



# Corrections to “New results on covers and partial spreads of polar spaces”

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## 1 Introduction

In the proof of Lemma 1.2 of [1] one case was missing and so the assertion of the lemma is not correct as it is stated in [1]. More precisely, the lower bound on  $|l \cap B|$  in the lemma is not correct. We prove a new version of the lemma with the correct lower bound. The lemma was used several times in [1], but only the proof of Proposition 3.2 does not remain true with the new bound. We therefore give a new proof of Proposition 3.2 using the correct and weaker version of Lemma 1.1. As a matter of fact, the two main results in [1] (stated in the introduction) remain true.

## 2 New proofs

Lemma 2.1 of [1] was stated in a wrong way. The following version is correct.

**Lemma 2.1.** *Consider in  $\text{PG}(4, q)$  a quadric that is a cone with vertex a point  $P$  over a non-degenerate elliptic quadric  $Q^-(3, q)$ . Suppose that  $B$  is a set of at most  $2q$  points contained in the quadric. If every solid of  $\text{PG}(4, q)$  meets  $B$ , then one of the following occurs:*

- (a) *Some line of the quadric is contained in  $B$ .*
- (b)  *$|B| > \frac{9}{5}q + 1$ ,  $P \in B$ , and there exists a unique line  $l$  of the quadric that meets  $B$  in at least  $2 + \frac{1}{2}(3q - |B|)$  points. This line has at most  $|B| - 1 - q$  points in  $B$ .*

*Proof.* “Case 2” in the proof in the original paper is only correct when  $3q - b - 2b_1 \geq 0$  (otherwise the estimate in the beginning of Case 2 is wrong). Under this restriction the original paper shows in Case 2 that  $|l_1| \geq 1 + \frac{1}{3}|B|$  and  $|B| > 1 + \frac{9}{5}q$ . This implies that  $|l_1| \geq 2 + \frac{1}{2}(3q - |B|)$ . It thus remains to consider the situation in Case 2 when  $2b_1 > 3q - b$ . Then (1) of the original proof implies that  $5q - 3b + 2b_i \leq 3q - b - 2b_1 < 0$  for  $i \geq 2$ . Therefore (4) of the original proof implies that

$$12q^3 \leq 12bq^2 - 5b^2q + b^3 + b_1^2(5q - 3b + 2b_1).$$

Using  $b_1 < b - q$ , this gives  $0 < q^2(5b - 9q)$ , that is  $b > \frac{9}{5}q$  and therefore  $|B| > 1 + \frac{9}{5}q$ . Also  $|l_1 \cap B| = 1 + b_1 > 1 + \frac{1}{2}(3q - b)$ . Using  $|B| = b + 1$ , this gives  $|l_1 \cap B| \geq 1 + \frac{1}{2}(3q + 1 - b) = 2 + \frac{1}{2}(3q - |B|)$ . As in the original proof we have  $b_1 < b - q$  and hence  $|l_1 \cap B| = 1 + b_1 < |B| - q$ .  $\square$

The proof of Theorem 3.1 in [1] used the wrong version of Lemma 2.1 and the (correct) Proposition 3.2. Lemma 2.1 was used to show that the set of holes is contained in the union of mutually skew lines  $l_1, \dots, l_s$ , all containing at least  $\frac{1}{3}|B| + 1$  holes. With the correct version of Lemma 2.1, the proof of Theorem 3.1 in [1] shows only that the lines  $l_i$  have at least  $2 + \frac{1}{2}(3q - |B|)$  holes. Then Proposition 3.2 of the original paper is not strong enough to finish the proof of Theorem 3.1, but has to be replaced by the following version.

**Proposition 3.2.** *Consider the elliptic quadric  $Q^-(5, q)$  and its ambient space  $\text{PG}(5, q)$ . Suppose that  $H$  is a set of  $\delta(q + 1)$  points of  $Q^-(5, q)$  with the following properties.*

- (a) *Every hyperplane of  $\text{PG}(5, q)$  meets  $H$  in  $\delta$  modulo  $q$  points.*
- (b) *There exist  $s$  mutually skew lines  $l_1, \dots, l_s$  of  $Q^-(5, q)$  such that  $H$  is contained in the union of the lines  $l_i$  and such that  $\frac{1}{2}(2q - \delta) + 2 \leq |l_i \cap H| \leq \delta - 1$  for  $i = 1, \dots, s$ .*

*Then  $\delta = 0$  or  $\delta \geq q$ .*

*Proof.* Assume on the contrary that  $1 \leq \delta \leq q - 1$ . We shall derive a contradiction. The points of  $H$  will be called *holes*. As  $\delta < q$ , hypothesis (a) implies that every hyperplane has at least  $\delta$  holes.

**Part 1.** In a hyperplane with  $rq + \delta$  holes, every solid has at least  $r$  holes. This was proved in the original paper.

**Part 2.** As the lines  $l_i$  are skew, the number  $s$  of lines  $l_i$  is upper bounded by

$$s < \frac{|H|}{\frac{1}{2}(2q - \delta)} < q + \delta.$$

Thus  $s \leq q + \delta - 1$ .

**Part 3.** In this part we shall show that every solid that is spanned by two of the lines  $l_i$  contains at least  $\frac{1}{4}(3q+7)$  of the lines  $l_i$ .

For this consider the lines  $l_1, l_2$  and the solid  $S$  they span. Since  $S$  contains skew generators, then  $S \cap Q^-(5, q) = Q^+(3, q)$ . Every hyperplane containing  $S$  meets the quadric in a parabolic quadric  $Q(4, q)$  and has  $\delta \pmod{q}$  points in  $H$ . Let  $rq + \delta$  be the smallest number of holes in a hyperplane on  $S$ , and let  $T$  be a hyperplane on  $S$  with exactly  $rq + \delta$  holes. Considering the hyperplanes on  $S$  gives

$$(q+1)(rq + \delta) \leq |H| + q|S \cap H|,$$

so  $|S \cap H| \geq r(q+1)$ . Hence, in  $T \setminus S$  there are at most  $\delta - r < q$  holes. Thus, if  $\pi$  is a plane of  $S$ , then some of the  $q$  solids  $\neq S$  of  $T$  on  $\pi$  does not contain a hole in  $T \setminus S$ , that is all its holes lie in  $\pi$ ; then Part 1 shows that  $\pi$  has at least  $r$  holes. Hence, every plane of  $S$  has at least  $r$  holes.

Let  $P_1$  be a point of  $l_1$  that is not a hole. As at least two points of  $l_2$  are not holes, we find a non-hole  $P_2$  on  $l_2$  such that  $P_1P_2$  is a secant line to the quadric. Then the line  $g := P_1P_2$  has no hole. This line lies in the plane  $\pi_1 := \langle g, l_1 \rangle$ , in the plane  $\pi_2 := \langle g, l_2 \rangle$ , and in  $q-1$  further planes of  $S$ , which all have at least  $r$  holes. It follows that

$$\begin{aligned} rq + \delta &\geq |S \cap H| \geq (q-1)r + |\pi_1 \cap H| + |\pi_2 \cap H| \\ \Rightarrow r + \delta &\geq |\pi_1 \cap H| + |\pi_2 \cap H|. \end{aligned}$$

Now  $\pi_i$  contains  $l_i$  and a second generator  $g_i$  of the hyperbolic quadric  $Q^+(3, q)$  in  $S$ . As  $|l_i \cap B| \geq \frac{1}{2}(2q-\delta)+2 \geq \frac{1}{2}(q+5)$ , this already implies that  $r+\delta \geq q+5$ , so that  $r \geq 6$ . Using

$$|\pi_i \cap H| \geq |l_i \cap H| + |g_i \cap H| - 1 \geq \frac{1}{2}(q+3) + |g_i \cap H|,$$

we find moreover that

$$|g_1 \cap H| + |g_2 \cap H| \leq r + \delta - q - 3 \leq r - 4.$$

We may assume that  $|g_1 \cap H| \leq \frac{1}{2}r - 2$ . Consider the two reguli  $R$  and  $R'$  of generators of the quadric  $S \cap Q^-(5, q) = Q^+(3, q)$ . We choose notations so that  $g_1 \in R'$ . Then  $l_1, l_2 \in R$ . As the lines  $l_i$  are skew, then no line  $l_i$  lies in  $R'$ . For every line  $l \in R$ , the set  $g_1 \cup l$  is the intersection of a plane of  $S$  with the hyperbolic quadric  $Q^+(3, q)$ . As every plane of  $S$  has at least  $r$  holes, it follows that every generator  $l$  of  $R$  has at least  $r - |g_1 \cap H| \geq \frac{1}{2}r + 2$  holes. If the generator  $l$  is not one of the lines  $l_i$ , then for each hole on  $l$ , the unique line  $l_i$  on this hole meets  $S$  only in this hole. Hence, if  $z$  is the number of lines  $l_i$  in  $R$ , then each of the other  $q+1-z$  lines  $l$  of  $R$  gives rise to at least  $\frac{1}{2}r + 2$  lines  $l_i$

that meet  $S$  in a point of  $l$ . Hence,  $z$  of the lines  $l_i$  are contained in  $S$  and at least  $(q+1-z)(\frac{1}{2}r+2)$  lines  $l_i$  meet  $S$  in just one point. Then Part 2 gives

$$z + (q+1-z)\left(\frac{1}{2}r+2\right) \leq q + \delta - 1.$$

As  $\delta \leq q-1$  and  $z \leq q+1$  and  $r \geq 6$ , this implies that  $z \geq \frac{1}{4}(3q+7)$ .

**Part 4.** We have seen in Part 3 that every two different lines  $l_i$  span a solid containing at least  $z_0 := \frac{1}{4}(3q+7)$  lines  $l_i$ . Consider such a solid  $S$ . Then  $S$  meets the quadric  $Q^-(5, q)$  in a hyperbolic quadric  $Q^+(3, q)$ . As the lines  $l_i$  are mutually skew, then  $S$  contains at most  $q+1$  of the lines  $l_i$ . As each line  $l_i$  has less than  $\delta$  holes, it follows that not all holes lie on a line  $l_i$  that is contained in  $S$ . Hence, there must be a line  $l_i$ , we may assume that this is  $l_1$ , that is not contained in  $S$ . Then  $l_1$  spans different solids with the different lines  $l_i$  that are contained in  $S$ . This gives rise to at least  $z_0$  different solids on  $l_1$ , which all contain at least  $z_0 - 1$  lines  $l_i$  other than  $l_1$ . As different solids on  $l_1$  can not share a second line  $l_i$ , this implies that these solids together contain at least  $1 + z_0(z_0 - 1)$  different lines  $l_i$ . Hence,  $1 + z_0(z_0 - 1) \leq q + \delta - 1 \leq 2(q - 1)$ . This is a contradiction.  $\square$

## References

- [1] **A. Klein** and **K. Metsch**, New results on covers and partial spreads of polar spaces, *Innov. Incidence Geom.* **1** (2005), 19–34.

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