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# Maximal Levi subgroups acting on the Euclidean building of $\mathrm{GL}_n(F)$

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**Abstract**

In this paper we give a complete invariant of the action of $\mathrm{GL}_n(F) \times \mathrm{GL}_m(F)$ on the Euclidean building $\mathcal{B}_e\mathrm{GL}_{n+m}(F)$, where $F$ is a discrete valuation field. We then use this invariant to give a natural metric on the resulting quotient space. In the special case of the torus acting on the tree $\mathcal{B}_e\mathrm{GL}_2(F)$, we obtain an algorithm for calculating the distance of any vertex in the tree to any fixed apartment.

Keywords: affine building, Euclidean building, Levi subgroup, group action

MSC 2000: 20E42, 20G25

## 1 Introduction

To understand distance in the 1-skeleton of a building $\mathcal{B}G$ associated to a reductive algebraic group $G$, one may look at a stabilizer $K$ of a point, and then study the action of $K$ on $\mathcal{B}G$. When working over a discrete valuation field vertices correspond to maximal compact subgroups. This analysis gives rise to information about $K\backslash G/K$, and therefore the Hecke algebra [4, 5].

In this paper we specialize to $G = \mathrm{GL}_n(F)$ and are interested in the double cosets $L\backslash G/K$, where $L \cong \mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F)$ is a maximal Levi subgroup of $G$. The study of the action of $L$ on the building $\mathcal{B}_e\mathrm{GL}_n(F)$ will lead to a description of distance from any vertex to a certain subbuilding stabilized by $L$. In the case when $n = 2$ and $L = T$ is a maximal split torus, our description gives a way of calculating the distance from a given point to a fixed apartment.

We also give a combinatorial description of the quotient space $L\backslash\mathcal{B}_e\mathrm{GL}_n(F)$ as follows. Let $A^n = \{(\alpha_i)_{i=1}^n \mid \alpha_i \in \mathbb{N}, \alpha_i \geq \alpha_{i+1}\}$. If $n_1 \leq n_2$ there is an graph isometry between $L\backslash\mathcal{B}_e\mathrm{GL}_n(F)$ and $A^{n_1}$ where $A^n$ is endowed with the following metric: $d(\alpha, \beta) = \max_{i=1}^{n} |\alpha_i - \beta_i|$ where $\alpha, \beta \in A^n$. This result shows that the 1-skeleton of the resulting quotient space only depends on $\min(n_1, n_2)$.

This paper is broken up into two main sections. The first gives a description of the building in terms of $\mathcal{O}$-lattices and describes an invariant of the action of $L$ on this building. The second section gives a geometric interpretation of this invariant, yielding a combinatorial description of the quotient space $L \backslash \mathcal{B}_e \mathrm{GL}_n(F)$.

# 2 Orbits of maximal Levi factors on $\mathcal{B}_e\mathrm{GL}(V)$

## 2.1 $\mathcal{O}$-lattices and $\mathcal{B}_e\mathrm{GL}(V)$

Throughout this paper let $F$ be a discrete valuation field with valuation $\upsilon$. We will denote the ring of integers in $F$ by $\mathcal{O}$, and fix once and for all a uniformizer $\varpi$ of $\mathcal{O}$. Let the unique maximal prime ideal be denoted as $\mathcal{P} = (\varpi)$, and the residue field $\mathcal{O}/\mathcal{P}$ will be denoted by $\mathfrak{k}$. Let $\mathcal{P}^k = (\varpi^k)$ for $k \in \mathbb{Z}$. Then $\log_{\mathcal{P}}(\mathcal{P}^k) = k$. Let $V$ be a finite dimensional vector space defined over $F$ of dimension $n$. We will describe the Euclidean building $\mathcal{B}_e\mathrm{GL}(V)$ associated to $\mathrm{GL}(V)$. For more details see [1]. Let $\Lambda \subset V$ be a finitely generated free $\mathcal{O}$-module of rank $n$. Denote by $[\Lambda]$ the homothety class of $\Lambda$, that is $[\Lambda] = \{a\Lambda \mid a \in F^\times\}$.

Homothety classes of lattices will form the vertices of $\mathcal{B}_e\mathrm{GL}(V)$. Two vertices $\lambda_1, \lambda_2 \in \mathcal{B}_e\mathrm{GL}(V)$ are incident if there are representatives $\Lambda_i \in \lambda_i$ so that $\varpi\Lambda_1 \subset \Lambda_2 \subset \Lambda_1$, i.e. $\Lambda_2/\varpi\Lambda_1$ is a $\mathfrak{k}$-subspace of $\Lambda_1/\varpi\Lambda_1$. The chambers in $\mathcal{B}_e\mathrm{GL}(V)$ are collections of maximally incident vertices. To put this more concretely, a chamber is a collection of $n$ vertices $\lambda_0 \cdots \lambda_{n-1}$ with representatives $\Lambda_0 \cdots \Lambda_{n-1}$ satisfying $\varpi\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_{n-1} \subsetneq \Lambda_0$. A wall of a chamber is any subset of $n-1$ vertices in the given chamber. We will denote by $\mathcal{B}_e\mathrm{GL}(V)^k$ the set of all facets of $\mathcal{B}_e\mathrm{GL}(V)$ of dimension $k$.

A frame $\mathcal{F}$ in $V$ is a collection of lines $l_1, \ldots, l_n \subset V$ which are linearly independent and span all of $V$. We now describe certain subcomplexes of $\mathcal{B}_e\mathrm{GL}(V)$. Define $\mathcal{A}_{\mathcal{F}}$ to be the subcomplex consisting of vertices $[\Lambda]$ of the following form:

$$\Lambda = \bigoplus_{i=1}^{n} \mathcal{O}e_i \tag{1}$$

where $e_i \in l_i \in \mathcal{F}$. Then $\mathcal{A}_{\mathcal{F}}$ is an apartment of $\mathcal{B}_e\mathrm{GL}(V)$, and every apartment is uniquely determined by a frame in this way.

The group $\mathrm{GL}(V)$ has a natural action of $\mathcal{B}_e\mathrm{GL}(V)$, namely the one induced from the action of $\mathrm{GL}(V)$ on $V$. This action preserves distance in the building.

A lemma which we will need later is the following.

**Lemma 2.1.** *Let $\Lambda, \Lambda'$ be $\mathcal{O}$-lattices of rank $n$ in $V$ with $\Lambda' \subset \Lambda$. Then the natural map from $\mathrm{GL}(\Lambda) \cap \mathrm{stab}(\Lambda')$ to $\mathrm{GL}(\Lambda/\Lambda')$ is surjective.*

*Proof.* This result appears to be well known, but the proof could not be found in the literature and so is given here. There is an $\mathcal{O}$-basis $\{e_1, \dots, e_n\}$ of $\Lambda$ so that $\{\varpi^{k_1} e_1, \dots, \varpi^{k_n} e_n\}$ with $k_i \in \mathbb{N}$ is an $\mathcal{O}$-basis of $\Lambda'$. This is equivalent to the statement that for any two vertices there is an apartment which contains them both. For $\overline{\sigma} \in \mathrm{GL}(\Lambda/\Lambda')$ we will construct $\sigma \in \mathrm{GL}(\Lambda) \cap \mathrm{stab}(\Lambda')$ which descends to $\overline{\sigma}$.

Let $\overline{e}_i$ be the image of $e_i$ in $\Lambda/\Lambda'$. Then

$$\overline{\sigma}(\overline{e}_i) = a_1^i \overline{e_1} + \cdots + a_n^i \overline{e}_n \tag{2}$$

where $a_j^i \in \mathcal{O}$. Observe that $a_j^i$ is unique modulo $\mathcal{P}^{k_j}$. Then define $\sigma$ on the $\mathcal{O}$-basis $\{e_1, \dots, e_n\}$ of $\Lambda$ as follows:

$$\sigma(e_i) = \begin{cases} \sum_{j=1}^n a_j^i e_j & \text{if } \overline{e}_i \neq 0, \\ e_i & \text{if } \overline{e}_i = 0. \end{cases} \tag{3}$$

What needs to be shown is that $\sigma$ is invertible and leaves $\Lambda'$ invariant.

First, we show $\sigma$ leaves $\Lambda'$ invariant.

$$0 = \overline{\sigma}(\varpi^{k_i} \overline{e}_i) = a_1^i \varpi^{k_i} \overline{e_1} + \cdots + a_n^i \varpi^{k_i} \overline{e}_n \tag{4}$$

This shows that $a_j^i \varpi^{k_i} \in \mathcal{P}^{k_j}$, and so $\sigma(\varpi^{k_i} e_i) \in \Lambda'$.

Next we show invertibility. Let $\sigma^*$ be the construction given above for $\overline{\sigma}^{-1}$, and let $\tau = \sigma \circ \sigma^*$. This will be a function which is a lift of the identity map in $\mathrm{GL}(\Lambda/\Lambda')$. Let $M = \mathrm{span}_{\mathcal{O}}\langle e_i \mid \overline{e}_i \neq 0 \rangle$ and let $M' = \mathrm{span}_{\mathcal{O}}\langle e_i \mid \overline{e}_i = 0 \rangle$. Then $\tau|_M = \mathrm{id} + E$ where $E \in \mathrm{Hom}_{\mathcal{O}}(M, \Lambda')$ and is $\mathrm{id}$ on $M'$. Any $\tau$ of this form is invertible and hence so is $\sigma$. □

## 2.2 $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ acting on $\mathcal{B}_e(\mathrm{GL}(W_1 \oplus W_2))$

Let $V$ be a vector space over $F$. Fix a maximal Levi subgroup $L$ of $\mathrm{GL}(V)$. Associated to $L$ are subspaces $W_1, W_2 \subset V$ satisfying $V = W_1 \oplus W_2$. Then $L \cong \mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$. In this section we will describe the orbits of the action of $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ on $\mathcal{B}_e \mathrm{GL}(V)^0$ in terms of an invariant $Q$. Additionally we will give a representative of each orbit.

Let $p_i$ be the projection of $V$ onto $W_i$ with respect to our given decomposition. We will use these maps to define invariants of the vertices and then show for our action that these invariants classify all orbits.

Let $\Lambda$ be an $\mathcal{O}$-lattice. We make the following definitions for $i = 1, 2$:

$$P_i(\Lambda) = \operatorname{Im}(p_i|_\Lambda), \tag{5}$$

$$K_i(\Lambda) = \operatorname{Ker}(p_{i'}|_\Lambda) = \Lambda \cap W_i, \tag{6}$$

where $i' = (i \bmod 2) + 1$.

These are lattices in $W_i$.

**Lemma 2.2.** $K_i(\Lambda) \subset P_i(\Lambda)$.

*Proof.* If $v \in K_i(\Lambda) = \Lambda \cap W_i$, then $v \in \Lambda$, so $p_i(v) \in P_i(\Lambda)$. But $p_i(v) = v$ since $v \in W_i$. □

By Lemma 2.2 we can define $Q_i(\Lambda) = P_i(\Lambda)/K_i(\Lambda)$. This is a finitely generated torsion $\mathcal{O}$-module.

**Proposition 2.3.** *$Q_1(\Lambda) \cong Q_2(\Lambda)$ as $\mathcal{O}$-modules. This isomorphism class will be denoted by $Q(\Lambda)$.*

*Proof.* We make slight modifications to the proof found in [2]. Let $p_i' \colon \Lambda \to Q_i(\Lambda)$ be the composition of $p_i$ with the natural projection map $\pi_i \colon P_i(\Lambda) \to Q_i(\Lambda)$. We define a map so that for all $v \in \Lambda$

$$\begin{aligned} \Theta \colon Q_1(\Lambda) &\to Q_2(\Lambda) \\ p_1'(v) &\mapsto p_2'(v). \end{aligned} \tag{7}$$

We will show that $\Theta$ is well defined, and is an isomorphism.

Let $w_1 + w_2, w_1' + w_2' \in \Lambda$ with $w_i, w_i' \in W_i$ and $\pi_1(w_1) = \pi_1(w_1')$. Then $\pi_1(w_1 - w_1') = 0$, and therefore $w_1 - w_1' \in K_1(\Lambda)$. Similarly $w_2 - w_2' \in K_2(\Lambda)$ and $\pi_2(w_2) = \pi_2(w_2')$ showing $\Theta$ is well defined. It is an isomorphism, because the map $\theta$, defined by reversing the roles of $1$ and $2$, is an inverse map. □

We now show that $Q$ is a complete invariant of the action of $L$ on $\mathcal{B}_e\mathrm{GL}(V)^0$.

**Theorem 2.4.** *Let $\Lambda, \Lambda'$ be $\mathcal{O}$-lattices. Then $\Lambda$ and $\Lambda'$ are in the same $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ orbit if and only if $Q(\Lambda) = Q(\Lambda')$.*

*Proof.* The class $Q(\Lambda)$ is a $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$-invariant since each factor of $\mathrm{GL}(W_i)$ commutes with the projection map $p_i$. We must show that if $Q(\Lambda) = Q(\Lambda')$ then there is a $g \in \mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ so that $\Lambda = g\Lambda'$.

We will need $g_1 \in \mathrm{GL}(W_1)$ and $g_2 \in \mathrm{GL}(W_2)$ so that $g_i P_i(\Lambda') = P_i(\Lambda)$ and $g_i K_i(\Lambda') = K_i(\Lambda)$ for $i = 1, 2$. There are certainly $g_i \in \mathrm{GL}(W_i)$ so that

$g_iP_i(\Lambda') = P_i(\Lambda)$. Then we may assume $K_i(\Lambda), K_i(\Lambda') \subset P_i(\Lambda)$. Since $Q(\Lambda) = Q(\Lambda')$ we know by the elementary divisor theorem there are bases

$$B_i = \{e_1, \ldots, e_{n_i}\} \text{ and } B_i' = \{e_1', \ldots, e_{n_i}'\}$$

of $P_i(\Lambda)$ so that $K_i(\Lambda)$ written in terms of $B_i$ has the same elementary divisors as $K_i(\Lambda')$ written in terms of $B_i'$. Let $h_i \in \mathrm{GL}(P_i(\Lambda))$ be the linear transformation which takes the basis $B_i$ to $B_i'$. Then $h_ig_i \in \mathrm{GL}(W_i)$ has the desired properties.

So we may replace $\Lambda'$ with $\Lambda'' = (h_1g_1, h_2g_2)\Lambda'$. Let $\Theta$ be the map from Proposition 2.3 associated to $\Lambda$, and $\Theta''$ associated to $\Lambda''$.

We claim $\Lambda = \Lambda''$ if and only if $\Theta = \Theta''$. To prove this we show that one can reconstruct $\Lambda$ from $\Theta$ (which implicitly encodes $Q_i(\Lambda)$ as the domain and range of the map), by taking

$$\Lambda_\Theta = \Big\{w_1 + w_2 \mid w_i \in P_i(\Lambda) \text{ and } \Theta(\pi_1(w_1)) = \pi_2(w_2)\Big\} \tag{8}$$

First, we show $\Lambda \subset \Lambda_\Theta$. Let $w = w_1 + w_2 \in \Lambda$, then by definition of $\Theta$ we have $\Theta(\pi_1(w_1)) = \pi_2(w_2)$. And so $v \in \Lambda_\Theta$. We now show $\Lambda_\Theta \subset \Lambda$. Let $w_1 + w_2 \in \Lambda_\Theta$. Then $w_1 \in P_1(\Lambda)$ so there is a $w_2' \in P_2(\Lambda)$ so that $w_1 + w_2' \in \Lambda \subset \Lambda_\Theta$. Then $0 + (w_2 - w_2') \in \Lambda_\Theta$. So $\pi_2(w_2 - w_2') = 0$ which implies $w_2 - w_2' \in K_2(\Lambda) \subset \Lambda$. Hence $w_1 + w_2 = (w_1 + w_2') + (w_2 - w_2') \in \Lambda$ as desired.

To complete the theorem, we will show there is an element $g \in \mathrm{stab}(P_2(\Lambda)) \cap \mathrm{stab}(K_2(\Lambda))$ which takes $\Theta''$ to $\Theta$. There is an $\overline{h} \in \mathrm{GL}(P_2(\Lambda)/K_2(\Lambda))$ so that $(1, \overline{h})\Theta'' = \Theta$. By Lemma 2.1 there is a pullback $h$ of $\overline{h}$ to $h \in \mathrm{stab}(P_2(\Lambda)) \cap \mathrm{stab}(K_2(\Lambda)) \in \mathrm{GL}(W_2)$ then $(1, h)\Lambda'' = \Lambda$. □

Now let $[\Lambda] \in \mathcal{B}_e\mathrm{GL}(V)^0$, and $c \in F^\times$. Since $Q(\Lambda) = Q(c\Lambda)$ we will abuse notation and write $Q([\Lambda]) = Q(\Lambda)$.

**Corollary 2.5.** *$Q([\Lambda])$ is a complete invariant of the action of $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ on the space of vertices in $\mathcal{B}_e(V)^0$.*

## 2.3 Orbit representatives

We now give a set representatives of each orbit. We first do this in the case when $V$ is 2-dimensional, and then use this case to determine representatives for higher dimensions.

### 2.3.1 $\dim(V) = 2$

Let $V$ be a two-dimensional vector space over $F$, with decomposition $V = W_1 \oplus W_2$. Assume that $W_i$ is spanned by the vector $e_i$. We then define the following class of lattices:

$$\Lambda^k = \mathrm{span}_{\mathcal{O}}\langle \varpi^k e_1, e_1 + e_2 \rangle. \tag{9}$$

**Proposition 2.6.** $Q([\Lambda^k]) \cong \mathcal{O}/\mathcal{P}^k$.

*Proof.* We have $P_1(\Lambda^k) = \langle e_1 \rangle$ and $K_1(\Lambda^k) = \langle \pi^k e_1 \rangle$. Therefore $Q(\Lambda) \cong \mathcal{O}/\mathcal{P}^k$. □

**Corollary 2.7.** $\{[\Lambda^k]\}_{k=0}^{\infty}$ *is a complete set of representatives for the action of* $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ *on* $\mathcal{B}_e\mathrm{GL}(V)^0$.

*Proof.* Let $[\Lambda] \in \mathcal{B}_e\mathrm{GL}(V)^0$. Then $Q([\Lambda]) \cong \mathcal{O}/\mathcal{P}^k$ for some $k \in \mathbb{N}$. By Theorem 2.4, $[\Lambda]$ is in the orbit of $\Lambda^k$. □

### 2.3.2 General *V*

We now describe representatives when $V$ is $n$-dimensional. We may assume that $\dim W_i = n_i$ and $n_1 \leq n_2$. Choose a basis $\{e_1, \ldots, e_{n_1}\}$ of $W_1$ and $\{f_1, \ldots, f_{n_2}\}$ of $W_2$, and let $Y_i = \mathrm{span}_F(e_i, f_i)$, for $1 \leq i \leq n_1$. Let $\alpha = (\alpha_i) \in \mathbb{N}^{n_1}$. Let $[\Lambda^{\alpha_i}] \in \mathcal{B}_e\mathrm{GL}(Y_i)$ defined as in equation (9) with respect to the basis $\{e_i, f_i\}$. This allows us to define the following class of lattices:

$$\Lambda^\alpha = \bigoplus_{i=1}^{n_1} \Lambda^{\alpha_i} \bigoplus_{i=n_1+1}^{n_2} \mathcal{O} f_i \tag{10}$$

**Proposition 2.8.** *Let* $A^n = \{\alpha = (\alpha_i) \in \mathbb{N}^n \mid \alpha_i \geq \alpha_{i+1}\}$. *Then* $[\Lambda^\alpha]_{\alpha \in A^{n_1}}$ *is a complete set of representatives of the orbits of* $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ *acting on* $\mathcal{B}_e\mathrm{GL}(V)^0$.

*Proof.* By the elementary divisor theorem $Q_1(\Lambda)$ decomposes into a direct sum of $\mathcal{O}$-modules as follows: $Q_1([\Lambda]) \cong \mathcal{O}^r \bigoplus_{i=1}^{n_1} \mathcal{O}/\mathcal{P}^{\alpha_i}$ where $\alpha_i \in \mathbb{N}$ and $r \in \mathbb{N}$. However, $r = 0$ since both $P_1(\Lambda)$ and $K_1(\Lambda)$ are rank $n_1$. We may assume $\alpha_i \geq \alpha_{i+1}$. Then by Theorem 2.4, $[\Lambda]$ is in the same orbit as $[\Lambda^\alpha]$. □

### 2.3.3 Double cosets

The description of orbits is equivalent to the space of double cosets $L\backslash\mathrm{GL}(V)/K$, where $K$ is the stabilizer of a vertex in $\mathcal{B}_e(\mathrm{GL}(V))$. We now give an explicit description of a set of double coset representatives.

The Levi subgroup $L$ is associated to a parabolic subgroup $P$ with a decomposition $P = LN$, where $N$ is the unipotent radical of $P$. The Iwasawa decomposition shows that $\mathrm{GL}(V) = PK$, and so we may choose the double coset representatives of $L\backslash\mathrm{GL}(V)/K$ to be in $N$.

We use the basis for $V$ of the previous section to identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(F)$. We will also let $K = Z(\mathrm{GL}_n(F))\mathrm{GL}_n(\mathcal{O})$. Then $N \cong M_{n_1\times n_2}(F)$, the $n_1 \times n_2$ matrices embedded in $\mathrm{GL}_n(F)$ as follows:

$$u\colon M_{n_1\times n_2}(F) \to N$$
$$B \mapsto \begin{pmatrix} I_{n_1} & B \\ 0 & I_{n_2} \end{pmatrix}.$$

Let $\alpha \in A^{n_1}$ and define $m^\alpha \in M_{n_1\times n_2}$ as follows:

$$[m^\alpha]_{ij} = \begin{cases} \varpi^{-\alpha_i} & \text{if } i = j \in \{1,\dots,n_1\}, \\ 0 & \text{else.} \end{cases} \tag{11}$$

Now let $n^\alpha = u(m^\alpha)$. Then we have the following proposition.

**Proposition 2.9.** *We may write $\mathrm{GL}_n(F)$ as a disjoint union*

$$\mathrm{GL}_n(F) = \coprod_{\alpha\in A^{n_1}} Ln^\alpha K.$$

*Proof.* Let $\alpha \in A^{n_1}$, and define $l^\alpha$ to be the linear transformation that sends $e_i$ to $e_i$ and $f_i$ to $\varpi^{-\alpha_i} f_i$ for $1 \le i \le n_1$, and $f_j$ to $f_j$ for $n_1 + 1 \le j \le n_2$. Note that $l^\alpha \in L$.

Let $\Lambda = \mathrm{span}_{\mathcal{O}}(e_1 \dots, e_{n_1}, f_1, \dots, f_{n_2})$, and notice that $K$ stabilizes $[\Lambda]$. Furthermore, we have $l^\alpha n^\alpha(\Lambda) = \Lambda^\alpha$. □

This double coset decomposition is in no way canonical, although it has some nice properties. All the $n^\alpha$ are supported on the span of root groups $U^{i,i+n_1}$ for $1 \le i \le n_1$, with the roots taken with respect to the diagonal torus. In fact, these root group form a set of maximally mutually orthogonal root groups in $N$. Any such set of root groups can be a support of coset representatives. This can easily be seen by having $W_{n_i}$ the Weyl groups of $\mathrm{GL}_{n_i}$ act on the $n^\alpha$. This leads to the following conjecture for more general groups.

**Conjecture 2.10.** *Let $G$ be a reductive group over $F$ and $P$ a parabolic subgroup with $P = LN$, and assume $N$ is abelian. Let $K$ be a maximal open, bounded subgroup of $G$. Then there is a discrete subset $N' \subset N$ and a maximally mutually orthogonal set of root groups $U^\alpha < N$ so that:*

1. *each $n \in N'$ is supported in the group generated by the $U^\alpha$;*
2. *$G = \coprod_{n\in N'} LnZ(G)K$.*

### 2.3.4 Stabilizers

We now wish to compute stabilizers for each orbit so that we may realize the orbits as homogeneous spaces. For spherical buildings knowing the stabilizers plays a role in representation theory, for instance [3]. For Euclidean buildings this may have applications to understanding cuspidal representations.

Fix a $\Lambda$ and let $S_i = \operatorname{stab}(P_i(\Lambda)) \cap \operatorname{stab}(K_i(\Lambda))$. Furthermore, let

$$T_i = \{I + A \mid A \in \operatorname{End}(W_i) \text{ and } A(P_i(\Lambda)) \subset K_i(\Lambda)\} \cap S_i. \tag{12}$$

Then $T_i \triangleleft S_i$ and $S_i/T_i \cong \operatorname{GL}(Q_i(\Lambda))$ by Lemma 2.1. Let

$$\overline{S_\Lambda} = \{(h_1, \Theta^*_\Lambda(h_1)) \mid h_1 \in \operatorname{GL}(Q_1(\Lambda))\} \subset (\operatorname{GL}(Q_1(\Lambda)) \times \operatorname{GL}(Q_2(\Lambda)) \tag{13}$$

where $\Theta^*_\Lambda$ is the isomorphism induced on $\operatorname{GL}(Q_1(\Lambda))$ from the isomorphism $\Theta_\Lambda \colon Q_1(\Lambda) \to Q_2(\Lambda)$ defined in equation (7). Finally, let $S_\Lambda$ be the pullback of $\overline{S_\Lambda}$ in $S_1 \times S_2$.

**Proposition 2.11.** $S_\Lambda = \operatorname{stab}_L(\Lambda)$.

*Proof.* Let $(A_1, A_2) \in S_\Lambda$ with $A_i \in \operatorname{GL}(W_i)$, and let $\Lambda' = (A_1, A_2) \cdot \Lambda$. Then because $A_i \in S_i$ we have $P_i(\Lambda) = P_i(\Lambda')$ and $K_i(\Lambda) = K_i(\Lambda')$. We now wish to show $\Theta_\Lambda = \Theta_{\Lambda'}$. By the proof of Theorem 2.4 this will show that $\Lambda = \Lambda'$. Let $B_i$ be the image of $A_i$ in $S_i/T_i \cong \operatorname{GL}(Q_i(\Lambda))$, and let $v \in Q_1(\Lambda)$ and $v' = B_1^{-1}v$. Then

$$\begin{aligned}
\Theta_{\Lambda'}(v) &= \Theta_{\Lambda'}(B_1 B_1^{-1} v) && (14)\\
&= B_2 \Theta_\Lambda(B_1^{-1} v) && (15)\\
&= \Theta^*_\Lambda(B_1)\Theta_\Lambda(B_1^{-1} v) && (16)\\
&= \Theta_\Lambda\left(B_1 \Theta_\Lambda^{-1}(\Theta_\Lambda(B_1^{-1} v))\right) && (17)\\
&= \Theta_\Lambda(v). && (18)
\end{aligned}$$

Line (15) follows from the action of $(A_1, A_2)$ on $\Lambda$, line (16) comes from the fact that $B_2 = \Theta^*_\Lambda(B_1)$, and line (17) is the definition of the induced map $\Theta^*_\Lambda$.

This proves $S_\Lambda \subset \operatorname{stab}_L(\Lambda)$. We now prove the other direction. Assume $(A_1, A_2) \in \operatorname{stab}_L(\Lambda)$, then $A_i \in S_i$. The calculation above shows that the projection of $A_2$ in $\operatorname{GL}(Q_2(\Lambda))$ has to equal the image of $A_1$ in $\operatorname{GL}(Q_2(\Lambda))$ under $\Theta^*_\Lambda$, proving the result. □

We end this section by giving an explicit description of $S_{\Lambda^\alpha}$, the stabilizers of our orbit representatives. Let $\alpha \in A^{n_1}$, then we define $\Lambda^\alpha$ as in section 2.3. By this definition $P_1(\Lambda^\alpha) = \mathrm{span}_{\mathcal{O}}\langle e_1, \ldots, e_{n_1}\rangle$ and $P_2(\Lambda^\alpha) = \mathrm{span}_{\mathcal{O}}\langle f_1, \ldots, f_{n_2}\rangle$. Also,

$$K_1(\Lambda^\alpha) = \mathrm{span}_{\mathcal{O}}\langle \varpi^{\alpha_1} e_1, \ldots, \varpi^{\alpha_{n_1}} e_{n_1}\rangle, \text{ and}$$
$$K_2(\Lambda^\alpha) = \mathrm{span}_{\mathcal{O}}\langle \varpi^{\alpha_1} f_1, \ldots, \varpi^{\alpha_{n_2}} f_{n_2}\rangle,$$

where $\alpha_j = 0$ if $j > n_1$. Then $S_i$ looks like

$$S_i = \begin{pmatrix} \mathcal{P}^{\beta_{11}} & \mathcal{P}^{\beta_{12}} & \mathcal{P}^{\beta_{13}} & \ldots & \mathcal{P}^{\beta_{1n_i}} \\ \mathcal{P}^{\beta_{21}} & \mathcal{P}^{\beta_{22}} & \mathcal{P}^{\beta_{23}} & \ldots & \mathcal{P}^{\beta_{2n_i}} \\ \vdots & & \ddots & & \vdots \\ \mathcal{P}^{\beta_{n_i1}} & \mathcal{P}^{\beta_{n_i2}} & \mathcal{P}^{\beta_{n_i3}} & \ldots & \mathcal{P}^{\beta_{n_in_i}} \end{pmatrix} \cap \mathrm{GL}_{n_i}(\mathcal{O}) \tag{19}$$

where $\beta_{ij} = \max(0, \alpha_i - \alpha_j)$.

Also, $T_i$ looks like

$$T_i = \begin{pmatrix} \mathcal{U}^{\alpha_1} & \mathcal{P}^{\alpha_1} & \mathcal{P}^{\alpha_1} & \ldots & \mathcal{P}^{\alpha_1} \\ \mathcal{P}^{\alpha_2} & \mathcal{U}^{\alpha_1} & \mathcal{P}^{\alpha_2} & \ldots & \mathcal{P}^{\alpha_2} \\ \vdots & & \ddots & & \vdots \\ \mathcal{P}^{\alpha_{n_i}} & \mathcal{P}^{\alpha_{n_i}} & \mathcal{P}^{\alpha_{n_i}} & \ldots & \mathcal{U}^{\alpha_{n_i}} \end{pmatrix} \tag{20}$$

where $\mathcal{U}^k = 1 + \mathcal{P}^k$ if $k \geq 1$ and $\mathcal{U}^0 = \mathcal{O}^\times$.

The other component to Proposition 2.11 has to do with the map $\Theta_\Lambda$. For $\Lambda^\alpha$ there is a life of this map $\overline{\Theta_{\Lambda^\alpha}}\colon P_1(\Lambda^\alpha) \to P_2(\Lambda^\alpha)$ which is independent of $\alpha$, and is given by $\overline{\Theta_{\Lambda^\alpha}}(e_i) = f_i$ for $1 \leq i \leq n_1$. So by Theorem 2.11, $S_{\Lambda^\alpha}$ is the product of the group

$$\left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{n_2-n_1} \end{pmatrix} \,\middle|\, A \in S_1 \right\} \tag{21}$$

with the group $T_1 \times T_2$ (embedded block diagonally into $\mathrm{GL}_{n_1+n_2}(F)$).

# 3 Geometric interpretation of $Q$

## 3.1 Distance between orbits

The main result of section 2.2 gives an invariant $Q$ of the action of $L = \mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ acting on $\mathcal{B}_e\mathrm{GL}(W_1 \oplus W_2)^0$. In this section we give a geometric interpretation of this invariant in terms of a distance between orbits.

By Proposition 2.8 we may identify the space of orbits $L\backslash\mathcal{B}_e\mathrm{GL}(V)$ with $A^{n_1}$. We define a function called the orbital distance as follows:

$$\begin{aligned} d_O\colon A^{n_1}\times A^{n_1} &\to \mathbb{N} \\ (\alpha,\beta) &\mapsto \max_{i=1,\dots,n_1}(|\alpha_i-\beta_i|). \end{aligned} \tag{22}$$

The main result of this section is that the name "orbital distance" is justified; that is, $d_O$ is actually the minimum distance between two orbits as measured in the 1-skeleton of the building $\mathcal{B}_e\mathrm{GL}(V)$.

For simplicity if $[\Lambda]\in\mathcal{B}_e(V)$ then let $L[\Lambda]$ denote the orbit of $[\Lambda]$ under $L$.

**Proposition 3.1.** *Let* $[\Lambda_1],[\Lambda_2]\in\mathcal{B}_e\mathrm{GL}(V)$ *be incident, then*

$$d_O(L[\Lambda_1],L[\Lambda_2])\le 1.$$

*Proof.* Let $[\Lambda_1],[\Lambda_2]$ be two incident vertices with $\varpi\Lambda_1\subset\Lambda_2\subset\Lambda_1$. Let $L[\Lambda_1]$ be identified with $\alpha\in A^{n_1}$ and $L[\Lambda_2]$ with $\beta\in A^{n_1}$. We have

$$\varpi P_i(\Lambda_1)\subset P_i(\Lambda_2)\subset P_i(\Lambda_1), \tag{23}$$

$$\varpi K_i(\Lambda_1)\subset K_i(\Lambda_2)\subset K_i(\Lambda_1). \tag{24}$$

There are two extreme cases. First $P_1(\Lambda_2)=P_1(\Lambda_1)$ and $K_1(\Lambda_2)=\varpi K_1(\Lambda_1)$. In this case $\beta_i=\alpha_i+1$ for all $i\in\{1,\dots n_1\}$.

In the second case $P_1(\Lambda_2)=\varpi P_1(\Lambda_1)$, and $K_1(\Lambda_2)=K_1(\Lambda_1)\cap\varpi P_1(\Lambda_1)\supset\varpi K_1(\Lambda_1)$. In this case $\alpha_i=\beta_i+1$ or $\alpha_i=\beta_i$.

The above argument shows that no matter what $P_1(\Lambda_2)$ and $K_1(\Lambda_2)$ are we have $|\alpha_i-\beta_i|\le 1$ as desired. □

Proposition 3.1 shows that if two incident vertices are in different orbits, then their $L$-orbits have orbital distance 1. To show $d_O$ is actually the proposed metric we need to show if two orbits have orbital distance 1, then there are incident representatives of each orbit. The following technical lemma proves this.

**Lemma 3.2.** *Let* $[\Lambda_1],[\Lambda_2]\in\mathcal{B}_e\mathrm{GL}(V)$. *Assume* $d_O(L[\Lambda_1],L[\Lambda_2])=k>0$. *Then there is an* $[\Lambda_3]\in\mathcal{B}_e\mathrm{GL}(V)$ *incident to* $[\Lambda_2]$ *so that* $d_O(L[\Lambda_1],L[\Lambda_3])=k-1$.

*Proof.* Let $[\Lambda_1],[\Lambda_2]$ be as in the statement of the lemma. Since we are working in $L$-orbits, and $L$ preserves distance in $\mathcal{B}_e\mathrm{GL}(V)$ we may choose any representatives for $[\Lambda_1]$ and $[\Lambda_2]$ that we like. In particular if $L[\Lambda_1],L[\Lambda_2]$ are identified with $\alpha,\beta\in A^{n_1}$ respectively, we may take for our representatives $\Lambda^\alpha,\Lambda^\beta$ respectively, as defined in Proposition 2.8.

Recall that if $W_1$ has basis $\{e_i\}_{i=1}^{n_1}$ and $W_2$ has basis $\{f_i\}_{i=1}^{n_2}$ then $\Lambda^\alpha = \bigoplus_{i=1}^{n_1} \Lambda^{\alpha_i} \bigoplus_{i=n_1+1}^{n_2} \mathcal{O} f_i$ where $\Lambda^{\alpha_i} = \langle \varpi^{\alpha_i} e_i, e_i + f_i \rangle$.

To find a $[\Lambda_3]$ with the desired property we need to show there exists $\gamma \in A^{n_1}$ so that $d_O(\alpha, \gamma) = k - 1$ and $d_O(\beta, \gamma) = 1$. To do this, we define $\gamma = (\gamma_i)$ where

$$\gamma_i = \begin{cases} \beta_i + 1 & \text{if } \alpha_i - \beta_i = k, \\ \beta_i - 1 & \text{if } \beta_i - \alpha_i = k, \\ \beta_i & \text{else.} \end{cases}$$

Let $S = \{i \mid \beta_i - \alpha_i = k\}$. We now define $\Lambda_3$ as follows:

$$\Lambda_3 = \bigoplus_{i \in S} \varpi \Lambda^{\gamma_i} \bigoplus_{i \in \{1,\ldots,n_1\} \setminus S} \Lambda^{\gamma_i} \bigoplus_{i=n_1+1}^{n_2} \mathcal{O} f_i. \tag{25}$$

By construction $d_O(L[\Lambda^\alpha], L[\Lambda_3]) = k - 1$. So all we need to show is that $[\Lambda^\beta]$ and $[\Lambda_3]$ are incident. This follows from the two-dimensional case and the fact that

$$\Lambda^k \supset \Lambda^{k+1} \supset \varpi \Lambda^k \tag{26}$$

and that

$$\Lambda^k \supset \varpi \Lambda^{k-1} \supset \varpi \Lambda^k. \tag{27}$$

□

Together Proposition 3.1 and Lemma 3.2 give us the following theorem.

**Theorem 3.3.** *Let $[\Lambda_1], [\Lambda_2] \in \mathcal{B}_e\mathrm{GL}(V)^0$. Then $d_O(L[\Lambda_1], L[\Lambda_2])$ is the minimal distance between any two representatives of the orbits as measured in the 1-skeleton of $\mathcal{B}_e\mathrm{GL}(V)^0$.*

Theorem 3.3 gives a complete combinatorial description of the geometry of the orbit space $L\mathcal{B}_e\mathrm{GL}(V)^0$. Figure 1 on the next page is the quotient space for $L\backslash\mathcal{B}_e\mathrm{GL}(V)$ when $V$ is 4-dimensional and $n_1 = n_2 = 2$.

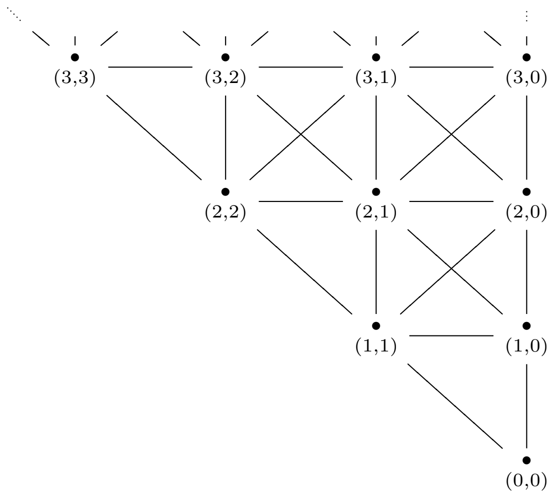
## 3.2 Distance to $\overline{\mathcal{A}_{\mathcal{F}_1 \cup \mathcal{F}_2}}$ in $\mathcal{B}_e(\mathrm{GL}(W_1 \oplus W_2))$

There is an important special case of Theorem 3.3. The orbit for which $Q(\Lambda) = 0$ is distinguished. In this section we give both a description of this orbit, as well as another description of the distance from a given point to this orbit.

Recall from section 1 that an apartment $\mathcal{A}_\mathcal{F}$ is specified by a frame $\mathcal{F}$ in $W_1 \oplus W_2$. Denote by $\mathrm{Frame}(V)$ the set of all frames in a vector space $V$. We will be interested in the following collection of apartments:

$$\overline{\mathcal{A}_{W_1 \oplus W_2}} = \bigcup_{\substack{\mathcal{F}_1 \in \mathrm{Frame}(W_1) \\ \mathcal{F}_2 \in \mathrm{Frame}(W_2)}} \mathcal{A}_{\mathcal{F}_1 \cup \mathcal{F}_2}. \tag{28}$$

Figure 1: The quotient space for $L\backslash\mathcal{B}_e\mathrm{GL}(V)$ $(\dim V = 4,\, n_1 = n_2 = 2)$.




**Proposition 3.4.** *$\overline{\mathcal{A}_{W_1\oplus W_2}}$ is a subbuilding of $\mathcal{B}_e\mathrm{GL}(V)$.*

*Proof.* Since $\overline{\mathcal{A}_{W_1\oplus W_2}}$ is a union of apartments from an actual building all that needs to be shown is that any two chambers $C_1, C_2 \in \overline{\mathcal{A}_{W_1\oplus W_2}}$ are in a common apartment. Let $\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset \varpi\Lambda_1$ be a chain of $\mathcal{O}$-lattices corresponding to a chamber $C \in \overline{\mathcal{A}_{W_1\oplus W_2}}$, and $M_1 \supset M_2 \supset \cdots \supset M_n \supset \varpi M_1$ a chain of lattices corresponding to a chamber $D \in \overline{\mathcal{A}_{W_1\oplus W_2}}$. Since each $[\Lambda_i] \in \overline{\mathcal{A}_{W_1\oplus W_2}}$ we can write $\Lambda_i = \Lambda_i^1 \oplus \Lambda_i^2$ with $[\Lambda_i^j] \in \mathcal{B}_e(\mathrm{GL}(W_j))$. Similarly for the $M_i$. The $\{[\Lambda_i^j]\}_{i=1}^n$,$\{[M_i^j]\}_{i=1}^n$ specify facets $C_j, D_j \in \mathcal{B}_e(\mathrm{GL}(W_j))$ since $\Lambda_1^j \supset \Lambda_i^j \supset \varpi\Lambda_1^j$ (it will be the case that some of the $\Lambda_i^j = \Lambda_{i+1}^j$ but this will not matter), and similarly for the $M_i^j$. Then there are common apartments $\mathcal{A}_j \subset \mathcal{B}_e\mathrm{GL}(W_j)$ which contain $C_j$ and $D_j$. Since each $\mathcal{A}_j$ is specified by a frame $\mathcal{F}_j$ in $W_j$ the apartment specified by $\mathcal{F}_1 \cup \mathcal{F}_2$, contains the chambers $C$ and $D$. $\square$

Now let $[\Lambda] \in \mathcal{B}_e\mathrm{GL}(V)^0$. We define a function on $\mathcal{B}_e\mathrm{GL}(V)^0$ as follows:

$$\begin{aligned} d_p\colon \mathcal{B}_e(\mathrm{GL}(W_1 \oplus W_2))^0 &\to \mathbb{N} \\ [\Lambda] &\mapsto \log_{\mathcal{P}}[\mathrm{Ann}(Q(\Lambda))]. \end{aligned} \tag{29}$$

Here $\mathrm{Ann}(Q(\Lambda)) = \{x \in \mathcal{O} \mid xQ(\Lambda) = 0\}$ is the annihilator of $Q(\Lambda)$ in $\mathcal{O}$. The $p$ subscript is because it turns out $d_p$ is distance it takes to project $[\Lambda]$ onto

$\overline{\mathcal{A}_{W_1\oplus W_2}}$. This follows from the fact $\overline{\mathcal{A}_{W_1\oplus W_2}}$ is the orbit where $Q(\Lambda)=0$. We have the following theorem.

**Theorem 3.5.** *Let* $[\Lambda]\in\mathcal{B}_e\mathrm{GL}(V)^0$ *then* $d_p([\Lambda])=d_O(L[\Lambda],\overline{\mathcal{A}_{W_1\oplus W_2}})$.

*Proof.* $d_O(L[\Lambda],\overline{\mathcal{A}_{W_1\oplus W_2}})=d_O(L[\Lambda],L[\Lambda^{(0)}])$, where $Q(\Lambda^{(0)})=0$. If $L[\Lambda]$ is the orbit associated $\alpha\in A^{n_1}$ then $d_O(L[\Lambda],L[\Lambda^{(0)}])=\max(\alpha_i)$ for $1\le i\le n_1$ and $\alpha_i\in\alpha$, but this is the same as $d_p([\Lambda])$. □

In the special case when $n_1=n_2=1$, $\overline{\mathcal{A}_{W_1\oplus W_2}}$ is just an apartment of $\mathcal{B}_e\mathrm{GL}(V)^0$. Then $d_p$ is just measuring the distance of a given point to a fixed apartment. This suggests that one may be able to find the distance of a vertex to a fixed apartment by studying the action of a maximal split torus on the building.

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