Corrections to “New results on covers and partial spreads of polar spaces”

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1. Introduction

In the proof of Lemma 1.2 of [1] one case was missing and so the assertion of the lemma is not correct as it is stated in [1]. More precisely, the lower bound on $|l \cap B|$ in the lemma is not correct. We prove a new version of the lemma with the correct lower bound. The lemma was used several times in [1], but only the proof of Proposition 3.2 does not remain true with the new bound. We therefore give a new proof of Proposition 3.2 using the correct and weaker version of Lemma 1.1. As a matter of fact, the two main results in [1] (stated in the introduction) remain true.

2. New proofs

Lemma 2.1 of [1] was stated in a wrong way. The following version is correct.

Lemma 2.1. *Consider in* $\mathsf{PG}(4,q)$ *a quadric that is a cone with vertex a point* P *over a non-degenerate elliptic quadric* $Q^-(3,q)$*. Suppose that* B *is a set of at most* $2q$ *points contained in the quadric. If every solid of* $\mathsf{PG}(4,q)$ *meets* B*, then one of the following occurs:*

(a) *Some line of the quadric is contained in* B*.*

(b) $|B| > \frac{9}{5}q+1$*,* $P \in B$*, and there exists a unique line* l *of the quadric that meets* B *in at least* $2+\frac{1}{2}(3q-|B|)$ *points. This line has at most* $|B|-1-q$ *points in* B*.*

Proof. "Case 2" in the proof in the original paper is only correct when $3q - b - 2b_1 \geq 0$ (otherwise the estimate in the beginning of Case 2 is wrong). Under this restriction the original paper shows in Case 2 that $|l_1| \geq 1 + \frac{1}{3}|B|$ and $|B| > 1 + \frac{9}{5}q$. This implies that $|l_1| \geq 2 + \frac{1}{2}(3q - |B|)$. It thus remains to consider the situation in Case 2 when $2b_1 > 3q - b$. Then (1) of the original proof implies that $5q - 3b + 2b_i \leq 3q - b - 2b_1 < 0$ for $i \geq 2$. Therefore (4) of the original proof implies that

$$12q^3 \leq 12bq^2 - 5b^2q + b^3 + b_1^2(5q - 3b + 2b_1).$$

Using $b_1 < b - q$, this gives $0 < q^2(5b - 9q)$, that is $b > \frac{9}{5}q$ and therefore $|B| > 1 + \frac{9}{5}q$. Also $|l_1 \cap B| = 1 + b_1 > 1 + \frac{1}{2}(3q - b)$. Using $|B| = b + 1$, this gives $|l_1 \cap B| \geq 1 + \frac{1}{2}(3q + 1 - b) = 2 + \frac{1}{2}(3q - |B|)$. As in the original proof we have $b_1 < b - q$ and hence $|l_1 \cap B| = 1 + b_1 < |B| - q$. □

The proof of Theorem 3.1 in [1] used the wrong version of Lemma 2.1 and the (correct) Proposition 3.2. Lemma 2.1 was used to show that the set of holes is contained in the union of mutually skew lines $l_1, \dots, l_s$, all containing at least $\frac{1}{3}|B| + 1$ holes. With the correct version of Lemma 2.1, the proof of Theorem 3.1 in [1] shows only that the lines l_i have at least $2 + \frac{1}{2}(3q - |B|)$ holes. Then Proposition 3.2 of the original paper is not strong enough to finish the proof of Theorem 3.1, but has to be replaced by the following version.

Proposition 3.2. *Consider the elliptic quadric $Q^-(5, q)$ and its ambient space* $\mathrm{PG}(5, q)$*. Suppose that H is a set of $\delta(q + 1)$ points of $Q^-(5, q)$ with the following properties.*

(a) *Every hyperplane of* $\mathrm{PG}(5, q)$ *meets H in δ modulo q points.*

(b) *There exist s mutually skew lines $l_1, \dots, l_s$ of $Q^-(5, q)$ such that H is contained in the union of the lines l_i and such that $\frac{1}{2}(2q - \delta) + 2 \leq |l_i \cap H| \leq \delta - 1$ for $i = 1, \dots, s$.*

Then $\delta = 0$ or $\delta \geq q$.

Proof. Assume on the contrary that $1 \leq \delta \leq q - 1$. We shall derive a contradiction. The points of H will be called *holes*. As $\delta < q$, hypothesis (a) implies that every hyperplane has at least δ holes.

Part 1. In a hyperplane with $rq + \delta$ holes, every solid has at least r holes. This was proved in the original paper.

Part 2. As the lines l_i are skew, the number s of lines l_i is upper bounded by

$$s < \frac{|H|}{\frac{1}{2}(2q - \delta)} < q + \delta.$$

Thus $s \leq q + \delta - 1$.

Part 3. In this part we shall show that every solid that is spanned by two of the lines l_i contains at least $\frac{1}{4}(3q+7)$ of the lines l_i.

For this consider the lines l_1, l_2 and the solid S they span. Since S contains skew generators, then $S \cap Q^-(5,q) = Q^+(3,q)$. Every hyperplane containing S meets the quadric in a parabolic quadric $Q(4,q)$ and has $\delta \pmod q$ points in H. Let $rq+\delta$ be the smallest number of holes in a hyperplane on S, and let T be a hyperplane on S with exactly $rq+\delta$ holes. Considering the hyperplanes on S gives

$$(q+1)(rq+\delta) \leq |H| + q|S \cap H|,$$

so $|S \cap H| \geq r(q+1)$. Hence, in $T \setminus S$ there are at most $\delta - r < q$ holes. Thus, if π is a plane of S, then some of the q solids $\neq S$ of T on π does not contain a hole in $T \setminus S$, that is all its holes lie in π; then Part 1 shows that π has at least r holes. Hence, every plane of S has at least r holes.

Let P_1 be a point of l_1 that is not a hole. As at least two points of l_2 are not holes, we find a non-hole P_2 on l_2 such that P_1P_2 is a secant line to the quadric. Then the line $g := P_1P_2$ has no hole. This line lies in the plane $\pi_1 := \langle g, l_1 \rangle$, in the plane $\pi_2 := \langle g, l_2 \rangle$, and in $q-1$ further planes of S, which all have at least r holes. It follows that

$$\begin{aligned} rq+\delta \geq |S \cap H| &\geq (q-1)r + |\pi_1 \cap H| + |\pi_2 \cap H| \\ \Rightarrow \quad r+\delta &\geq |\pi_1 \cap H| + |\pi_2 \cap H|. \end{aligned}$$

Now π_i contains l_i and a second generator g_i of the hyperbolic quadric $Q^+(3,q)$ in S. As $|l_i \cap B| \geq \frac{1}{2}(2q-\delta)+2 \geq \frac{1}{2}(q+5)$, this already implies that $r+\delta \geq q+5$, so that $r \geq 6$. Using

$$|\pi_i \cap H| \geq |l_i \cap H| + |g_i \cap H| - 1 \geq \frac{1}{2}(q+3) + |g_i \cap H|,$$

we find moreover that

$$|g_1 \cap H| + |g_2 \cap H| \leq r + \delta - q - 3 \leq r - 4.$$

We may assume that $|g_1 \cap H| \leq \frac{1}{2}r - 2$. Consider the two reguli R and R' of generators of the quadric $S \cap Q^-(5,q) = Q^+(3,q)$. We choose notations so that $g_1 \in R'$. Then $l_1, l_2 \in R$. As the lines l_i are skew, then no line l_i lies in R'. For every line $l \in R$, the set $g_1 \cup l$ is the intersection of a plane of S with the hyperbolic quadric $Q^+(3,q)$. As every plane of S has at least r holes, it follows that every generator l of R has at least $r - |g_1 \cap H| \geq \frac{1}{2}r + 2$ holes. If the generator l is not one of the lines l_i, then for each hole on l, the unique line l_i on this hole meets S only in this hole. Hence, if z is the number of lines l_i in R, then each of the other $q+1-z$ lines l of R gives rise to at least $\frac{1}{2}r+2$ lines l_i

that meet S in a point of l. Hence, z of the lines l_i are contained in S and at least $(q+1-z)(\frac{1}{2}r+2)$ lines l_i meet S in just one point. Then Part 2 gives

$$z+(q+1-z)\Big(\frac{1}{2}r+2\Big)\le q+\delta-1.$$

As $\delta \le q-1$ and $z \le q+1$ and $r \ge 6$, this implies that $z \ge \frac{1}{4}(3q+7)$.

Part 4. We have seen in Part 3 that every two different lines l_i span a solid containing at least $z_0 := \frac{1}{4}(3q+7)$ lines l_i. Consider such a solid S. Then S meets the quadric $Q^-(5,q)$ in a hyperbolic quadric $Q^+(3,q)$. As the lines l_i are mutually skew, then S contains at most $q+1$ of the lines l_i. As each line l_i has less than δ holes, it follows that not all holes lie on a line l_i that is contained in S. Hence, there must be a line l_i, we may assume that this is l_1, that is not contained in S. Then l_1 spans different solids with the different lines l_i that are contained in S. This gives rise to at least z_0 different solids on l_1, which all contain at least z_0-1 lines l_i other than l_1. As different solids on l_1 can not share a second line l_i, this implies that these solids together contain at least $1+z_0(z_0-1)$ different lines l_i. Hence, $1+z_0(z_0-1)\le q+\delta-1\le 2(q-1)$. This is a contradiction. $\square$

References

[1] **A. Klein** and **K. Metsch**, New results on covers and partial spreads of polar spaces, *Innov. Incidence Geom.* **1** (2005), 19–34.

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