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16-dimensional compact projective planes with a large group fixing two points and only one line

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Abstract

We complete the determination of all pairs $(\mathcal{P},\Delta)$, where $\mathcal{P}$ is a compact projective plane with a 16-dimensional point set, Δ is an automorphism group of $\mathcal{P}$ of dimension at least 35, and Δ does not fix exactly one point and one line. If Δ fixes two points and only one line, then Δ contains a 15-dimensional translation group and a compact subgroup $\mathsf{Spin}_7\mathbb{R}$; hence $\dim\Delta \geq 36$. The planes are described by their coordinatizing Cartesian fields, more explicitly for $\dim\Delta > 36$.

Keywords: compact projective plane, 16-dimensional plane, Cartesian field, translation group
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1 Introduction

Let $\mathcal{P} = (P,\mathfrak{L})$ be a topological projective plane with a compact point set P of finite (covering) dimension $d = \dim P > 0$. A systematic treatment of such planes can be found in the book *Compact Projective Planes* [18]. Each line $L \in \mathfrak{L}$ is homotopy equivalent to a sphere $\mathbb{S}_\ell$ with $\ell \,|\, 8$, and $d = 2\ell$, see [18, (54.11)]. In all known examples, L is in fact homeomorphic to $\mathbb{S}_\ell$. Taken with the compact-open topology, the automorphism group $\Sigma = \operatorname{Aut}\mathcal{P}$ (of all continuous collineations) is a locally compact transformation group of P with a countable basis, the dimension $\dim\Sigma$ is finite, cf. [18, (44.3 and 83.2)].

For $\ell \leq 4$, all sufficiently homogeneous planes are known explicitly, see [18, Chaps. 7, 8]. In the case $\ell = 8$ the aim is to determine all pairs $(\mathcal{P},\Delta)$, where Δ is a connected closed subgroup of Σ and $\dim\Delta \geq b$ for a suitable bound b.

(If $\dim \Delta \geq 27$, then Δ is always a Lie group [13].) Here, we deal with the case that $b = 35$ and Δ fixes exactly 3 elements (say two points and one line). This completes the classification for $b = 35$ and all groups Δ which do not fix exactly two elements (a point and a line), cf. [17] for the other possible configurations of fixed elements.

Theorem 1.1. *If Δ fixes exactly 2 points and one line and if $\dim \Delta \geq 34$, then the group T of translations in Δ is at least 15-dimensional.*

Either Δ has a subgroup $\Upsilon \cong \mathsf{Spin}_7\mathbb{R}$ and $\dim \Delta \geq 36$, or T is transitive, a maximal semi-simple subgroup of Δ is isomorphic to $\mathsf{SU}_4\mathbb{C} \cong \mathsf{Spin}_6\mathbb{R}$, and $\dim \Delta = 34$.

All planes satisfying the hypotheses of Theorem 1.1 with $\dim \Delta \geq 35$ will be described by coordinate methods in Theorems 3.1 and 3.3.

2 Structure of the group

Essential for the proof is the so-called *stiffness*:

The stabilizer of a quadrangle has dimension at most 14; see [18, (83.23)].

Particularly important is Bödi's improvement [1]:

($\Diamond$) *If the fixed elements of the connected Lie group Λ form a connected subplane $\mathcal{E}$, then Λ is isomorphic to the 14-dimensional compact group G_2 or its subgroup $\mathsf{SU}_3\mathbb{C}$, or $\dim \Lambda < 8$. If $\mathcal{E}$ is a Baer subplane ($\dim \mathcal{E} = 8$), then Λ is a subgroup of $\mathsf{SU}_2\mathbb{C}$. Moreover, $\Lambda \cong \mathsf{G}_2$ implies $\dim \mathcal{E} = 2$.*

If Δ fixes 2 distinct points and $\dim \Delta > 30$, then it follows from other classification results ([11, 12, 15]) that Δ is not semi-simple and has no normal torus subgroup. The main result of [16] can now be stated in the following form:

Lemma 2.1. *If Δ fixes exactly one line W and at least 2 points on W, and if $\dim \Delta \geq 33$, then Δ has a minimal normal subgroup $\mathsf{M} \cong \mathbb{R}^{\bar{t}}$ consisting of translations with axis W.*

Two more facts will be needed repeatedly:

Lemma 2.2. *Assume that Γ is a solvable Lie subgroup of Δ. Then Γ has a chain of normal subgroups Γ_κ with $\dim \Gamma_{\kappa+1}/\Gamma_\kappa \leq 2$; see* [2, I § 5, Th. 1, Cor. 4, p. 46]. *If κ is the largest index such that $a^{\Gamma_\kappa} = a$, if $\mathsf{N} = \Gamma_{\kappa+1}$ and $a \neq x \in a^{\mathsf{N}}$, then $\dim x^{\Gamma_a} \leq 2$. In fact, $x^{\Gamma_a} \subseteq a^{\mathsf{N}}$ and $\dim x^{\Gamma_a} \leq \dim \mathsf{N}/\mathsf{N}_a \leq \dim \mathsf{N}/\Gamma_\kappa$.*

Notation. The connected component of a group Γ will be denoted by Γ^1. Let u and v be the two fixed points of Δ. For a point $a \notin W = uv$ we put $\nabla = (\Delta_a)^1$. By Lemma 2.1 there exists a minimal ∇-invariant vector subgroup $\Theta \cong \mathbb{R}^t$ consisting of translations in M. The *radical* $\mathsf{P} = \sqrt{\Delta}$ is the largest solvable normal subgroup of Δ. We write $\Delta : \Gamma = \dim \Delta - \dim \Gamma$ and $\Gamma|_M$ for the group induced by Γ on the Γ-invariant set M.

The *dimension formula* $\dim \Gamma = \dim \Gamma_x + \dim x^\Gamma$ holds for any closed subgroup Γ of Δ, see [18, (96.10)]. This fact will often be used without mention.

Lemma 2.3. *If a maximal semi-simple subgroup Ψ of Δ or of ∇ (a Levi complement of the radical) has a subgroup $\Lambda \cong \mathsf{G}_2$, then Ψ is almost simple, and $\Psi = \Lambda$ or there is a group $\Upsilon \cong \mathsf{Spin}_7\mathbb{R}$ with $\Lambda < \Upsilon \leq \Psi$. The central involution $\alpha \in \Upsilon$ is a reflection.*

Proof. This follows from ($\Diamond$) and the observation that (in the relevant dimension range) each simple group which contains G_2 is of type B or D or G_2, see [7] for details. By [18, (55.40)], any action of $\mathsf{SO}_5\mathbb{R}$ on a compact projective plane is trivial. Hence $\Psi \not\cong \mathsf{SO}_7\mathbb{R}$ and α is not planar. $\square$

Proof of Theorem 1.1. Recall that there exists a minimal ∇-invariant subgroup $\Theta \cong \mathbb{R}^t$ which is contained in the group T of translations with axis W. But for the last step, we may assume that $\dim \mathsf{T} < 16$.

1) *The elements of Θ have center u or center v, and we may assume $\Theta \leq \mathsf{T}_{[v]}$.*

In fact, for $v \in L \neq W$ the stabilizer Θ_L consists of translations with center v. The action of Θ on the pencil $\mathfrak{L}_v$ shows that $\dim \Theta_{[v]} \geq t - 8$, cf. [18, (61.11a)], and $\dim \Theta_{[v]} = 0$ or $\Theta = \Theta_{[v]}$ by minimality. Therefore $t \leq 8$. Assume that $\mathbb{1} \neq \vartheta \in \Theta_{[z]}$ for some center $z \neq u, v$, and note that $\Theta_{[z]}$ is connected by [18, (61.9)]. Choose any point $a \notin W$. If $\mathbb{R} \cong \Pi \leq \Theta$ and $\vartheta \in \Pi$, then the connected component Λ of Δ_{a,a^ϑ} centralizes each translation in Π because $\vartheta^\Lambda = \vartheta$ and Λ acts linearly on Θ. Thus, Λ fixes the orbit a^Π pointwise and the fixed elements of Λ form a connected subplane $\mathcal{E}$. Moreover, $\nabla\colon \Lambda = \dim (a^\vartheta)^\nabla \leq \dim a^\Theta \leq 8$ and $\dim \Lambda \geq 18 - t$. Hence the stiffness theorem ($\Diamond$) shows that $\Lambda \cong \mathsf{G}_2$. Consequently, $t \geq 4$ and Λ acts non-trivially on Θ by the last part of ($\Diamond$). The action of any compact or semi-simple Lie group on a real vector space is completely reducible, and each irreducible module of G_2 on $\mathbb{R}^{16}$ has a dimension divisible by 7, see [18, (95.10)]. Since $\Pi^\Lambda = \Pi$, we conclude that $t = 8$ and $\dim \nabla \leq 22$. Because Θ is minimal, ∇ acts irreducibly on Θ. By Lemma 2.3, the group ∇ has a subgroup $\Upsilon \cong \mathsf{Spin}_7\mathbb{R}$. The central involution $\alpha \in \Upsilon$ is a reflection and inverts each translation in Θ. Thus, α has axis W and some center, which may be chosen as a. Now

$\alpha^\Delta\alpha \subseteq \mathsf{T}$ and $\dim \mathsf{T} = \dim a^\Delta \geq 12$, see [18, (61.19)]. The group Υ acts faithfully on each invariant subgroup of T. This implies $\mathsf{T}_{[u]} \cong \mathsf{T}_{[v]} \cong \mathbb{R}^8$ (cf. [18, (95.10)]) and then $\mathcal{P}$ is the classical Moufang plane $\mathcal{O}$ over the octonions by [18, (81.17)], but we have assumed that $\dim \mathsf{T} < 16$.

Before continuing the proof of Theorem 1.1, we now prove the following lemma.

Lemma 2.4. *For the connected component* Λ *of the stabilizer of some quadrangle containing* u, v*, and an arbitrary point* a*, the radical* P *of* Δ *satisfies* $\mathsf{P} : (\Lambda \cap \mathsf{P}) \leq 20$*. If* $\dim \Lambda \geq 8$*, then* $\Lambda \cap \mathsf{P} = \mathbb{1}$*; in this case,* $\dim \mathsf{P} = 20$ *implies* $\dim \Theta \geq 2$ *and* $\dim \mathsf{P}_a = 4$.

Proof. Lemma 2.2, applied to the action of P on the line pencil $\mathcal{L}_v$ yields a group $\mathsf{X} \leq \mathsf{P}$ fixing two lines av and bv such that $\mathsf{P} : \mathsf{X} \leq 10$. Analogously, the action of X on the line av provides a point c with $\mathsf{X} : \mathsf{X}_{a,c} \leq 10$. As P is solvable and $\Theta^{\mathsf{P}_a} = \Theta$ by step 1), there exists a minimal X_a-invariant vector subgroup $\mathsf{N} \leq \Theta$ of dimension at most 2, and the argument of Lemma 2.2 shows that c can be chosen in a^{N}. The fixed elements of $\Lambda = (\mathsf{P}_{a,c,bv})^1$ form a connected subplane $\mathcal{E}$ since Λ acts linearly on N and centralizes the translation $\xi \in \mathsf{N}$ with $a^\xi = c$. If $\dim \Lambda \geq 8$, then Λ is simple by ($\Diamond$) and $\Lambda \cap \mathsf{P}$ is a solvable normal subgroup of Λ, hence trivial. $\square$

2) *Our aim is to show that one of the groups* $\mathsf{T}_{[u]}$ *or* $\mathsf{T}_{[v]}$ *is linearly transitive.*

This will be accomplished in steps 2) – 15). Again let $\Theta \leq \mathsf{T}_{[v]}$. For $a \notin W$ and $w \in W \setminus \{u, v\}$, consider the connected component Ω of ∇_w. The dimension formula gives $\dim \Omega \geq 10$. As above, let $\mathbb{R} \cong \Pi \leq \Theta$, $\mathbb{1} \neq \rho \in \Pi$, $c = a^\rho$, and put $\Lambda = (\Omega_c)^1$. Then $\Omega : \Lambda = \dim c^\Omega \leq \dim a^\Theta$. Because the action of ∇ on Θ is linear, $\Lambda \leq \mathrm{Cs}\,\Pi$ and ($\Diamond$) applies.

3) For $t = 1$ this gives $\Lambda \cong \mathsf{G}_2$. Put $\Delta = \mathsf{P}\Psi$, where $\mathsf{P} = \sqrt{\Delta}$ is the radical and Ψ is a maximal semi-simple subgroup of Δ. Lemma 2.4 shows that $\dim \mathsf{P} \leq 19$; consequently, $\dim \Psi > 14$. According to Lemma 2.3 the Levi complement Ψ has a subgroup $\Upsilon \cong \mathsf{Spin}_7\mathbb{R}$. For $t < 8$ the central involution $\alpha \in \Upsilon$ acts trivially on Θ by [18, (95.10)] and α is a reflection whose axis is a line through v and whose center is u. We may choose a on this axis. By the dual of [18, (61.19b)] we get $\dim \mathsf{T}_{[u]} = \dim (av)^\Delta > 0$. The reflection α inverts the elements of $\mathsf{T}_{[u]}$, and the representation of Υ on $\mathsf{T}_{[u]}$ is faithful. This implies that $\mathsf{T}_{[u]} \cong \mathbb{R}^8$ is linearly transitive as claimed. Moreover, $\mathsf{T}_{[u]}$ is a minimal normal subgroup of Δ. The action of Υ on av is equivalent to a linear action, see [18, (96.36)]. Hence $\Upsilon \leq \nabla$ for a suitable choice of a, so that ∇ acts irreducibly on $\mathsf{T}_{[u]}$.

4) *From* $t = 2$ *it would follow that* $\dim \mathsf{T} = 16$*, contrary to the general assump-*

tion.

If $a \neq c \in a^\Theta$, then $\Gamma = (\nabla_c)^1$ satisfies $\dim \Gamma \geq 16$. Consider a point $w \in W \setminus \{u, v\}$ and the connected component Λ of the stabilizer Γ_w, and note that $\dim \Lambda \geq 8$. By ($\Diamond$) the group Λ is almost simple and hence acts trivially on a^Θ. Therefore, $\Lambda \not\cong \mathsf{G}_2$ and $\Lambda \cong \mathsf{SU}_3\mathbb{C}$. This implies that Γ acts faithfully and transitively on $W \setminus \{u, v\}$, see [18, (96.11)]. According to [15, Lemma 5], the group Γ has a compact subgroup $\Phi \cong \mathsf{SU}_4\mathbb{C}$ of codimension 1. Consequently, Γ is not semi-simple and the commutator subgroup Γ' coincides with Φ. Moreover, $\dim \nabla = 18$ and the group Δ is transitive outside of W. Since Γ' acts trivially on Θ, the central involution α of Γ' is a reflection with axis av. (Note that $\Gamma'/\langle\alpha\rangle \cong \mathsf{SO}_6\mathbb{R}$ cannot act on a Baer subplane.) As before, $\mathsf{T}_{[u]} \cong \mathbb{R}^8$ and Γ' acts faithfully on $\mathsf{T}_{[u]}$. By [18, (95.6b)], the centralizer $\nabla \cap \mathrm{Cs}\,\mathsf{T}_{[u]}$ has positive dimension. Hence ∇ contains homologies with center v. The dual of [18, (61.20b)] shows that $\mathsf{T}_{[v]}$ is also linearly transitive.

5) *The cases* $3 \leq t \leq 6$ *lead to a contradiction*.

Consider the subplane $\mathcal{F} = \langle a^\Theta, u, v, w\rangle$; either $\mathcal{F} = \mathcal{P}$ and $\Omega = (\nabla_w)^1$ acts faithfully on Θ, or $\mathcal{F}$ is a Baer subplane. In the latter case we write $\Omega|_\mathcal{F} = \Omega/\mathsf{K}$, where K denotes the kernel of the action of Ω on $\mathcal{F}$. Recall from ($\Diamond$) that K is a compact group of dimension 3 or at most 1. The different possibilities will be discussed separately. As before, Λ denotes the connected component of the stabilizer of w, a and $c \in a^\Theta$, and $\dim \Lambda \geq 10 - t$.

6) If $t = 3$ and $\mathcal{F} = \mathcal{P}$, then Ω would be embeddable into $\mathsf{GL}_3\mathbb{R}$. Hence $t = 3$ implies $\mathcal{F} \neq \mathcal{P}$. A group Λ of dimension ≥ 8 would act trivially on Θ and on $\mathcal{F}$, but this is impossible. Therefore, $\dim \Lambda = 7$ and $\dim \Omega = 10$; moreover, Ω acts transitively on $\Theta \setminus \{\mathbb{1}\}$ and Ω/K has a subgroup $\mathsf{SO}_3\mathbb{R}$. The stiffness result [18, (83.15)] shows that $\Lambda : \mathsf{K} \leq 5$. Consequently, $\dim \mathsf{K} = 3$ and Ω/K is a 7-dimensional subgroup of $\mathsf{GL}_3\mathbb{R}$. However, such a subgroup does not exist because $\mathsf{SO}_3\mathbb{R}$ is a maximal subgroup of $\mathsf{SL}_3\mathbb{R}$, see [18, (94.34)].

7) Now let $t = 4$ and $\mathcal{F} = \mathcal{P}$. If Ω is not transitive on $\Theta \setminus \{\mathbb{1}\}$, then it follows from ($\Diamond$) that there is an orbit of dimension 3, and suitable stabilizers fix subplanes of dimensions 4 and 8. By [18, (83.9)] and [5, XI.9.6], this implies that Λ is a compact Lie group of rank at most 2, in fact, $\Lambda \cong \mathsf{SU}_3\mathbb{C}$, $\mathsf{SO}_4\mathbb{R}$, or $\dim \Lambda \leq 4$, see [14, (2.1)]. On the other hand, $\dim \Lambda \geq 6$ and Λ acts faithfully on Θ and fixes a one-parameter subgroup. This is a contradiction. Hence Ω is transitive on $\Theta \setminus \{\mathbb{1}\}$, and $\Omega' \cong \mathsf{Sp}_4\mathbb{R}$, see [21] or [18, (95.10)]. In particular, Ω contains a central involution α, and α cannot be planar, since the stabilizer of a degenerate quadrangle in an 8-dimensional plane has dimension at most 7, see [18, (83.17)]. Therefore, α is a reflection

with axis W, and $\alpha^{\Delta}\alpha \subseteq \mathsf{T}$, cf. [18, (23.20)]. Moreover, $\dim \Omega \leq 11$ and $\dim \nabla \leq 19$. The dimension formula yields $\dim \mathsf{T} \geq \dim a^{\Delta} \geq 15$. The reflection α acts on T as $-\mathbb{1}$. Because Ω is connected, α induces on T a map of determinant 1; consequently, $\mathsf{T} \cong \mathbb{R}^{16}$.

8) If $t = 4$ and $\mathcal{F} \neq \mathcal{P}$, the stiffness results [18, (83.17 and 22)] imply $\dim \Omega/\mathsf{K} \leq 7$ and $\dim \mathsf{K} \leq 3$, hence $\dim \Omega = 10$ and $\dim \nabla = 18$. Therefore, $\dim w^{\nabla} = 8$ for each choice of w, and ∇ is transitive on $S = W \setminus \{u, v\}$. According to [5, XI.9.5], the group Λ/K is compact, and then we have $\Lambda/\mathsf{K} \cong \mathsf{SO}_3\mathbb{R}$ and $\Lambda \cong \mathsf{SO}_4\mathbb{R}$, cf. [14, (2.1)]. In particular, $\dim \Lambda = 6$, $\dim \nabla_c = 14$, and $\dim w^{\nabla_c} = 8$, so that ∇_c is also transitive on S. Let Φ be a maximal compact subgroup of ∇_c containing Λ and note that S is homotopy equivalent to $\mathbb{S}_7$. The exact homotopy sequence

$$\cdots \to \pi_{q+1}S \to \pi_q\Lambda \to \pi_q\Phi \to \pi_qS \to \pi_{q-1}\Lambda \to \ldots$$

shows that $\pi_1\Phi \cong \mathbb{Z}_2$, $\pi_3\Phi \cong \mathbb{Z}^2$, $\pi_5\Phi \cong \mathbb{Z}_2^{\,2}$, and that $\pi_7\Phi$ is infinite. By [18, (94.36)], this implies that Φ is a semi-simple group having exactly two almost simple factors. Moreover, $\Phi \neq \Lambda$ because $\pi_7\Lambda$ is finite. Since $\dim \Phi < \dim \nabla_c$ and $\pi_5\mathsf{SU}_3\mathbb{C} \cong \mathbb{Z}$, the group Φ has a factor $\mathsf{B} \cong \mathsf{U}_2\mathbb{H}$, cf. [18, (94.33)] and note that $\mathsf{SO}_5\mathbb{R}$ cannot act on a plane. For the same reason, the central involution $\beta \in \mathsf{B}$ is a reflection; its axis is av, since, obviously, $[\mathsf{B}, \Theta] = \mathbb{1}$. From $\dim a^{\Delta} = 16$ we infer that $\beta^{\Delta}\beta = \mathsf{T}_{[u]}$ is linearly transitive. Either ∇ acts faithfully on $\mathsf{T}_{[u]}$ or ∇ contains homologies with axis au. In the second case, $\mathsf{T}_{[v]}$ is also linearly transitive, see [18, (61.20)], but then the representation of B on $\mathsf{T}_{[v]}$ would be trivial (use [18, (95.10)] and note that $[\mathsf{B}, \Theta] = \mathbb{1}$) and B would consist of homologies with center u. Consequently, ∇ acts on $\mathsf{T}_{[u]}$ as a transitive subgroup of $\mathsf{GL}_8\mathbb{R}$, and [21] shows that ∇ has a transitive factor $\mathsf{X} \cong \mathsf{SL}_2\mathbb{H}$. The stabilizer $\mathsf{X}_w = \mathsf{X} \cap \Omega$ is a 7-dimensional group which fixes $\mathcal{F}$ pointwise, a contradiction to $(\lozenge)$.

9) Thus the cases $2 \leq t \leq 4$ cannot arise. Therefore, $t > 4$ and $\mathcal{F} = \mathcal{P}$. For $t < 7$, we have $\Lambda \not\cong \mathsf{SU}_3\mathbb{C}$ and hence $10 \leq \dim \Omega < t + 8$. Since Θ is a minimal ∇-invariant vector group, ∇ induces on Θ an irreducible group $\widetilde{\nabla}$ of dimension $\dim \widetilde{\nabla} \geq \dim \Omega \geq 10$.

10) Let $t = 5$. By [18, (95.6 and 10)], the commutator group $\widetilde{\nabla}'$ is an almost simple group of dimension 10 or 24. In the latter case the dimension of ∇ would be too large. Hence $\widetilde{\nabla}'$ is locally isomorphic to a group $\mathsf{O}'_5(\mathbb{R}, r)$ and $\dim \widetilde{\nabla} \leq 11$. Because of Brouwer's Theorem [18, (96.30)] or [8], an almost simple group of dimension > 3 has no subgroup of codimension 1. Consequently, $\Omega' \cong \widetilde{\nabla}' \cong \mathsf{O}'_5(\mathbb{R}, r)$, and [18, (55.40)] implies $r > 0$. In the notation of step 2), there is some $\rho \in \Theta$ such that Λ has a subgroup $\mathsf{SO}_3\mathbb{R}$. By [18, (83.10)], the group Λ is then compact, and [14, (2.1)] shows $\Lambda \cong \mathsf{SO}_4\mathbb{R}$

(note that $4 < \dim \mathsf{\Lambda} < 8$). Hence Ω' is a hyperbolic motion group of the 4-dimensional projective space $P\Theta$. The stabilizer E of an exterior point of $P\Theta$ is not compact, but E contains a group $\mathsf{SO}_3\mathbb{R}$; therefore, E has to be compact for the same reason as $\mathsf{\Lambda}$, a contradiction.

11) Suppose that $t = 6$ and that Ω acts irreducibly on Θ. The stiffness result ($\Diamond$) implies $\dim \mathsf{\Lambda} < 8$ and $10 \leq \dim \Omega \leq 13$. With [18, (95.5 and 6)] it follows that either $\dim \Omega' = 8$ and the center $\mathsf{Z}(\Omega)$ is isomorphic to $\mathbb{C}^\times$, or the action of Ω' on Θ can be understood as the tensor product of the natural representations of $\mathsf{A} = \mathsf{SL}_2\mathbb{R}$ and $\mathsf{B} = \mathsf{SL}_3\mathbb{R}$ and $\Omega' \cong \mathsf{A}\times\mathsf{B}$. In both cases, Ω contains a central involution ω. On a Baer subplane, Ω would induce a group of dimension at most 7, see [18, (83.17)]. Therefore, ω is a reflection with axis uv and center a. We have $\dim \nabla \leq 21$. The hypothesis together with [18, (61.19)] implies $13 \leq \dim a^{\mathsf{\Delta}} = \dim \mathsf{T} < 16$. Consequently $\dim \nabla > 18$, $\dim \Omega > 10$ and then $\dim \Omega' = 11$. Because ω belongs to a connected group and acts as $-\mathbb{1}$ on T, both $\mathsf{T}_{[u]}$ and $\mathsf{T}_{[v]}$ have even dimension, and $\mathsf{T} \cong \mathbb{R}^{14}$. Hence one of the groups $\mathsf{T}_{[u]}$ and $\mathsf{T}_{[v]}$ is linearly transitive. Recall that $\Theta \leq \mathsf{T}_{[v]}$. By complete reducibility and [18, (95.10)], either B acts irreducibly on $\mathsf{T}_{[u]} \cong \mathbb{R}^8$ or B centralizes a 2- dimensional subgroup of T. In the latter case, the fixed elements of B would form a connected subplane contrary to ($\Diamond$). Since Ω fixes u and w, the factor A acts faithfully on $\mathsf{T}_{[u]}$. This contradicts the irreducibility of B, see [18, (95.4)].

12) If $t = 6$ and there is a minimal Ω-invariant vector subgroup $\mathsf{H} < \Theta$, and if $\mathsf{\Lambda} = (\Omega_c)^1$ for some $c \in a^{\mathsf{H}} \setminus \{a\}$, then $10 - \dim \mathsf{H} \leq \dim \mathsf{\Lambda} < 8$ by ($\Diamond$). Consider the action of Ω on the subplane $\mathcal{F}_{\mathsf{H}} = \langle a^{\mathsf{H}}, u, v, w \rangle$ and the connected component Φ of the kernel of this action. If $\dim \mathsf{H} \leq 4$, then it follows as in steps 6) and 7) that $\mathcal{F}_{\mathsf{H}}$ is an (Ω-invariant) Baer subplane of $\mathcal{P}$. Now $\dim \Omega/\Phi \leq 7$ by [18, (83.17)], and then [18, (83.22)] implies $\Phi \cong \mathsf{SU}_2\mathbb{C}$. Recall from step 5) that Ω acts faithfully on Θ. Since the action of Φ on Θ is completely reducible, Φ acts faithfully on a complement of H in Θ, but $\mathsf{SU}_2\mathbb{C}$ has no faithful representation in dimension < 4. Therefore, $\dim \mathsf{H} = 5$ and the commutator group Ω' is semi-simple and irreducible on H, see [18, (95.6b)]. Inspection of the list [18, (95.10)] shows $\Omega' \cong \mathsf{O}_5'(\mathbb{R}, r)$, and then Ω' would centralize a complement of H in Θ in contradiction to ($\Diamond$). Hence $t \neq 6$.

13) Steps 3) – 12) yield the following conclusion.

Conclusion. *If $\mathcal{P}$ is not a translation plane and if $\Theta \cong \mathbb{R}^t$ is a minimal ∇-invariant subgroup of $\mathsf{T}_{[v]}$, then either $t \geq 7$, or $t = 1$ and $\mathsf{T}_{[u]} \cong \mathbb{R}^8$ is a minimal normal subgroup of $\mathsf{\Delta}$.*

14) Now let $t = 7$ and assume first that Ω acts irreducibly on Θ for each choice

of w. By [18, (95.6)], the commutator group Ω' is almost simple. Moreover, $9 \leq \dim \Omega' \leq 15$ (since $\mathsf{\Lambda} \not\cong \mathsf{G}_2$). The list [18, (95.10)] shows that $\dim \Omega' = 14$ and that Ω' has torus rank 2. Because t is odd, each torus subgroup of Ω' fixes a non-trivial vector $\rho \in \Theta$, and [18, (83.10)] implies that the corresponding stabilizer $\mathsf{\Lambda}$ is compact. It follows that $\mathsf{\Lambda} \cong \mathsf{SU}_3\mathbb{C}$ and then $\Omega' \cong \mathsf{G}_2$ is also compact. Hence $\mathsf{\Lambda} \cong \mathsf{SU}_3\mathbb{C}$ for each $c = a^\rho$ and arbitrary w. Suppose that Ω' is a Levi complement of $\mathsf{P} = \sqrt{\Delta}$. Then Lemma 2.4 shows that $\dim \mathsf{P} = 20$ and $\dim \mathsf{P}_a = 4$. This implies that $[\mathsf{P}_a, \Omega'] = \mathbb{1} = \mathsf{P}_a \cap \Omega'$. The fixed elements of $\Omega' \cong \mathsf{G}_2$ form a 2-dimensional subplane $\mathcal{E}$ by [18, (96.35)] and P_a acts effectively on $\mathcal{E}$, but the stabilizer of a triangle in $\mathcal{E}$ is only 2-dimensional, see [18, (33.10)]. Hence Ω' is not a Levi complement of the radical. By Lemma 2.3, the group Δ has a subgroup $\Upsilon \cong \mathsf{Spin}_7\mathbb{R}$. Since Υ induces the group $\mathsf{SO}_7\mathbb{R}$ on $\Theta \cong \mathbb{R}^7$, the central involution $\alpha \in \Upsilon$ is a reflection with axis av and center u. As in step 3) it follows that $\mathsf{T}_{[u]} \cong \mathbb{R}^8$ is linearly transitive and is a minimal normal subgroup of Δ, and we may assume that ∇ acts irreducibly on $\mathsf{T}_{[u]}$.

15) Last alternative: $t = 7$ and there is a minimal Ω-invariant vector subgroup $\mathsf{H} < \Theta$. The proof follows a similar scheme as in the case of the action of ∇ on Θ. We have $1 \leq s := \dim \mathsf{H} < 7$. If $s = 1$, then $\dim \mathsf{\Lambda} \geq 9$ and $\mathsf{\Lambda} \cong \mathsf{G}_2$. As G_2 has no representation in dimension < 7, the group $\mathsf{\Lambda}$ would act trivially on Θ and hence on $\langle a^\Theta, u, w\rangle = \mathcal{P}$, a contradiction. In the case $s = 2$, the stiffness theorem ($\Diamond$) implies $\mathsf{\Lambda} \cong \mathsf{SU}_3\mathbb{C}$. Again $\mathsf{\Lambda}$ would act trivially on Θ, see [18, (95.3 and 10)]. The arguments of step 6) with H instead of Θ show that $s \neq 3$. Next, let $s = 4$ and assume first that Ω acts faithfully on H as an irreducible subgroup of $\mathsf{GL}_4\mathbb{R}$. Then Ω' is a semi-simple group of dimension ≥ 8, see [18, (95.6b)]. Hence Ω' is isomorphic to $\mathsf{Sp}_4\mathbb{R}$ or to $\mathsf{SL}_4\mathbb{R}$. The action of Ω' on Θ is completely reducible, and H has an Ω'-invariant complement $\mathsf{X} \cong \mathbb{R}^3$ in Θ. Consequently Ω' induces the identity on the subplane $\langle a^{\mathsf{X}}, u, w\rangle$, but this contradicts ($\Diamond$). Therefore $\langle a^{\mathsf{H}}, u, w\rangle$ is a Baer subplane of $\mathcal{P}$ and Ω induces on H a group Ω/K, where K^1 is isomorphic to a subgroup of $\mathsf{SU}_2\mathbb{C}$. Either $\mathsf{K}^1 \cong \mathsf{SU}_2\mathbb{C}$ or $\dim \mathsf{K} \leq 1$. In both cases, the semi-simple group Ω' fixes a complement X of H in Θ and $\dim \Omega' \geq 8$. If $\mathsf{K}^1 \cong \mathsf{SU}_2\mathbb{C}$, then $\mathsf{K}^1|_{\mathsf{X}} \cong \mathsf{SO}_3\mathbb{R}$, which is a maximal subgroup of $\mathsf{SL}_3\mathbb{R}$, cf. [18, (94.34)]. Accordingly, $\Omega'|_{\mathsf{X}} \cong \mathsf{SL}_3\mathbb{R}$, a contradiction. If $\dim \mathsf{K} \leq 1$, then $\dim \Omega'|_{\mathsf{H}} > 7$ and Ω' contains the group $\mathsf{Sp}_4\mathbb{R}$. This is again impossible. It follows that $s > 4$ and that Ω acts faithfully on H. For $s = 5$, representation theory shows that $\Omega' \cong \mathsf{O}'_5(\mathbb{R}, r)$, see [18, (95.10)], and Ω' would act trivially on a complement of H in Θ, a contradiction to ($\Diamond$). In the case $s = 6$, finally, the semi-simple group Ω' fixes a unique complement X of H, and X is even Ω-invariant. This has been excluded at the beginning of step 15).

16) In any case, one of the groups $\mathsf{T}_{[u]}$ or $\mathsf{T}_{[v]}$ is linearly transitive, and we may assume that $\Theta = \mathsf{T}_{[v]} \cong \mathbb{R}^8$ and that ∇ induces an irreducible group on Θ. By [5, XI.9.5 and 6], the stabilizer of an arbitrary quadrangle is compact and Λ is always a compact connected Lie group of torus rank at most 2. If $4 < \dim \Lambda < 8$, then $\Lambda \cong \mathsf{SO}_4\mathbb{R}$, see [14, (2.1)] or [5, XI.9.9].

17) Put $\Gamma = \Delta_{au}$. Because Θ is transitive on $av \setminus \{v\}$, it follows that $\Delta = \Gamma\Theta$ and that Γ acts irreducibly on Θ. If $\dim \Delta \geq 40$, then $\dim \mathsf{T} = 16$ or $\mathcal{P}$ is the classical Moufang plane according to [18, (87.7)]. Hence our assumptions imply $26 \leq \dim \Gamma \leq 31$. The centralizer $\Gamma \cap \mathrm{Cs}\,\Theta$ fixes each line in $\mathfrak{L}_u$ and consists of collineations with center u.

18) *Let G be a closed, connected irreducible subgroup of* $\mathsf{SL}_8\mathbb{R}$. *If* $\dim G \geq 18$, *then* G' *is isomorphic to an almost direct product* $\mathsf{SL}_2\mathbb{R} \cdot \mathsf{SL}_4\mathbb{R}$ *or* $\mathsf{SU}_2\mathbb{C} \cdot \mathsf{SL}_2\mathbb{H}$, *or to one of the almost simple groups* $\mathsf{Sp}_4\mathbb{C}$, $\mathsf{Spin}_7(\mathbb{R}, r)$ *with* $(r = 0, 3)$, $\mathsf{O}'_8(\mathbb{R}, r)$, $\mathsf{SL}_4\mathbb{C}$, *or* $\dim G' \geq 36$.

   In fact, G' is semi-simple and $\dim G' \geq 16$ by [18, (95.6)]. Suppose that $G' = \mathsf{AB}$ is an almost direct product where A has minimal dimension. If B acts irreducibly on $V = \mathbb{R}^8$, then $\mathsf{A} \cong \mathbb{H}'$ and $\mathsf{B} \leq \mathsf{SL}_2\mathbb{H}$. In the other case, $\dim \mathsf{B} \geq 8$, and Clifford's Lemma [18, (95.5)] shows that B acts faithfully and irreducibly on a subspace U such that $V = U \oplus U^\alpha$ for some $\alpha \in \mathsf{A}$. By [18, (95.10)], it follows that $\dim \mathsf{B} \neq 8$. Therefore, $\dim \mathsf{B} > 9$, and B contains a group $\mathsf{Sp}_4\mathbb{R}$. If $0 \neq x \in U$, then the fixed points of B_x form a 1-dimensional subspace of U, and $\langle x, x^\alpha \rangle \cong \mathbb{R}^2$ is A-invariant. Consequently, $\mathsf{A} \cong \mathsf{SL}_2\mathbb{R}$ and $\dim \mathsf{B} = 15$. All possibilities for an almost simple group G' are listed in [18, (95.10)].

19) If $\Gamma_{[u]} = \mathbb{1}$, then Γ acts faithfully on Θ; hence Γ' is semi-simple and $\dim \Gamma' \geq 24$, see [18, (95.6)]. By the last step, $\Gamma' \cong \mathsf{SL}_4\mathbb{C}$ or $\Gamma' \cong \mathsf{O}'_8(\mathbb{R}, r)$. In the first case, the involution $\beta = \mathrm{diag}(\mathbb{1}, -\mathbb{1}) \in \mathsf{SL}_4\mathbb{C}$ is not a reflection and hence fixes a Baer subplane $\mathcal{B}$ pointwise, cf. [18, (55.29)]. The group $\mathsf{B} = (\mathbb{1}, \mathsf{SL}_2\mathbb{C}) \leq \mathrm{Cs}\,\beta$ would induce on $\mathcal{B}$ a group of central collineations with center u, but this is impossible by [18, (61.20)], as B is semi-simple. If $\Gamma \cong \mathsf{O}'_8(\mathbb{R}, r)$, the diagonal involution $\mathrm{diag}(1, 1, \ldots, 1, -1, -1)$ would fix a 6-dimensional subset of $\mathfrak{L}_u$ and hence would be neither a reflection nor a Baer involution. This contradicts [18, (55.29)].

20) In the previous step it has been proved that $\Gamma_{[u]} \neq \mathbb{1}$. Assume first that $\Gamma_{[u]}$ contains homologies. We may choose a in such a way that $\Gamma_{[u,av]} \neq \mathbb{1}$. From the dual of [18, (61.20b)] it follows that $s := \dim \mathsf{T}_{[u]} = \dim a^\Gamma = \dim \Gamma - \dim \nabla$, and, hence, $\Gamma = \nabla \mathsf{T}^1_{[u]}$. Moreover, this is also the dimension of the set of all axes of homologies in Γ with center u. We choose $b \in a^{\mathsf{T}_{[u]}} \setminus \{a\}$

and $c \in av \smallsetminus \{a\}$ and put $\mathsf{\Lambda} = (\nabla_{b,c})^1$. Then

$$26 \leq \dim \mathsf{\Gamma} = \dim \nabla + s \leq \dim \mathsf{\Lambda} + 8 + 2s \leq 22 + 2s \;\text{ and }\; 1 < s < 8.$$

The assumption $s \leq 5$ implies successively $\dim \mathsf{\Lambda} \geq 8$, $\mathsf{\Lambda} \cong \mathsf{SU}_3\mathbb{C}$ or $\mathsf{\Lambda} \cong \mathsf{G}_2$, $\mathsf{\Lambda}$ acts trivially on $\mathsf{T}_{[u]}$, $\mathsf{\Lambda} \not\cong \mathsf{G}_2$, $s = 5$, $\mathsf{\Lambda} \not\cong \mathsf{SU}_3\mathbb{C}$, a contradiction. Asssume that $s = 6$. Then $\mathsf{\Lambda} \not\cong \mathsf{SU}_3\mathbb{C}$ because $\mathsf{\Lambda}$ fixes some elements of $\mathsf{T}_{[u]}$. Hence $\mathsf{\Lambda} \cong \mathsf{SO}_4\mathbb{R}$ by step 16), and $\dim \nabla = 20$. For any admissible b, the dimension formula gives

$$12 \leq \dim \nabla_c = \dim b^{\nabla_c} + \dim \mathsf{\Lambda} \leq s + 6 = 12\,,$$

and $\dim \nabla_c = 12$, $\dim b^{\nabla_c} = 6$. By [18, (96.11a)], the group ∇_c acts transitively on $\mathsf{T}^1_{[u]} \cong \mathbb{R}^6$. The action is also effective since its kernel is trivial on $\langle a^{\mathsf{T}^1_{[u]}}, c, v\rangle = \mathcal{P}$. On the other hand, the results in [21] (or in [18, (96.19–22)]) show that a transitive subgroup $G \leq \mathsf{GL}_6\mathbb{R}$ satisfies $\dim G \leq 10$ or $\dim G \geq 16$. Therefore, $s = 7$ and $\dim \mathsf{T} = 15$.

21) Now let $\mathsf{\Gamma}_{[u]} = \mathsf{T}_{[u]} := \mathsf{H}$. If $\dim \mathsf{H} = 1$ and if $a \neq b \in a^{\mathsf{H}}$, then $\dim \mathsf{\Gamma}_{a,b} \geq 17$, and ($\Diamond$) implies that $\mathsf{\Gamma}$ has a subgroup $\mathsf{\Lambda} \cong \mathsf{G}_2$. From the fact that

$$\dim(\mathsf{\Gamma} \cap \mathrm{Cs}\,\Theta) = \dim \mathsf{H} = 1\,,$$

it follows with [18, (95.6)] that a maximal semi-simple subgroup $\mathsf{\Psi}$ of $\mathsf{\Gamma}$ acts irreducibly on Θ, and that $\dim \mathsf{\Psi} \geq 23$. Because $\mathsf{\Gamma}$ contains G_2 but has no subgroup $\mathsf{SO}_5\mathbb{R}$ by [18, (55.40)], step 18) shows that $\mathsf{\Psi} \cong \mathsf{Spin}_8(\mathbb{R}, r)$ with $r \leq 1$, and $\mathsf{\Psi}$ induces on Θ a group $\mathsf{O}'_8(\mathbb{R}, r)$ by [18, (95.10)]. Consequently, $\mathsf{\Gamma}$ would contain a reflection with axis av, a possibility which has been dealt with in step 20). Thus, we may assume that $\dim \mathsf{H} = s > 1$; recall that $s < 8$ by the assumption made at the beginning of the proof. As $\mathsf{\Lambda}$ fixes a subspace of H and G_2 has no non-trivial representation in dimension < 7, we conclude that $\mathsf{\Lambda} \not\cong \mathsf{G}_2$, $\dim \mathsf{\Lambda} \leq 8$ and $\dim \nabla \leq 23$. The group ∇ acts faithfully and irreducibly on $\Theta \cong \mathbb{R}^8$. All possibilities for the semi-simple group ∇' have been listed in step 18). Only the first 5 groups of this list have a dimension at most 23 and we conclude that $18 \leq \dim \nabla' \leq 21$. If $\dim \nabla' > 18$, then ∇' is almost simple and the representation of ∇' on H shows that either $s = 7$, or ∇' fixes a^{H} pointwise, but in the latter case $\dim \nabla' \leq 8 + \dim \mathsf{\Lambda}$, which is a contradiction. If $\dim \nabla' = 18$, then $\dim \nabla \leq 19$. We consider the group $\widetilde{\mathsf{\Gamma}} \cong \mathsf{\Gamma}/\mathsf{H}$ induced by $\mathsf{\Gamma}$ on Θ, which contains ∇. From 18) and the inequalities

$$26 \leq \dim \mathsf{\Gamma} \leq 19 + 8 \quad \text{and} \quad \dim \widetilde{\mathsf{\Gamma}} \leq 27 - s$$

it follows that $\dim \widetilde{\mathsf{\Gamma}}' \leq 21$. Assume that ∇' is a proper subgroup of $\widetilde{\mathsf{\Gamma}}'$. Then $\widetilde{\mathsf{\Gamma}}'$ is isomorphic to $\mathsf{Spin}_7(\mathbb{R}, r)$ or $\mathsf{Sp}_4\mathbb{C}$, and a maximal compact subgroup

K of $\widetilde{\Gamma}'$ acts in the canonical way on the homogeneous space $M = \widetilde{\Gamma}'/\nabla'$, but this would imply $\dim \mathsf{K} \le 6$ by [18, (96.13)]. (Note that the kernel N of the action of K on M is contained in the intersection of all conjugates of ∇' in $\widetilde{\Gamma}'$, a proper normal subgroup of $\widetilde{\Gamma}'$; hence $\dim \mathsf{N} = 0$.) Consequently, $\dim \widetilde{\Gamma} \le 19$ and then $s \ge 7$. Steps 19) – 21) complete the proof of the first part of Theorem 1.1.

22) Assume now that $\mathsf{H} = \mathsf{T}^1_{[u]} \cong \mathbb{R}^7$. We will show that a maximal semi-simple subgroup of Δ is isomorphic to $\mathsf{Spin}_7\mathbb{R}$. With the rôles of u and v interchanged, the Conclusion implies that either some 1-dimensional subgroup $\Pi < \mathsf{H}$ is ∇-invariant or ∇ acts irreducibly on H. By hypothesis $\dim \nabla \ge 18$. Let $\nabla = \Psi\mathsf{P}$, where Ψ is a maximal semi-simple subgroup of ∇ and $\mathsf{P} = \sqrt{\nabla}$. In the first case, the stabilizer Λ of a suitable quadrangle has dimension at least 9; hence $\Lambda \cong \mathsf{G}_2$ by ($\Diamond$), and $\Psi \ne \Lambda$ since ∇ acts irreducibly on Θ. Lemma 2.3 implies that Ψ has a subgroup $\Upsilon \cong \mathsf{Spin}_7\mathbb{R}$. In the second case, ∇ induces an irreducible group ∇/N on Θ and an irreducible group ∇/K on H. By [18, (95.6)] we have $\mathsf{P} : (\mathsf{N} \cap \mathsf{P}) \le 2$ and $\mathsf{P} : (\mathsf{K} \cap \mathsf{P}) \le 1$, hence $\dim \mathsf{P} \le 3$ and $\dim \Psi \ge 15$. As $\dim \mathsf{K} \le 8$ and $\widehat{\Psi} = \Psi/(\mathsf{K} \cap \Psi)$ is almost simple by [18, (95.5)], the list [18, (95.10)] shows that $\widehat{\Psi}$ is a simple group of type G_2 or $\widehat{\Psi} \cong \mathsf{O}'_7(\mathbb{R}, r)$. The kernel $\mathsf{N} \cap \Psi$ is a product of some of the almost simple factors of Ψ, and $\mathsf{N} \cap \Psi$ acts freely on H. Consequently, $\dim(\mathsf{N} \cap \Psi) = 0$ or $\mathsf{N} \cap \Psi \cong \widehat{\Psi}$, but the latter is impossible for reasons of dimension. In particular, $\mathsf{N}^1 \le \mathsf{P}$ and $\dim \mathsf{N} \le 1$ as N^1 injects into the centralizer of $\widehat{\Psi}$ in its representation on H. If $\dim \widehat{\Psi} = 14$, then Ψ has a proper factor of type G_2, but this contradicts the fact that Ψ acts irreducibly on Θ. It follows that $\dim \Psi \ge 21$, and then $\Psi \cong \mathsf{Spin}_7(\mathbb{R}, r)$ with $r = 0, 3$ by step 18). The group Ψ is transitive neither on Θ nor on H. Therefore $\dim \Lambda \ge 8$ for a suitable quadrangle, and Λ contains a group $\mathsf{SU}_3\mathbb{C}$. This excludes the case $r = 3$.

Let $\overline{\Psi}$ be a Levi complement of $\sqrt{\Delta}$. From $\dim \mathsf{T} = 15$ and Theorem [18, (87.5)] it follows that $\dim \Delta < 40$ and $\dim \overline{\Psi} \le 24$. If $\dim \overline{\Psi} > 21$, then $\overline{\Psi} = \Upsilon\mathsf{X}$, where $\Upsilon \cong \mathsf{Spin}_7\mathbb{R}$ and the 3-dimensional almost simple factor X centralizes Υ. We may assume that $\Upsilon \le \Psi$. Then X fixes the axis av of the reflection in Υ and the unique fixed point a of Υ on a^Θ. By [18, (95.6)] the group X would induce the identity both on a^Θ and a^H, a contradiction.

23) Finally, let $\mathsf{T} \cong \mathbb{R}^{16}$. By step 16), we may assume that the complement $\nabla = \Delta_a$ of T acts irreducibly on $\Theta = \mathsf{T}_{[v]}$. Moreover, $\dim \nabla \ge 18$ by hypothesis. Because of Lemma 2.3, the assertion is true whenever ∇ has a subgroup G_2, in particular, if $\dim \nabla > 24$. In the case $\dim \nabla = 24$, it follows from [18, (87.7)] that Δ does not have two fixed points. Therefore, attention can be restricted to $\dim \nabla \le 23$. If ∇ has no subgroup G_2, we exploit the fact that in a translation plane a maximal compact subgroup Φ of ∇ has codi-

mension at most 2 and is normal in ∇, see [18, (81.8)]. Consequently, $\dim \Phi \geq 16$. Consider the kernel $\mathsf{N} = \nabla \cap \mathrm{Cs}\,\Theta = \nabla_{[u]}$ of the action of ∇ on Θ and the irreducible subgroup $\widetilde{\nabla} = \nabla/\nabla_{[u]}$ of $\mathrm{Aut}\,\Theta$. It is a special feature of 16-dimensional translation planes that $\Phi_{[u]}$ is finite, see [18, (81.20)]. Hence $\widetilde{\Phi} = \Phi/\Phi_{[u]}$ satisfies $\dim \widetilde{\Phi} = \dim \Phi$. The large subgroups in the maximal compact subgroup $\mathsf{SO}_8\mathbb{R}$ of $\mathrm{Aut}\,\Theta$ are listed in [18, (95.12)]. Since $\mathsf{G}_2 \not\hookrightarrow \nabla$, we conclude that $\dim \Phi = 16$ and that $\Phi' \cong \mathsf{SU}_4\mathbb{C}$ (recall from step 21) that $\mathsf{SO}_5\mathbb{R} \not\hookrightarrow \Phi$). Moreover, Φ' acts faithfully and irreducibly on Θ, see [18, (95.12c)]. Hence $\Phi \cong \mathsf{U}_4\mathbb{C}$, $\dim \nabla = 18$, and $\dim \Delta = 34$. This completes the proof of Theorem 1.1. □

3 The planes and their automorphism groups

Now let $\dim \Delta \geq 35$. If T is transitive, then $\dim \Sigma_{[a]} > 0$ and the existence of a subgroup $\mathsf{Spin}_7\mathbb{R}$ in Δ implies $\dim \Sigma \geq 38$. All such planes are described in [18, (82.5)]. We may assume, therefore, that $\mathsf{T}_{[u]} \cong \mathbb{R}^7$ and $\mathsf{T}_{[v]} \cong \mathbb{R}^8$, cf. also [18, (61.12)]. The plane $\mathcal{P}$ can then be coordinatized by a 'Cartesian field' $(\mathbb{O}, +, \bullet)$, cf. [5, XI.4.2] or [18, (24.4)]. (Such linear ternary fields with associative addition have also been called *Cartesian groups* even though they are like rings rather than groups.) If the lines of the form $y = s{\bullet}x + t$ together with the 'verticals' form an affine plane and if multiplication is continuous, then, by [18, (43.6)], the Cartesian field indeed yields a compact projective plane.

Theorem 3.1. *Consider a topological Cartesian field* $(\mathbb{R}, +, *, 1)$ *with unit element, and assume that* $(-r)*s = -(r*s)$ *holds identically. Let* $\rho : [0,\infty) \approx [0,\infty)$ *be a homeomorphism with* $\rho(1) = 1$. *Write each octonion* $x \in \mathbb{O}$ *in the form* $x = \xi + \mathfrak{x}$, *where* $\xi = \mathrm{Re}\,x = \frac{1}{2}(x + \overline{x})$ *and* $\mathfrak{x} = \mathrm{Pu}\,x = \frac{1}{2}(x - \overline{x})$, *and define a new multiplication on* $\mathbb{O}$ *by*

$$s \diamond x = |s|^{-1} s \left(|s| * \xi + \rho(|s|) \cdot \mathfrak{x} \right) \ \ \textit{for} \ \ s \neq 0 \ \ \textit{and} \ \ 0 \diamond x = 0\,.$$

Then $\mathbb{O}_\diamond = (\mathbb{O}, +, \diamond, 1)$ *is a topological Cartesian field with unit element* 1. *A plane* $\mathcal{P}$ *can be coordinatized by such a Cartesian field if and only if* $\mathcal{P}$ *satisfies the hypotheses of Theorem* 1.1 *with* $\dim \Delta \geq 35$.

Remark 3.2. 1) An analogous construction can be applied to $\mathbb{C}$ and to $\mathbb{H}$ instead of $\mathbb{O}$.

2) Obviously, the multiplications $\diamond$ and $*$ coincide on $\mathbb{R}$. It follows that $\mathbb{O}_\diamond$ is a quasi-field if and only if $*$ is the ordinary multiplication of the reals. These quasifields and the corresponding translation planes are discussed in [6] and in [18, (82.4 and 5)].

Proof of Theorem 3.1. **Part A.** Suppose first that $\mathcal{P}$ has the properties of Theorem 1.1 without being a translation plane. Then $\dim \mathsf{T} = 15$ and Δ has a subgroup $\Upsilon \cong \mathsf{Spin}_7\mathbb{R}$.

1) We may assume that $\Delta = \mathsf{T}\Upsilon$ and that the translation group $\mathsf{T}_{[v]}$ with center v is transitive. As remarked above, the affine plane $\mathcal{P}^W$ can then be coordinatized with respect to any quadrangle $0 = a, u, v, e$ in the usual way (as in [18, § 22]) by a Cartesian field $\mathbb{O}_\diamond = (\mathbb{O}, +, \diamond)$, where $+$ denotes the ordinary addition of the octonions. (Call to mind that each translation can be written in the form $(x, y) \mapsto (x+a, y+b)$; hence $(\mathbb{O}, +) \cong \mathsf{T}_{[v]} \cong \mathbb{R}^8$.)

2) If u is the other fixed point of Δ, then $\Xi := \mathsf{T}_{[u]} \cong \mathbb{R}^7$ is Υ-invariant. Thus, there is a 7-dimensional vector subgroup V of $(\mathbb{O}, +)$ such that

$$\Xi = \{(x, y) \mapsto (x+c, y) \mid c \in V\}.$$

3) *The group Υ fixes a triangle and may be identified with $\nabla = \Delta_a$.* Indeed, $\nabla \cong \Delta_a/\mathsf{T}_a$ is isomorphic to a subgroup of $\Delta/\mathsf{T} \cong \Upsilon$. Since $\dim \nabla \geq 20$ and Υ has no proper subgroups of small codimension, $\nabla \cong \Upsilon$. By the Mal'cev–Iwasawa Theorem [18, (93.10)], Υ and ∇ are conjugate in Δ.

4) Because Υ induces on Ξ the group $\mathsf{SO}_7\mathbb{R}$, the central involution $\alpha \in \Upsilon$ fixes the orbit a^Ξ pointwise and α is a reflection with axis au, cf. [18, (55.29)]. In coordinates, α has the form $(x, y) \mapsto (x, -y)$ since α inverts each translation in $\mathsf{T}_{[v]}$. This implies that $(-s) \diamond x = -(s \diamond x)$ holds identically in $\mathbb{O}_\diamond$.

5) According to [18, (96.36)], the action of Υ on the (invariant) line au is equivalent to a linear action, and the fixed point set is homeomorphic to $\mathbb{S}_1$. Moreover, Υ acts trivially on the 1-dimensional quotient space au/Ξ. Therefore, each Ξ-orbit in $au \setminus \{u\}$ is Υ-invariant and contains a unique fixed point of Υ.

6) Since α has center v, the group Υ acts faithfully on av. The faithful representation of $\mathsf{Spin}_7\mathbb{R}$ on $\mathbb{R}^8$ being unique up to a linear transformation of $\mathbb{R}^8$, the line $av \setminus \{v\}$ can be identified with $\{0\}\times\mathbb{O}$ in such a way that Υ preserves the ordinary norm of $\mathbb{O}$.

7) Let e be chosen on a fixed line of Υ in the pencil $\mathfrak{L}_v$ such that a, u, v, e is a nondegenerate quadrangle. Then the stabilizer $\Lambda = \Upsilon_e$ is isomorphic to G_2, and Λ fixes a one-parameter subgroup $(\mathbb{R}, +)$ of the vector group $\mathbb{O}$, corresponding to a transitive group of 'vertical' translations of the 2-dimensional plane $\mathcal{E}$ consisting of the fixed elements of Λ. Consequently, $\mathcal{E}$ is coordinatized by a Cartesian field $\mathbb{R}_* = (\mathbb{R}, +, *)$. In fact, $\mathbb{R}_*$ is a Cartesian subfield of $\mathbb{O}_\diamond$, and $*$ is the restriction of the multiplication $\diamond$ to $\mathbb{R}$. In particular, $(-s) * x = -(s * x)$ holds for all $s, x \in \mathbb{R}$. Since Λ fixes the coordinate quadrangle, Λ is a group of automorphisms of $\mathbb{O}_\diamond$.

8) In the coordinates introduced in 1), the line ae is given by the equation $y = x$. Because the group Λ fixes this line, Λ acts in the same way on both the coordinate axes. From $\Xi^{\Lambda} \subseteq \Xi^{\Upsilon} = \Xi$ it follows that V is Λ-invariant. In fact, V is the unique Λ-invariant complement of $\mathbb{R}$ in $\mathbb{O}$. Hence V coincides with the vector space $\mathrm{Pu}\,\mathbb{O}$ of the pure elements in $\mathbb{O}$. The fixed point set of Λ in its action on $\mathbb{O}$ is $\mathbb{R}$. Consequently, 5) implies that the fixed point set of Υ on $\mathbb{O}\times\{0\}$ is $\mathbb{R}\times\{0\}$.

9) For $s \neq 0$, consider the line L_s of slope s with the equation $y = s\diamond x$ and note that $s \diamond 1 = s$ and that $x \mapsto s \diamond x$ is a homeomorphism of $\mathbb{O}$. If $s \in \mathbb{R}$, then $(1, s)$ is a fixed point of Λ and the line L_s is Λ-invariant. Therefore, also the stabilizer $\mathsf{H} = \mathsf{T}_{L_s}$ is Λ- invariant. It is isomorphic to $\mathbb{R}^7$ by [18, (61.11c)] and has the form

$$\{(x, y) \mapsto (x + c,\, y + \zeta(c)) \mid c \in \mathrm{Pu}\,\mathbb{O}\},$$

where ζ is an $\mathbb{R}$-linear endomorphism of $\mathrm{Pu}\,\mathbb{O}$ centralizing Λ. Since the centralizer of Λ is isomorphic to $\mathbb{R}$ by Schur's Lemma, there is a number $\rho(s) \in \mathbb{R}^{\times}$ such that

$$\mathsf{H} = \{(x, y) \mapsto (x + \mathfrak{c},\, y + \rho(s)\cdot\mathfrak{c}) \mid \mathfrak{c} \in \mathrm{Pu}\,\mathbb{O}\}.$$

10) For $s \in \mathbb{R}$, each point $(\xi, s * \xi)$ with $\xi \in \mathbb{R}$ belongs to L_s by 7). Hence step 9) yields

$$L_s = \{(\xi + \mathfrak{x},\, s * \xi + \rho(s)\cdot\mathfrak{x} \mid \xi \in \mathbb{R} \wedge \mathfrak{x} \in \mathrm{Pu}\,\mathbb{O}\}\,.$$

In the following, the other lines will be obtained by applying transformations $\varphi \in \Upsilon$ to the lines L_s with real s.

11) The group Υ acts on $\mathbb{O}\times\mathbb{O}$ in the same way as on the Moufang plane with the same point set. By 6) this is true for $\{0\}\times\mathbb{O}$ because $\mathbb{R}^8$ and $\mathbb{O}$ have been identified accordingly. The subgroup Λ acts identically on $\{0\}\times\mathbb{O}$ and $\mathbb{O}\times\{0\}$, see 8). Since the centralizer of the action of Λ on $\mathrm{Pu}\,\mathbb{O}$ is the center of $\mathsf{GL}_7\mathbb{R}$, the action of Υ on $\mathbb{O}\times\{0\}$ is uniquely determined by the restriction to Λ and the fact that Υ fixes $\mathbb{R}\times\{0\}$.

12) The group Υ is transitive on the spheres of constant norm in $\{0\}\times\mathbb{O}$, and for any $s \neq 0$ there is some $\varphi \in \Upsilon$ such that $\varphi(e) = (1, |s|^{-1}s)$. The map φ has the form $(x, y) \mapsto (Ax, By)$ with $A, B \in \mathsf{SO}_8\mathbb{R}$ such that for some $C \in \mathsf{SO}_8\mathbb{R}$ the equation $B(s\cdot x) = Cs\cdot Ax$ holds identically with respect to the ordinary multiplication $\cdot$ of the octonions, see [18, (17.12–16)]. Hence $Bx = |s|^{-1}s \cdot Ax$ and φ maps $L_{|s|}$ onto the set

$$\big\{(\xi + A\mathfrak{x},\, |s|^{-1}s\,(|s| * \xi + \rho(|s|)\cdot A\mathfrak{x}) \mid \xi \in \mathbb{R} \wedge \mathfrak{x} \in \mathrm{Pu}\,\mathbb{O}\big\}\,.$$

Writing $\mathfrak{x}$ instead of $A\mathfrak{x}$, we obtain for L_s the equation $y = s \diamond x$ as claimed.

Part B. *The construction in Theorem* 3.1 *always yields a topological Cartesian field.*

Obviously, the multiplication $\mathbb{O}\times\mathbb{O} \to \mathbb{O} : (a, x) \mapsto a \diamond x$ is continuous. By [18, (43.6)] it suffices, therefore, to show that for $a \neq b$ the maps

$$\lambda_{a,b} : x \mapsto -a \diamond x + b \diamond x \quad \text{and} \quad \mu_{a,b} : x \mapsto x \diamond a - x \diamond b$$

are bijections of $\mathbb{O}$. For each $x \in \mathbb{O}$ we write $x = |x|\, x_1 = \xi + \mathfrak{x}$.

1) For $c = |c|\, c_1 \in \mathbb{O}$ the equation $\mu_{a,b}(x) = c$ has a unique solution: in fact, by taking norms in $\mathbb{O}$, we get the condition

$$(|x| * \alpha - |x| * \beta)^2 + \rho(|x|)^2 \cdot |\, \mathfrak{a} - \mathfrak{b}\,|^2 = |c|^2.$$

The left hand side is monotone in $|x|$ since $(\mathbb{R}, +, *)$ is a topological Cartesian field and therefore $r \mapsto r * \alpha - r * \beta$ is either a continuous bijection of $\mathbb{R}$ or constant. Consequently, $|x|$ is uniquely determined by c, in particular, $c = 0$ implies $x = 0$. In all other cases, x can be obtained from $|x|$ and c. (Note that $x_1(\, |x| * \alpha - |x| * \beta + \rho(|x|)(\mathfrak{a} - \mathfrak{b})\,)_1 = c_1$.)

2) Injectivity of $\lambda_{a,b}$ means $-a \diamond x + b \diamond x = -a \diamond y + b \diamond y \Rightarrow a = b \vee x = y$, and this is equivalent to injectivity of $\mu_{x,y}$.

3) In order to obtain surjectivity, we will show in the next steps that

$$\lim\nolimits_{x\to\infty} \lambda_{a,b}(x) = \infty \tag{†}$$

in the one-point compactification $\widehat{\mathbb{O}}$ of $\mathbb{O}$, i.e., that $\lambda_{a,b}$ has a continuous injective extension to $\widehat{\mathbb{O}}$. Such an extension is necessarily a homeomorphism, cf. also [18, (51.19)].

4) Condition (†) is true in the Cartesian field $(\mathbb{R}, +, *)$. Hence $|a| < |b|$ implies

$$\lim\nolimits_{\xi\to\infty}(|b| * \xi - |a| * \xi) = \infty.$$

5) It can easily be seen that (†) holds in each of the following cases:

$$a = 0 \vee b = 0, \quad |a| = |b|, \quad a_1 = \pm b_1.$$

6) If (†) is not true in general, then there is a sequence x_ν such that $\lim_{\nu\to\infty} x_\nu = \infty$ and for some $a, b \in \mathbb{O}$ with $|a| < |b|$ the sequence $\lambda_{a,b}(x_\nu)$ is bounded. Here

$$\lambda_{a,b}(x_\nu) = b_1(|b| * \xi_\nu + \rho(|b|) \cdot \mathfrak{x}_\nu) - a_1(|a| * \xi_\nu + \rho(|a|) \cdot \mathfrak{x}_\nu)\,.$$

7) Suppose that the sequence $\mathfrak{x}_\nu$ is bounded. Then $\lim_{\nu\to\infty} \xi_\nu = \infty$, and 6) yields $\lim_{\nu\to\infty}(|a| * \xi_\nu)(|b| * \xi_\nu)^{-1} = a_1^{-1} b_1$. This is a positive number of norm 1. Hence $a_1 = b_1$ contrary to step 5). An analogous argument shows that the ξ_ν are unbounded. Therefore we may assume that the ξ_ν as well as the $\mathfrak{x}_\nu$ converge to ∞ in $\widehat{\mathbb{O}}$.

8) The problem can be reduced to the 2-dimensional case as follows: we have $a^{-1}b \notin \mathbb{R}$ by step 5). The automorphism group of $\mathbb{O}$ is transitive on the sphere $\{\mathfrak{x} \in \mathbb{O} \mid \mathfrak{x}^2 = -1\}$ in $\mathrm{Pu}\,\mathbb{O}$, and we can arrange that $\overline{a}_1 b_1 = c \in \mathbb{C}$. Write each element $x \in \mathbb{O}$ as $x = x' + x''$ with $x' \in \mathbb{C}$ and $x'' \in \mathbb{C}^{\perp}$, the orthogonal complement of $\mathbb{C}$ in $\mathbb{O}$. Then

$$\overline{a}_1 \lambda_{a,b}(x_\nu) = \\ c\,(|b| * \xi_\nu) - |a| * \xi_\nu + (c\rho(|b|) - \rho(|a|)) \cdot \mathfrak{x}'_\nu + (c\rho(|b|) - \rho(|a|)) \cdot \mathfrak{x}''_\nu$$

is a bounded sequence. Hence also the sequence $(c\rho(|b|) - \rho(|a|)) \cdot \mathfrak{x}''_\nu \in \mathbb{C}^{\perp}$ is bounded and therefore $\lim_{\nu\to\infty} \mathfrak{x}'_\nu = \infty$ by step 7).

9) Let $c = p + iq$ with $p^2 + q^2 = 1$ and put $\mathfrak{x}'_\nu = i\eta_\nu$. Taking conjugates if necessary and selecting suitable subsequences, the possibilities can be reduced to $\lim_{\nu\to\infty} \eta_\nu = +\infty$ and the following cases: $\lim_{\nu\to\infty} \xi_\nu = +\infty$ or $\lim_{\nu\to\infty} \xi_\nu = -\infty$. The sequence

$$p\,(|b| * \xi_\nu) - |a| * \xi_\nu - q\,\rho(|b|)\,\eta_\nu + i\left(q\,(|b| * \xi_\nu) + p\,\rho(|b|)\,\eta_\nu - \rho(|a|)\,\eta_\nu\right)$$

is bounded, and so are the real and the imaginary part and the following linear combinations of these:

$$|b| * \xi_\nu - p\,(|a| * \xi_\nu) - q\,\rho(|a|)\,\eta_\nu \tag{1}$$

$$\text{and}\quad q\,(|a| * \xi_\nu) + (\rho(|b|) - p\,\rho(|a|))\,\eta_\nu\,. \tag{2}$$

Since $\rho(|b|) - p\,\rho(|a|) > 0$, boundedness of (2) implies $\lim_{\nu\to\infty} q\,\xi_\nu = -\infty$, but then the sequence (1) would not be bounded. This proves the claim of Part B.

Part C. Consider a projective plane $\mathcal{P}$ coordinatized by a topological Cartesian field $\mathbb{O}_\diamond = (\mathbb{O}, +, \diamond)$ as described in Theorem 3.1. It remains to show that $\mathrm{Aut}\,\mathcal{P}$ contains a group Δ fixing exactly two points such that $\dim \Delta \geq 35$.

1) Obviously, $\{(x, y) \mapsto (x + \mathfrak{c},\, y + d) \mid \mathfrak{c} \in \mathrm{Pu}\,\mathbb{O},\, d \in \mathbb{O}\} \leq \mathsf{T}$ and $\dim \mathsf{T} \geq 15$.

2) The maps $(x, y) \mapsto (Ax, By)$ of $\mathbb{O}\times\mathbb{O}$ such that $A, B \in \mathsf{Spin}_7\mathbb{R}$ and identically $B(s\cdot x) = Bs\cdot Ax$ form a group Υ of automorphisms of the Moufang plane, they satisfy $A1 = 1$ and hence fix the set $\mathbb{R}\times\{0\}$, cf. **A**), step 9) or [18, (17.14)]. The involution $(x, y) \mapsto (x, -y)$ is a reflection in $\Upsilon_{[v]}$. Consequently, $\Upsilon \cong \mathsf{Spin}_7\mathbb{R}$ acts faithfully on $\{0\}\times\mathbb{O}$ and induces on $\mathrm{Pu}\,\mathbb{O}\times\{0\}$ the group $\mathsf{SO}_7\mathbb{R}$. It follows that

$$B(s \diamond x) = Bs_1(|s| * \xi + \rho(|s|) \cdot A\mathfrak{x}) = Bs \diamond Ax\,.$$

Therefore $\Upsilon \leq \mathrm{Aut}\,\mathcal{P}$, the group $\Delta = \Upsilon\mathsf{T}$ fixes exactly the points u, v, and $\dim \Delta = 36$. □

Theorem 3.3 (Automorphism groups). *Assume that the plane $\mathcal{P}$ satisfies the hypotheses of Theorem* 1.1 *with* $\dim \Delta \geq 35$ *and let* $\Sigma = \operatorname{Aut} \mathcal{P}$ *be the full automorphism group,* Σ^1 *its connected component. If $\mathcal{P}$ is not the classical Moufang plane, then*

(a) $\dim \Sigma < 40$ *and each of the two fixed points of* Δ *is also a fixed point of* Σ. *Any subgroup* $\Upsilon \cong \mathsf{Spin}_7\mathbb{R}$ *of* Σ *fixes some point* $a \notin uv$.

(b) *If* $\dim \Sigma = 39$*, then $\mathcal{P}$ is a translation plane.*

(c) *The plane $\mathcal{P}$ is a translation plane if, and only if, it can be coordinatized by a quasi-field $\mathbb{O}_\diamond$ as in Theorem* 3.1 *where $*$ is the ordinary multiplication of the reals. In this case* $\dim \Sigma = 39$ *if, and only if, ρ is a multiplicative homomorphism; otherwise* $\dim \Sigma = 38$.

If $\mathcal{P}$ is not a translation plane, then the following holds:

(d) $\dim \Sigma \leq 38$ *and* $\Sigma = \mathsf{T}^1\Upsilon\mathsf{Z}$*, where* Z *denotes the centralizer of* Υ *in* Σ.

(e) $\dim \Sigma = 38$ *if, and only if, $\mathcal{P}$ can be coordinatized by a Cartesian field $\mathbb{O}_\diamond$ as in Theorem* 3.1 *where*

$$r * s = \begin{cases} rs & (s \geq 0) \\ |r|^\gamma rs & (s < 0) \end{cases} \qquad \textit{for some } \gamma > 0\,,$$

and $\rho : [0,\infty) \to [0,\infty)$ *is a multiplicative homomorphism.*

Proof. (a) If $\dim \Sigma \geq 40$, then $\mathcal{P}$ can be coordinatized by a mutation of the octonions and Σ has no subgroup $\mathsf{Spin}_7\mathbb{R}$, see [18, (82.29) and (87.7)]. We use the same notation as in the proof of Theorem 3.1. If $W^\sigma \neq W$ for some $\sigma \in \Sigma$, then $\Sigma\!:\!\Delta \geq \dim W^{\sigma\mathsf{T}} \geq 7$ and $\dim \Sigma \geq 43$. Hence $W^\Sigma = W$. The group $\Upsilon < \Delta$ acts effectively on W and each point $z \in W \setminus \{u,v\}$ has an orbit $z^\Upsilon \approx \mathbb{S}_7$. Therefore $v^\Sigma \in \{u,v\}$, or again $\dim \Sigma \geq 43$. If some $\sigma \in \Sigma$ interchanges u and v, then $\mathcal{P}$ is a translation plane. Consider a Levi complement Ψ in a maximal compact subgroup of Σ^1. All such groups are conjugate in Σ^1, see [18, (93.10) and (94.28)]. Therefore, Ψ contains conjugates of Υ and of Υ^σ. The first acts effectively on the pencil $\mathfrak{L}_u \cong \mathbb{R}^8$, the second induces a group $\mathsf{SO}_7\mathbb{R}$ on $\mathfrak{L}_u$. The central involutions in these groups are reflections with centers v and u respectively, their axes are Ψ-invariant, or else Ψ would contain translations by the dual of [18, (23.20)]. Consequently, Ψ fixes some point $a \notin W$, and the kernel $\Psi_{[u]}$ of the action of Ψ on $\mathfrak{L}_u$ is finite by [18, (81.20)]. It follows that Ψ is almost simple (cf. step 18) above) and has a proper subgroup $\mathsf{Spin}_7\mathbb{R}$. The list [18, (95.10)] shows that $\dim \Psi = 28$ and then $\dim \Sigma \geq 44$, a contradiction. Therefore Σ fixes u and v. If $\mathsf{Spin}_7\mathbb{R} \cong \Upsilon < \Sigma$, then the central involution in Υ is a reflection and Υ fixes its axis X. Any action of the group Υ

on a space X homeomorphic to $\mathbb{R}^8$ is equivalent to a linear action ([18, (96.36)]). Hence Υ has a fixed point $a \in X$.

(b) We have $\Upsilon \le \nabla := \Sigma_a^1$ and $\dim \nabla \le 24$. Put $\mathsf{X} = \nabla \cap \mathrm{Cs}\,\Upsilon$. The representation of Υ on the Lie algebra of ∇ shows that $\nabla = \Upsilon\mathsf{X}$. The group X acts effectively on the two-dimensional plane $\mathcal{E}$ of the fixed elements of a subgroup $\Lambda \cong \mathsf{G}_2$ of Υ. By [18, (32.10)] and the dimension formula, $\dim \mathsf{X} \le 2$, $\dim \nabla = 23$, and $\dim a^\Sigma = 16$. Since the centralizer of $\mathsf{Spin}_7\mathbb{R}$ in $\mathsf{GL}_8\mathbb{R}$ is isomorphic to $\mathbb{R}^\times$ (cf. [18, (95.10)]), the action of ∇ on av has a kernel $\nabla_{[u]}$ of positive dimension. By the dual of [18, (61.20b)] it follows that $\dim \mathsf{T}_{[u]} = 8$.

(c) See [18, (82.5)].

(d) For each $\sigma \in \Sigma$ there is some $\tau \in \mathsf{T}^1$ such that $a^{\sigma\tau}$ is Υ-invariant, cf. step 5) of the proof of Theorem 3.1. Put $\sigma\tau = \omega^{-1}$. It follows that $\Upsilon^\omega \le \nabla$. Since $\nabla = \Upsilon\mathsf{X}$ and all Levi complements in a connected group are conjugate (cf. [18, (94.28c)]), we have $\Upsilon^\omega = \Upsilon$. Each automorphism of Υ is an inner automorphism (see [20, 6.]). Consequently, $\omega \in \Upsilon\mathsf{Z}$.

(e) Consider $\Lambda < \Upsilon$ and the subplane $\mathcal{E}$ consisting of the fixed elements of Λ as in step 7) of the proof of Theorem 3.1. Suppose that $\dim \Sigma = 38$. Then $\dim \mathsf{Z} = 2$ by part (d), and $\dim \mathrm{Cs}\,\Lambda = 3$ as Λ also centralizes the vertical translations of $\mathcal{E}$. Moreover, $\mathrm{Cs}_\Delta\,\Lambda$ contains the central reflection $\alpha \in \Upsilon$ (with axis au). It follows from ($\Diamond$) that $\mathrm{Cs}\,\Lambda$ acts effectively on $\mathcal{E}$. By assumption, $\mathcal{P}$ is not a translation plane; hence $*$ is not the ordinary multiplication and $\mathcal{E}$ is not classical. All planes $\mathcal{E}$ admitting a 3-dimensional group are known explicitly; this classification is summarized in [18, (38.1)], details are given in [18, §§ 34–37]. As the group fixes the points u and v, the results just mentioned show that $\mathcal{E}$ is a plane over a Cartesian field of the kind described in [18, (37.3)], which includes the Moulton planes. The reflection α induces on $\mathcal{E}$ the map $(x,y) \mapsto (x,-y)$. This is a collineation of $\mathcal{E}$ if and only if $(-s) * x = -(s * x)$ holds identically in $\mathbb{R}$. An easy calculation shows that the multiplication $*$ of [18, (37.3)] has indeed the form given in (e), cf. also [18, (37.4 and 6)]. In particular, $\mathcal{E}$ is not a Moulton plane. Note that the product $*$ is associative whenever the right or the middle factor is positive.

The group Z^1 induces on $\mathcal{E}$ the maps $(x,y) \mapsto ((r * x)\cdot s, y\cdot s)$ with $r, s > 0$. It can easily be seen that $(x,y) \mapsto (x\cdot s, y\cdot s)$, $s < 0$, $x, y \in \mathbb{O}$ yields always an automorphism of $\mathcal{P}$. An element $\zeta \in \mathsf{Z}$ which induces on $\mathcal{E}$ a map $(x,y) \mapsto (r * x, y)$ has necessarily the form $(x,y) \mapsto (\varphi_r(x), y)$ because Υ acts irreducibly on $\mathsf{T}_{[v]} \cong \mathbb{R}^8$. This means that ζ is a homology with axis av. Hence $\zeta(x,y) = (r \diamond x, y)$. This map is a collineation if and only if

$a \diamond (r \diamond x) = (a \diamond r) \diamond x$ for all $a, x \in \mathbb{O}$. Equivalently (since $|a| * r = |ar|$),

$$|a| * (r * \xi) + \rho(|a|)\rho(r)\,\mathfrak{x} = (|a| * r) * \xi + \rho(|ar|)\,\mathfrak{x}\,.$$

Thus ρ is multiplicative. Conversely, the conditions in (e) imply $\dim \mathsf{Z} = 2$ and hence $\dim \Sigma = 38$. If ρ is not multiplicative, then $\dim \Sigma = 37$. □

The case $\dim \boldsymbol{\Sigma} = 37$. With the same notation as before, we have $\dim \Sigma = 37$ if and only if $\mathrm{Cs}\,\Lambda$ acts on $\mathcal{E}$ as a 2-dimensional group with 2 fixed points. All planes over a proper Cartesian field $(\mathbb{R}, +, *)$ admitting such a group have been described. They depend on the choice of some suitable real functions rather than a few real parameters. By [18, (32.8)], a quasi-field $(\mathbb{R}, +, *)$ is in fact a field; therefore, $\mathcal{E}$ is not a translation plane. Only the Cartesian fields of those planes $\mathcal{E}$ can be used which admit a reflection with an axis au. The connected component Γ of $\mathrm{Cs}\,\Lambda$ is isomorphic to $\mathbb{R}^2$ or to the linear group

$$\mathrm{L}_2 := \{(t \mapsto at + b) : \mathbb{R} \to \mathbb{R} \mid a > 0\}.$$

In the first case, Γ_{au} fixes each line of $\mathcal{E}$ through the point u, because Γ contains all translations of $\mathcal{E}$ with center v. As $\mathcal{E}$ is not a translation plane, Γ_{au} induces a one-parameter group of homologies of $\mathcal{E}$ with center u and a common axis. The point a may be chosen on this axis; then Γ fixes exactly the elements u, v, av, uv of $\mathcal{E}$, and av is the axis of the elements of Γ_{au}. The planes $\mathcal{E}$ of this type have been determined by Groh [4], cf. [10, *2.7.11.3*].

Homologies of $\mathcal{E}$ with axis av have the form $\gamma_r : (x, y) \mapsto (r * x, y)$. The group Γ_{au} coincides with the connected component Z^1 of $\mathsf{Z} = \mathrm{Cs}\,\Upsilon$ because Z fixes the axis au of the unique central involution $\alpha \in \Upsilon$, and we have $\mathsf{Z}^1 \leq \Gamma$ and $\dim \mathsf{Z} = \dim \Gamma_{au}$. An element $\zeta_r \in \mathrm{Cs}\,\Upsilon$ which induces on $\mathcal{E}$ the homology γ_r fixes necessarily each point on the line av because the centralizer of the representation of Υ on $\mathbb{R}^8$ consists of real dilatations. Consequently ζ_r can be written as $(x, y) \mapsto (r \diamond x, y)$, and the product $\diamond$ is associative whenever the middle factor is a positive real number. The latter condition reduces to the identity $\rho(r * s) = \rho(r)\rho(s)$ for $r, s > 0$. An admissible multiplication $*$ and a homeomorphism ρ yield a plane $\mathcal{P}$ with $\dim \Sigma \geq 37$ if and only if ρ satisfies this identity.

If $\Gamma \cong \mathrm{L}_2$, there are the following possibilities:

(a) Γ acts transitively on the set of points not on uv,

(b) Γ fixes exactly two points and two lines,

(c) Γ fixes exactly two lines and more than two points, or dually

($\tilde{\text{c}}$) Γ fixes exactly the points u and v and more than two lines through v.

(a) Planes with a group Γ satisfying (a) have been studied by Groh [3], cf. [10, *2.7.5.2*]. Those planes $\mathcal{E}$ which are symmetric with respect to a horizontal line can be described in the half-plane $(0,\infty)\times\mathbb{R}$ as follows: Let L be the graph of a strictly convex continuous function $f\colon(0,\infty)\to\mathbb{R}$ such that

$$\lim_{x\to 0} f(x)=\infty,\ \ \lim_{x\to\infty} f(x)=-\infty,\ \ \lim_{x\to\infty} f'(x)=0\,.$$

Then the images of L under the maps $(x,y)\mapsto(rx,ry{+}b)$, $r\in\mathbb{R}^\times$, $b\in\mathbb{R}$ together with the horizontals and verticals are the lines of an affine plane of type (a). This can easily be translated into a representation in $\mathbb{R}^2$ by means of a Cartesian field $\mathbb{R}_*$. In the latter representation Γ contains a one-parameter subgroup of maps $\gamma_t:(x,y)\mapsto(\varphi_t(x),\,e^t y)$ acting transitively on the X-axis. A line of slope s is mapped by γ_t onto a line of slope $\sigma_t(s)$. The fact that γ_t is a collineation of $\mathcal{E}$ is equivalent to the identity

$$e^t(s*x)=\sigma_t(s)*\varphi_t(x)-\sigma_t(s)*\varphi_t(0)\,. \tag{$*$}$$

It remains to find a necessary and sufficient condition for γ_t to be induced by a map ζ_t of $\mathbb{O}^2$ in Z. (Note that again Γ_{au} is the connected component of Z $=\mathrm{Cs}\,\Upsilon$ since $\mathsf{Z}^1\le\Gamma_{au}$ and both groups are homeomorphic to $\mathbb{R}$.) From $\zeta_t\in\mathrm{Cs}\,\Upsilon$ it follows that ζ_t has the form $(x,y)\mapsto(\varphi_t(\xi)+e^{\kappa t}\mathfrak{x},\,e^t y)$. Expressing the fact that the line $y=s\diamond x$ is mapped to a line

$$e^t y=c\diamond(\varphi_t(\xi)+e^{\kappa t}\mathfrak{x})-d$$

yields the condition

$$e^t|s|^{-1}s\big(|s|*\xi+\rho(|s|)\,\mathfrak{x}\big)=|c|^{-1}c\big(|c|*\varphi_t(\xi)-|c|*\varphi_t(0)+e^{\kappa t}\rho(|c|)\,\mathfrak{x}\big).$$

If $0<s\in\mathbb{R}$, then $|s|=s$ and $c=\sigma_t(|s|)=|c|$; comparison of the pure components of the condition above gives

$$e^t\rho(|s|)=e^{\kappa t}\rho(\sigma_t(|s|))\,. \tag{†}$$

In general, we obtain in the same way that $e^t|s|^{-1}s\,\rho(|s|)=|c|^{-1}c\,e^{\kappa t}\rho(|c|)$, which by (†) means $|s|^{-1}s\,e^{\kappa t}\,\rho(\sigma_t(|s|))=|c|^{-1}c\,e^{\kappa t}\rho(|c|)$. Passing to absolute values, one obtains $|c|=\sigma_t(|s|)$ and then $|s|^{-1}s=|c|^{-1}c$, so that finally $c=\sigma_t(|s|)|s|^{-1}s$. Because of ($*$) and (†), the condition above is then satisfied.

We remark that $\kappa\neq 1$, or else $\sigma_t(s)=s$ for all $s>0$ and then also for all $s<0$, and $\mathcal{E}$ would be a translation plane. In particular, ρ is uniquely determined by $\mathcal{E}$.

(b) The classification of these planes has been obtained by Schellhammer [19], cf. [10, *2.7.11.4*]. For each multiplication $*$ defining such a plane there exists a one-parameter group of automorphisms $\gamma_t : (x,y) \mapsto (\varphi_t(x), e^t y)$ of $\mathcal{E}$ fixing a and mapping a line of slope s to a line of slope $\sigma_t(s)$, where $e^t(s * x) = \sigma_t(s) * \varphi_t(x)$. An extension of γ_t to a map $\zeta_t \in \mathrm{Cs}\,\Upsilon$ has again the form $(x,y) \mapsto (\varphi_t(\xi) + e^{\kappa t}\mathfrak{x},\ e^t y)$. As before, this is a collineation of $\mathcal{P}$ if and only if condition (†) holds. Each pair of an admissible multiplication $*$ and a homeomorphism ρ which satisfies (†) yields a plane $\mathcal{P}$ with $\dim\Sigma \geq 37$.

(c) The description of the possible planes $\mathcal{E}$ is due to Pohl [9], cf. [10, *2.7.11.5*]. The same calculations as in case (b) lead once more to condition (†). By assumption there is some slope $r > 0$ such that $\sigma_t(r) = r$. It follows that $\kappa = 1$ and then $\sigma_t(|s|) = |s|$ for each s. As $\Upsilon\Gamma_a \leq \nabla$, the central involution $\alpha \in \Upsilon$ (with axis au) commutes with the maps γ_t. Consequently, γ_t also fixes the negative real slopes, and Γ_a induces homologies of $\mathcal{E}$. Thus, planes with $\dim\Sigma \geq 37$ can be obtained in case (c) if and only if Γ fixes the line uv pointwise; there is no condition on the homeomorphism ρ. The orbits of Γ_a in $\mathcal{E}$ are rays beginning at the origin in the real affine plane. It follows that $\mathcal{E}$ can be described by a Cartesian field multiplication of the form $s*x = sx$ for $x \geq 0$ and $s * x = \mu(s)x$ for $x < 0$, where $\mu : \mathbb{R} \approx \mathbb{R}$ with $\mu(-s) = -\mu(s)$ and $\mu(1) = 1$. Planes of this kind have been called generalized Moulton planes.

($\tilde{\mathrm{c}}$) Though the planes $\mathcal{E}$ are dual to those of case (c), the conclusions are not because of the different rôles of the central reflection $\alpha \in \Upsilon$. As in the previous cases, the conditions $e^t(s * x) = \sigma_t(s) * \varphi_t(x)$ and (†) must be satisfied. In case ($\tilde{\mathrm{c}}$) we may assume that $\varphi_t(1) = 1$. Then we obtain $\sigma_t(s) = e^t s$ for all $s \in \mathbb{R}$, and (†) reduces to the condition that ρ is a multiplicative homomorphism.

Examples are given by the multiplications

$$s * x = \begin{cases} sx & (x \leq 1) \\ s(|s|^m x + 1 - |s|^m) & (x \geq 1)\,, \end{cases} \qquad (m > 0).$$

In fact, $\varphi_t(x) = x$ for $x \leq 1$ and $\varphi_t(x) = e^{-mt}x + 1 - e^{-mt}$ for $x \geq 1$.

Thus in each of the cases there are large families of planes $\mathcal{P}$ with a group of dimension 37 fixing exactly two points and the line joining them.

References

[1] **R. Bödi**, On the dimensions of automorphism groups of eight-dimensional ternary fields, *Geom. Dedicata* **53** (1994), 201–216.

[2] **N. Bourbaki**, *Lie Groups and Lie Algebras. Chapters 1–3*, Elem. Math. (Berlin), Springer, 1989.

[3] **H. Groh**, Point homogeneous flat affine planes, *J. Geom.* **8** (1976), 145–162.

[4] ———, Pasting of $\mathbb{R}^2$-planes, *Geom. Dedicata* **11** (1981), 69–98.

[5] **Th. Grundhöfer** and **H. Salzmann**, Locally compact double loops and ternary fields, **in** *Quasigroups and Loops: Theory and Applications*, O. Chein, H. O. Pflugfelder, J. D. H. Smith (eds.), Chapter XI, pp. 313–355, Berlin, Heldermann, 1990.

[6] **H. Hähl**, Sechzehndimensionale lokalkompakte Translationsebenen mit $\mathsf{Spin}(7)$ als Kollineationsgruppe, *Arch. Math.* **48** (1987), 267–276.

[7] **H. Hähl** and **H. Salzmann**, 16-dimensional compact projective planes with a large group fixing two points and two lines, *Arch. Math.* **85** (2005), 89–100.

[8] **K. H. Hofmann**, Lie algebras with subalgebras of codimension one, *Illinois J. Math.* **9** (1965), 636–643.

[9] **H.-J. Pohl**, Flat projective planes with 2-dimensional collineation group fixing at least two lines and more than two points, *J. Geom.* **38** (1990), 107–157.

[10] **B. Polster** and **G. F. Steinke**, *Geometries on Surfaces*, Encyclopedia Math. Appl. **84**, Cambridge Univ. Press, 2001.

[11] **B. Priwitzer**, Large semisimple groups on 16-dimensional compact projective planes are almost simple, *Arch. Math.* **68** (1997), 430–440.

[12] ———, Large almost simple groups acting on 16-dimensional compact projective planes, *Monatsh. Math.* **127** (1999), 67–82.

[13] **B. Priwitzer** and **H. Salzmann**, Large automorphism groups of 16-dimensional planes are Lie groups, *J. Lie Theory* **8** (1998), 83–93.

[14] **H. Salzmann**, Automorphismengruppen 8-dimensionaler Ternärkörper, *Math. Z.* **166** (1979), 265–275.

[15] ———, Characterization of 16-dimensional Hughes planes, *Arch. Math.* **71** (1998), 249–256.

[16] ———, On the classification of 16-dimensional planes, *Beiträge Algebra Geom.* **41** (2000), 557–568.

[17] ———, 16-dimensional compact projective planes with 3 fixed points, Special issue dedicated to Adriano Barlotti, *Adv. Geom.* (2003), suppl., S153–S157.

[18] **H. Salzmann**, **D. Betten**, **T. Grundhöfer**, **H. Hähl**, **R. Löwen** and **M. Stroppel**, *Compact Projective Planes. With an introduction to octonion geometry*, de Gruyter Exp. Math. **21**, W. de Gruyter & Co, Berlin, 1995.

[19] **I. Schellhammer**, *Einige Klassen von ebenen projektiven Ebenen*, Diplomarbeit, Tübingen, 1981.

[20] **J. Tits**, *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Lecture Notes in Math. **40**, Springer-Verlag, Berlin-New York, 1967.

[21] **H. Völklein**, Transitivitätsfragen bei linearen Liegruppen, *Arch. Math.* **36** (1981), 23–34.

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