N is fixed point free on $\mathcal{O}$ (the set of fixed points of N is mapped onto itself by G). Therefore N has orbits whose lengths are powers of p; hence $p \mid n + 1$, and so p is odd. As a consequence, p does not divide $n/2$, nor $n - 1$. Then p does not divide $n(n-1)/2$. Since this number is the number of exterior lines to $\mathcal{O}$, then N fixes some exterior line. Let $\mathcal{F}$ be the set of fixed lines of N. Since $N \trianglelefteq G$, then G fixes $\mathcal{F}$. Moreover, as G is transitive on $\mathcal{E}$, the following condition must be satisfied:

The number of points that are on the lines of $\mathcal{F}$ *is greater than or equal to* $n^2 - 1$. $(*)$

We examine the possibilities for $\mathcal{F}$.

(a) $\mathcal{F}$ *consists of one line.*

Because of $(*)$, $n + 1 \geq n^2 - 1$. So $n = 2$. Clearly, $\Pi = \mathsf{PG}(2, 2)$ and $\mathcal{O}$ is a conic. Moreover, G is the symmetric group on 3 elements and N is the alternating group.

(b) *$\mathcal{F}$ consists of $k \geq 2$ lines through a point P.*

In this case, the point P should belong to $\mathcal{E}$, and this is absurd, as G is transitive on $\mathcal{E}$.

(c) *$\mathcal{F}$ consists of three lines not on the same point.*

By condition $(*)$, we have $3(n-2)+3 \geq n^2-1$. So $n=2$, $\Pi = \mathsf{PG}(2,2)$ and $\mathcal{O}$ is a conic.

(d) *$\mathcal{F}$ is the set of lines of a subplane π_0.*

Since G fixes $\mathcal{F}$, then G fixes π_0. So, because of $(*)$, π_0 must be a Baer subplane. Moreover $\pi_0 \cap \mathcal{O} = \emptyset$. Let $g \in N$ be a p-element. If $A \in \mathcal{O}$, then the line AA^g meets π_0 in a fixed point of N. So $A^{g^2} = A$. Hence $g^2 = 1$ and so $p = 2$, which contradicts the fact that p is odd. $\square$

3 Two orbits

As before Π is a projective plane of even order n, $\mathcal{O}$ is an oval with nucleus K and G a collineation group fixing $\mathcal{O}$. Here we consider the case where G has two orbits on $\mathcal{E}$, the set of points not in $\mathcal{O} \cup \{K\}$.

Theorem 3.1. *Assume that G has two orbits on $\mathcal{E}$. Then either*

(1) *G fixes a point $P_\infty \in \mathcal{O}$ and acts 2-transitively on $\mathcal{O} \setminus \{P_\infty\}$; or*

(2) *$n = 2^{2(2e+1)}$, where $e \geq 1$, Π is the dual Lüneburg plane of order n and $G \trianglerighteq \mathsf{Sz}(\sqrt{n})$; or*

(3) *$n = 4$, $\Pi = \mathsf{PG}(2,4)$, $\mathcal{O}$ is the set of points of an irreducible conic and $G \trianglerighteq \mathsf{PSL}(2,4)$.*

Proof. We begin with the following lemma.

Lemma 3.2. *Let H be a collineation group fixing $\mathcal{O}$. If $|H|$ is odd, then H fixes a line.*

Proof. By the Feit–Thompson theorem, H is soluble. Therefore H contains a normal subgroup N, which is an elementary abelian p-group, where p is an odd prime. Let $\mathrm{Fix}(N)$ be the set of points fixed by N. Since N is a normal subgroup of H, then H maps $\mathrm{Fix}(N)$ onto itself. In its action on $\mathcal{O}$, there are two possibilities for N: either

(a) $\mathrm{Fix}(N) \cap \mathcal{O} = \emptyset$; or

(b) $\mathrm{Fix}(N) \cap \mathcal{O} \neq \emptyset$.

We prove that in case (a) there is precisely one fixed line which is exterior to $\mathcal{O}$.

Since N fixes no point of $\mathcal{O}$, then $p \mid n+1$, and so p does not divide $n(n-1)/2$, which is the number of exterior lines. Therefore N fixes some exterior line. If N could fix two exterior lines, then their point of intersection P would be a fixed point, distinct from K. Therefore the tangent line KP would meet $\mathcal{O}$ in a fixed point; a contradiction.

As for case (b), there are the following subcases to examine:

(i) $|\mathrm{Fix}(N) \cap \mathcal{O}| = 1$. In this case N fixes the tangent line through the unique fixed point of N.

(ii) $|\mathrm{Fix}(N) \cap \mathcal{O}| = 2$. The secant line joining the two fixed points is clearly a fixed line.

(iii) $|\mathrm{Fix}(N) \cap \mathcal{O}| \geq 3$. As N fixes the nucleus K, then N fixes a quadrangle. Thus the fixed points and lines of N form a subplane π_0. Let m be the order of π_0. Since K is a fixed point, then on each of the $m+1$ lines of π_0 passing through K there is a point of $\mathcal{O}$, which is then a fixed point of N. Therefore the set $\mathcal{O}_0 = \pi_0 \cap \mathcal{O}$ has size $m+1$ and so is an oval of π_0, on which N acts trivially. If H acts trivially on π_0, then H has fixed lines. Assume that H acts non-trivially on π_0. Then H/N induces a collineation group of π_0 which fixes $\mathcal{O}_0$. Now H/N has odd order, so we can repeat the same argument, as at begin of the proof, with π_0, H/N and $\mathcal{O}_0$ instead of Π, H and $\mathcal{O}$, until we will find a subplane and a collineation group acting trivially on it. $\square$

We note *en passant* the following result, which seems interesting.

Corollary 3.3. *If H has no fixed point on $\mathcal{O}$ and $|H|$ is odd, then H fixes precisely one line, which is exterior to $\mathcal{O}$.*

Now we can conclude the proof of Theorem 3.1.

Since G has two orbits on $\mathcal{E}$ and $|\mathcal{E}| = n^2 - 1$ is odd, then one orbit has even length. Therefore G has even order, and so has involutions. We aim to prove that G has involutory elations.

By way of contradiction, assume that G has only Baer involutions. Therefore $G \cong O(G) \rtimes S_2$, where S_2 is a cyclic Sylow 2-subgroup (see [1, Proposition 5]) and $O(G)$ is the largest subgroup of G of odd order. Clearly $O(G)$ also acts on $\mathcal{O}$, and since $|O(G)|$ is odd, it has fixed lines, because of the previous lemma. There are several cases to treat.

1. $O(G)$ *fixes an exterior line*.

   Since $O(G) \trianglelefteq G$, the points of this line constitute one of the two orbits of G on $\mathcal{E}$. Therefore G is transitive on $\mathcal{O}$ and case (1) of Theorem 2.1 applies. But in this case G has no Baer involution; so this case cannot hold.

2. $O(G)$ *fixes a secant line*.

   Let s be this secant line, let $s \cap \mathcal{O} = \{A, B\}$, and let t_A and t_B be the tangent lines at A and B respectively. The group $O(G)$ fixes the two points of intersection of this secant line or interchanges them. Since $O(G)$ is odd, then $O(G)$ must fix each of the two points. Therefore G would have at least three orbits on $\mathcal{E}$: the points of $s \setminus \{A, B\}$, the points of $t_A \setminus \{A\}$ and the points of $t_B \setminus \{B\}$. This is a contradiction.

3. $O(G)$ *fixes a tangent line* t.

   Let A be the point of tangency. Then $t \setminus \{A, K\}$ is one of the two orbits of G on $\mathcal{E}$. Let $\pi_1, \pi_2, \ldots, \pi_k$ be the Baer subplanes determined by S_2 and its conjugates. All these planes share the line t, and for the corresponding affine planes having t as line at infinity we have (letting π_i^t and Π^t to denote the set of points of the affine planes)

   $$\bigcup_{i=1}^{k} \pi_i^t = \Pi^t . \tag{1}$$

   We claim that (1) is a partition. For if C is, for example, a common point of π_1^t and π_2^t, then C should be fixed by the corresponding Baer involutions: $C^{\gamma_1} = C^{\gamma_2}$. So $C^{\gamma_1\gamma_2} = C$, and $\gamma_1\gamma_2 \in O(G)$. Therefore $O(G)$ fixes a point not in t, and so $O(G)$ must fix a secant line. This case has been proven to be impossible.

   The partition (1) induces a partition of $\mathcal{O} \setminus \{t\}$. Therefore we have two equations involving n and k

   $$\begin{cases} k\sqrt{n} = n \\ n^2 - n = k(n - \sqrt{n}) , \end{cases}$$

   which are clearly incompatible.

4. $O(G)$ *fixes a subplane* π_0.

   Since $G \cong O(G) \rtimes S_2$, then S_2 induces a collineation group on π_0 and the Baer involution γ of S_2 is either a Baer involution on π_0 or the identity (cf. [4, Proposition 2.3]). Let γ be the identity on π_0. Let C be any point of π_0 and let ℓ be a line of π_0 exterior to $\mathcal{O}_0 = \mathcal{O} \cap \pi_0$. Then ℓ, as a line of Π, is fixed by γ, which is a Baer involution. Therefore ℓ is secant to $\mathcal{O}$, and so $\ell \cap \mathcal{O}$ is a pair of points fixed by $O(G)$. Therefore $O(G)$ fixes a secant line to $\mathcal{O}$; this case has been proven to be impossible (cf. case 2.).

Therefore γ is a Baer involution of π_0, and so defines a Baer subplane of π_0. We repeat the same argument as before until G will have at least three fixed points on $\mathcal{O}$. This will give rise to a contradiction, as G would have at least three orbits on $\mathcal{E}$.

In conclusion G has some involutorial elation. In particular then $n = 2$ or $n \equiv 0 \pmod 4$ ([6, Chapter 4, sec. 1, result 12]). Let Λ be the set of all centres of these elations. Since $\Lambda \subseteq \mathcal{E}$, then Λ is one of the two orbits of G on $\mathcal{E}$. We have to examine two alternatives for Λ: either

(I) Λ is contained in one tangent line; or

(II) Λ is contained in more than one tangent line.

Case (I). Let t be the tangent line containing Λ. Clearly $\Lambda = t \setminus \{K, P_\infty\}$, where P_∞ is the point of tangency. We prove that G acts 2-transitively on $\mathcal{O} \setminus \{P_\infty\}$. It suffices to show that for $A, B, B' \in \mathcal{O} \setminus \{P_\infty\}$ there is an element $g \in G$ which fixes A and maps B to B'. To prove this, let $U = P_\infty B \cap KA$ and $U' = P_\infty B' \cap KA$. Since U and U' are in $\mathcal{E} \setminus \Lambda$, there is $g \in G$ mapping U to U'. Clearly this element maps, by construction, B to B'. We proved that case (I) corresponds to case (1) of the theorem.

Case (II). Every tangent line contains an element of Λ, that is, every tangent is the axis of a nontrivial elation. Therefore G acts transitively on $\mathcal{O}$. Since G has even order, Theorem 2.1 applies. The case that G contains $\mathsf{PSL}(2, q)$ does not hold, since in this case G would have only one orbit on $\mathcal{E}$. To exclude case (1) of Theorem 2.1 we prove that if $n > 4$ then G cannot fix any line. Clearly G cannot fix a secant line (otherwise it would have more than two orbits on $\mathcal{E}$). We argue by contradiction and assume that G fixes an exterior line ℓ. First of all we prove:

Lemma 3.4. *G is 2-transitive on $\mathcal{O}$.*

Proof. G has four orbits of points: $\mathcal{O}$, $\{K\}$, the set of points of ℓ and $\mathcal{E} \setminus \ell$. Therefore G has four orbits of lines. The tangent lines and $\{\ell\}$ are two orbits, the other two are clearly the set of secant lines to $\mathcal{O}$ and the set of exterior lines minus ℓ. Hence G acts transitively on the secant lines, and so it is 2-homogeneous. Since $|G|$ is even, then G is 2-transitive on $\mathcal{O}$ (see [8, Theorem 1, case (i)] or [7, Theorem 9.4B]). □

Lemma 3.5. *Let t be a tangent line and T its point of tangency. If $n > 2$, then G_T acts transitively on $t \setminus \{K, T, C\}$, where $C = \ell \cap t$.*

Proof. Let $A, B \in t \setminus \{K, T, C\}$. Since $A, B \in \mathcal{L} \setminus \ell$, there is $g \in G$ mapping A to B. Therefore g fixes the tangent line t and so $g \in G_T$. □

Because of Lemma 3.5, if $n > 2$, then $n-2$ divides $|G|$. By Lemma 3.4, G is 2-transitive, and so its order is $|G| = n(n+1)a$, for some integer $a \geq 1$. Now $(n(n+1), n-2)$ is either 2 or 6. Therefore the order of G is either

$$[n(n+1)(n-2)/2]b \text{ or}$$
$$[n(n+1)(n-2)/6]c$$

and in this case $n \geq 8$. By the main result of [3], we have $n+1 = q = p^d$, where p is an odd prime, and G is isomorphic to a subgroup of $\mathsf{A\Gamma L}(1,q)$. Now

$$|\mathsf{A\Gamma L}(1,q)| = q(q-1)d\,.$$

Since $|G|$ divides $q(q-1)d$ and $n = q-1$, we get either

$$(q-3)b/2 \leq d \text{ or} \tag{2}$$
$$(q-3)c/6 \leq d\,. \tag{3}$$

Case (2) is possible only if $q = 5$, $b = 1$ and $d = 1$; hence $n = 4$, the plane is desarguesian and $\mathcal{O}$ is a conic. Case (3) holds only if $q = 9$, $c = 1$ and $d = 2$; hence $n = 8$, the plane is desarguesian and $\mathcal{O}$ is a conic, but there is no collineation group acting 2-transitively on $\mathcal{O}$ and fixing an exterior line.

We conclude that if $n > 4$, then G fixes no line. Therefore case (2) of Theorem 2.1 applies, and so $n = 2^{2(2e+1)}$, where $e \geq 1$, the plane Π is the dual Lüneburg plane of order n and $G \trianglerighteq \mathsf{Sz}(\sqrt{n})$. □

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