

page 1 / 23

go back

full screen

close

quit

16-dimensional compact projective planes with a large group fixing two points and only one line

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Abstract

We complete the determination of all pairs (\mathcal{P}, Δ) , where \mathcal{P} is a compact projective plane with a 16-dimensional point set, Δ is an automorphism group of \mathcal{P} of dimension at least 35, and Δ does not fix exactly one point and one line. If Δ fixes two points and only one line, then Δ contains a 15-dimensional translation group and a compact subgroup $\text{Spin}_7\mathbb{R}$; hence $\dim \Delta \geq 36$. The planes are described by their coordinatizing Cartesian fields, more explicitly for $\dim \Delta > 36$.

Keywords: compact projective plane, 16-dimensional plane, Cartesian field, translation group

MSC 2000: 51H10

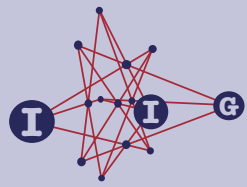
1. Introduction

Let $\mathcal{P} = (P, \mathcal{L})$ be a topological projective plane with a compact point set P of finite (covering) dimension $d = \dim P > 0$. A systematic treatment of such planes can be found in the book *Compact Projective Planes* [18]. Each line $L \in \mathcal{L}$ is homotopy equivalent to a sphere \mathbb{S}_ℓ with $\ell \mid 8$, and $d = 2\ell$, see [18, (54.11)]. In all known examples, L is in fact homeomorphic to \mathbb{S}_ℓ . Taken with the compact-open topology, the automorphism group $\Sigma = \text{Aut } \mathcal{P}$ (of all continuous collineations) is a locally compact transformation group of P with a countable basis, the dimension $\dim \Sigma$ is finite, cf. [18, (44.3 and 83.2)].

For $\ell \leq 4$, all sufficiently homogeneous planes are known explicitly, see [18, Chaps. 7, 8]. In the case $\ell = 8$ the aim is to determine all pairs (\mathcal{P}, Δ) , where Δ is a connected closed subgroup of Σ and $\dim \Delta \geq b$ for a suitable bound b .

ACADEMIA
PRESS





page 2 / 23

go back

full screen

close

quit

(If $\dim \Delta \geq 27$, then Δ is always a Lie group [13].) Here, we deal with the case that $b = 35$ and Δ fixes exactly 3 elements (say two points and one line). This completes the classification for $b = 35$ and all groups Δ which do not fix exactly two elements (a point and a line), cf. [17] for the other possible configurations of fixed elements.

Theorem 1.1. *If Δ fixes exactly 2 points and one line and if $\dim \Delta \geq 34$, then the group Γ of translations in Δ is at least 15-dimensional.*

Either Δ has a subgroup $\Upsilon \cong \text{Spin}_7\mathbb{R}$ and $\dim \Delta \geq 36$, or Γ is transitive, a maximal semi-simple subgroup of Δ is isomorphic to $\text{SU}_4\mathbb{C} \cong \text{Spin}_6\mathbb{R}$, and $\dim \Delta = 34$.

All planes satisfying the hypotheses of Theorem 1.1 with $\dim \Delta \geq 35$ will be described by coordinate methods in Theorems 3.1 and 3.3.

2. Structure of the group

Essential for the proof is the so-called *stiffness*:

The stabilizer of a quadrangle has dimension at most 14; see [18, (83.23)].

Particularly important is Bödi's improvement [1]:

(\diamond) *If the fixed elements of the connected Lie group Λ form a connected subplane \mathcal{E} , then Λ is isomorphic to the 14-dimensional compact group G_2 or its subgroup $\text{SU}_3\mathbb{C}$, or $\dim \Lambda < 8$. If \mathcal{E} is a Baer subplane ($\dim \mathcal{E} = 8$), then Λ is a subgroup of $\text{SU}_2\mathbb{C}$. Moreover, $\Lambda \cong G_2$ implies $\dim \mathcal{E} = 2$.*

If Δ fixes 2 distinct points and $\dim \Delta > 30$, then it follows from other classification results ([11, 12, 15]) that Δ is not semi-simple and has no normal torus subgroup. The main result of [16] can now be stated in the following form:

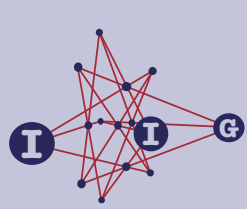
Lemma 2.1. *If Δ fixes exactly one line W and at least 2 points on W , and if $\dim \Delta \geq 33$, then Δ has a minimal normal subgroup $M \cong \mathbb{R}^t$ consisting of translations with axis W .*

Two more facts will be needed repeatedly:

Lemma 2.2. *Assume that Γ is a solvable Lie subgroup of Δ . Then Γ has a chain of normal subgroups Γ_κ with $\dim \Gamma_{\kappa+1}/\Gamma_\kappa \leq 2$; see [2, I § 5, Th. 1, Cor. 4, p. 46]. If κ is the largest index such that $a^{\Gamma_\kappa} = a$, if $N = \Gamma_{\kappa+1}$ and $a \neq x \in a^N$, then $\dim x^{\Gamma_a} \leq 2$. In fact, $x^{\Gamma_a} \subseteq a^N$ and $\dim x^{\Gamma_a} \leq \dim N/N_a \leq \dim N/\Gamma_\kappa$.*

ACADEMIA
PRESS





page 3 / 23

go back

full screen

close

quit

Notation. The connected component of a group Γ will be denoted by Γ^1 . Let u and v be the two fixed points of Δ . For a point $a \notin W = uv$ we put $\nabla = (\Delta_a)^1$. By Lemma 2.1 there exists a minimal ∇ -invariant vector subgroup $\Theta \cong \mathbb{R}^t$ consisting of translations in M . The radical $P = \sqrt{\Delta}$ is the largest solvable normal subgroup of Δ . We write $\Delta : \Gamma = \dim \Delta - \dim \Gamma$ and $\Gamma|_M$ for the group induced by Γ on the Γ -invariant set M .

The dimension formula $\dim \Gamma = \dim \Gamma_x + \dim x^\Gamma$ holds for any closed subgroup Γ of Δ , see [18, (96.10)]. This fact will often be used without mention.

Lemma 2.3. *If a maximal semi-simple subgroup Ψ of Δ or of ∇ (a Levi complement of the radical) has a subgroup $\Lambda \cong G_2$, then Ψ is almost simple, and $\Psi = \Lambda$ or there is a group $\Upsilon \cong \text{Spin}_7\mathbb{R}$ with $\Lambda < \Upsilon \leq \Psi$. The central involution $\alpha \in \Upsilon$ is a reflection.*

Proof. This follows from (\diamond) and the observation that (in the relevant dimension range) each simple group which contains G_2 is of type B or D or G_2 , see [7] for details. By [18, (55.40)], any action of $\text{SO}_5\mathbb{R}$ on a compact projective plane is trivial. Hence $\Psi \not\cong \text{SO}_7\mathbb{R}$ and α is not planar. \square

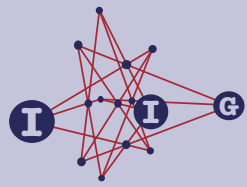
Proof of Theorem 1.1. Recall that there exists a minimal ∇ -invariant subgroup $\Theta \cong \mathbb{R}^t$ which is contained in the group T of translations with axis W . But for the last step, we may assume that $\dim T < 16$.

1) *The elements of Θ have center u or center v , and we may assume $\Theta \leq T_{[v]}$.*

In fact, for $v \in L \neq W$ the stabilizer Θ_L consists of translations with center v . The action of Θ on the pencil \mathcal{L}_v shows that $\dim \Theta_{[v]} \geq t - 8$, cf. [18, (61.11a)], and $\dim \Theta_{[v]} = 0$ or $\Theta = \Theta_{[v]}$ by minimality. Therefore $t \leq 8$. Assume that $\mathbb{1} \neq \vartheta \in \Theta_{[z]}$ for some center $z \neq u, v$, and note that $\Theta_{[z]}$ is connected by [18, (61.9)]. Choose any point $a \notin W$. If $\mathbb{R} \cong \Pi \leq \Theta$ and $\vartheta \in \Pi$, then the connected component Λ of Δ_{a, a^ϑ} centralizes each translation in Π because $\vartheta^\Lambda = \vartheta$ and Λ acts linearly on Θ . Thus, Λ fixes the orbit a^Π pointwise and the fixed elements of Λ form a connected subplane \mathcal{E} . Moreover, $\nabla : \Lambda = \dim(a^\vartheta)^\nabla \leq \dim a^\Theta \leq 8$ and $\dim \Lambda \geq 18 - t$. Hence the stiffness theorem (\diamond) shows that $\Lambda \cong G_2$. Consequently, $t \geq 4$ and Λ acts non-trivially on Θ by the last part of (\diamond) . The action of any compact or semi-simple Lie group on a real vector space is completely reducible, and each irreducible module of G_2 on \mathbb{R}^{16} has a dimension divisible by 7, see [18, (95.10)]. Since $\Pi^\Lambda = \Pi$, we conclude that $t = 8$ and $\dim \nabla \leq 22$. Because Θ is minimal, ∇ acts irreducibly on Θ . By Lemma 2.3, the group ∇ has a subgroup $\Upsilon \cong \text{Spin}_7\mathbb{R}$. The central involution $\alpha \in \Upsilon$ is a reflection and inverts each translation in Θ . Thus, α has axis W and some center, which may be chosen as a . Now

ACADEMIA
PRESS





page 4 / 23

go back

full screen

close

quit

$\alpha^\Delta \alpha \subseteq T$ and $\dim T = \dim a^\Delta \geq 12$, see [18, (61.19)]. The group Υ acts faithfully on each invariant subgroup of T . This implies $T_{[u]} \cong T_{[v]} \cong \mathbb{R}^8$ (cf. [18, (95.10)]) and then \mathcal{P} is the classical Moufang plane \mathcal{O} over the octonions by [18, (81.17)], but we have assumed that $\dim T < 16$.

Before continuing the proof of Theorem 1.1, we now prove the following lemma.

Lemma 2.4. *For the connected component Λ of the stabilizer of some quadrangle containing u, v , and an arbitrary point a , the radical P of Δ satisfies $P : (\Lambda \cap P) \leq 20$. If $\dim \Lambda \geq 8$, then $\Lambda \cap P = \mathbb{1}$; in this case, $\dim P = 20$ implies $\dim \Theta \geq 2$ and $\dim P_a = 4$.*

Proof. Lemma 2.2, applied to the action of P on the line pencil \mathcal{L}_v yields a group $X \leq P$ fixing two lines av and bv such that $P : X \leq 10$. Analogously, the action of X on the line av provides a point c with $X : X_{a,c} \leq 10$. As P is solvable and $\Theta^{P_a} = \Theta$ by step 1), there exists a minimal X_a -invariant vector subgroup $N \leq \Theta$ of dimension at most 2, and the argument of Lemma 2.2 shows that c can be chosen in a^N . The fixed elements of $\Lambda = (P_{a,c,bv})^1$ form a connected subplane \mathcal{E} since Λ acts linearly on N and centralizes the translation $\xi \in N$ with $a^\xi = c$. If $\dim \Lambda \geq 8$, then Λ is simple by (\diamond) and $\Lambda \cap P$ is a solvable normal subgroup of Λ , hence trivial. \square

2) Our aim is to show that one of the groups $T_{[u]}$ or $T_{[v]}$ is linearly transitive.

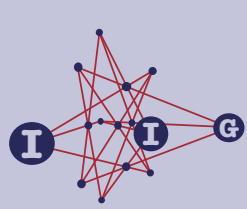
This will be accomplished in steps 2) – 15). Again let $\Theta \leq T_{[v]}$. For $a \notin W$ and $w \in W \setminus \{u, v\}$, consider the connected component Ω of ∇_w . The dimension formula gives $\dim \Omega \geq 10$. As above, let $\mathbb{R} \cong \Pi \leq \Theta$, $\mathbb{1} \neq \rho \in \Pi$, $c = a^\rho$, and put $\Lambda = (\Omega_c)^1$. Then $\Omega : \Lambda = \dim c^\Omega \leq \dim a^\Theta$. Because the action of ∇ on Θ is linear, $\Lambda \leq C_s \Pi$ and (\diamond) applies.

3) For $t = 1$ this gives $\Lambda \cong G_2$. Put $\Delta = P\Psi$, where $P = \sqrt{\Delta}$ is the radical and Ψ is a maximal semi-simple subgroup of Δ . Lemma 2.4 shows that $\dim P \leq 19$; consequently, $\dim \Psi > 14$. According to Lemma 2.3 the Levi complement Ψ has a subgroup $\Upsilon \cong \text{Spin}_7 \mathbb{R}$. For $t < 8$ the central involution $\alpha \in \Upsilon$ acts trivially on Θ by [18, (95.10)] and α is a reflection whose axis is a line through v and whose center is u . We may choose a on this axis. By the dual of [18, (61.19b)] we get $\dim T_{[u]} = \dim(av)^\Delta > 0$. The reflection α inverts the elements of $T_{[u]}$, and the representation of Υ on $T_{[u]}$ is faithful. This implies that $T_{[u]} \cong \mathbb{R}^8$ is linearly transitive as claimed. Moreover, $T_{[u]}$ is a minimal normal subgroup of Δ . The action of Υ on av is equivalent to a linear action, see [18, (96.36)]. Hence $\Upsilon \leq \nabla$ for a suitable choice of a , so that ∇ acts irreducibly on $T_{[u]}$.

4) From $t = 2$ it would follow that $\dim T = 16$, contrary to the general assump-

ACADEMIA
PRESS





page 5 / 23

go back

full screen

close

quit

tion.

If $a \neq c \in a^\ominus$, then $\Gamma = (\nabla_c)^1$ satisfies $\dim \Gamma \geq 16$. Consider a point $w \in W \setminus \{u, v\}$ and the connected component Λ of the stabilizer Γ_w , and note that $\dim \Lambda \geq 8$. By (\diamond) the group Λ is almost simple and hence acts trivially on a^\ominus . Therefore, $\Lambda \not\cong G_2$ and $\Lambda \cong \text{SU}_3\mathbb{C}$. This implies that Γ acts faithfully and transitively on $W \setminus \{u, v\}$, see [18, (96.11)]. According to [15, Lemma 5], the group Γ has a compact subgroup $\Phi \cong \text{SU}_4\mathbb{C}$ of codimension 1. Consequently, Γ is not semi-simple and the commutator subgroup Γ' coincides with Φ . Moreover, $\dim \nabla = 18$ and the group Δ is transitive outside of W . Since Γ' acts trivially on Θ , the central involution α of Γ' is a reflection with axis av . (Note that $\Gamma'/\langle\alpha\rangle \cong \text{SO}_6\mathbb{R}$ cannot act on a Baer subplane.) As before, $T_{[u]} \cong \mathbb{R}^8$ and Γ' acts faithfully on $T_{[u]}$. By [18, (95.6b)], the centralizer $\nabla \cap \text{Cs } T_{[u]}$ has positive dimension. Hence ∇ contains homologies with center v . The dual of [18, (61.20b)] shows that $T_{[v]}$ is also linearly transitive.

5) The cases $3 \leq t \leq 6$ lead to a contradiction.

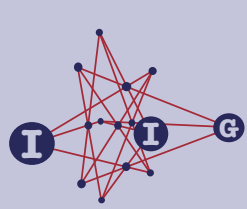
Consider the subplane $\mathcal{F} = \langle a^\ominus, u, v, w \rangle$; either $\mathcal{F} = \mathcal{P}$ and $\Omega = (\nabla_w)^1$ acts faithfully on Θ , or \mathcal{F} is a Baer subplane. In the latter case we write $\Omega|_{\mathcal{F}} = \Omega/K$, where K denotes the kernel of the action of Ω on \mathcal{F} . Recall from (\diamond) that K is a compact group of dimension 3 or at most 1. The different possibilities will be discussed separately. As before, Λ denotes the connected component of the stabilizer of w , a and $c \in a^\ominus$, and $\dim \Lambda \geq 10 - t$.

6) If $t = 3$ and $\mathcal{F} = \mathcal{P}$, then Ω would be embeddable into $\text{GL}_3\mathbb{R}$. Hence $t = 3$ implies $\mathcal{F} \neq \mathcal{P}$. A group Λ of dimension ≥ 8 would act trivially on Θ and on \mathcal{F} , but this is impossible. Therefore, $\dim \Lambda = 7$ and $\dim \Omega = 10$; moreover, Ω acts transitively on $\Theta \setminus \{1\}$ and Ω/K has a subgroup $\text{SO}_3\mathbb{R}$. The stiffness result [18, (83.15)] shows that $\Lambda : K \leq 5$. Consequently, $\dim K = 3$ and Ω/K is a 7-dimensional subgroup of $\text{GL}_3\mathbb{R}$. However, such a subgroup does not exist because $\text{SO}_3\mathbb{R}$ is a maximal subgroup of $\text{SL}_3\mathbb{R}$, see [18, (94.34)].

7) Now let $t = 4$ and $\mathcal{F} = \mathcal{P}$. If Ω is not transitive on $\Theta \setminus \{1\}$, then it follows from (\diamond) that there is an orbit of dimension 3, and suitable stabilizers fix subplanes of dimensions 4 and 8. By [18, (83.9)] and [5, XI.9.6], this implies that Λ is a compact Lie group of rank at most 2, in fact, $\Lambda \cong \text{SU}_3\mathbb{C}$, $\text{SO}_4\mathbb{R}$, or $\dim \Lambda \leq 4$, see [14, (2.1)]. On the other hand, $\dim \Lambda \geq 6$ and Λ acts faithfully on Θ and fixes a one-parameter subgroup. This is a contradiction. Hence Ω is transitive on $\Theta \setminus \{1\}$, and $\Omega' \cong \text{Sp}_4\mathbb{R}$, see [21] or [18, (95.10)]. In particular, Ω contains a central involution α , and α cannot be planar, since the stabilizer of a degenerate quadrangle in an 8-dimensional plane has dimension at most 7, see [18, (83.17)]. Therefore, α is a reflection

ACADEMIA
PRESS





page 6 / 23

go back

full screen

close

quit

with axis W , and $\alpha^\Delta \alpha \subseteq T$, cf. [18, (23.20)]. Moreover, $\dim \Omega \leq 11$ and $\dim \nabla \leq 19$. The dimension formula yields $\dim T \geq \dim a^\Delta \geq 15$. The reflection α acts on T as $-\mathbb{1}$. Because Ω is connected, α induces on T a map of determinant 1; consequently, $T \cong \mathbb{R}^{16}$.

- 8) If $t = 4$ and $\mathcal{F} \neq \mathcal{P}$, the stiffness results [18, (83.17 and 22)] imply $\dim \Omega/K \leq 7$ and $\dim K \leq 3$, hence $\dim \Omega = 10$ and $\dim \nabla = 18$. Therefore, $\dim w^\nabla = 8$ for each choice of w , and ∇ is transitive on $S = W \setminus \{u, v\}$. According to [5, XI.9.5], the group Λ/K is compact, and then we have $\Lambda/K \cong \text{SO}_3\mathbb{R}$ and $\Lambda \cong \text{SO}_4\mathbb{R}$, cf. [14, (2.1)]. In particular, $\dim \Lambda = 6$, $\dim \nabla_c = 14$, and $\dim w^{\nabla_c} = 8$, so that ∇_c is also transitive on S . Let Φ be a maximal compact subgroup of ∇_c containing Λ and note that S is homotopy equivalent to \mathbb{S}_7 . The exact homotopy sequence

$$\cdots \rightarrow \pi_{q+1}S \rightarrow \pi_q\Lambda \rightarrow \pi_q\Phi \rightarrow \pi_qS \rightarrow \pi_{q-1}\Lambda \rightarrow \cdots$$

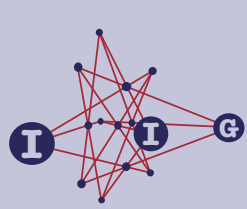
shows that $\pi_1\Phi \cong \mathbb{Z}_2$, $\pi_3\Phi \cong \mathbb{Z}^2$, $\pi_5\Phi \cong \mathbb{Z}_2^2$, and that $\pi_7\Phi$ is infinite. By [18, (94.36)], this implies that Φ is a semi-simple group having exactly two almost simple factors. Moreover, $\Phi \neq \Lambda$ because $\pi_7\Lambda$ is finite. Since $\dim \Phi < \dim \nabla_c$ and $\pi_5\text{SU}_3\mathbb{C} \cong \mathbb{Z}$, the group Φ has a factor $B \cong \text{U}_2\mathbb{H}$, cf. [18, (94.33)] and note that $\text{SO}_5\mathbb{R}$ cannot act on a plane. For the same reason, the central involution $\beta \in B$ is a reflection; its axis is av , since, obviously, $[B, \Theta] = \mathbb{1}$. From $\dim a^\Delta = 16$ we infer that $\beta^\Delta \beta = T_{[u]}$ is linearly transitive. Either ∇ acts faithfully on $T_{[u]}$ or ∇ contains homologies with axis au . In the second case, $T_{[v]}$ is also linearly transitive, see [18, (61.20)], but then the representation of B on $T_{[v]}$ would be trivial (use [18, (95.10)] and note that $[B, \Theta] = \mathbb{1}$) and B would consist of homologies with center u . Consequently, ∇ acts on $T_{[u]}$ as a transitive subgroup of $\text{GL}_8\mathbb{R}$, and [21] shows that ∇ has a transitive factor $X \cong \text{SL}_2\mathbb{H}$. The stabilizer $X_w = X \cap \Omega$ is a 7-dimensional group which fixes \mathcal{F} pointwise, a contradiction to (\diamond) .

- 9) Thus the cases $2 \leq t \leq 4$ cannot arise. Therefore, $t > 4$ and $\mathcal{F} = \mathcal{P}$. For $t < 7$, we have $\Lambda \not\cong \text{SU}_3\mathbb{C}$ and hence $10 \leq \dim \Omega < t + 8$. Since Θ is a minimal ∇ -invariant vector group, ∇ induces on Θ an irreducible group $\tilde{\nabla}$ of dimension $\dim \tilde{\nabla} \geq \dim \Omega \geq 10$.

- 10) Let $t = 5$. By [18, (95.6 and 10)], the commutator group $\tilde{\nabla}'$ is an almost simple group of dimension 10 or 24. In the latter case the dimension of ∇ would be too large. Hence $\tilde{\nabla}'$ is locally isomorphic to a group $\text{O}'_5(\mathbb{R}, r)$ and $\dim \tilde{\nabla} \leq 11$. Because of Brouwer's Theorem [18, (96.30)] or [8], an almost simple group of dimension > 3 has no subgroup of codimension 1. Consequently, $\Omega' \cong \tilde{\nabla}' \cong \text{O}'_5(\mathbb{R}, r)$, and [18, (55.40)] implies $r > 0$. In the notation of step 2), there is some $\rho \in \Theta$ such that Λ has a subgroup $\text{SO}_3\mathbb{R}$. By [18, (83.10)], the group Λ is then compact, and [14, (2.1)] shows $\Lambda \cong \text{SO}_4\mathbb{R}$

ACADEMIA
PRESS





page 7 / 23

go back

full screen

close

quit

(note that $4 < \dim \Lambda < 8$). Hence Ω' is a hyperbolic motion group of the 4-dimensional projective space $P\Theta$. The stabilizer E of an exterior point of $P\Theta$ is not compact, but E contains a group $SO_3\mathbb{R}$; therefore, E has to be compact for the same reason as Λ , a contradiction.

11) Suppose that $t = 6$ and that Ω acts irreducibly on Θ . The stiffness result (\diamond) implies $\dim \Lambda < 8$ and $10 \leq \dim \Omega \leq 13$. With [18, (95.5 and 6)] it follows that either $\dim \Omega' = 8$ and the center $Z(\Omega)$ is isomorphic to \mathbb{C}^\times , or the action of Ω' on Θ can be understood as the tensor product of the natural representations of $A = SL_2\mathbb{R}$ and $B = SL_3\mathbb{R}$ and $\Omega' \cong A \times B$. In both cases, Ω contains a central involution ω . On a Baer subplane, Ω would induce a group of dimension at most 7, see [18, (83.17)]. Therefore, ω is a reflection with axis uv and center a . We have $\dim \nabla \leq 21$. The hypothesis together with [18, (61.19)] implies $13 \leq \dim a^\Delta = \dim T < 16$. Consequently $\dim \nabla > 18$, $\dim \Omega > 10$ and then $\dim \Omega' = 11$. Because ω belongs to a connected group and acts as $-\mathbb{1}$ on T , both $T_{[u]}$ and $T_{[v]}$ have even dimension, and $T \cong \mathbb{R}^{14}$. Hence one of the groups $T_{[u]}$ and $T_{[v]}$ is linearly transitive. Recall that $\Theta \leq T_{[v]}$. By complete reducibility and [18, (95.10)], either B acts irreducibly on $T_{[u]} \cong \mathbb{R}^8$ or B centralizes a 2-dimensional subgroup of T . In the latter case, the fixed elements of B would form a connected subplane contrary to (\diamond). Since Ω fixes u and w , the factor A acts faithfully on $T_{[u]}$. This contradicts the irreducibility of B , see [18, (95.4)].

12) If $t = 6$ and there is a minimal Ω -invariant vector subgroup $H < \Theta$, and if $\Lambda = (\Omega_c)^1$ for some $c \in a^H \setminus \{a\}$, then $10 - \dim H \leq \dim \Lambda < 8$ by (\diamond). Consider the action of Ω on the subplane $\mathcal{F}_H = \langle a^H, u, v, w \rangle$ and the connected component Φ of the kernel of this action. If $\dim H \leq 4$, then it follows as in steps 6) and 7) that \mathcal{F}_H is an (Ω -invariant) Baer subplane of \mathcal{P} . Now $\dim \Omega/\Phi \leq 7$ by [18, (83.17)], and then [18, (83.22)] implies $\Phi \cong SU_2\mathbb{C}$. Recall from step 5) that Ω acts faithfully on Θ . Since the action of Φ on Θ is completely reducible, Φ acts faithfully on a complement of H in Θ , but $SU_2\mathbb{C}$ has no faithful representation in dimension < 4 . Therefore, $\dim H = 5$ and the commutator group Ω' is semi-simple and irreducible on H , see [18, (95.6b)]. Inspection of the list [18, (95.10)] shows $\Omega' \cong O'_5(\mathbb{R}, r)$, and then Ω' would centralize a complement of H in Θ in contradiction to (\diamond). Hence $t \neq 6$.

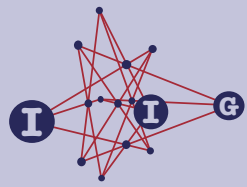
13) Steps 3) – 12) yield the following conclusion.

Conclusion. *If \mathcal{P} is not a translation plane and if $\Theta \cong \mathbb{R}^t$ is a minimal ∇ -invariant subgroup of $T_{[v]}$, then either $t \geq 7$, or $t = 1$ and $T_{[u]} \cong \mathbb{R}^8$ is a minimal normal subgroup of Δ .*

14) Now let $t = 7$ and assume first that Ω acts irreducibly on Θ for each choice

ACADEMIA
PRESS





page 8 / 23

go back

full screen

close

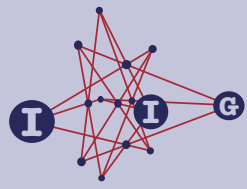
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of w . By [18, (95.6)], the commutator group Ω' is almost simple. Moreover, $9 \leq \dim \Omega' \leq 15$ (since $\Lambda \not\cong G_2$). The list [18, (95.10)] shows that $\dim \Omega' = 14$ and that Ω' has torus rank 2. Because t is odd, each torus subgroup of Ω' fixes a non-trivial vector $\rho \in \Theta$, and [18, (83.10)] implies that the corresponding stabilizer Λ is compact. It follows that $\Lambda \cong \text{SU}_3\mathbb{C}$ and then $\Omega' \cong G_2$ is also compact. Hence $\Lambda \cong \text{SU}_3\mathbb{C}$ for each $c = a^\rho$ and arbitrary w . Suppose that Ω' is a Levi complement of $P = \sqrt{\Delta}$. Then Lemma 2.4 shows that $\dim P = 20$ and $\dim P_a = 4$. This implies that $[P_a, \Omega'] = \mathbb{1} = P_a \cap \Omega'$. The fixed elements of $\Omega' \cong G_2$ form a 2-dimensional subplane \mathcal{E} by [18, (96.35)] and P_a acts effectively on \mathcal{E} , but the stabilizer of a triangle in \mathcal{E} is only 2-dimensional, see [18, (33.10)]. Hence Ω' is not a Levi complement of the radical. By Lemma 2.3, the group Δ has a subgroup $\Upsilon \cong \text{Spin}_7\mathbb{R}$. Since Υ induces the group $\text{SO}_7\mathbb{R}$ on $\Theta \cong \mathbb{R}^7$, the central involution $\alpha \in \Upsilon$ is a reflection with axis av and center u . As in step 3) it follows that $T_{[u]} \cong \mathbb{R}^8$ is linearly transitive and is a minimal normal subgroup of Δ , and we may assume that ∇ acts irreducibly on $T_{[u]}$.

- 15) Last alternative: $t = 7$ and there is a minimal Ω -invariant vector subgroup $H < \Theta$. The proof follows a similar scheme as in the case of the action of ∇ on Θ . We have $1 \leq s := \dim H < 7$. If $s = 1$, then $\dim \Lambda \geq 9$ and $\Lambda \cong G_2$. As G_2 has no representation in dimension < 7 , the group Λ would act trivially on Θ and hence on $\langle a^\Theta, u, w \rangle = \mathcal{P}$, a contradiction. In the case $s = 2$, the stiffness theorem (\diamond) implies $\Lambda \cong \text{SU}_3\mathbb{C}$. Again Λ would act trivially on Θ , see [18, (95.3 and 10)]. The arguments of step 6) with H instead of Θ show that $s \neq 3$. Next, let $s = 4$ and assume first that Ω acts faithfully on H as an irreducible subgroup of $\text{GL}_4\mathbb{R}$. Then Ω' is a semi-simple group of dimension ≥ 8 , see [18, (95.6b)]. Hence Ω' is isomorphic to $\text{Sp}_4\mathbb{R}$ or to $\text{SL}_4\mathbb{R}$. The action of Ω' on Θ is completely reducible, and H has an Ω' -invariant complement $X \cong \mathbb{R}^3$ in Θ . Consequently Ω' induces the identity on the subplane $\langle a^X, u, w \rangle$, but this contradicts (\diamond). Therefore $\langle a^H, u, w \rangle$ is a Baer subplane of \mathcal{P} and Ω induces on H a group Ω/K , where K^1 is isomorphic to a subgroup of $\text{SU}_2\mathbb{C}$. Either $K^1 \cong \text{SU}_2\mathbb{C}$ or $\dim K \leq 1$. In both cases, the semi-simple group Ω' fixes a complement X of H in Θ and $\dim \Omega' \geq 8$. If $K^1 \cong \text{SU}_2\mathbb{C}$, then $K^1|_X \cong \text{SO}_3\mathbb{R}$, which is a maximal subgroup of $\text{SL}_3\mathbb{R}$, cf. [18, (94.34)]. Accordingly, $\Omega'|_X \cong \text{SL}_3\mathbb{R}$, a contradiction. If $\dim K \leq 1$, then $\dim \Omega'|_H > 7$ and Ω' contains the group $\text{Sp}_4\mathbb{R}$. This is again impossible. It follows that $s > 4$ and that Ω acts faithfully on H . For $s = 5$, representation theory shows that $\Omega' \cong O'_5(\mathbb{R}, r)$, see [18, (95.10)], and Ω' would act trivially on a complement of H in Θ , a contradiction to (\diamond). In the case $s = 6$, finally, the semi-simple group Ω' fixes a unique complement X of H , and X is even Ω -invariant. This has been excluded at the beginning of step 15).

ACADEMIA
PRESS





page 9 / 23

go back

full screen

close

quit

16) In any case, one of the groups $T_{[u]}$ or $T_{[v]}$ is linearly transitive, and we may assume that $\Theta = T_{[v]} \cong \mathbb{R}^8$ and that ∇ induces an irreducible group on Θ . By [5, XI.9.5 and 6], the stabilizer of an arbitrary quadrangle is compact and Λ is always a compact connected Lie group of torus rank at most 2. If $4 < \dim \Lambda < 8$, then $\Lambda \cong \text{SO}_4\mathbb{R}$, see [14, (2.1)] or [5, XI.9.9].

17) Put $\Gamma = \Delta_{au}$. Because Θ is transitive on $av \setminus \{v\}$, it follows that $\Delta = \Gamma\Theta$ and that Γ acts irreducibly on Θ . If $\dim \Delta \geq 40$, then $\dim T = 16$ or \mathcal{P} is the classical Moufang plane according to [18, (87.7)]. Hence our assumptions imply $26 \leq \dim \Gamma \leq 31$. The centralizer $\Gamma \cap C_s \Theta$ fixes each line in \mathcal{L}_u and consists of collineations with center u .

18) Let G be a closed, connected irreducible subgroup of $\text{SL}_8\mathbb{R}$. If $\dim G \geq 18$, then G' is isomorphic to an almost direct product $\text{SL}_2\mathbb{R} \cdot \text{SL}_4\mathbb{R}$ or $\text{SU}_2\mathbb{C} \cdot \text{SL}_2\mathbb{H}$, or to one of the almost simple groups $\text{Sp}_4\mathbb{C}$, $\text{Spin}_7(\mathbb{R}, r)$ with $(r = 0, 3)$, $O'_8(\mathbb{R}, r)$, $\text{SL}_4\mathbb{C}$, or $\dim G' \geq 36$.

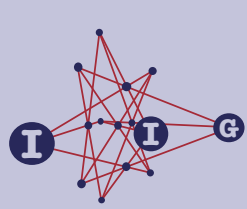
In fact, G' is semi-simple and $\dim G' \geq 16$ by [18, (95.6)]. Suppose that $G' = AB$ is an almost direct product where A has minimal dimension. If B acts irreducibly on $V = \mathbb{R}^8$, then $A \cong \mathbb{H}'$ and $B \leq \text{SL}_2\mathbb{H}$. In the other case, $\dim B \geq 8$, and Clifford's Lemma [18, (95.5)] shows that B acts faithfully and irreducibly on a subspace U such that $V = U \oplus U^\alpha$ for some $\alpha \in A$. By [18, (95.10)], it follows that $\dim B \neq 8$. Therefore, $\dim B > 9$, and B contains a group $\text{Sp}_4\mathbb{R}$. If $0 \neq x \in U$, then the fixed points of B_x form a 1-dimensional subspace of U , and $\langle x, x^\alpha \rangle \cong \mathbb{R}^2$ is A -invariant. Consequently, $A \cong \text{SL}_2\mathbb{R}$ and $\dim B = 15$. All possibilities for an almost simple group G' are listed in [18, (95.10)].

19) If $\Gamma_{[u]} = \mathbb{1}$, then Γ acts faithfully on Θ ; hence Γ' is semi-simple and $\dim \Gamma' \geq 24$, see [18, (95.6)]. By the last step, $\Gamma' \cong \text{SL}_4\mathbb{C}$ or $\Gamma' \cong O'_8(\mathbb{R}, r)$. In the first case, the involution $\beta = \text{diag}(\mathbb{1}, -\mathbb{1}) \in \text{SL}_4\mathbb{C}$ is not a reflection and hence fixes a Baer subplane \mathcal{B} pointwise, cf. [18, (55.29)]. The group $B = (\mathbb{1}, \text{SL}_2\mathbb{C}) \leq C_s \beta$ would induce on \mathcal{B} a group of central collineations with center u , but this is impossible by [18, (61.20)], as B is semi-simple. If $\Gamma \cong O'_8(\mathbb{R}, r)$, the diagonal involution $\text{diag}(1, 1, \dots, 1, -1, -1)$ would fix a 6-dimensional subset of \mathcal{L}_u and hence would be neither a reflection nor a Baer involution. This contradicts [18, (55.29)].

20) In the previous step it has been proved that $\Gamma_{[u]} \neq \mathbb{1}$. Assume first that $\Gamma_{[u]}$ contains homologies. We may choose a in such a way that $\Gamma_{[u,av]} \neq \mathbb{1}$. From the dual of [18, (61.20b)] it follows that $s := \dim T_{[u]} = \dim a^\Gamma = \dim \Gamma - \dim \nabla$, and, hence, $\Gamma = \nabla T_{[u]}^1$. Moreover, this is also the dimension of the set of all axes of homologies in Γ with center u . We choose $b \in a^{T_{[u]}} \setminus \{a\}$

ACADEMIA
PRESS





page 10 / 23

go back

full screen

close

quit

and $c \in av \setminus \{a\}$ and put $\Lambda = (\nabla_{b,c})^1$. Then

$$26 \leq \dim \Gamma = \dim \nabla + s \leq \dim \Lambda + 8 + 2s \leq 22 + 2s \quad \text{and} \quad 1 < s < 8.$$

The assumption $s \leq 5$ implies successively $\dim \Lambda \geq 8$, $\Lambda \cong \text{SU}_3\mathbb{C}$ or $\Lambda \cong \text{G}_2$, Λ acts trivially on $T_{[u]}$, $\Lambda \not\cong \text{G}_2$, $s = 5$, $\Lambda \not\cong \text{SU}_3\mathbb{C}$, a contradiction. Assume that $s = 6$. Then $\Lambda \not\cong \text{SU}_3\mathbb{C}$ because Λ fixes some elements of $T_{[u]}$. Hence $\Lambda \cong \text{SO}_4\mathbb{R}$ by step 16), and $\dim \nabla = 20$. For any admissible b , the dimension formula gives

$$12 \leq \dim \nabla_c = \dim b^{\nabla_c} + \dim \Lambda \leq s + 6 = 12,$$

and $\dim \nabla_c = 12$, $\dim b^{\nabla_c} = 6$. By [18, (96.11a)], the group ∇_c acts transitively on $T_{[u]}^1 \cong \mathbb{R}^6$. The action is also effective since its kernel is trivial on $\langle a^{T_{[u]}^1}, c, v \rangle = \mathcal{P}$. On the other hand, the results in [21] (or in [18, (96.19–22)]) show that a transitive subgroup $G \leq \text{GL}_6\mathbb{R}$ satisfies $\dim G \leq 10$ or $\dim G \geq 16$. Therefore, $s = 7$ and $\dim \Gamma = 15$.

21) Now let $\Gamma_{[u]} = T_{[u]} := \text{H}$. If $\dim \text{H} = 1$ and if $a \neq b \in a^{\text{H}}$, then $\dim \Gamma_{a,b} \geq 17$, and (\diamond) implies that Γ has a subgroup $\Lambda \cong \text{G}_2$. From the fact that

$$\dim(\Gamma \cap \text{Cs } \Theta) = \dim \text{H} = 1,$$

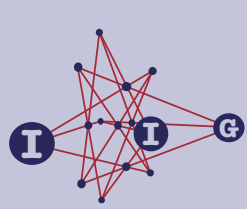
it follows with [18, (95.6)] that a maximal semi-simple subgroup Ψ of Γ acts irreducibly on Θ , and that $\dim \Psi \geq 23$. Because Γ contains G_2 but has no subgroup $\text{SO}_5\mathbb{R}$ by [18, (55.40)], step 18) shows that $\Psi \cong \text{Spin}_8(\mathbb{R}, r)$ with $r \leq 1$, and Ψ induces on Θ a group $\text{O}'_8(\mathbb{R}, r)$ by [18, (95.10)]. Consequently, Γ would contain a reflection with axis av , a possibility which has been dealt with in step 20). Thus, we may assume that $\dim \text{H} = s > 1$; recall that $s < 8$ by the assumption made at the beginning of the proof. As Λ fixes a subspace of H and G_2 has no non-trivial representation in dimension < 7 , we conclude that $\Lambda \not\cong \text{G}_2$, $\dim \Lambda \leq 8$ and $\dim \nabla \leq 23$. The group ∇ acts faithfully and irreducibly on $\Theta \cong \mathbb{R}^8$. All possibilities for the semi-simple group ∇' have been listed in step 18). Only the first 5 groups of this list have a dimension at most 23 and we conclude that $18 \leq \dim \nabla' \leq 21$. If $\dim \nabla' > 18$, then ∇' is almost simple and the representation of ∇' on H shows that either $s = 7$, or ∇' fixes a^{H} pointwise, but in the latter case $\dim \nabla' \leq 8 + \dim \Lambda$, which is a contradiction. If $\dim \nabla' = 18$, then $\dim \nabla \leq 19$. We consider the group $\tilde{\Gamma} \cong \Gamma/\text{H}$ induced by Γ on Θ , which contains ∇ . From 18) and the inequalities

$$26 \leq \dim \Gamma \leq 19 + 8 \quad \text{and} \quad \dim \tilde{\Gamma} \leq 27 - s$$

it follows that $\dim \tilde{\Gamma}' \leq 21$. Assume that ∇' is a proper subgroup of $\tilde{\Gamma}'$. Then $\tilde{\Gamma}'$ is isomorphic to $\text{Spin}_7(\mathbb{R}, r)$ or $\text{Sp}_4\mathbb{C}$, and a maximal compact subgroup

ACADEMIA
PRESS





page 11 / 23

go back

full screen

close

quit

K of $\tilde{\Gamma}'$ acts in the canonical way on the homogeneous space $M = \tilde{\Gamma}'/\nabla'$, but this would imply $\dim K \leq 6$ by [18, (96.13)]. (Note that the kernel N of the action of K on M is contained in the intersection of all conjugates of ∇' in $\tilde{\Gamma}'$, a proper normal subgroup of $\tilde{\Gamma}'$; hence $\dim N = 0$.) Consequently, $\dim \tilde{\Gamma} \leq 19$ and then $s \geq 7$. Steps 19) – 21) complete the proof of the first part of Theorem 1.1.

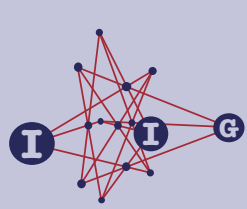
- 22) Assume now that $H = T_{[u]}^1 \cong \mathbb{R}^7$. We will show that a maximal semi-simple subgroup of Δ is isomorphic to $\text{Spin}_7\mathbb{R}$. With the rôles of u and v interchanged, the Conclusion implies that either some 1-dimensional subgroup $\Pi < H$ is ∇ -invariant or ∇ acts irreducibly on H . By hypothesis $\dim \nabla \geq 18$. Let $\nabla = \Psi P$, where Ψ is a maximal semi-simple subgroup of ∇ and $P = \sqrt{\nabla}$. In the first case, the stabilizer Λ of a suitable quadrangle has dimension at least 9; hence $\Lambda \cong G_2$ by (\diamond), and $\Psi \neq \Lambda$ since ∇ acts irreducibly on Θ . Lemma 2.3 implies that Ψ has a subgroup $\Upsilon \cong \text{Spin}_7\mathbb{R}$. In the second case, ∇ induces an irreducible group ∇/N on Θ and an irreducible group ∇/K on H . By [18, (95.6)] we have $P : (N \cap P) \leq 2$ and $P : (K \cap P) \leq 1$, hence $\dim P \leq 3$ and $\dim \Psi \geq 15$. As $\dim K \leq 8$ and $\hat{\Psi} = \Psi / (K \cap \Psi)$ is almost simple by [18, (95.5)], the list [18, (95.10)] shows that $\hat{\Psi}$ is a simple group of type G_2 or $\hat{\Psi} \cong O_7^{\epsilon}(\mathbb{R}, r)$. The kernel $N \cap \Psi$ is a product of some of the almost simple factors of Ψ , and $N \cap \Psi$ acts freely on H . Consequently, $\dim(N \cap \Psi) = 0$ or $N \cap \Psi \cong \hat{\Psi}$, but the latter is impossible for reasons of dimension. In particular, $N^1 \leq P$ and $\dim N \leq 1$ as N^1 injects into the centralizer of $\hat{\Psi}$ in its representation on H . If $\dim \hat{\Psi} = 14$, then Ψ has a proper factor of type G_2 , but this contradicts the fact that Ψ acts irreducibly on Θ . It follows that $\dim \Psi \geq 21$, and then $\Psi \cong \text{Spin}_7(\mathbb{R}, r)$ with $r = 0, 3$ by step 18). The group Ψ is transitive neither on Θ nor on H . Therefore $\dim \Lambda \geq 8$ for a suitable quadrangle, and Λ contains a group $\text{SU}_3\mathbb{C}$. This excludes the case $r = 3$.

Let $\bar{\Psi}$ be a Levi complement of $\sqrt{\Delta}$. From $\dim T = 15$ and Theorem [18, (87.5)] it follows that $\dim \Delta < 40$ and $\dim \bar{\Psi} \leq 24$. If $\dim \bar{\Psi} > 21$, then $\bar{\Psi} = \Upsilon X$, where $\Upsilon \cong \text{Spin}_7\mathbb{R}$ and the 3-dimensional almost simple factor X centralizes Υ . We may assume that $\Upsilon \leq \Psi$. Then X fixes the axis av of the reflection in Υ and the unique fixed point a of Υ on a^{Θ} . By [18, (95.6)] the group X would induce the identity both on a^{Θ} and a^H , a contradiction.

- 23) Finally, let $T \cong \mathbb{R}^{16}$. By step 16), we may assume that the complement $\nabla = \Delta_a$ of T acts irreducibly on $\Theta = T_{[v]}$. Moreover, $\dim \nabla \geq 18$ by hypothesis. Because of Lemma 2.3, the assertion is true whenever ∇ has a subgroup G_2 , in particular, if $\dim \nabla > 24$. In the case $\dim \nabla = 24$, it follows from [18, (87.7)] that Δ does not have two fixed points. Therefore, attention can be restricted to $\dim \nabla \leq 23$. If ∇ has no subgroup G_2 , we exploit the fact that in a translation plane a maximal compact subgroup Φ of ∇ has codi-

ACADEMIA
PRESS





page 12 / 23

go back

full screen

close

quit

mension at most 2 and is normal in ∇ , see [18, (81.8)]. Consequently, $\dim \Phi \geq 16$. Consider the kernel $N = \nabla \cap C_s \Theta = \nabla_{[u]}$ of the action of ∇ on Θ and the irreducible subgroup $\tilde{\nabla} = \nabla / \nabla_{[u]}$ of $\text{Aut } \Theta$. It is a special feature of 16-dimensional translation planes that $\Phi_{[u]}$ is finite, see [18, (81.20)]. Hence $\tilde{\Phi} = \Phi / \Phi_{[u]}$ satisfies $\dim \tilde{\Phi} = \dim \Phi$. The large subgroups in the maximal compact subgroup $\text{SO}_8 \mathbb{R}$ of $\text{Aut } \Theta$ are listed in [18, (95.12)]. Since $G_2 \not\curvearrowright \nabla$, we conclude that $\dim \Phi = 16$ and that $\Phi' \cong \text{SU}_4 \mathbb{C}$ (recall from step 21) that $\text{SO}_5 \mathbb{R} \not\curvearrowright \Phi$). Moreover, Φ' acts faithfully and irreducibly on Θ , see [18, (95.12c)]. Hence $\Phi \cong \text{U}_4 \mathbb{C}$, $\dim \nabla = 18$, and $\dim \Delta = 34$. This completes the proof of Theorem 1.1. \square

3. The planes and their automorphism groups

Now let $\dim \Delta \geq 35$. If T is transitive, then $\dim \Sigma_{[a]} > 0$ and the existence of a subgroup $\text{Spin}_7 \mathbb{R}$ in Δ implies $\dim \Sigma \geq 38$. All such planes are described in [18, (82.5)]. We may assume, therefore, that $T_{[u]} \cong \mathbb{R}^7$ and $T_{[v]} \cong \mathbb{R}^8$, cf. also [18, (61.12)]. The plane \mathcal{P} can then be coordinatized by a ‘Cartesian field’ $(\mathbb{O}, +, \cdot)$, cf. [5, XI.4.2] or [18, (24.4)]. (Such linear ternary fields with associative addition have also been called *Cartesian groups* even though they are like rings rather than groups.) If the lines of the form $y = s \cdot x + t$ together with the ‘verticals’ form an affine plane and if multiplication is continuous, then, by [18, (43.6)], the Cartesian field indeed yields a compact projective plane.

Theorem 3.1. *Consider a topological Cartesian field $(\mathbb{R}, +, *, 1)$ with unit element, and assume that $(-r) * s = -(r * s)$ holds identically. Let $\rho : [0, \infty) \approx [0, \infty)$ be a homeomorphism with $\rho(1) = 1$. Write each octonion $x \in \mathbb{O}$ in the form $x = \xi + \mathfrak{x}$, where $\xi = \text{Re } x = \frac{1}{2}(x + \bar{x})$ and $\mathfrak{x} = \text{Pu } x = \frac{1}{2}(x - \bar{x})$, and define a new multiplication on \mathbb{O} by*

$$s \diamond x = |s|^{-1} s (|s| * \xi + \rho(|s|) \cdot \mathfrak{x}) \text{ for } s \neq 0 \text{ and } 0 \diamond x = 0.$$

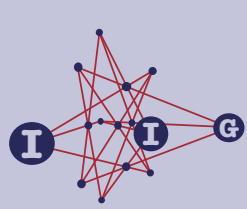
Then $\mathbb{O}_{\diamond} = (\mathbb{O}, +, \diamond, 1)$ is a topological Cartesian field with unit element 1. A plane \mathcal{P} can be coordinatized by such a Cartesian field if and only if \mathcal{P} satisfies the hypotheses of Theorem 1.1 with $\dim \Delta \geq 35$.

Remark 3.2. 1) An analogous construction can be applied to \mathbb{C} and to \mathbb{H} instead of \mathbb{O} .

2) Obviously, the multiplications \diamond and $*$ coincide on \mathbb{R} . It follows that \mathbb{O}_{\diamond} is a quasi-field if and only if $*$ is the ordinary multiplication of the reals. These quasifields and the corresponding translation planes are discussed in [6] and in [18, (82.4 and 5)].

ACADEMIA
PRESS





page 13 / 23

go back

full screen

close

quit

Proof of Theorem 3.1. Part A. Suppose first that \mathcal{P} has the properties of Theorem 1.1 without being a translation plane. Then $\dim \mathbb{T} = 15$ and Δ has a subgroup $\Upsilon \cong \text{Spin}_7\mathbb{R}$.

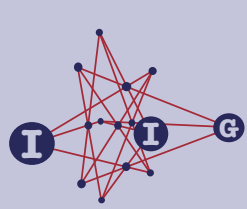
- 1) We may assume that $\Delta = \mathbb{T}\Upsilon$ and that the translation group $\mathbb{T}_{[v]}$ with center v is transitive. As remarked above, the affine plane \mathcal{P}^W can then be coordinatized with respect to any quadrangle $0 = a, u, v, e$ in the usual way (as in [18, § 22]) by a Cartesian field $\mathbb{O}_\diamond = (\mathbb{O}, +, \diamond)$, where $+$ denotes the ordinary addition of the octonions. (Call to mind that each translation can be written in the form $(x, y) \mapsto (x+a, y+b)$; hence $(\mathbb{O}, +) \cong \mathbb{T}_{[v]} \cong \mathbb{R}^8$.)
- 2) If u is the other fixed point of Δ , then $\Xi := \mathbb{T}_{[u]} \cong \mathbb{R}^7$ is Υ -invariant. Thus, there is a 7-dimensional vector subgroup V of $(\mathbb{O}, +)$ such that

$$\Xi = \{(x, y) \mapsto (x+c, y) \mid c \in V\}.$$

- 3) The group Υ fixes a triangle and may be identified with $\nabla = \Delta_a$. Indeed, $\nabla \cong \Delta_a/\mathbb{T}_a$ is isomorphic to a subgroup of $\Delta/\mathbb{T} \cong \Upsilon$. Since $\dim \nabla \geq 20$ and Υ has no proper subgroups of small codimension, $\nabla \cong \Upsilon$. By the Mal'cev-Iwasawa Theorem [18, (93.10)], Υ and ∇ are conjugate in Δ .
- 4) Because Υ induces on Ξ the group $\text{SO}_7\mathbb{R}$, the central involution $\alpha \in \Upsilon$ fixes the orbit $a\Xi$ pointwise and α is a reflection with axis au , cf. [18, (55.29)]. In coordinates, α has the form $(x, y) \mapsto (x, -y)$ since α inverts each translation in $\mathbb{T}_{[v]}$. This implies that $(-s) \diamond x = -(s \diamond x)$ holds identically in \mathbb{O}_\diamond .
- 5) According to [18, (96.36)], the action of Υ on the (invariant) line au is equivalent to a linear action, and the fixed point set is homeomorphic to \mathbb{S}_1 . Moreover, Υ acts trivially on the 1-dimensional quotient space au/Ξ . Therefore, each Ξ -orbit in $au \setminus \{u\}$ is Υ -invariant and contains a unique fixed point of Υ .
- 6) Since α has center v , the group Υ acts faithfully on av . The faithful representation of $\text{Spin}_7\mathbb{R}$ on \mathbb{R}^8 being unique up to a linear transformation of \mathbb{R}^8 , the line $av \setminus \{v\}$ can be identified with $\{0\} \times \mathbb{O}$ in such a way that Υ preserves the ordinary norm of \mathbb{O} .
- 7) Let e be chosen on a fixed line of Υ in the pencil \mathcal{L}_v such that a, u, v, e is a nondegenerate quadrangle. Then the stabilizer $\Lambda = \Upsilon_e$ is isomorphic to G_2 , and Λ fixes a one-parameter subgroup $(\mathbb{R}, +)$ of the vector group \mathbb{O} , corresponding to a transitive group of 'vertical' translations of the 2-dimensional plane \mathcal{E} consisting of the fixed elements of Λ . Consequently, \mathcal{E} is coordinatized by a Cartesian field $\mathbb{R}_* = (\mathbb{R}, +, *)$. In fact, \mathbb{R}_* is a Cartesian subfield of \mathbb{O}_\diamond , and $*$ is the restriction of the multiplication \diamond to \mathbb{R} . In particular, $(-s) * x = -(s * x)$ holds for all $s, x \in \mathbb{R}$. Since Λ fixes the coordinate quadrangle, Λ is a group of automorphisms of \mathbb{O}_\diamond .

ACADEMIA
PRESS





page 14 / 23

go back

full screen

close

quit

8) In the coordinates introduced in 1), the line ae is given by the equation $y = x$. Because the group Λ fixes this line, Λ acts in the same way on both the coordinate axes. From $\Xi^\Lambda \subseteq \Xi^\Upsilon = \Xi$ it follows that V is Λ -invariant. In fact, V is the unique Λ -invariant complement of \mathbb{R} in \mathbb{O} . Hence V coincides with the vector space $\text{Pu } \mathbb{O}$ of the pure elements in \mathbb{O} . The fixed point set of Λ in its action on \mathbb{O} is \mathbb{R} . Consequently, 5) implies that the fixed point set of Υ on $\mathbb{O} \times \{0\}$ is $\mathbb{R} \times \{0\}$.

9) For $s \neq 0$, consider the line L_s of slope s with the equation $y = s \diamond x$ and note that $s \diamond 1 = s$ and that $x \mapsto s \diamond x$ is a homeomorphism of \mathbb{O} . If $s \in \mathbb{R}$, then $(1, s)$ is a fixed point of Λ and the line L_s is Λ -invariant. Therefore, also the stabilizer $H = T_{L_s}$ is Λ -invariant. It is isomorphic to \mathbb{R}^7 by [18, (61.11c)] and has the form

$$\{(x, y) \mapsto (x + c, y + \zeta(c)) \mid c \in \text{Pu } \mathbb{O}\},$$

where ζ is an \mathbb{R} -linear endomorphism of $\text{Pu } \mathbb{O}$ centralizing Λ . Since the centralizer of Λ is isomorphic to \mathbb{R} by Schur's Lemma, there is a number $\rho(s) \in \mathbb{R}^\times$ such that

$$H = \{(x, y) \mapsto (x + c, y + \rho(s) \cdot c) \mid c \in \text{Pu } \mathbb{O}\}.$$

10) For $s \in \mathbb{R}$, each point $(\xi, s * \xi)$ with $\xi \in \mathbb{R}$ belongs to L_s by 7). Hence step 9) yields

$$L_s = \{(\xi + \mathfrak{x}, s * \xi + \rho(s) \cdot \mathfrak{x}) \mid \xi \in \mathbb{R} \wedge \mathfrak{x} \in \text{Pu } \mathbb{O}\}.$$

In the following, the other lines will be obtained by applying transformations $\varphi \in \Upsilon$ to the lines L_s with real s .

11) The group Υ acts on $\mathbb{O} \times \mathbb{O}$ in the same way as on the Moufang plane with the same point set. By 6) this is true for $\{0\} \times \mathbb{O}$ because \mathbb{R}^8 and \mathbb{O} have been identified accordingly. The subgroup Λ acts identically on $\{0\} \times \mathbb{O}$ and $\mathbb{O} \times \{0\}$, see 8). Since the centralizer of the action of Λ on $\text{Pu } \mathbb{O}$ is the center of $\text{GL}_7 \mathbb{R}$, the action of Υ on $\mathbb{O} \times \{0\}$ is uniquely determined by the restriction to Λ and the fact that Υ fixes $\mathbb{R} \times \{0\}$.

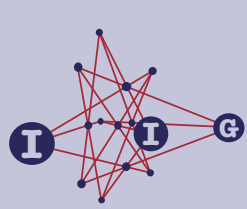
12) The group Υ is transitive on the spheres of constant norm in $\{0\} \times \mathbb{O}$, and for any $s \neq 0$ there is some $\varphi \in \Upsilon$ such that $\varphi(e) = (1, |s|^{-1}s)$. The map φ has the form $(x, y) \mapsto (Ax, By)$ with $A, B \in \text{SO}_8 \mathbb{R}$ such that for some $C \in \text{SO}_8 \mathbb{R}$ the equation $B(s \cdot x) = Cs \cdot Ax$ holds identically with respect to the ordinary multiplication \cdot of the octonions, see [18, (17.12–16)]. Hence $Bx = |s|^{-1}s \cdot Ax$ and φ maps $L_{|s|}$ onto the set

$$\{(\xi + A\mathfrak{x}, |s|^{-1}s (|s| * \xi + \rho(|s|) \cdot A\mathfrak{x})) \mid \xi \in \mathbb{R} \wedge \mathfrak{x} \in \text{Pu } \mathbb{O}\}.$$

Writing \mathfrak{x} instead of $A\mathfrak{x}$, we obtain for L_s the equation $y = s \diamond x$ as claimed.

ACADEMIA
PRESS





page 15 / 23

go back

full screen

close

quit

Part B. The construction in Theorem 3.1 always yields a topological Cartesian field.

Obviously, the multiplication $\mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O} : (a, x) \mapsto a \diamond x$ is continuous. By [18, (43.6)] it suffices, therefore, to show that for $a \neq b$ the maps

$$\lambda_{a,b} : x \mapsto -a \diamond x + b \diamond x \quad \text{and} \quad \mu_{a,b} : x \mapsto x \diamond a - x \diamond b$$

are bijections of \mathbb{O} . For each $x \in \mathbb{O}$ we write $x = |x| x_1 = \xi + \mathfrak{r}$.

- 1) For $c = |c| c_1 \in \mathbb{O}$ the equation $\mu_{a,b}(x) = c$ has a unique solution: in fact, by taking norms in \mathbb{O} , we get the condition

$$(|x| * \alpha - |x| * \beta)^2 + \rho(|x|)^2 \cdot |a - b|^2 = |c|^2.$$

The left hand side is monotone in $|x|$ since $(\mathbb{R}, +, *)$ is a topological Cartesian field and therefore $r \mapsto r * \alpha - r * \beta$ is either a continuous bijection of \mathbb{R} or constant. Consequently, $|x|$ is uniquely determined by c , in particular, $c = 0$ implies $x = 0$. In all other cases, x can be obtained from $|x|$ and c . (Note that $x_1(|x| * \alpha - |x| * \beta + \rho(|x|)(a - b))_1 = c_1$.)

- 2) Injectivity of $\lambda_{a,b}$ means $-a \diamond x + b \diamond x = -a \diamond y + b \diamond y \Rightarrow a = b \vee x = y$, and this is equivalent to injectivity of $\mu_{x,y}$.
- 3) In order to obtain surjectivity, we will show in the next steps that

$$\lim_{x \rightarrow \infty} \lambda_{a,b}(x) = \infty \quad (\dagger)$$

in the one-point compactification $\widehat{\mathbb{O}}$ of \mathbb{O} , i.e., that $\lambda_{a,b}$ has a continuous injective extension to $\widehat{\mathbb{O}}$. Such an extension is necessarily a homeomorphism, cf. also [18, (51.19)].

- 4) Condition (\dagger) is true in the Cartesian field $(\mathbb{R}, +, *)$. Hence $|a| < |b|$ implies

$$\lim_{\xi \rightarrow \infty} (|b| * \xi - |a| * \xi) = \infty.$$

- 5) It can easily be seen that (\dagger) holds in each of the following cases:

$$a = 0 \vee b = 0, \quad |a| = |b|, \quad a_1 = \pm b_1.$$

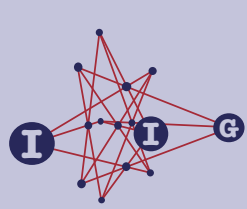
- 6) If (\dagger) is not true in general, then there is a sequence x_ν such that $\lim_{\nu \rightarrow \infty} x_\nu = \infty$ and for some $a, b \in \mathbb{O}$ with $|a| < |b|$ the sequence $\lambda_{a,b}(x_\nu)$ is bounded. Here

$$\lambda_{a,b}(x_\nu) = b_1(|b| * \xi_\nu + \rho(|b|) \cdot \mathfrak{r}_\nu) - a_1(|a| * \xi_\nu + \rho(|a|) \cdot \mathfrak{r}_\nu).$$

- 7) Suppose that the sequence \mathfrak{r}_ν is bounded. Then $\lim_{\nu \rightarrow \infty} \xi_\nu = \infty$, and 6) yields $\lim_{\nu \rightarrow \infty} (|a| * \xi_\nu)(|b| * \xi_\nu)^{-1} = a_1^{-1} b_1$. This is a positive number of norm 1. Hence $a_1 = b_1$ contrary to step 5). An analogous argument shows that the ξ_ν are unbounded. Therefore we may assume that the ξ_ν as well as the \mathfrak{r}_ν converge to ∞ in $\widehat{\mathbb{O}}$.

ACADEMIA
PRESS





page 16 / 23

go back

full screen

close

quit

- 8) The problem can be reduced to the 2-dimensional case as follows: we have $a^{-1}b \notin \mathbb{R}$ by step 5). The automorphism group of \mathbb{O} is transitive on the sphere $\{\mathfrak{x} \in \mathbb{O} \mid \mathfrak{x}^2 = -1\}$ in $\text{Pu } \mathbb{O}$, and we can arrange that $\bar{a}_1 b_1 = c \in \mathbb{C}$. Write each element $x \in \mathbb{O}$ as $x = x' + x''$ with $x' \in \mathbb{C}$ and $x'' \in \mathbb{C}^\perp$, the orthogonal complement of \mathbb{C} in \mathbb{O} . Then

$$\bar{a}_1 \lambda_{a,b}(x_\nu) = c(|b| * \xi_\nu) - |a| * \xi_\nu + (c\rho(|b|) - \rho(|a|)) \cdot \mathfrak{x}'_\nu + (c\rho(|b|) - \rho(|a|)) \cdot \mathfrak{x}''_\nu$$

is a bounded sequence. Hence also the sequence $(c\rho(|b|) - \rho(|a|)) \cdot \mathfrak{x}''_\nu \in \mathbb{C}^\perp$ is bounded and therefore $\lim_{\nu \rightarrow \infty} \mathfrak{x}'_\nu = \infty$ by step 7).

- 9) Let $c = p + iq$ with $p^2 + q^2 = 1$ and put $\mathfrak{x}'_\nu = i\eta_\nu$. Taking conjugates if necessary and selecting suitable subsequences, the possibilities can be reduced to $\lim_{\nu \rightarrow \infty} \eta_\nu = +\infty$ and the following cases: $\lim_{\nu \rightarrow \infty} \xi_\nu = +\infty$ or $\lim_{\nu \rightarrow \infty} \xi_\nu = -\infty$. The sequence

$$p(|b| * \xi_\nu) - |a| * \xi_\nu - q\rho(|b|)\eta_\nu + i(q(|b| * \xi_\nu) + p\rho(|b|)\eta_\nu - \rho(|a|)\eta_\nu)$$

is bounded, and so are the real and the imaginary part and the following linear combinations of these:

$$|b| * \xi_\nu - p(|a| * \xi_\nu) - q\rho(|a|)\eta_\nu \tag{1}$$

$$\text{and } q(|a| * \xi_\nu) + (\rho(|b|) - p\rho(|a|))\eta_\nu. \tag{2}$$

Since $\rho(|b|) - p\rho(|a|) > 0$, boundedness of (2) implies $\lim_{\nu \rightarrow \infty} q\xi_\nu = -\infty$, but then the sequence (1) would not be bounded. This proves the claim of Part B.

Part C. Consider a projective plane \mathcal{P} coordinatized by a topological Cartesian field $\mathbb{O}_\diamond = (\mathbb{O}, +, \diamond)$ as described in Theorem 3.1. It remains to show that $\text{Aut } \mathcal{P}$ contains a group Δ fixing exactly two points such that $\dim \Delta \geq 35$.

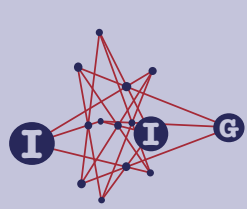
- 1) Obviously, $\{(x, y) \mapsto (x + \mathfrak{c}, y + d) \mid \mathfrak{c} \in \text{Pu } \mathbb{O}, d \in \mathbb{O}\} \leq \mathbb{T}$ and $\dim \mathbb{T} \geq 15$.
- 2) The maps $(x, y) \mapsto (Ax, By)$ of $\mathbb{O} \times \mathbb{O}$ such that $A, B \in \text{Spin}_7\mathbb{R}$ and identically $B(s \cdot x) = Bs \cdot Ax$ form a group Υ of automorphisms of the Moufang plane, they satisfy $A1 = 1$ and hence fix the set $\mathbb{R} \times \{0\}$, cf. **A**), step 9) or [18, (17.14)]. The involution $(x, y) \mapsto (x, -y)$ is a reflection in $\Upsilon_{[v]}$. Consequently, $\Upsilon \cong \text{Spin}_7\mathbb{R}$ acts faithfully on $\{0\} \times \mathbb{O}$ and induces on $\text{Pu } \mathbb{O} \times \{0\}$ the group $\text{SO}_7\mathbb{R}$. It follows that

$$B(s \diamond x) = Bs_1(|s| * \xi + \rho(|s|) \cdot A\mathfrak{x}) = Bs \diamond Ax.$$

Therefore $\Upsilon \leq \text{Aut } \mathcal{P}$, the group $\Delta = \Upsilon\mathbb{T}$ fixes exactly the points u, v , and $\dim \Delta = 36$. \square

ACADEMIA
PRESS





page 17 / 23

go back

full screen

close

quit

Theorem 3.3 (Automorphism groups). *Assume that the plane \mathcal{P} satisfies the hypotheses of Theorem 1.1 with $\dim \Delta \geq 35$ and let $\Sigma = \text{Aut } \mathcal{P}$ be the full automorphism group, Σ^1 its connected component. If \mathcal{P} is not the classical Moufang plane, then*

- (a) $\dim \Sigma < 40$ and each of the two fixed points of Δ is also a fixed point of Σ . Any subgroup $\Upsilon \cong \text{Spin}_7\mathbb{R}$ of Σ fixes some point $a \notin uv$.
- (b) If $\dim \Sigma = 39$, then \mathcal{P} is a translation plane.
- (c) The plane \mathcal{P} is a translation plane if, and only if, it can be coordinatized by a quasi-field \mathbb{O}_\diamond as in Theorem 3.1 where $*$ is the ordinary multiplication of the reals. In this case $\dim \Sigma = 39$ if, and only if, ρ is a multiplicative homomorphism; otherwise $\dim \Sigma = 38$.

If \mathcal{P} is not a translation plane, then the following holds:

- (d) $\dim \Sigma \leq 38$ and $\Sigma = \mathbb{T}^1 \Upsilon Z$, where Z denotes the centralizer of Υ in Σ .
- (e) $\dim \Sigma = 38$ if, and only if, \mathcal{P} can be coordinatized by a Cartesian field \mathbb{O}_\diamond as in Theorem 3.1 where

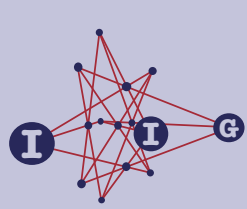
$$r * s = \begin{cases} rs & (s \geq 0) \\ |r|^\gamma rs & (s < 0) \end{cases} \quad \text{for some } \gamma > 0,$$

and $\rho : [0, \infty) \rightarrow [0, \infty)$ is a multiplicative homomorphism.

Proof. (a) If $\dim \Sigma \geq 40$, then \mathcal{P} can be coordinatized by a mutation of the octonions and Σ has no subgroup $\text{Spin}_7\mathbb{R}$, see [18, (82.29) and (87.7)]. We use the same notation as in the proof of Theorem 3.1. If $W^\sigma \neq W$ for some $\sigma \in \Sigma$, then $\Sigma : \Delta \geq \dim W^{\sigma^T} \geq 7$ and $\dim \Sigma \geq 43$. Hence $W^\Sigma = W$. The group $\Upsilon < \Delta$ acts effectively on W and each point $z \in W \setminus \{u, v\}$ has an orbit $z^\Upsilon \approx \mathbb{S}_7$. Therefore $v^\Sigma \in \{u, v\}$, or again $\dim \Sigma \geq 43$. If some $\sigma \in \Sigma$ interchanges u and v , then \mathcal{P} is a translation plane. Consider a Levi complement Ψ in a maximal compact subgroup of Σ^1 . All such groups are conjugate in Σ^1 , see [18, (93.10) and (94.28)]. Therefore, Ψ contains conjugates of Υ and of Υ^σ . The first acts effectively on the pencil $\mathcal{L}_u \cong \mathbb{R}^8$, the second induces a group $\text{SO}_7\mathbb{R}$ on \mathcal{L}_u . The central involutions in these groups are reflections with centers v and u respectively, their axes are Ψ -invariant, or else Ψ would contain translations by the dual of [18, (23.20)]. Consequently, Ψ fixes some point $a \notin W$, and the kernel $\Psi_{[u]}$ of the action of Ψ on \mathcal{L}_u is finite by [18, (81.20)]. It follows that Ψ is almost simple (cf. step 18) above) and has a proper subgroup $\text{Spin}_7\mathbb{R}$. The list [18, (95.10)] shows that $\dim \Psi = 28$ and then $\dim \Sigma \geq 44$, a contradiction. Therefore Σ fixes u and v . If $\text{Spin}_7\mathbb{R} \cong \Upsilon < \Sigma$, then the central involution

ACADEMIA
PRESS





page 18 / 23

go back

full screen

close

quit

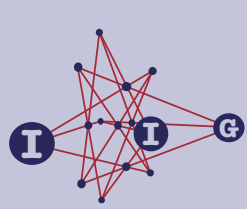
in Υ is a reflection and Υ fixes its axis X . Any action of the group Υ on a space X homeomorphic to \mathbb{R}^8 is equivalent to a linear action ([18, (96.36)]). Hence Υ has a fixed point $a \in X$.

- (b) We have $\Upsilon \leq \nabla := \Sigma_a^1$ and $\dim \nabla \leq 24$. Put $X = \nabla \cap \text{Cs } \Upsilon$. The representation of Υ on the Lie algebra of ∇ shows that $\nabla = \Upsilon X$. The group X acts effectively on the two-dimensional plane \mathcal{E} of the fixed elements of a subgroup $\Lambda \cong G_2$ of Υ . By [18, (32.10)] and the dimension formula, $\dim X \leq 2$, $\dim \nabla = 23$, and $\dim a^\Sigma = 16$. Since the centralizer of $\text{Spin}_7 \mathbb{R}$ in $\text{GL}_8 \mathbb{R}$ is isomorphic to \mathbb{R}^\times (cf. [18, (95.10)]), the action of ∇ on av has a kernel $\nabla_{[u]}$ of positive dimension. By the dual of [18, (61.20b)] it follows that $\dim \Upsilon_{[u]} = 8$.
- (c) See [18, (82.5)].
- (d) For each $\sigma \in \Sigma$ there is some $\tau \in \Upsilon^1$ such that $a^{\sigma\tau}$ is Υ -invariant, cf. step 5) of the proof of Theorem 3.1. Put $\sigma\tau = \omega^{-1}$. It follows that $\Upsilon^\omega \leq \nabla$. Since $\nabla = \Upsilon X$ and all Levi complements in a connected group are conjugate (cf. [18, (94.28c)]), we have $\Upsilon^\omega = \Upsilon$. Each automorphism of Υ is an inner automorphism (see [20, 6.]). Consequently, $\omega \in \Upsilon Z$.
- (e) Consider $\Lambda < \Upsilon$ and the subplane \mathcal{E} consisting of the fixed elements of Λ as in step 7) of the proof of Theorem 3.1. Suppose that $\dim \Sigma = 38$. Then $\dim Z = 2$ by part (d), and $\dim \text{Cs } \Lambda = 3$ as Λ also centralizes the vertical translations of \mathcal{E} . Moreover, $\text{Cs}_\Delta \Lambda$ contains the central reflection $\alpha \in \Upsilon$ (with axis av). It follows from (\diamond) that $\text{Cs } \Lambda$ acts effectively on \mathcal{E} . By assumption, \mathcal{P} is not a translation plane; hence $*$ is not the ordinary multiplication and \mathcal{E} is not classical. All planes \mathcal{E} admitting a 3-dimensional group are known explicitly; this classification is summarized in [18, (38.1)], details are given in [18, §§ 34–37]. As the group fixes the points u and v , the results just mentioned show that \mathcal{E} is a plane over a Cartesian field of the kind described in [18, (37.3)], which includes the Moulton planes. The reflection α induces on \mathcal{E} the map $(x, y) \mapsto (x, -y)$. This is a collineation of \mathcal{E} if and only if $(-s) * x = -(s * x)$ holds identically in \mathbb{R} . An easy calculation shows that the multiplication $*$ of [18, (37.3)] has indeed the form given in (e), cf. also [18, (37.4 and 6)]. In particular, \mathcal{E} is not a Moulton plane. Note that the product $*$ is associative whenever the right or the middle factor is positive.

The group Z^1 induces on \mathcal{E} the maps $(x, y) \mapsto ((r * x) \cdot s, y \cdot s)$ with $r, s > 0$. It can easily be seen that $(x, y) \mapsto (x \cdot s, y \cdot s)$, $s < 0$, $x, y \in \mathbb{O}$ yields always an automorphism of \mathcal{P} . An element $\zeta \in Z$ which induces on \mathcal{E} a map $(x, y) \mapsto (r * x, y)$ has necessarily the form $(x, y) \mapsto (\varphi_r(x), y)$ because Υ acts irreducibly on $\Upsilon_{[v]} \cong \mathbb{R}^8$. This means that ζ is a homology with axis av . Hence $\zeta(x, y) = (r \diamond x, y)$. This map is a collineation if and only if

ACADEMIA
PRESS





page 19 / 23

go back

full screen

close

quit

$a \diamond (r \diamond x) = (a \diamond r) \diamond x$ for all $a, x \in \mathbb{O}$. Equivalently (since $|a| * r = |ar|$),

$$|a| * (r * \xi) + \rho(|a|)\rho(r) \mathfrak{x} = (|a| * r) * \xi + \rho(|ar|) \mathfrak{x}.$$

Thus ρ is multiplicative. Conversely, the conditions in (e) imply $\dim Z = 2$ and hence $\dim \Sigma = 38$. If ρ is not multiplicative, then $\dim \Sigma = 37$. \square

The case $\dim \Sigma = 37$. With the same notation as before, we have $\dim \Sigma = 37$ if and only if $Cs\Lambda$ acts on \mathcal{E} as a 2-dimensional group with 2 fixed points. All planes over a proper Cartesian field $(\mathbb{R}, +, *)$ admitting such a group have been described. They depend on the choice of some suitable real functions rather than a few real parameters. By [18, (32.8)], a quasi-field $(\mathbb{R}, +, *)$ is in fact a field; therefore, \mathcal{E} is not a translation plane. Only the Cartesian fields of those planes \mathcal{E} can be used which admit a reflection with an axis au . The connected component Γ of $Cs\Lambda$ is isomorphic to \mathbb{R}^2 or to the linear group

$$L_2 := \{(t \mapsto at + b) : \mathbb{R} \rightarrow \mathbb{R} \mid a > 0\}.$$

In the first case, Γ_{au} fixes each line of \mathcal{E} through the point u , because Γ contains all translations of \mathcal{E} with center v . As \mathcal{E} is not a translation plane, Γ_{au} induces a one-parameter group of homologies of \mathcal{E} with center u and a common axis. The point a may be chosen on this axis; then Γ fixes exactly the elements u, v, av, uv of \mathcal{E} , and av is the axis of the elements of Γ_{au} . The planes \mathcal{E} of this type have been determined by Groh [4], cf. [10, 2.7.11.3].

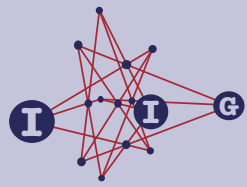
Homologies of \mathcal{E} with axis av have the form $\gamma_r : (x, y) \mapsto (r * x, y)$. The group Γ_{au} coincides with the connected component Z^1 of $Z = Cs\Upsilon$ because Z fixes the axis au of the unique central involution $\alpha \in \Upsilon$, and we have $Z^1 \leq \Gamma$ and $\dim Z = \dim \Gamma_{au}$. An element $\zeta_r \in Cs\Upsilon$ which induces on \mathcal{E} the homology γ_r fixes necessarily each point on the line av because the centralizer of the representation of Υ on \mathbb{R}^8 consists of real dilatations. Consequently ζ_r can be written as $(x, y) \mapsto (r \diamond x, y)$, and the product \diamond is associative whenever the middle factor is a positive real number. The latter condition reduces to the identity $\rho(r * s) = \rho(r)\rho(s)$ for $r, s > 0$. An admissible multiplication $*$ and a homeomorphism ρ yield a plane \mathcal{P} with $\dim \Sigma \geq 37$ if and only if ρ satisfies this identity.

If $\Gamma \cong L_2$, there are the following possibilities:

- (a) Γ acts transitively on the set of points not on uv ,
- (b) Γ fixes exactly two points and two lines,
- (c) Γ fixes exactly two lines and more than two points, or dually
- (c̃) Γ fixes exactly the points u and v and more than two lines through v .

ACADEMIA
PRESS





page 20 / 23

go back

full screen

close

quit

- (a) Planes with a group Γ satisfying (a) have been studied by Groh [3], cf. [10, 2.7.5.2]. Those planes \mathcal{E} which are symmetric with respect to a horizontal line can be described in the half-plane $(0, \infty) \times \mathbb{R}$ as follows: Let L be the graph of a strictly convex continuous function $f: (0, \infty) \rightarrow \mathbb{R}$ such that

$$\lim_{x \rightarrow 0} f(x) = \infty, \quad \lim_{x \rightarrow \infty} f(x) = -\infty, \quad \lim_{x \rightarrow \infty} f'(x) = 0.$$

Then the images of L under the maps $(x, y) \mapsto (rx, ry+b)$, $r \in \mathbb{R}^\times$, $b \in \mathbb{R}$ together with the horizontals and verticals are the lines of an affine plane of type (a). This can easily be translated into a representation in \mathbb{R}^2 by means of a Cartesian field \mathbb{R}_* . In the latter representation Γ contains a one-parameter subgroup of maps $\gamma_t: (x, y) \mapsto (\varphi_t(x), e^t y)$ acting transitively on the X -axis. A line of slope s is mapped by γ_t onto a line of slope $\sigma_t(s)$. The fact that γ_t is a collineation of \mathcal{E} is equivalent to the identity

$$e^t(s * x) = \sigma_t(s) * \varphi_t(x) - \sigma_t(s) * \varphi_t(0). \quad (*)$$

It remains to find a necessary and sufficient condition for γ_t to be induced by a map ζ_t of \mathbb{O}^2 in Z . (Note that again Γ_{au} is the connected component of $Z = Cs \Upsilon$ since $Z^1 \leq \Gamma_{au}$ and both groups are homeomorphic to \mathbb{R} .) From $\zeta_t \in Cs \Upsilon$ it follows that ζ_t has the form $(x, y) \mapsto (\varphi_t(\xi) + e^{\kappa t} \mathfrak{x}, e^t y)$. Expressing the fact that the line $y = s \diamond x$ is mapped to a line

$$e^t y = c \diamond (\varphi_t(\xi) + e^{\kappa t} \mathfrak{x}) - d$$

yields the condition

$$e^t |s|^{-1} s (|s| * \xi + \rho(|s|) \mathfrak{x}) = |c|^{-1} c (|c| * \varphi_t(\xi) - |c| * \varphi_t(0) + e^{\kappa t} \rho(|c|) \mathfrak{x}).$$

If $0 < s \in \mathbb{R}$, then $|s| = s$ and $c = \sigma_t(|s|) = |c|$; comparison of the pure components of the condition above gives

$$e^t \rho(|s|) = e^{\kappa t} \rho(\sigma_t(|s|)). \quad (\dagger)$$

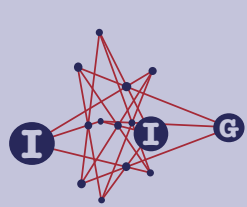
In general, we obtain in the same way that $e^t |s|^{-1} s \rho(|s|) = |c|^{-1} c e^{\kappa t} \rho(|c|)$, which by (\dagger) means $|s|^{-1} s e^{\kappa t} \rho(\sigma_t(|s|)) = |c|^{-1} c e^{\kappa t} \rho(|c|)$. Passing to absolute values, one obtains $|c| = \sigma_t(|s|)$ and then $|s|^{-1} s = |c|^{-1} c$, so that finally $c = \sigma_t(|s|) |s|^{-1} s$. Because of $(*)$ and (\dagger) , the condition above is then satisfied.

We remark that $\kappa \neq 1$, or else $\sigma_t(s) = s$ for all $s > 0$ and then also for all $s < 0$, and \mathcal{E} would be a translation plane. In particular, ρ is uniquely determined by \mathcal{E} .

- (b) The classification of these planes has been obtained by Schellhammer [19], cf. [10, 2.7.11.4]. For each multiplication $*$ defining such a plane there

ACADEMIA
PRESS





page 21 / 23

go back

full screen

close

quit

exists a one-parameter group of automorphisms $\gamma_t : (x, y) \mapsto (\varphi_t(x), e^t y)$ of \mathcal{E} fixing a and mapping a line of slope s to a line of slope $\sigma_t(s)$, where $e^t(s * x) = \sigma_t(s) * \varphi_t(x)$. An extension of γ_t to a map $\zeta_t \in \text{Cs } \Upsilon$ has again the form $(x, y) \mapsto (\varphi_t(\xi) + e^{\kappa t} \xi, e^t y)$. As before, this is a collineation of \mathcal{P} if and only if condition (\dagger) holds. Each pair of an admissible multiplication $*$ and a homeomorphism ρ which satisfies (\dagger) yields a plane \mathcal{P} with $\dim \Sigma \geq 37$.

(c) The description of the possible planes \mathcal{E} is due to Pohl [9], cf. [10, 2.7.11.5]. The same calculations as in case (b) lead once more to condition (\dagger) . By assumption there is some slope $r > 0$ such that $\sigma_t(r) = r$. It follows that $\kappa = 1$ and then $\sigma_t(|s|) = |s|$ for each s . As $\Upsilon \Gamma_a \leq \nabla$, the central involution $\alpha \in \Upsilon$ (with axis au) commutes with the maps γ_t . Consequently, γ_t also fixes the negative real slopes, and Γ_a induces homologies of \mathcal{E} . Thus, planes with $\dim \Sigma \geq 37$ can be obtained in case (c) if and only if Γ fixes the line uv pointwise; there is no condition on the homeomorphism ρ . The orbits of Γ_a in \mathcal{E} are rays beginning at the origin in the real affine plane. It follows that \mathcal{E} can be described by a Cartesian field multiplication of the form $s * x = sx$ for $x \geq 0$ and $s * x = \mu(s)x$ for $x < 0$, where $\mu : \mathbb{R} \approx \mathbb{R}$ with $\mu(-s) = -\mu(s)$ and $\mu(1) = 1$. Planes of this kind have been called generalized Moulton planes.

(c̃) Though the planes \mathcal{E} are dual to those of case (c), the conclusions are not because of the different rôles of the central reflection $\alpha \in \Upsilon$. As in the previous cases, the conditions $e^t(s * x) = \sigma_t(s) * \varphi_t(x)$ and (\dagger) must be satisfied. In case (c̃) we may assume that $\varphi_t(1) = 1$. Then we obtain $\sigma_t(s) = e^t s$ for all $s \in \mathbb{R}$, and (\dagger) reduces to the condition that ρ is a multiplicative homomorphism.

Examples are given by the multiplications

$$s * x = \begin{cases} sx & (x \leq 1) \\ s(|s|^m x + 1 - |s|^m) & (x \geq 1), \end{cases} \quad (m > 0).$$

In fact, $\varphi_t(x) = x$ for $x \leq 1$ and $\varphi_t(x) = e^{-mt} x + 1 - e^{-mt}$ for $x \geq 1$.

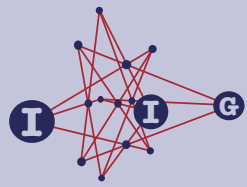
Thus in each of the cases there are large families of planes \mathcal{P} with a group of dimension 37 fixing exactly two points and the line joining them.

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page 22 / 23

go back

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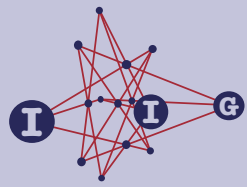
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page 23 / 23

go back

full screen

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