Maximal Levi subgroups acting on the Euclidean building of $\mathrm{GL}_n(F)$

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Abstract

In this paper we give a complete invariant of the action of $\mathrm{GL}_n(F) \times \mathrm{GL}_m(F)$ on the Euclidean building $\mathcal{B}_e\mathrm{GL}_{n+m}(F)$, where F is a discrete valuation field. We then use this invariant to give a natural metric on the resulting quotient space. In the special case of the torus acting on the tree $\mathcal{B}_e\mathrm{GL}_2(F)$, we obtain an algorithm for calculating the distance of any vertex in the tree to any fixed apartment.

`Keywords:` affine building, Euclidean building, Levi subgroup, group action

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1. Introduction

To understand distance in the 1-skeleton of a building $\mathcal{B}G$ associated to a reductive algebraic group G, one may look at a stabilizer K of a point, and then study the action of K on $\mathcal{B}G$. When working over a discrete valuation field vertices correspond to maximal compact subgroups. This analysis gives rise to information about $K\backslash G/K$, and therefore the Hecke algebra [4, 5].

In this paper we specialize to $G = \mathrm{GL}_n(F)$ and are interested in the double cosets $L\backslash G/K$, where $L \cong \mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F)$ is a maximal Levi subgroup of G. The study of the action of L on the building $\mathcal{B}_e\mathrm{GL}_n(F)$ will lead to a description of distance from any vertex to a certain subbuilding stabilized by L. In the case when $n = 2$ and $L = T$ is a maximal split torus, our description gives a way of calculating the distance from a given point to a fixed apartment.

We also give a combinatorial description of the quotient space $L\backslash\mathcal{B}_e\mathrm{GL}_n(F)$ as follows. Let $A^n = \{(\alpha_i)_{i=1}^n \mid \alpha_i \in \mathbb{N}, \alpha_i \geq \alpha_{i+1}\}$. If $n_1 \leq n_2$ there is an graph isometry between $L\backslash\mathcal{B}_e\mathrm{GL}_n(F)$ and A^{n_1} where A^n is endowed with the

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following metric: $d(\alpha,\beta) = \max_{i=1}^{n} |\alpha_i - \beta_i|$ where $\alpha, \beta \in A^n$. This result shows that the 1-skeleton of the resulting quotient space only depends on $\min(n_1, n_2)$.

This paper is broken up into two main sections. The first gives a description of the building in terms of $\mathcal{O}$-lattices and describes an invariant of the action of L on this building. The second section gives a geometric interpretation of this invariant, yielding a combinatorial description of the quotient space $L\backslash\mathcal{B}_e\mathrm{GL}_n(F)$.

2. Orbits of maximal Levi factors on $\mathcal{B}_e\mathrm{GL}(V)$

2.1. $\mathcal{O}$-lattices and $\mathcal{B}_e\mathrm{GL}(V)$

Throughout this paper let F be a discrete valuation field with valuation v. We will denote the ring of integers in F by $\mathcal{O}$, and fix once and for all a uniformizer ϖ of $\mathcal{O}$. Let the unique maximal prime ideal be denoted as $\mathcal{P} = (\varpi)$, and the residue field $\mathcal{O}/\mathcal{P}$ will be denoted by $\mathfrak{k}$. Let $\mathcal{P}^k = (\varpi^k)$ for $k \in \mathbb{Z}$. Then $\log_{\mathcal{P}}(\mathcal{P}^k) = k$. Let V be a finite dimensional vector space defined over F of dimension n. We will describe the Euclidean building $\mathcal{B}_e\mathrm{GL}(V)$ associated to $\mathrm{GL}(V)$. For more details see [1]. Let $\Lambda \subset V$ be a finitely generated free $\mathcal{O}$-module of rank n. Denote by $[\Lambda]$ the homothety class of Λ, that is $[\Lambda] = \{a\Lambda \mid a \in F^\times\}$.

Homothety classes of lattices will form the vertices of $\mathcal{B}_e\mathrm{GL}(V)$. Two vertices $\lambda_1, \lambda_2 \in \mathcal{B}_e\mathrm{GL}(V)$ are incident if there are representatives $\Lambda_i \in \lambda_i$ so that $\varpi\Lambda_1 \subset \Lambda_2 \subset \Lambda_1$, i.e. $\Lambda_2/\varpi\Lambda_1$ is a $\mathfrak{k}$-subspace of $\Lambda_1/\varpi\Lambda_1$. The chambers in $\mathcal{B}_e\mathrm{GL}(V)$ are collections of maximally incident vertices. To put this more concretely, a chamber is a collection of n vertices $\lambda_0 \cdots \lambda_{n-1}$ with representatives $\Lambda_0 \cdots \Lambda_{n-1}$ satisfying $\varpi\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_{n-1} \subsetneq \Lambda_0$. A wall of a chamber is any subset of $n-1$ vertices in the given chamber. We will denote by $\mathcal{B}_e\mathrm{GL}(V)^k$ the set of all facets of $\mathcal{B}_e\mathrm{GL}(V)$ of dimension k.

A frame $\mathcal{F}$ in V is a collection of lines $l_1, \ldots, l_n \subset V$ which are linearly independent and span all of V. We now describe certain subcomplexes of $\mathcal{B}_e\mathrm{GL}(V)$. Define $\mathcal{A}_{\mathcal{F}}$ to be the subcomplex consisting of vertices $[\Lambda]$ of the following form:

$$\Lambda = \bigoplus_{i=1}^{n} \mathcal{O}e_i \tag{1}$$

where $e_i \in l_i \in \mathcal{F}$. Then $\mathcal{A}_{\mathcal{F}}$ is an apartment of $\mathcal{B}_e\mathrm{GL}(V)$, and every apartment is uniquely determined by a frame in this way.

The group $\mathrm{GL}(V)$ has a natural action of $\mathcal{B}_e\mathrm{GL}(V)$, namely the one induced from the action of $\mathrm{GL}(V)$ on V. This action preserves distance in the building.

A lemma which we will need later is the following.

Lemma 2.1. *Let* Λ, Λ' *be* $\mathcal{O}$*-lattices of rank* n *in* V *with* $\Lambda' \subset \Lambda$*. Then the natural map from* $\mathrm{GL}(\Lambda) \cap \mathrm{stab}(\Lambda')$ *to* $\mathrm{GL}(\Lambda/\Lambda')$ *is surjective.*

Proof. This result appears to be well known, but the proof could not be found in the literature and so is given here. There is an $\mathcal{O}$-basis $\{e_1, \ldots, e_n\}$ of Λ so that $\{\varpi^{k_1}e_1, \ldots, \varpi^{k_n}e_n\}$ with $k_i \in \mathbb{N}$ is an $\mathcal{O}$-basis of Λ'. This is equivalent to the statement that for any two vertices there is an apartment which contains them both. For $\overline{\sigma} \in \mathrm{GL}(\Lambda/\Lambda')$ we will construct $\sigma \in \mathrm{GL}(\Lambda) \cap \mathrm{stab}(\Lambda')$ which descends to $\overline{\sigma}$.

Let $\overline{e}_i$ be the image of e_i in Λ/Λ'. Then

$$\overline{\sigma}(\overline{e}_i) = a_1^i\overline{e_1} + \cdots + a_n^i\overline{e}_n \tag{2}$$

where $a_j^i \in \mathcal{O}$. Observe that a_j^i is unique modulo $\mathcal{P}^{k_j}$. Then define σ on the $\mathcal{O}$-basis $\{e_1, \ldots, e_n\}$ of Λ as follows:

$$\sigma(e_i) = \begin{cases} \sum_{j=1}^n a_j^i e_j & \text{if } \overline{e}_i \neq 0, \\ e_i & \text{if } \overline{e}_i = 0. \end{cases} \tag{3}$$

What needs to be shown is that σ is invertible and leaves Λ' invariant.

First, we show σ leaves Λ' invariant.

$$0 = \overline{\sigma}(\varpi^{k_i}\overline{e}_i) = a_1^i\varpi^{k_i}\overline{e_1} + \cdots + a_n^i\varpi^{k_i}\overline{e}_n \tag{4}$$

This shows that $a_j^i\varpi^{k_i} \in \mathcal{P}^{k_j}$, and so $\sigma(\varpi^{k_i}e_i) \in \Lambda'$.

Next we show invertibility. Let σ^* be the construction given above for $\overline{\sigma}^{-1}$, and let $\tau = \sigma \circ \sigma^*$. This will be a function which is a lift of the identity map in $\mathrm{GL}(\Lambda/\Lambda')$. Let $M = \mathrm{span}_{\mathcal{O}}\langle e_i \mid \overline{e}_i \neq 0\rangle$ and let $M' = \mathrm{span}_{\mathcal{O}}\langle e_i \mid \overline{e}_i = 0\rangle$. Then $\tau|_M = \mathrm{id} + E$ where $E \in \mathrm{Hom}_{\mathcal{O}}(M, \Lambda')$ and is id on M'. Any τ of this form is invertible and hence so is σ. □

2.2. $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ acting on $\mathcal{B}_e(\mathrm{GL}(W_1 \oplus W_2))$

Let V be a vector space over F. Fix a maximal Levi subgroup L of $\mathrm{GL}(V)$. Associated to L are subspaces $W_1, W_2 \subset V$ satisfying $V = W_1 \oplus W_2$. Then $L \cong \mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$. In this section we will describe the orbits of the action of $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ on $\mathcal{B}_e\mathrm{GL}(V)^0$ in terms of an invariant Q. Additionally we will give a representative of each orbit.

Let p_i be the projection of V onto W_i with respect to our given decomposition. We will use these maps to define invariants of the vertices and then show for our action that these invariants classify all orbits.

Let Λ be an $\mathcal{O}$-lattice. We make the following definitions for $i = 1, 2$:

$$P_i(\Lambda) = \operatorname{Im}(p_i|_\Lambda), \tag{5}$$
$$K_i(\Lambda) = \operatorname{Ker}(p_{i'}|_\Lambda) = \Lambda \cap W_i, \tag{6}$$

where $i' = (i \bmod 2) + 1$.

These are lattices in W_i.

Lemma 2.2. $K_i(\Lambda) \subset P_i(\Lambda)$.

Proof. If $v \in K_i(\Lambda) = \Lambda \cap W_i$, then $v \in \Lambda$, so $p_i(v) \in P_i(\Lambda)$. But $p_i(v) = v$ since $v \in W_i$. □

By Lemma 2.2 we can define $Q_i(\Lambda) = P_i(\Lambda)/K_i(\Lambda)$. This is a finitely generated torsion $\mathcal{O}$-module.

Proposition 2.3. $Q_1(\Lambda) \cong Q_2(\Lambda)$ *as* $\mathcal{O}$*-modules. This isomorphism class will be denoted by* $Q(\Lambda)$.

Proof. We make slight modifications to the proof found in [2]. Let $p_i' \colon \Lambda \to Q_i(\Lambda)$ be the composition of p_i with the natural projection map $\pi_i \colon P_i(\Lambda) \to Q_i(\Lambda)$. We define a map so that for all $v \in \Lambda$

$$\begin{aligned} \Theta \colon Q_1(\Lambda) &\to Q_2(\Lambda) \\ p_1'(v) &\mapsto p_2'(v). \end{aligned} \tag{7}$$

We will show that Θ is well defined, and is an isomorphism.

Let $w_1 + w_2, w_1' + w_2' \in \Lambda$ with $w_i, w_i' \in W_i$ and $\pi_1(w_1) = \pi_1(w_1')$. Then $\pi_1(w_1 - w_1') = 0$, and therefore $w_1 - w_1' \in K_1(\Lambda)$. Similarly $w_2 - w_2' \in K_2(\Lambda)$ and $\pi_2(w_2) = \pi_2(w_2')$ showing Θ is well defined. It is an isomorphism, because the map θ, defined by reversing the roles of 1 and 2, is an inverse map. □

We now show that Q is a complete invariant of the action of L on $\mathcal{B}_e\mathrm{GL}(V)^0$.

Theorem 2.4. *Let* Λ, Λ' *be* $\mathcal{O}$*-lattices. Then* Λ *and* Λ' *are in the same* $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ *orbit if and only if* $Q(\Lambda) = Q(\Lambda')$.

Proof. The class $Q(\Lambda)$ is a $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$-invariant since each factor of $\mathrm{GL}(W_i)$ commutes with the projection map p_i. We must show that if $Q(\Lambda) = Q(\Lambda')$ then there is a $g \in \mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ so that $\Lambda = g\Lambda'$.

We will need $g_1 \in \mathrm{GL}(W_1)$ and $g_2 \in \mathrm{GL}(W_2)$ so that $g_i P_i(\Lambda') = P_i(\Lambda)$ and $g_i K_i(\Lambda') = K_i(\Lambda)$ for $i = 1, 2$. There are certainly $g_i \in \mathrm{GL}(W_i)$ so that

$g_iP_i(\Lambda') = P_i(\Lambda)$. Then we may assume $K_i(\Lambda), K_i(\Lambda') \subset P_i(\Lambda)$. Since $Q(\Lambda) = Q(\Lambda')$ we know by the elementary divisor theorem there are bases

$$B_i = \{e_1, \ldots, e_{n_i}\} \text{ and } B_i' = \{e_1', \ldots, e_{n_i}'\}$$

of $P_i(\Lambda)$ so that $K_i(\Lambda)$ written in terms of B_i has the same elementary divisors as $K_i(\Lambda')$ written in terms of B_i'. Let $h_i \in \mathrm{GL}(P_i(\Lambda))$ be the linear transformation which takes the basis B_i to B_i'. Then $h_ig_i \in \mathrm{GL}(W_i)$ has the desired properties.

So we may replace Λ' with $\Lambda'' = (h_1g_1, h_2g_2)\Lambda'$. Let Θ be the map from Proposition 2.3 associated to Λ, and Θ'' associated to Λ''.

We claim $\Lambda = \Lambda''$ if and only if $\Theta = \Theta''$. To prove this we show that one can reconstruct Λ from Θ (which implicitly encodes $Q_i(\Lambda)$ as the domain and range of the map), by taking

$$\Lambda_\Theta = \left\{ w_1 + w_2 \mid w_i \in P_i(\Lambda) \text{ and } \Theta(\pi_1(w_1)) = \pi_2(w_2) \right\} \tag{8}$$

First, we show $\Lambda \subset \Lambda_\Theta$. Let $w = w_1 + w_2 \in \Lambda$, then by definition of Θ we have $\Theta(\pi_1(w_1)) = \pi_2(w_2)$. And so $v \in \Lambda_\Theta$. We now show $\Lambda_\Theta \subset \Lambda$. Let $w_1 + w_2 \in \Lambda_\Theta$. Then $w_1 \in P_1(\Lambda)$ so there is a $w_2' \in P_2(\Lambda)$ so that $w_1 + w_2' \in \Lambda \subset \Lambda_\Theta$. Then $0 + (w_2 - w_2') \in \Lambda_\Theta$. So $\pi_2(w_2 - w_2') = 0$ which implies $w_2 - w_2' \in K_2(\Lambda) \subset \Lambda$. Hence $w_1 + w_2 = (w_1 + w_2') + (w_2 - w_2') \in \Lambda$ as desired.

To complete the theorem, we will show there is an element $g \in \mathrm{stab}(P_2(\Lambda)) \cap \mathrm{stab}(K_2(\Lambda))$ which takes Θ'' to Θ. There is an $\overline{h} \in \mathrm{GL}(P_2(\Lambda)/K_2(\Lambda))$ so that $(1, \overline{h})\Theta'' = \Theta$. By Lemma 2.1 there is a pullback h of $\overline{h}$ to $h \in \mathrm{stab}(P_2(\Lambda)) \cap \mathrm{stab}(K_2(\Lambda)) \in \mathrm{GL}(W_2)$ then $(1, h)\Lambda'' = \Lambda$. □

Now let $[\Lambda] \in \mathcal{B}_e\mathrm{GL}(V)^0$, and $c \in F^\times$. Since $Q(\Lambda) = Q(c\Lambda)$ we will abuse notation and write $Q([\Lambda]) = Q(\Lambda)$.

Corollary 2.5. *$Q([\Lambda])$ is a complete invariant of the action of $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ on the space of vertices in $\mathcal{B}_e(V)^0$.*

2.3. Orbit representatives

We now give a set representatives of each orbit. We first do this in the case when V is 2-dimensional, and then use this case to determine representatives for higher dimensions.

2.3.1. $\dim(V) = 2$

Let V be a two-dimensional vector space over F, with decomposition $V = W_1 \oplus W_2$. Assume that W_i is spanned by the vector e_i. We then define the following class of lattices:

$$\Lambda^k = \mathrm{span}_{\mathcal{O}}\langle \varpi^k e_1, e_1 + e_2\rangle. \tag{9}$$

Proposition 2.6. $Q([\Lambda^k]) \cong \mathcal{O}/\mathcal{P}^k$.

Proof. We have $P_1(\Lambda^k) = \langle e_1\rangle$ and $K_1(\Lambda^k) = \langle \pi^k e_1\rangle$. Therefore $Q(\Lambda) \cong \mathcal{O}/\mathcal{P}^k$. □

Corollary 2.7. *$\{[\Lambda^k]\}_{k=0}^{\infty}$ is a complete set of representatives for the action of $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ on $\mathcal{B}_e\mathrm{GL}(V)^0$.*

Proof. Let $[\Lambda] \in \mathcal{B}_e\mathrm{GL}(V)^0$. Then $Q([\Lambda]) \cong \mathcal{O}/\mathcal{P}^k$ for some $k \in \mathbb{N}$. By Theorem 2.4, $[\Lambda]$ is in the orbit of Λ^k. □

2.3.2. General V

We now describe representatives when V is n-dimensional. We may assume that $\dim W_i = n_i$ and $n_1 \leq n_2$. Choose a basis $\{e_1, \ldots, e_{n_1}\}$ of W_1 and $\{f_1, \ldots, f_{n_2}\}$ of W_2, and let $Y_i = \mathrm{span}_F(e_i, f_i)$, for $1 \leq i \leq n_1$. Let $\alpha = (\alpha_i) \in \mathbb{N}^{n_1}$. Let $[\Lambda^{\alpha_i}] \in \mathcal{B}_e\mathrm{GL}(Y_i)$ defined as in equation (9) with respect to the basis $\{e_i, f_i\}$. This allows us to define the following class of lattices:

$$\Lambda^\alpha = \bigoplus_{i=1}^{n_1} \Lambda^{\alpha_i} \bigoplus_{i=n_1+1}^{n_2} \mathcal{O} f_i \tag{10}$$

Proposition 2.8. *Let $A^n = \{\alpha = (\alpha_i) \in \mathbb{N}^n \mid \alpha_i \geq \alpha_{i+1}\}$. Then $[\Lambda^\alpha]_{\alpha \in A^{n_1}}$ is a complete set of representatives of the orbits of $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ acting on $\mathcal{B}_e\mathrm{GL}(V)^0$.*

Proof. By the elementary divisor theorem $Q_1(\Lambda)$ decomposes into a direct sum of $\mathcal{O}$-modules as follows: $Q_1([\Lambda]) \cong \mathcal{O}^r \bigoplus_{i=1}^{n_1} \mathcal{O}/\mathcal{P}^{\alpha_i}$ where $\alpha_i \in \mathbb{N}$ and $r \in \mathbb{N}$. However, $r = 0$ since both $P_1(\Lambda)$ and $K_1(\Lambda)$ are rank n_1. We may assume $\alpha_i \geq \alpha_{i+1}$. Then by Theorem 2.4, $[\Lambda]$ is in the same orbit as $[\Lambda^\alpha]$. □

2.3.3. Double cosets

The description of orbits is equivalent to the space of double cosets $L\backslash\mathrm{GL}(V)/K$, where K is the stabilizer of a vertex in $\mathcal{B}_e(\mathrm{GL}(V))$. We now give an explicit description of a set of double coset representatives.

The Levi subgroup L is associated to a parabolic subgroup P with a decomposition $P = LN$, where N is the unipotent radical of P. The Iwasawa decomposition shows that $\mathrm{GL}(V) = PK$, and so we may choose the double coset representatives of $L\backslash\mathrm{GL}(V)/K$ to be in N.

We use the basis for V of the previous section to identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(F)$. We will also let $K = Z(\mathrm{GL}_n(F))\mathrm{GL}_n(\mathcal{O})$. Then $N \cong M_{n_1 \times n_2}(F)$, the $n_1 \times n_2$ matrices embedded in $\mathrm{GL}_n(F)$ as follows:

$$u\colon M_{n_1\times n_2}(F) \to N$$
$$B \mapsto \begin{pmatrix} I_{n_1} & B \\ 0 & I_{n_2} \end{pmatrix}.$$

Let $\alpha \in A^{n_1}$ and define $m^\alpha \in M_{n_1\times n_2}$ as follows:

$$[m^\alpha]_{ij} = \begin{cases} \varpi^{-\alpha_i} & \text{if } i = j \in \{1, \dots, n_1\}, \\ 0 & \text{else.} \end{cases} \tag{11}$$

Now let $n^\alpha = u(m^\alpha)$. Then we have the following proposition.

Proposition 2.9. *We may write $\mathrm{GL}_n(F)$ as a disjoint union*

$$\mathrm{GL}_n(F) = \coprod_{\alpha \in A^{n_1}} L n^\alpha K.$$

Proof. Let $\alpha \in A^{n_1}$, and define l^α to be the linear transformation that sends e_i to e_i and f_i to $\varpi^{-\alpha_i} f_i$ for $1 \le i \le n_1$, and f_j to f_j for $n_1 + 1 \le j \le n_2$. Note that $l^\alpha \in L$.

Let $\Lambda = \mathrm{span}_{\mathcal{O}}(e_1 \dots, e_{n_1}, f_1, \dots, f_{n_2})$, and notice that K stabilizes $[\Lambda]$. Furthermore, we have $l^\alpha n^\alpha(\Lambda) = \Lambda^\alpha$. □

This double coset decomposition is in no way canonical, although it has some nice properties. All the n^α are supported on the span of root groups $U^{i,i+n_1}$ for $1 \le i \le n_1$, with the roots taken with respect to the diagonal torus. In fact, these root group form a set of maximally mutually orthogonal root groups in N. Any such set of root groups can be a support of coset representatives. This can easily be seen by having W_{n_i} the Weyl groups of GL_{n_i} act on the n^α. This leads to the following conjecture for more general groups.

Conjecture 2.10. *Let G be a reductive group over F and P a parabolic subgroup with $P = LN$, and assume N is abelian. Let K be a maximal open, bounded subgroup of G. Then there is a discrete subset $N' \subset N$ and a maximally mutually orthogonal set of root groups $U^\alpha < N$ so that:*

1. *each $n \in N'$ is supported in the group generated by the U^α;*
2. *$G = \coprod_{n \in N'} LnZ(G)K$.*

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2.3.4. Stabilizers

We now wish to compute stabilizers for each orbit so that we may realize the orbits as homogeneous spaces. For spherical buildings knowing the stabilizers plays a role in representation theory, for instance [3]. For Euclidean buildings this may have applications to understanding cuspidal representations.

Fix a Λ and let $S_i = \operatorname{stab}(P_i(\Lambda)) \cap \operatorname{stab}(K_i(\Lambda))$. Furthermore, let

$$T_i = \{I + A \mid A \in \operatorname{End}(W_i) \text{ and } A(P_i(\Lambda)) \subset K_i(\Lambda)\} \cap S_i. \tag{12}$$

Then $T_i \triangleleft S_i$ and $S_i/T_i \cong \mathrm{GL}(Q_i(\Lambda))$ by Lemma 2.1. Let

$$\overline{S_\Lambda} = \{(h_1, \Theta^*_\Lambda(h_1)) \mid h_1 \in \mathrm{GL}(Q_1(\Lambda))\} \subset (\mathrm{GL}(Q_1(\Lambda)) \times \mathrm{GL}(Q_2(\Lambda)) \tag{13}$$

where Θ^*_Λ is the isomorphism induced on $\mathrm{GL}(Q_1(\Lambda))$ from the isomorphism $\Theta_\Lambda \colon Q_1(\Lambda) \to Q_2(\Lambda)$ defined in equation (7). Finally, let S_Λ be the pullback of $\overline{S_\Lambda}$ in $S_1 \times S_2$.

Proposition 2.11. $S_\Lambda = \operatorname{stab}_L(\Lambda)$.

Proof. Let $(A_1, A_2) \in S_\Lambda$ with $A_i \in \mathrm{GL}(W_i)$, and let $\Lambda' = (A_1, A_2) \cdot \Lambda$. Then because $A_i \in S_i$ we have $P_i(\Lambda) = P_i(\Lambda')$ and $K_i(\Lambda) = K_i(\Lambda')$. We now wish to show $\Theta_\Lambda = \Theta_{\Lambda'}$. By the proof of Theorem 2.4 this will show that $\Lambda = \Lambda'$. Let B_i be the image of A_i in $S_i/T_i \cong \mathrm{GL}(Q_i(\Lambda))$, and let $v \in Q_1(\Lambda)$ and $v' = B_1^{-1}v$. Then

$$\begin{aligned} \Theta_{\Lambda'}(v) &= \Theta_{\Lambda'}(B_1 B_1^{-1} v) && (14)\\ &= B_2 \Theta_\Lambda(B_1^{-1} v) && (15)\\ &= \Theta^*_\Lambda(B_1)\Theta_\Lambda(B_1^{-1} v) && (16)\\ &= \Theta_\Lambda\left(B_1 \Theta_\Lambda^{-1}(\Theta_\Lambda(B_1^{-1} v))\right) && (17)\\ &= \Theta_\Lambda(v). && (18) \end{aligned}$$

Line (15) follows from the action of (A_1, A_2) on Λ, line (16) comes from the fact that $B_2 = \Theta^*_\Lambda(B_1)$, and line (17) is the definition of the induced map Θ^*_Λ.

This proves $S_\Lambda \subset \operatorname{stab}_L(\Lambda)$. We now prove the other direction. Assume $(A_1, A_2) \in \operatorname{stab}_L(\Lambda)$, then $A_i \in S_i$. The calculation above shows that the projection of A_2 in $\mathrm{GL}(Q_2(\Lambda))$ has to equal the image of A_1 in $\mathrm{GL}(Q_2(\Lambda))$ under Θ^*_Λ, proving the result. □

We end this section by giving an explicit description of S_{Λ^α}, the stabilizers of our orbit representatives. Let $\alpha \in A^{n_1}$, then we define Λ^α as in section 2.3. By this definition $P_1(\Lambda^\alpha) = \mathrm{span}_{\mathcal{O}}\langle e_1, \ldots, e_{n_1}\rangle$ and $P_2(\Lambda^\alpha) = \mathrm{span}_{\mathcal{O}}\langle f_1, \ldots, f_{n_2}\rangle$. Also,

$$K_1(\Lambda^\alpha) = \mathrm{span}_{\mathcal{O}}\langle \varpi^{\alpha_1} e_1, \ldots, \varpi^{\alpha_{n_1}} e_{n_1}\rangle, \text{ and}$$
$$K_2(\Lambda^\alpha) = \mathrm{span}_{\mathcal{O}}\langle \varpi^{\alpha_1} f_1, \ldots, \varpi^{\alpha_{n_2}} f_{n_2}\rangle,$$

where $\alpha_j = 0$ if $j > n_1$. Then S_i looks like

$$S_i = \begin{pmatrix} \mathcal{P}^{\beta_{11}} & \mathcal{P}^{\beta_{12}} & \mathcal{P}^{\beta_{13}} & \ldots & \mathcal{P}^{\beta_{1n_i}} \\ \mathcal{P}^{\beta_{21}} & \mathcal{P}^{\beta_{22}} & \mathcal{P}^{\beta_{23}} & \ldots & \mathcal{P}^{\beta_{2n_i}} \\ \vdots & & \ddots & & \vdots \\ \mathcal{P}^{\beta_{n_i1}} & \mathcal{P}^{\beta_{n_i2}} & \mathcal{P}^{\beta_{n_i3}} & \ldots & \mathcal{P}^{\beta_{n_in_i}} \end{pmatrix} \cap \mathrm{GL}_{n_i}(\mathcal{O}) \tag{19}$$

where $\beta_{ij} = \max(0, \alpha_i - \alpha_j)$.

Also, T_i looks like

$$T_i = \begin{pmatrix} \mathcal{U}^{\alpha_1} & \mathcal{P}^{\alpha_1} & \mathcal{P}^{\alpha_1} & \ldots & \mathcal{P}^{\alpha_1} \\ \mathcal{P}^{\alpha_2} & \mathcal{U}^{\alpha_1} & \mathcal{P}^{\alpha_2} & \ldots & \mathcal{P}^{\alpha_2} \\ \vdots & & \ddots & & \vdots \\ \mathcal{P}^{\alpha_{n_i}} & \mathcal{P}^{\alpha_{n_i}} & \mathcal{P}^{\alpha_{n_i}} & \ldots & \mathcal{U}^{\alpha_{n_i}} \end{pmatrix} \tag{20}$$

where $\mathcal{U}^k = 1 + \mathcal{P}^k$ if $k \geq 1$ and $\mathcal{U}^0 = \mathcal{O}^\times$.

The other component to Proposition 2.11 has to do with the map Θ_Λ. For Λ^α there is a life of this map $\overline{\Theta_{\Lambda^\alpha}}\colon P_1(\Lambda^\alpha) \to P_2(\Lambda^\alpha)$ which is independent of α, and is given by $\overline{\Theta_{\Lambda^\alpha}}(e_i) = f_i$ for $1 \leq i \leq n_1$. So by Theorem 2.11, S_{Λ^α} is the product of the group

$$\left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{n_2-n_1} \end{pmatrix} \middle| A \in S_1 \right\} \tag{21}$$

with the group $T_1 \times T_2$ (embedded block diagonally into $\mathrm{GL}_{n_1+n_2}(F)$).

3. Geometric interpretation of Q

3.1. Distance between orbits

The main result of section 2.2 gives an invariant Q of the action of $L = \mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ acting on $\mathcal{B}_e\mathrm{GL}(W_1 \oplus W_2)^0$. In this section we give a geometric interpretation of this invariant in terms of a distance between orbits.

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By Proposition 2.8 we may identify the space of orbits $L\backslash\mathcal{B}_e\mathrm{GL}(V)$ with A^{n_1}. We define a function called the orbital distance as follows:

$$\begin{aligned} d_O\colon A^{n_1} \times A^{n_1} &\to \mathbb{N} \\ (\alpha,\beta) &\mapsto \max_{i=1,\dots,n_1}(|\alpha_i-\beta_i|). \end{aligned} \tag{22}$$

The main result of this section is that the name "orbital distance" is justified; that is, d_O is actually the minimum distance between two orbits as measured in the 1-skeleton of the building $\mathcal{B}_e\mathrm{GL}(V)$.

For simplicity if $[\Lambda] \in \mathcal{B}_e(V)$ then let $L[\Lambda]$ denote the orbit of $[\Lambda]$ under L.

Proposition 3.1. *Let $[\Lambda_1],[\Lambda_2] \in \mathcal{B}_e\mathrm{GL}(V)$ be incident, then*

$$d_O(L[\Lambda_1],L[\Lambda_2]) \le 1.$$

Proof. Let $[\Lambda_1],[\Lambda_2]$ be two incident vertices with $\varpi\Lambda_1 \subset \Lambda_2 \subset \Lambda_1$. Let $L[\Lambda_1]$ be identified with $\alpha \in A^{n_1}$ and $L[\Lambda_2]$ with $\beta \in A^{n_1}$. We have

$$\varpi P_i(\Lambda_1) \subset P_i(\Lambda_2) \subset P_i(\Lambda_1), \tag{23}$$
$$\varpi K_i(\Lambda_1) \subset K_i(\Lambda_2) \subset K_i(\Lambda_1). \tag{24}$$

There are two extreme cases. First $P_1(\Lambda_2) = P_1(\Lambda_1)$ and $K_1(\Lambda_2) = \varpi K_1(\Lambda_1)$. In this case $\beta_i = \alpha_i + 1$ for all $i \in \{1,\dots n_1\}$.

In the second case $P_1(\Lambda_2) = \varpi P_1(\Lambda_1)$, and $K_1(\Lambda_2) = K_1(\Lambda_1) \cap \varpi P_1(\Lambda_1) \supset \varpi K_1(\Lambda_1)$. In this case $\alpha_i = \beta_i + 1$ or $\alpha_i = \beta_i$.

The above argument shows that no matter what $P_1(\Lambda_2)$ and $K_1(\Lambda_2)$ are we have $|\alpha_i - \beta_i| \le 1$ as desired. □

Proposition 3.1 shows that if two incident vertices are in different orbits, then their L-orbits have orbital distance 1. To show d_O is actually the proposed metric we need to show if two orbits have orbital distance 1, then there are incident representatives of each orbit. The following technical lemma proves this.

Lemma 3.2. *Let $[\Lambda_1],[\Lambda_2] \in \mathcal{B}_e\mathrm{GL}(V)$. Assume $d_O(L[\Lambda_1],L[\Lambda_2]) = k > 0$. Then there is an $[\Lambda_3] \in \mathcal{B}_e\mathrm{GL}(V)$ incident to $[\Lambda_2]$ so that $d_O(L[\Lambda_1],L[\Lambda_3]) = k-1$.*

Proof. Let $[\Lambda_1],[\Lambda_2]$ be as in the statement of the lemma. Since we are working in L-orbits, and L preserves distance in $\mathcal{B}_e\mathrm{GL}(V)$ we may choose any representatives for $[\Lambda_1]$ and $[\Lambda_2]$ that we like. In particular if $L[\Lambda_1], L[\Lambda_2]$ are identified with $\alpha,\beta \in A^{n_1}$ respectively, we may take for our representatives $\Lambda^\alpha,\Lambda^\beta$ respectively, as defined in Proposition 2.8.

Recall that if W_1 has basis $\{e_i\}_{i=1}^{n_1}$ and W_2 has basis $\{f_i\}_{i=1}^{n_2}$ then $\Lambda^\alpha = \bigoplus_{i=1}^{n_1} \Lambda^{\alpha_i} \bigoplus_{i=n_1+1}^{n_2} \mathcal{O} f_i$ where $\Lambda^{\alpha_i} = \langle \varpi^{\alpha_i} e_i, e_i + f_i \rangle$.

To find a $[\Lambda_3]$ with the desired property we need to show there exists $\gamma \in A^{n_1}$ so that $d_O(\alpha, \gamma) = k - 1$ and $d_O(\beta, \gamma) = 1$. To do this, we define $\gamma = (\gamma_i)$ where

$$\gamma_i = \begin{cases} \beta_i + 1 & \text{if } \alpha_i - \beta_i = k, \\ \beta_i - 1 & \text{if } \beta_i - \alpha_i = k, \\ \beta_i & \text{else.} \end{cases}$$

Let $S = \{i \mid \beta_i - \alpha_i = k\}$. We now define Λ_3 as follows:

$$\Lambda_3 = \bigoplus_{i \in S} \varpi \Lambda^{\gamma_i} \bigoplus_{i \in \{1,\ldots,n_1\} \setminus S} \Lambda^{\gamma_i} \bigoplus_{i=n_1+1}^{n_2} \mathcal{O} f_i. \tag{25}$$

By construction $d_O(L[\Lambda^\alpha], L[\Lambda_3]) = k - 1$. So all we need to show is that $[\Lambda^\beta]$ and $[\Lambda_3]$ are incident. This follows from the two-dimensional case and the fact that

$$\Lambda^k \supset \Lambda^{k+1} \supset \varpi \Lambda^k \tag{26}$$

and that

$$\Lambda^k \supset \varpi \Lambda^{k-1} \supset \varpi \Lambda^k. \tag{27}$$

□

Together Proposition 3.1 and Lemma 3.2 give us the following theorem.

Theorem 3.3. *Let $[\Lambda_1], [\Lambda_2] \in \mathcal{B}_e \mathrm{GL}(V)^0$. Then $d_O(L[\Lambda_1], L[\Lambda_2])$ is the minimal distance between any two representatives of the orbits as measured in the 1-skeleton of $\mathcal{B}_e \mathrm{GL}(V)^0$.*

Theorem 3.3 gives a complete combinatorial description of the geometry of the orbit space $L\mathcal{B}_e \mathrm{GL}(V)^0$. Figure 1 on the next page is the quotient space for $L \backslash \mathcal{B}_e \mathrm{GL}(V)$ when V is 4-dimensional and $n_1 = n_2 = 2$.

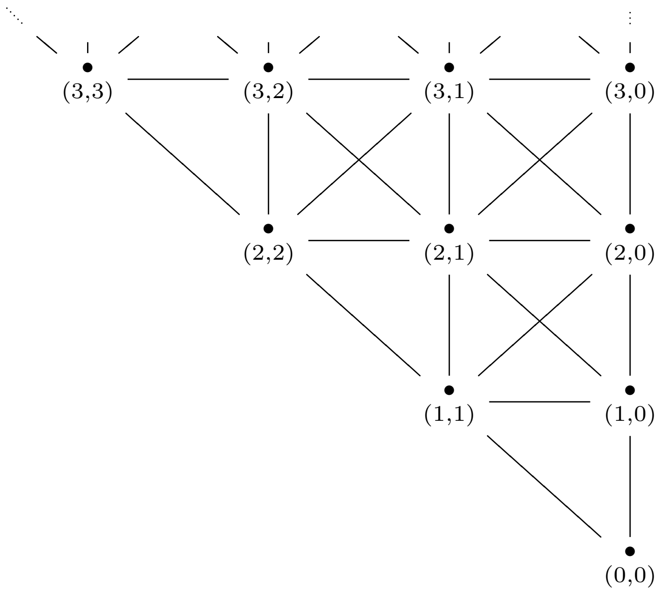
3.2. Distance to $\overline{\mathcal{A}_{\mathcal{F}_1 \cup \mathcal{F}_2}}$ in $\mathcal{B}_e(\mathrm{GL}(W_1 \oplus W_2))$

There is an important special case of Theorem 3.3. The orbit for which $Q(\Lambda) = 0$ is distinguished. In this section we give both a description of this orbit, as well as another description of the distance from a given point to this orbit.

Recall from section 1 that an apartment $\mathcal{A}_\mathcal{F}$ is specified by a frame $\mathcal{F}$ in $W_1 \oplus W_2$. Denote by $\mathrm{Frame}(V)$ the set of all frames in a vector space V. We will be interested in the following collection of apartments:

$$\overline{\mathcal{A}_{W_1 \oplus W_2}} = \bigcup_{\substack{\mathcal{F}_1 \in \mathrm{Frame}(W_1) \\ \mathcal{F}_2 \in \mathrm{Frame}(W_2)}} \mathcal{A}_{\mathcal{F}_1 \cup \mathcal{F}_2}. \tag{28}$$

Figure 1: The quotient space for $L\backslash\mathcal{B}_e\mathrm{GL}(V)$ ($\dim V = 4$, $n_1 = n_2 = 2$).



Proposition 3.4. *$\overline{\mathcal{A}_{W_1\oplus W_2}}$ is a subbuilding of $\mathcal{B}_e\mathrm{GL}(V)$.*

Proof. Since $\overline{\mathcal{A}_{W_1\oplus W_2}}$ is a union of apartments from an actual building all that needs to be shown is that any two chambers $C_1, C_2 \in \overline{\mathcal{A}_{W_1\oplus W_2}}$ are in a common apartment. Let $\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset \varpi\Lambda_1$ be a chain of $\mathcal{O}$-lattices corresponding to a chamber $C \in \overline{\mathcal{A}_{W_1 W_2}}$, and $M_1 \supset M_2 \supset \cdots \supset M_n \supset \varpi M_1$ a chain of lattices corresponding to a chamber $D \in \overline{\mathcal{A}_{W_1\oplus W_2}}$. Since each $[\Lambda_i] \in \overline{\mathcal{A}_{W_1\oplus W_2}}$ we can write $\Lambda_i = \Lambda_i^1 \oplus \Lambda_i^2$ with $[\Lambda_i^j] \in \mathcal{B}_e(\mathrm{GL}(W_j))$. Similarly for the M_i. The $\{[\Lambda_i^j]\}_{i=1}^n$,$\{[M_i^j]\}_{i=1}^n$ specify facets $C_j, D_j \in \mathcal{B}_e(\mathrm{GL}(W_j))$ since $\Lambda_1^j \supset \Lambda_i^j \supset \varpi\Lambda_1^j$ (it will be the case that some of the $\Lambda_i^j = \Lambda_{i+1}^j$ but this will not matter), and similarly for the M_i^j. Then there are common apartments $\mathcal{A}_j \subset \mathcal{B}_e\mathrm{GL}(W_j)$ which contain C_j and D_j. Since each $\mathcal{A}_j$ is specified by a frame $\mathcal{F}_j$ in W_j the apartment specified by $\mathcal{F}_1 \cup \mathcal{F}_2$, contains the chambers C and D. $\square$

Now let $[\Lambda] \in \mathcal{B}_e\mathrm{GL}(V)^0$. We define a function on $\mathcal{B}_e\mathrm{GL}(V)^0$ as follows:

$$\begin{aligned} d_p\colon \mathcal{B}_e(\mathrm{GL}(W_1 \oplus W_2))^0 &\to \mathbb{N} \\ [\Lambda] &\mapsto \log_{\mathcal{P}}[\mathrm{Ann}(Q(\Lambda))]. \end{aligned} \tag{29}$$

Here $\mathrm{Ann}(Q(\Lambda)) = \{x \in \mathcal{O} \mid xQ(\Lambda) = 0\}$ is the annihilator of $Q(\Lambda)$ in $\mathcal{O}$. The p subscript is because it turns out d_p is distance it takes to project $[\Lambda]$ onto

$\overline{\mathcal{A}_{W_1 \oplus W_2}}$. This follows from the fact $\overline{\mathcal{A}_{W_1 \oplus W_2}}$ is the orbit where $Q(\Lambda) = 0$. We have the following theorem.

Theorem 3.5. *Let* $[\Lambda] \in \mathcal{B}_e \mathrm{GL}(V)^0$ *then* $d_p([\Lambda]) = d_O(L[\Lambda], \overline{\mathcal{A}_{W_1 \oplus W_2}})$.

Proof. $d_O(L[\Lambda], \overline{\mathcal{A}_{W_1 \oplus W_2}}) = d_O(L[\Lambda], L[\Lambda^{(0)}])$, where $Q(\Lambda^{(0)}) = 0$. If $L[\Lambda]$ is the orbit associated $\alpha \in A^{n_1}$ then $d_O(L[\Lambda], L[\Lambda^{(0)}]) = \max(\alpha_i)$ for $1 \leq i \leq n_1$ and $\alpha_i \in \alpha$, but this is the same as $d_p([\Lambda])$. □

In the special case when $n_1 = n_2 = 1$, $\overline{\mathcal{A}_{W_1 \oplus W_2}}$ is just an apartment of $\mathcal{B}_e \mathrm{GL}(V)^0$. Then d_p is just measuring the distance of a given point to a fixed apartment. This suggests that one may be able to find the distance of a vertex to a fixed apartment by studying the action of a maximal split torus on the building.

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