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Generalized Clifford parallelisms

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Dedicated to Mario Marchi on the occasion of his 70th birthday

Abstract

We define generalized Clifford parallelisms in $\mathsf{PG}(3, F)$ with the help of a quaternion skew field H over a field F of arbitrary characteristic. Moreover we give a geometric description of such parallelisms involving hyperbolic quadrics in projective spaces over suitable quadratic extensions of F.

Keywords: Clifford parallelism, quadratic forms, quadrics, quaternions

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1 Introduction

It is known that the three dimensional real projective space $\mathsf{PG}(3, \mathbb{R})$ can be endowed with two projectively equivalent parallelisms, namely the *left* and *right Clifford parallelisms*, related to left and right multiplications in the Hamilton quaternion algebra $\mathbb{H}(\mathbb{R})$ (see e.g. [15]). For these parallelisms there are many equivalent geometric representations (see e.g. [22, Sec. 142], [8, 12 A], [16, Chapter 14]). In particular each parallel class can be described considering the lines that meet a fixed imaginary line (and its conjugate) belonging to one of the two reguli of a complex hyperbolic quadric whose points do not belong to $\mathsf{PG}(3, \mathbb{R})$.

The aim of this work is to extend these notions to the projective 3-space over a general (commutative) field of arbitrary characteristic. This can be done in several ways, using constructions that involve either rings of *generalized quaternions*, or the notions of *Baer subspace* of a projective space and *indicator set* of

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a spread of lines (see [12], [13]). All these constructions always give rise to regular parallelisms.

In particular we consider these constructions over a field which admits different quadratic extensions.

For the description of the parallelisms our starting point in Section 4 is the context of projective kinematic spaces (see [15]). Here the lines through the point $\mathbf{1}$ are the maximal commutative subalgebras of H. The set of these lines is partitioned into conjugacy classes with respect to the quaternion multiplication. We show that each conjugacy class of lines corresponds to a different quadratic extension L of the field F and determines by right and left cosets respectively two sets of mutually disjoint spreads which are some of the right and left Clifford parallel classes. Such spreads are indicated by the lines of the two reguli of a quadric in $\mathsf{PG}(3, L)$ with no points in $\mathsf{PG}(3, F)$ (Theorem 4.7). Following this procedure for all quadratic extensions L/F the whole line set of $\mathsf{PG}(3, F)$ is covered, thus obtaining the complete right and left Clifford parallelisms with respect to the given quaternion skew field H (see Theorem 4.10). We remark that different Clifford parallelisms corresponding to different quaternion skew fields over F are not projectively equivalent. Moreover new "non-Clifford" regular parallelisms can be obtained, using a method which has no equivalent in the classical case (see Remark 4.13).

2 Quadratic spaces and quaternion algebras

Recall from [19] that a *quadratic space* $(V(F), q)$ is an n-dimensional vector space V over a field F of arbitrary characteristic, endowed with a quadratic form q. The *bilinear form* corresponding to q is $b_q(v, w) = q(v+w) - q(v) - q(w)$ for all $v, w \in V$, and the quadratic space $(V(F), q)$ is said to be *regular* or *non-degenerate* if b_q is non-degenerate, *singular* otherwise.

If $(V(F), q)$ is a quadratic space, a nonzero vector $v \in V$ is said to be *isotropic* if $q(v) = 0$ and *anisotropic* otherwise. We say that $(V(F), q)$ is *isotropic* if it contains an isotropic vector, *anisotropic* otherwise; a subspace W of V is said to be *totally isotropic* if $b_q(W, W) = 0$; an isotropic n-dimensional quadratic form is said to be *hyperbolic* if n is even and V is a direct sum of two totally isotropic $(n/2)$-dimensional subspaces.

If $(V(F), q)$ is a regular n-dimensional quadratic space and $\operatorname{char} F \neq 2$, the *discriminant* of q is $d(q) = (-1)^{n(n-1)/2} \det(q)$ considered as an element of $F^*/(F^*)^2$, where we denote by $(F^*)^2$ the group of squares of F^*. If, on the contrary, $\operatorname{char} F = 2$, then the symmetric bilinear form b_q is in fact alternating, thus one can fix a symplectic basis $\big(e_1, \dots, e_{n/2}, f_1, \dots, f_{n/2}\big)$ of $(V(F), q)$

(that is a basis such that $b_q(e_i, f_i) = 1$ for every $i = 1, 2, \dots, n/2$, and all other pairs of vectors are orthogonal) and define the discriminant to be $d(q) = q(e_1)q(f_1) + \cdots + q(e_{n/2})q(f_{n/2})$ considered as an element of $F/\{x + x^2 \mid x \in F\}$ (see e.g. [19, 9.4.2]).

If $(V(F), q)$ is an n-dimensional regular quadratic space over the field F, then associated to q there is a non-degenerate quadric $\mathcal{Q}$ in the projective space $\mathsf{PG}(n-1, F)$, namely the quadric whose points are represented by the isotropic vectors of q ($\mathcal{Q} = \emptyset$ if q is anisotropic). Conversely, given any non-degenerate quadric in $\mathsf{PG}(n-1, F)$, its equation gives rise to a family of *similar* regular quadratic forms q_ρ over F^n, i.e. quadratic forms whose elements differ in a proportional factor $\rho \in F^*$. Note that the quadratic spaces (F^n, q_ρ) in general are not isometric, but they correspond to the same quadric $\mathcal{Q}$ in $\mathsf{PG}(n-1, F)$ and their discriminant is the same.

If K is any commutative field extension of F, then a quadratic form q defined in $V(F)$ can be regarded also as a quadratic form denoted by q_K over the extended vector space $V(K)$, and the quadric $\mathcal{Q}$ associated to q in $\mathsf{PG}(3, F)$ as a quadric in $\mathsf{PG}(3, K)$, denoted by $\mathcal{Q}_K$.

Let K be a separable quadratic extension of a non-separably closed field F, denote by $x \mapsto \overline{x}$ the unique non trivial element of $\mathrm{Gal}(K/F)$ and fix an element $b \in F^*$. Then, according to [21], the *quaternion algebra* $H = (K/F, b)$ is the subring of $\mathrm{M}_2(K)$ consisting of matrices of the form

$$\begin{pmatrix} x & y \\ b\overline{y} & \overline{x} \end{pmatrix}$$

and it is a central simple algebra over F; if F is separably closed, then the quaternion algebra H over F is $\mathrm{M}_2(F)$. In both cases the ground field F can be identified with the subalgebra of scalar matrices. For a quaternion $h \in H$ we define the *conjugate of* h to be the quaternion

$$\overline{h} := \begin{pmatrix} \overline{x} & -y \\ -b\overline{y} & x \end{pmatrix},$$

the *norm of* h to be $n(h) := h\overline{h} = \det(h) \in F$ and the *trace of* h to be $t(h) := h + \overline{h} = \mathrm{tr}(h) \in F$. Then each $h \in H$ satisfies the quadratic equation $h^2 - t(h)h + n(h) = 0$. Note that K, embedded into H as the subring of all matrices

$$\begin{pmatrix} x & 0 \\ 0 & \overline{x} \end{pmatrix},$$

is invariant under the conjugation of H, which, restricted to K, coincides with the conjugation associated to the field extension K/F. Note also that n is a non-degenerate quadratic form on the vector space F^4, which in the sequel will be called the *norm form* of H, thus (F^4, n) is a quadratic space that turns out to be regular. The following holds true.

2.1 Theorem ([18, Chapter 3, Theorem 2.7], [10, Chapter 11]). *For $H = (K/F, b)$ the following statements are equivalent.*

(i) $H \cong \mathrm{M}_2(F)$.

(ii) *H is not a division algebra.*

(iii) *(H, n) is isotropic as a quadratic space.*

(iv) *(H, n) is hyperbolic as a quadratic space.*

(v) *$b \in N(K)$, where N is the norm of K/F.*

If any of these conditions holds for the algebra H, then we say that H *splits* over F, or equivalently that H is a *split* quaternion algebra over F or that F is a *splitting field* for H.

Recall that the quaternion algebra $H = (K/F, b)$ is a 4-dimensional vector space over F, and it is always possible to find a basis $(1, i, j, ij)$ of H such that $K = F(i)$ and

$$\begin{array}{lll} i^2 = a & & \\ j^2 = b & \text{if } \operatorname{char} F \neq 2, & \\ ij = -ji & & \end{array} \qquad \text{or} \qquad \begin{array}{ll} i^2 + i = a & \\ j^2 = b & \text{if } \operatorname{char} F = 2. \\ ij = j(i+1) & \end{array}$$

Moreover, if $\operatorname{char} F = 2$, then $K' := F(j)$ is an inseparable field extension of F contained in H.

The following theorem is a characterization of those quadratic field extensions of the given field F which are subalgebras (indeed maximal commutative subfields) of a fixed quaternion skew field H over F in terms of algebraic properties of the norm of H which correspond to geometric properties of the quadric associated to the norm form.

2.2 Theorem. *Let F be a field of any characteristic, and let H be a quaternion skew field over F, with norm form n. Let L be a quadratic extension of F. Then the following are equivalent:*

(i) *L is a subalgebra of H.*

(ii) *n_L is isotropic.*

(iii) *n_L is hyperbolic.*

The equivalence (i) $\Leftrightarrow$ (ii) is a well known result, see e.g. [23, Chapter I, Theorem 2.8], [19, Chapter 8, Theorem 5.4] and also [10, 11.A]; in the particular case $\operatorname{char} F \neq 2$ we provide a simple and direct proof in [2]. Equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 2.1.

2.3 Proposition. *Let q be a 4-dimensional regular quadratic form over a field F. If $d(q)$ is trivial, then q is similar to the norm form of a suitable quaternion algebra H over F, and H is a skew field if and only if q is anisotropic over F. If $d(q)$ is non-trivial, then there exists a quadratic extension F' of F (namely the discriminant extension) such that $q_{F'}$ is similar to the norm form of a quaternion algebra H over F', and H is a skew field if and only if $q_{F'}$ is anisotropic over F'. Conversely if q is isometric to the norm form of a quaternion algebra H over a field F, then $d(q)$ is trivial.*

Proof. Assume $d(q)$ is trivial. Then by [6, Lemma 4.4] q is either hyperbolic or similar to the orthogonal sum of quadratic forms $s_1 N \perp s_2 N$ for suitable $s_1, s_2 \in F^*$, where N is the norm of a separable quadratic extension K/F. In the first case q is similar to the norm form of a split quaternion algebra, and this happens exactly when q is isotropic over F. If, on the contrary, q is anisotropic over F, then q is similar to the norm form of the quaternion algebra $H = (K/F, s_1^{-1} s_2)$. If $d(q)$ is non-trivial and d is a representative for the square class of $d(q)$, the same as above holds over the discriminant quadratic extension $F' = F(\sqrt{d})$ of F. The converse is obvious. □

2.4 Proposition. *Let $\mathcal{Q}$ be a quadric of $\mathsf{PG}(3, F)$ and write q for a representative of the similarity class of quadratic forms associated to $\mathcal{Q}$ in the vector space F^4. Then the following hold:*

(i) *$\mathcal{Q}$ is hyperbolic if and only if q is isometric to the norm form of the split quaternion algebra $\mathrm{M}_2(F)$ over F.*

(ii) *$\mathcal{Q}$ has no points in $\mathsf{PG}(3, F)$ and there exists a separable quadratic extension K of F such that $\mathcal{Q}_K$ is hyperbolic in $\mathsf{PG}(3, K)$ if and only if q is isometric to the norm form of a quaternion skew field H over F.*

Proof. (i) is obvious. To prove claim (ii) note that, if q is anisotropic and q_K hyperbolic, by [6, Lemma 4.2], q is similar to the quadratic form $s_1 N \perp s_2 N$, where N is the norm of the extension K/F, and thus, as in the proof of the previous proposition, q is similar to the norm form of a suitable quaternion algebra H. Since q is anisotropic over F, H is a division algebra.

Conversely if q is isometric to the norm form of a quaternion skew field H over F, then $\mathcal{Q}$ does not have points in $\mathsf{PG}(3, F)$ by Theorem 2.1. Moreover H contains a maximal separable subfield K, so q_K is hyperbolic by Theorem 2.2 and $\mathcal{Q}_K$ is then hyperbolic in $\mathsf{PG}(3, K)$. □

2.5 Remark. Note that Propositions 2.3 and 2.4(ii) characterize the quadrics $\mathcal{Q}$ without points in PG$(3, F)$ which determine a quaternion skew field H over F as those quadrics whose similarity class of associated quadratic forms consists of anisotropic forms with trivial discriminant.

3 Spreads and parallelisms in 3-space

Let $\mathbb{P} = (\mathcal{P}, \mathcal{L})$ be a 3-dimensional projective space. If $\mathcal{M} \subseteq \mathcal{L}$, then a line $L \in \mathcal{L}$ is said to be *transversal to* $\mathcal{M}$ if L meets each $M \in \mathcal{M}$ in a unique point.

Using the notion of transversals, we can now state the definition of a regulus in $\mathbb{P}$, due to B. Segre [20, Chapter 18] (see also [14]):

Let $T_0, T_1, T_2 \in \mathcal{L}$ be pairwise skew. Then the set

$$\mathcal{R} := \{L \in \mathcal{L} \mid L \text{ transversal to } T_0, T_1, T_2\} \tag{3.1}$$

is called a *regulus*.

The elements of a regulus $\mathcal{R}$ must be pairwise skew, because otherwise the three lines $T_0, T_1, T_2 \in \mathcal{L}$ that determine $\mathcal{R}$ could not be pairwise skew.

Given three pairwise skew lines R_0, R_1, R_2, there is always at least one regulus containing them, since R_0, R_1, R_2 possess at least three transversals T_0, T_1, T_2, that are also pairwise skew and hence determine a regulus that of course must contain R_0, R_1, R_2. This regulus is unique, i.e., does not depend on the choice of the transversals T_0, T_1, T_2, if and only if the field F is commutative. See [14, Chapter 4].

Given a regulus $\mathcal{R}$, we consider

$$\mathcal{R}_{\text{opp}} := \{T \in \mathcal{L} \mid T \text{ transversal to } \mathcal{R}\}.$$

By the above, $\mathcal{R}_{\text{opp}}$ is a regulus if and only if F is commutative. In this case, we call $\mathcal{R}_{\text{opp}}$ the regulus *opposite* to $\mathcal{R}$. Note that in this situation a regulus and its opposite cover the same set of points, namely, a hyperbolic quadric $\mathcal{Q}_{\mathcal{R}}$ in PG$(3, F)$ (see [4], [5, Chapter 4]).

In pappian spaces, one can also define reguli as follows (see, e.g., [9, 17]): A regulus $\mathcal{R}$ is a set of pairwise skew lines, such that each line that meets three lines of $\mathcal{R}$, is a transversal of $\mathcal{R}$, and each point on a transversal of $\mathcal{R}$ lies on an element of $\mathcal{R}$.

Recall that a set $\mathcal{S} \subseteq \mathcal{L}$ is called a *spread* of $\mathbb{P}$, if each point $p \in \mathcal{P}$ lies on exactly one line $L \in \mathcal{S}$. So $\mathcal{S}$ is a partition of the point set $\mathcal{P}$ into lines. In particular, any two elements of a spread $\mathcal{S}$ are skew. A spread $\mathcal{S}$ in a pappian

3-space is called *regular*, if with any three pairwise skew lines $S_1, S_2, S_3 \in \mathcal{S}$ also all other lines of the unique regulus through S_1, S_2, S_3 belong to $\mathcal{S}$.

A *partial parallelism* of $\mathbb{P}$ is a set of mutually disjoint spreads. A *parallelism* of $\mathbb{P}$ is a partial parallelism which covers the whole line set $\mathcal{L}$ of the projective space; a *regular parallelism* is a parallelism consisting of regular spreads only. Given a parallelism of $\mathbb{P}$ we say that two lines $L, M \in \mathcal{L}$ are *parallel* ($L \parallel M$) if they belong to the same spread.

From now on we study the pappian projective space $\mathbb{P} = \mathsf{PG}(3, F)$ over a commutative field F.

By [1], there is a regular spread in $\mathsf{PG}(3, F)$ if and only if the field F admits a quadratic extension K. In this case $\mathsf{PG}(3, F) = (\mathcal{P}, \mathcal{L})$ can be embedded in $\mathsf{PG}(3, K) = (\mathcal{P}', \mathcal{L}')$ as a *Baer subspace*, i.e. a projective subgeometry such that each point $p \in \mathcal{P}'$ is incident with at least one line $L \in \mathcal{L}$. This means that, given any line $I \in \mathcal{L}'$ not intersecting $\mathcal{P}$, through any point p of I there is exactly one line $L_p \in \mathcal{L}$; so I defines the set of lines

$$\mathcal{S}(I) := \{L_p \mid p \in I\} \subseteq \mathcal{L}$$

which turns out to be a regular spread of $\mathsf{PG}(3, F)$ (see [1, Theorem 3.6]).

Conversely, any regular spread of $\mathsf{PG}(3, F)$ can be obtained as a set $\mathcal{S}(I)$ as above, where K is a suitable quadratic extension of F. We say that the line I *indicates*, or is an *indicator set* of, the spread $\mathcal{S}(I)$ (see e.g. [11]). If K/F is a separable field extension and $\overline{I}$ denotes the line (which is necessarily skew to I) conjugate to I with respect to this extension, then $\mathcal{S}(I) = \mathcal{S}(\overline{I})$. Note that different spreads of $\mathsf{PG}(3, F)$ may give rise to different quadratic extensions of the ground field F (see [3]).

4 Clifford parallelisms

Let now $H = (K/F, b)$ be a quaternion skew field over F. We consider $H \cong F^4$ as the underlying vector space of $\mathsf{PG}(3, F)$. The group H^* acts on H via right (or left) multiplications. Since these are F-linear bijections they induce collineations of $\mathsf{PG}(3, F)$. Now the *right* and *left Clifford parallelisms* on $\mathsf{PG}(3, F)$ can be defined as follows:

$$L \parallel_r M :\Leftrightarrow L = Mh \text{ for some } h \in H^* \quad (\textit{right Clifford parallel})$$
$$L \parallel_\ell M :\Leftrightarrow L = hM \text{ for some } h \in H^* \quad (\textit{left Clifford parallel})$$

One can easily check that these two relations are in fact parallelisms. By definition, right multiplications map each line to a right parallel one. Moreover, left multiplications map right parallel classes to right parallel classes. These same holds if "right" and "left" are interchanged. Altogether, PG$(3, F)$, endowed with the two Clifford parallelisms, is an example of a *kinematic space* (for details see [15]).

Since each (right or left) parallel class is a spread, we know that it has a unique representative passing through the point $\mathbf{1} := F$. The lines through $\mathbf{1}$ are exactly the 2-dimensional subspaces $F + Fx$, $x \in H \setminus F$, and these are exactly the maximal commutative subfields of H, which are certain quadratic field extensions of F.

First we study the *split quaternion algebra* (by Theorem 2.2) H_L over one of such extension fields L. As mentioned above, then H_L is isomorphic to the algebra $\mathrm{M}_2(L)$ of 2×2 matrices over L and the norm is nothing else but the determinant.

We consider $H_L = \mathrm{M}_2(L)$ also as the endomorphism ring of the vector space L^2, where the matrices are supposed to act from the right. For each $U \leq L^2$ we introduce the following notation:

$$I_U := \big\{M \in \mathrm{M}_2(L) \mid U \subseteq \ker M\big\},\ I^U := \big\{M \in \mathrm{M}_2(L) \mid \mathrm{im}\, M \subseteq U\big\}.$$

We study the quadric $\mathcal{Q}_L$ associated to $n_L = \det$ in PG$(3, L)$, i.e.

$$\mathcal{Q}_L = \big\{LM \mid M \in \mathrm{M}_2(L), M \neq 0, \det M = 0\big\}.$$

By Proposition 2.4, the quadric $\mathcal{Q}_L$ is hyperbolic. In addition, the following holds.

4.1 Proposition. *The two reguli on $\mathcal{Q}_L$ are the sets*

$$\mathcal{R}_L = \{I_U \mid U \leq L^2, \dim U = 1\}$$

and

$$(\mathcal{R}_L)_{\mathrm{opp}} = \{I^U \mid U \leq L^2, \dim U = 1\}.$$

Proof. For each $U \leq L^2$ with $\dim U = 1$ the set I_U is a 2-dimensional subspace of $\mathrm{M}_2(L)$, since for a fixed $u \in U \setminus \{0\}$ the mapping $\mathrm{M}_2(L) \to L^2 : M \mapsto uM$ is linear and surjective with kernel I_U. Similarly, one can see that I^U is a 2-dimensional subspace of $\mathrm{M}_2(L)$. So all I_U and all I^U are lines in $\mathcal{Q}_L$. Clearly, two such lines meet if and only if they are of different types, and each point LM in $\mathcal{Q}_L$ belongs to a line of each type. □

Now we turn to the quaternion skew field $H = (K/F, b)$ over F and consider a maximal commutative subfield of H, i.e. a quadratic field extension L of F with $F \subseteq L \subseteq H$. Then the following holds true.

4.2 Proposition ([7, p. 104]). *Let H be a quaternion algebra over F and L any quadratic subfield of H.*

(i) *If L/F is separable, then there exists $d \in F^*$ such that $H \cong (L/F, d)$.*

(ii) *If $\operatorname{char} F = 2$ and $L = F(h)$ is inseparable, then there exists a separable quadratic extension $F \subseteq L' \subseteq H$ such that $H \cong (L'/F, c)$, where $c = n(h)$.*

According to this result in the following, whenever we consider a quadratic subfield L of H, me may assume without loss of generality $L = K = F + Fi$ if L/F is a separable extension, or $L = K' := F + Fj$ if L/F is an inseparable extension.

4.3 Lemma. *Consider the quaternion skew field $H = (K/F, b)$, write $K' = F + Fj$ and consider the matrix algebras $H_K = \mathrm{M}_2(K)$ over K and $H_{K'} = \mathrm{M}_2(K')$ over K'.*

(i) *The elements of $H \subseteq H_K$ are exactly those matrices that are fixed by the bijection*

$$\kappa : \begin{pmatrix} x & y \\ z & t \end{pmatrix} \mapsto \begin{pmatrix} \bar{t} & b^{-1}\bar{z} \\ b\bar{y} & \bar{x}, \end{pmatrix},$$

which is involutory and K-semilinear with respect to conjugation.

(ii) *If $\operatorname{char} F = 2$, the F-algebra H can be embedded in the K'-algebra $H_{K'}$ via the correspondence φ mapping*

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} a+1 & a \\ a & a \end{pmatrix},$$
$$j \mapsto \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad ij \mapsto j \begin{pmatrix} a & a+1 \\ a & a \end{pmatrix}.$$

(iii) *If $\operatorname{char} F = 2$, the embedding φ described above is an isometry, i.e. for each $h \in H$, $n(h) = \det \varphi(h)$.*

Proof. Direct computation. □

4.4 Remark. Since K/F is quadratic, the projective space $\mathsf{PG}(3, F)$ is a Baer subspace of $\mathsf{PG}(3, K)$. Describe the projective spaces with the help of the 4-dimensional vector spaces H and H_K, respectively. Then, the collineation $\tilde{\kappa}$ induced by κ is a Baer collineation of $\mathsf{PG}(3, K)$, fixing exactly the points and lines of $\mathsf{PG}(3, F)$. The quadric $\mathcal{Q}_K$ is invariant under $\tilde{\kappa}$. In particular, lines in $\mathcal{Q}_K$ are mapped to lines in $\mathcal{Q}_K$. Assume that for a line $R \in \mathcal{R}_K$ we have $R^{\tilde{\kappa}} \in (\mathcal{R}_K)_{\mathrm{opp}}$. Then R and $R^{\tilde{\kappa}}$ meet in a point, which must belong to $PG(3, F)$. But $\mathcal{Q}_K$ contains no points of $PG(3, F)$ since the norm of H is anisotropic, a contradiction. Hence the reguli $\mathcal{R}_K$ and $(\mathcal{R}_K)_{\mathrm{opp}}$ are invariant under $\tilde{\kappa}$. Each line of $\mathcal{R}_K$ (and, similarly, each line of $(\mathcal{R}_K)_{\mathrm{opp}}$) indicates a regular spread of $\mathsf{PG}(3, F)$. The spreads indicated by lines $I, I' \in \mathcal{R}_K$ with $I' \neq I, I^{\tilde{\kappa}}$ are disjoint (this can be shown analogously to [2, Proposition 3.1]); so $\mathcal{R}_K$ (and also $(\mathcal{R}_K)_{\mathrm{opp}}$) gives rise to a regular partial parallelism.

In the case of characteristic 2, if we consider the inseparable extension K'/F, conjugation is trivial for points in $PG(3, K')$ and so we do not have a Baer collineation.

In what follows we will use the notion of conjugation with respect to multiplication in H, which here means automorphisms of type $x \mapsto c^{-1}xc$, $c \in H^*$, rather than the anti-automorphism $x \mapsto \bar{x}$ from before.

4.5 Remark. Let x, y be elements of an arbitrary quaternion algebra H. Then x is *conjugate* to y (i.e. $y = c^{-1}xc$ for some $c \in H^*$) if and only if as matrices in $\mathrm{M}_2(K)$, they have the same characteristic and minimal polynomials, if and only if $t(x) = t(y)$ and $n(x) = n(y)$. This simple characterization of conjugate elements entails that, considering a quaternion algebra H over a field F and the algebra H_L over any extension field L/F, any two elements $x, y \in H \subset H_L$ are conjugate with respect to multiplication in H_L if and only if they are so in H.

4.6 Lemma. *Let R be any line of $\mathsf{PG}(3, F)$ through the point $\mathbf{1}$. Then R, considered as a line of $\mathsf{PG}(3, L)$, meets $\mathcal{Q}_L$ if and only if R is conjugate to L.*

Proof. Let R be the line through the points $\mathbf{1} = F$ and Fh, for some $h \in H \setminus F$. Since by Lemma 4.3 the quaternions 1 and h are matrices of $\mathrm{M}_2(L)$, R meets $\mathcal{Q}_L$ if and only if there is an $l \in L$ such that $\det(-l1 + h) = 0$. This in turn is equivalent to the statement that the characteristic polynomial $p_h(X) = X^2 - \mathrm{tr}(h)X + \det h = X^2 - t(h)X + n(h)$ of the matrix h has a root $l \in L \setminus \{0\}$. Since $h \in H \setminus F$ the polynomial $p_h(X) \in F[X]$ is irreducible over F, and since l is a root of $p_h(X)$, it is the minimal polynomial of l over F, hence $l \in L \setminus F$. Moreover, since $p_h(X)$ has degree 2, it is also the characteristic polynomial of l, thus the quaternions h and l are conjugate as elements of $\mathrm{M}_2(L)$ and $R = F + Fh$ is conjugate to $F + Fl = L$. □

Now we study the partial parallelisms mentioned in Remark 4.4. We show that the spreads indicated by lines on $\mathcal{Q}_L$ are Clifford parallel classes. By Lemma 4.6 a line through $\mathbf{1}$ which is not conjugate to L does not meet $\mathcal{Q}_L$ and hence its (right or left) parallel class cannot be indicated by a line on $\mathcal{Q}_L$.

4.7 Theorem. *The regular spreads $\mathcal{S}(I)$ indicated by lines $I \in \mathcal{R}_L$ (or $(\mathcal{R}_L)_{\mathrm{opp}}$, respectively) are exactly the right (left) parallel classes of lines R through $\mathbf{1}$ that are conjugate to L.*

Proof. We distinguish the cases $L = K = F + Fi$ and $L = K' = F + Fj$. First consider the line $L = K$. Each line of the right parallel class of K has the form Kh for some $h \in H^*$, thus it is spanned by $h = \begin{pmatrix} x & y \\ b\bar{y} & \bar{x} \end{pmatrix}$ and $ih = \begin{pmatrix} ix & iy \\ b\bar{i}\bar{y} & \bar{i}\bar{x} \end{pmatrix}$. The points of intersection of Kh and $\mathcal{Q}_K$ are exactly the points $p = KM$, where M is a non-invertible K-linear combination of these two matrices. One obtains the two solutions

$$p = KM, \quad \text{with } M = -\bar{i}\begin{pmatrix} x & y \\ b\bar{y} & \bar{x} \end{pmatrix} + \begin{pmatrix} ix & iy \\ b\bar{i}\bar{y} & \bar{i}\bar{x} \end{pmatrix} = (i - \bar{i})\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \quad \text{and}$$

$$p' = KM', \text{ with } M' = -i\begin{pmatrix} x & y \\ b\bar{y} & \bar{x} \end{pmatrix} + \begin{pmatrix} ix & iy \\ b\bar{i}\bar{y} & \bar{i}\bar{x} \end{pmatrix} = (\bar{i} - i)\begin{pmatrix} 0 & 0 \\ b\bar{y} & \bar{x} \end{pmatrix}.$$

Obviously (cfr. Proposition 4.1), the matrices M from above, with $x, y \in K$, are exactly the elements of $I = I_U \in \mathcal{R}_K$, where $U = K(0,1)$. In particular, the right parallel class of K is indicated by I. Similarly, one can show that the left parallel class of K is indicated by $J = I^V \in (\mathcal{R}_K)_{\mathrm{opp}}$, where $V = K(1,0)$.

Assume now $\operatorname{char} F = 2$ and consider the inseparable extension $L = K' = F + Fj$. Again each line of the right parallel class of K' is spanned by h and jh for a suitable $h \in H^*$. We consider H as a subring of $H_{K'}$ (cfr. Lemma 4.3(ii)), and thus $h = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ and $jh = \begin{pmatrix} jz & jt \\ jx & jy \end{pmatrix}$. The points of intersection of $K'h$ and $\mathcal{Q}_{K'}$ are the points $p = K'M$ where M is a non-invertible K'-linear combination of h and jh. A straightforward computation shows that, assuming as a consequence of $h \in H^*$ that $\det(h) \neq 0$, the only solution is

$$p = K'M, \text{ with } M = j\begin{pmatrix} x+z & y+t \\ x+z & y+t \end{pmatrix},$$

and, again by Proposition 4.1, the matrices M of this form are the elements of $I = I_U \in \mathcal{R}_{K'}$, where $U = K'(1,1)$. Similarly one can show that the left parallel class of K' is indicated by $J = I^U \in (\mathcal{R}_{K'})_{\mathrm{opp}}$.

In the remainder of this proof there is no need to distinguish any more the separable case from the inseparable one, thus, from now on, writing L we mean either K or K'. We consider now $R = c^{-1}Lc$, $c \in H^*$. Let α be the collineation of $\mathsf{PG}(3, L)$ induced by the conjugation $x \mapsto c^{-1}xc$. Then α leaves both $\mathsf{PG}(3, F)$ and $\mathcal{Q}_L$ invariant. Moreover,

$$u \in \ker M \iff uc \in \ker c^{-1}Mc, \quad u \in \operatorname{im} M \iff uc \in \operatorname{im} c^{-1}Mc \qquad (4.1)$$

implies that α leaves $\mathcal{R}_L$ and $(\mathcal{R}_L)_{\mathrm{opp}}$ invariant. Since α maps the right (left) parallel class of L to the right (left) parallel class of R we conclude that the right (left) parallel class of R is indicated by $I^\alpha \in \mathcal{R}_L$ (or $J^\alpha \in (\mathcal{R}_L)_{\mathrm{opp}}$, respectively).

It remains to show that all spreads indicated by a line of $\mathcal{R}_L$ (or $(\mathcal{R}_L)_{\text{opp}}$) are right (left) parallel classes of lines $c^{-1}Lc$. But this follows from the above by (4.1), since for each 1-dimensional subspace $W \leq L^2$ there is a $c \in H^*$ with $W = Uc$ (or $W = Vc$, respectively). $\square$

Note that, in the proof above, in the separable case we have $p' = p^{\tilde{\kappa}}$ (see Remark 4.4). So the right parallel class of K is also indicated by $I' = I^{\tilde{\kappa}}$.

Since each (right or left) parallel class has a representative through $\mathbf{1}$, which is a maximal commutative subfield of H, by Proposition 4.2 we can describe it as $\mathcal{S}(I)$, with I a line in an appropriate Baer superspace of $\mathsf{PG}(3, F)$. In particular, we get the following.

4.8 Corollary. *The right and left Clifford parallelisms are regular.*

In the special case that F admits only one quadratic extension K (and hence K/F is separable, thus $\operatorname{char} F \neq 2$), the Clifford parallelisms are indicated by exactly the lines of a regulus and its opposite in $\mathsf{PG}(3, K)$. For $F = \mathbb{R}$ and $K = \mathbb{C}$ this is well known, see e.g. [8, 12 A]. This observation leads to the following corollary.

4.9 Corollary. *Let F, K and H be as before. Then all quadratic extensions of F in H are conjugate to K if and only if there exists a hyperbolic quadric $\mathcal{Q}$ in* $\mathsf{PG}(3, K)$ *having no points in* $\mathsf{PG}(3, F)$ *and incident with every line of* $\mathsf{PG}(3, F)$.

In general, however, we need more than one Baer superspace $\mathsf{PG}(3, L)$. In order to get a unified description of the entire parallelisms, we proceed as follows: Let $\widehat{F}$ be the quadratic closure of F. Then all quadratic extensions L of F are contained in $\widehat{F}$. We consider $\mathsf{PG}(3, F)$ and all $\mathsf{PG}(3, L)$ as subspaces of $\mathsf{PG}(3, \widehat{F})$. More explicitly, we take H as underlying vector space of $\mathsf{PG}(3, F)$ and H_L or $H_{\widehat{F}}$ (with the same basis $(1, i, j, ij)$) as underlying vector spaces of $\mathsf{PG}(3, L)$ or $\mathsf{PG}(3, \widehat{F})$, respectively. In particular, for any two distinct separable quadratic extensions L, L' we have $\mathsf{PG}(3, L) \cap \mathsf{PG}(3, L') = \mathsf{PG}(3, F)$.

In addition, we consider in $\mathsf{PG}(3, \widehat{F})$ and in all $\mathsf{PG}(3, L)$ the quadrics $\mathcal{Q}_{\widehat{F}}$ and $\mathcal{Q}_L$ associated to the norm of H. By Theorem 2.2 the quadric $\mathcal{Q}_L$ is empty exactly if L is not a subalgebra of H, and hyperbolic otherwise. This implies that also $\mathcal{Q}_{\widehat{F}}$ is hyperbolic. Let the reguli $\mathcal{R}_{\widehat{F}}$ on $\mathcal{Q}_{\widehat{F}}$ and $\mathcal{R}_L$ on $\mathcal{Q}_L$ be defined as in Proposition 4.1. Then $\mathcal{R}_L$ consists exactly of those lines of $\mathcal{R}_{\widehat{F}}$ that belong to $\mathsf{PG}(3, L)$, the same holds for the opposite reguli. In case that $\mathcal{Q}_L = \emptyset$ we set $\mathcal{R}_L := \emptyset =: (\mathcal{R}_L)_{\text{opp}}$.

For a line I on $\mathcal{Q}_{\widehat{F}}$ we can define $\mathcal{S}(I)$ only if I belongs to some $\mathsf{PG}(3, L)$; note that of course we then only consider the points of I that are points of $\mathsf{PG}(3, L)$.

4.10 Theorem. *Let F be a (commutative) field, H be a quaternion skew field over F and, for any quadratic extension L of F, let $\mathcal{Q}_L$ be the quadric of $\mathsf{PG}(3, L)$ associated to the norm of H and $\mathcal{R}_L$ and $(\mathcal{R}_L)_{\mathrm{opp}}$ the reguli of $\mathcal{Q}_L$ defined as above. Then the set*

$$\{\mathcal{S}(I) \mid I \in \mathcal{R}_L,\ L/F \text{ quadratic extension}\}$$

is the right Clifford parallelism of $\mathsf{PG}(3, F)$. *Analogously,*

$$\{\mathcal{S}(I) \mid I \in (\mathcal{R}_L)_{\mathrm{opp}},\ L/F \text{ quadratic extension}\}$$

is the left Clifford parallelism of $\mathsf{PG}(3, F)$.

Proof. This follows from Theorem 4.7. The change of basis we employed in order to prove that theorem (depending on L, and writing the elements of H and of H_L as matrices) does not affect the statements needed. □

4.11 Remark. Note that, according to Proposition 2.4, any quadric of $\mathsf{PG}(3, F)$ which has no points in $\mathsf{PG}(3, F)$ and is hyperbolic in a quadratic field extension of F is in fact the quadric associated to the norm form of a quaternion skew field over F, and thus, according to the previous theorem, it defines a Clifford parallelism in $\mathsf{PG}(3, F)$. Moreover any two such quadrics which are not projectively equivalent on F (i.e. such that there is no projective collineation of $\mathsf{PG}(3, \widehat{F})$ with coefficients in F mapping one to the other) define non projectively equivalent Clifford parallelisms in $\mathsf{PG}(3, F)$.

4.12 Example. In order to illustrate that in fact many different Baer superspaces may be needed, we consider an example. First, we make some general observations: Consider a field F of characteristic different from 2. A quadratic extension $K = F(\sqrt{c})$ of F (with $c \in F$ a non-square) appears as a maximal commutative subfield of H, if $K = F + Fx$ with $t(x) = 0$ and $n(x) = -c$. Two subfields $F + Fx$, $F + Fy$ of H with $t(x) = 0 = t(y)$ are isomorphic as F-algebras if and only if they are conjugate in H (by the classical Skolem-Noether Theorem), i.e., if and only if $n(x)$ and $n(y)$ are in the same square class of F^*. So the conjugacy classes of maximal commutative subfields of H are in 1-1 correspondence with the square classes of the subgroup $\{n(x) \mid x \in H^*, t(x) = 0\}$ of F^*.

Let us take the special case of the ordinary rational quaternions, i.e., $F = \mathbb{Q}$ and $H = (K/\mathbb{Q}, b)$, where $K = \mathbb{Q}(i)$ with $i^2 = -1 = b$. Then for $x \in H$ with $t(x) = 0$ we have $n(x) = x_2^2 + x_3^2 + x_4^2$, whence each field $\mathbb{Q}(\sqrt{-d})$, with $d \in \mathbb{Q}$ sum of three squares, appears as a subfield of H. Among many others, H contains the non-conjugate subfields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$, each of which gives rise to (only) a part of the Clifford parallelisms.

4.13 Remark. Assume that the 2-dimensional subalgebras of a quaternion skew field H over F do not belong to the same conjugacy class, or equivalently that the field F has non isomorphic quadratic extensions that are subalgebras of H. In this case the projective space $\mathsf{PG}(3,F)$ can be endowed with new parallelisms in the following way. Consider the right and left Clifford parallelisms defined in Theorem 4.10, fix a family $\mathscr{F}$ of quadratic extensions L of F with $F \subseteq L \subseteq H$ and define the following family of spreads:

$$\mathscr{C}(\mathscr{F}) := \{\mathcal{S}(I) \mid I \in \mathcal{R}_L,\ L/F \text{ quadratic extension, } L \notin \mathscr{F}\} \\ \cup \{\mathcal{S}(I) \mid I \in (\mathcal{R}_L)_{\text{opp}},\ L \in \mathscr{F}\}.$$

Then $\mathscr{C}(\mathscr{F})$ is a covering of the line set of $\mathsf{PG}(3,F)$, and any two spreads of this family are disjoint, for, a line Rh through a generic point h of $\mathsf{PG}(3,F)$ belongs to a left parallel class if and only if there exists a line R' through $\mathbf{1}$ such that $Rh = hR'$, and hence if and only if R and R' are conjugate in H. By Theorem 4.7 this happens if and only if the parallel classes are indicated by lines belonging to the same quadratic extension L of F.

These new "Clifford-like" parallelisms will be the target of more investigations in forthcoming papers.

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