Maximal Levi subgroups acting on the Euclidean building of $GL_n(F)$

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Abstract

In this paper we give a complete invariant of the action of $GL_n(F) \times GL_m(F)$ on the Euclidean building $B_eGL_{n+m}(F)$, where $F$ is a discrete valuation field. We then use this invariant to give a natural metric on the resulting quotient space. In the special case of the torus acting on the tree $B_eGL_2(F)$, we obtain an algorithm for calculating the distance of any vertex in the tree to any fixed apartment.

Keywords: affine building, Euclidean building, Levi subgroup, group action

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1. Introduction

To understand distance in the 1-skeleton of a building $BG$ associated to a reductive algebraic group $G$, one may look at a stabilizer $K$ of a point, and then study the action of $K$ on $BG$. When working over a discrete valuation field vertices correspond to maximal compact subgroups. This analysis gives rise to information about $K\backslash G/K$, and therefore the Hecke algebra [4, 5].

In this paper we specialize to $G = GL_n(F)$ and are interested in the double cosets $L\backslash G/K$, where $L \cong GL_{n_1}(F) \times GL_{n_2}(F)$ is a maximal Levi subgroup of $G$. The study of the action of $L$ on the building $B_eGL_n(F)$ will lead to a description of distance from any vertex to a certain subbuilding stabilized by $L$. In the case when $n = 2$ and $L = T$ is a maximal split torus, our description gives a way of calculating the distance from a given point to a fixed apartment.

We also give a combinatorial description of the quotient space $L\backslash B_eGL_n(F)$ as follows. Let $A^n = \{(\alpha_i)_{i=1}^n | \alpha_i \in \mathbb{N}, \alpha_i \geq \alpha_{i+1}\}$. If $n_1 \leq n_2$ there is an graph isometry between $L\backslash B_eGL_n(F)$ and $A^{n_1}$ where $A^n$ is endowed with the
following metric: \( d(\alpha, \beta) = \max_{i=1}^{n} |\alpha_i - \beta_i| \) where \( \alpha, \beta \in A^n \). This result shows that the 1-skeleton of the resulting quotient space only depends on \( \min(n_1, n_2) \).

This paper is broken up into two main sections. The first gives a description of the building in terms of \( O \)-lattices and describes an invariant of the action of \( L \) on this building. The second section gives a geometric interpretation of this invariant, yielding a combinatorial description of the quotient space \( L \backslash B_e \text{GL}_n(F) \).

2. Orbits of maximal Levi factors on \( B_e \text{GL}(V) \)

2.1. \( O \)-lattices and \( B_e \text{GL}(V) \)

Throughout this paper let \( F \) be a discrete valuation field with valuation \( \upsilon \). We will denote the ring of integers in \( F \) by \( O \), and fix once and for all a uniformizer \( \varpi \) of \( O \). Let the unique maximal prime ideal be denoted as \( \mathcal{P} = (\varpi) \), and the residue field \( O/\mathcal{P} \) will be denoted by \( k \). Let \( \mathcal{P}^k = (\varpi^k) \) for \( k \in \mathbb{Z} \).

Then \( \log_\mathcal{P}(\mathcal{P}^k) = k \). Let \( V \) be a finite dimensional vector space defined over \( F \) of dimension \( n \). We will describe the Euclidean building \( B_e \text{GL}(V) \) associated to \( \text{GL}(V) \). For more details see [1]. Let \( \Lambda \subset V \) be a finitely generated free \( O \)-module of rank \( n \). Denote by \([\Lambda]\) the homothety class of \( \Lambda \), that is \([\Lambda] = \{a\Lambda \mid a \in F^\times\}\).

Homothety classes of lattices will form the vertices of \( B_e \text{GL}(V) \). Two vertices \( \lambda_1, \lambda_2 \in B_e \text{GL}(V) \) are incident if there are representatives \( \lambda_i \in \lambda_i \) so that \( \varpi \lambda_1 \subset \lambda_2 \subset \lambda_1 \), i.e. \( \lambda_2/\varpi \lambda_1 \) is a \( t \)-subspace of \( \lambda_1/\varpi \lambda_1 \). The chambers in \( B_e \text{GL}(V) \) are collections of maximally incident vertices. To put this more concretely, a chamber is a collection of \( n \) vertices \( \lambda_0 \cdots \lambda_{n-1} \) with representatives \( \Lambda_0 \cdots \Lambda_{n-1} \) satisfying \( \varpi \Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_{n-1} \subsetneq \Lambda_0 \). A wall of a chamber is any subset of \( n-1 \) vertices in the given chamber. We will denote by \( B_e \text{GL}(V)^k \) the set of all facets of \( B_e \text{GL}(V) \) of dimension \( k \).

A frame \( F \) in \( V \) is a collection of lines \( l_1, \ldots, l_n \subset V \) which are linearly independent and span all of \( V \). We now describe certain subcomplexes of \( B_e \text{GL}(V) \). Define \( A_F \) to be the subcomplex consisting of vertices \([\Lambda]\) of the following form:

\[
\Lambda = \bigoplus_{i=1}^{n} O e_i
\]

where \( e_i \in l_i \in F \). Then \( A_F \) is an apartment of \( B_e \text{GL}(V) \), and every apartment is uniquely determined by a frame in this way.

The group \( \text{GL}(V) \) has a natural action of \( B_e \text{GL}(V) \), namely the one induced from the action of \( \text{GL}(V) \) on \( V \). This action preserves distance in the building.
A lemma which we will need later is the following.

**Lemma 2.1.** Let $\Lambda, \Lambda'$ be $O$-lattices of rank $n$ in $V$ with $\Lambda' \subset \Lambda$. Then the natural map from $GL(\Lambda) \cap \text{stab}(\Lambda')$ to $GL(\Lambda/\Lambda')$ is surjective.

**Proof.** This result appears to be well known, but the proof could not be found in the literature and so is given here. There is an $O$-basis $\{e_1, \ldots, e_n\}$ of $\Lambda$ so that $\{w^{k_1}e_1, \ldots, w^{k_n}e_n\}$ with $k_i \in \mathbb{N}$ is an $O$-basis of $\Lambda'$. This is equivalent to the statement that for any two vertices there is an apartment which contains them both. For $\bar{\sigma} \in GL(\Lambda/\Lambda')$ we will construct $\sigma \in GL(\Lambda) \cap \text{stab}(\Lambda')$ which descends to $\bar{\sigma}$.

Let $\bar{e}_i$ be the image of $e_i$ in $\Lambda/\Lambda'$. Then

$$\bar{\sigma}(\bar{e}_i) = a_1^i \bar{e}_1 + \cdots + a_n^i \bar{e}_n$$

(2)

where $a_j^i \in O$. Observe that $a_j^i$ is unique modulo $P^i_j$. Then define $\sigma$ on the $O$-basis $\{e_1, \ldots, e_n\}$ of $\Lambda$ as follows:

$$\sigma(e_i) = \begin{cases} \sum_{j=1}^{n} a_j^i e_j & \text{if } \bar{e}_i \neq 0, \\ e_i & \text{if } \bar{e}_i = 0. \end{cases}$$

(3)

What needs to be shown is that $\sigma$ is invertible and leaves $\Lambda'$ invariant.

First, we show $\sigma$ leaves $\Lambda'$ invariant.

$$0 = \bar{\sigma}(w^{k_i} \bar{e}_i) = a_1^i w^{k_i} \bar{e}_1 + \cdots + a_n^i w^{k_i} \bar{e}_n$$

(4)

This shows that $a_j^i w^{k_i} \in P^i_j$, and so $\sigma(w^{k_i} e_i) \in \Lambda'$.

Next we show invertibility. Let $\sigma^*$ be the construction given above for $\bar{\sigma}^{-1}$, and let $\tau = \sigma \circ \sigma^*$. This will be a function which is a lift of the identity map in $GL(\Lambda/\Lambda')$. Let $M = \text{span}_O(\bar{e}_i \mid \bar{e}_i \neq 0)$ and let $M' = \text{span}_O(e_i \mid \bar{e}_i = 0)$. Then $\tau|_M = \text{id} + E$ where $E \in \text{Hom}_O(M, \Lambda')$ and is id on $M'$. Any $\tau$ of this form is invertible and hence so is $\sigma$. \qed

### 2.2. $GL(W_1) \times GL(W_2)$ acting on $B_e(GL(W_1 \oplus W_2))$

Let $V$ be a vector space over $F$. Fix a maximal Levi subgroup $L$ of $GL(V)$. Associated to $L$ are subspaces $W_1, W_2 \subset V$ satisfying $V = W_1 \oplus W_2$. Then $L \cong GL(W_1) \times GL(W_2)$. In this section we will describe the orbits of the action of $GL(W_1) \times GL(W_2)$ on $B_e(GL(V))$ in terms of an invariant $Q$. Additionally we will give a representative of each orbit.

Let $p_i$ be the projection of $V$ onto $W_i$ with respect to our given decomposition. We will use these maps to define invariants of the vertices and then show for our action that these invariants classify all orbits.
Let $\Lambda$ be an $O$-lattice. We make the following definitions for $i = 1, 2$:

\begin{align}
P_i(\Lambda) &= \text{Im}(p_i|_{\Lambda}), \\
K_i(\Lambda) &= \text{Ker}(p_i'|_{\Lambda}) = \Lambda \cap W_i,
\end{align}

where $i' = (i \mod 2) + 1$.

These are lattices in $W_i$.

**Lemma 2.2.** $K_i(\Lambda) \subset P_i(\Lambda)$.

**Proof.** If $v \in K_i(\Lambda) = \Lambda \cap W_i$, then $v \in \Lambda$, so $p_i(v) \in P_i(\Lambda)$. But $p_i(v) = v$ since $v \in W_i$. □

By Lemma 2.2 we can define $Q_i(\Lambda) = P_i(\Lambda)/K_i(\Lambda)$. This is a finitely generated torsion $O$-module.

**Proposition 2.3.** $Q_1(\Lambda) \cong Q_2(\Lambda)$ as $O$-modules. This isomorphism class will be denoted by $Q(\Lambda)$.

**Proof.** We make slight modifications to the proof found in [2]. Let $p_i': \Lambda \to Q_i(\Lambda)$ be the composition of $p_i$ with the natural projection map $\pi_i: P_i(\Lambda) \to Q_i(\Lambda)$. We define a map so that for all $v \in \Lambda$

\begin{align}
\Theta: Q_1(\Lambda) &\to Q_2(\Lambda) \\
p_i'(v) &\mapsto p_2'(v). 
\end{align}

We will show that $\Theta$ is well defined, and is an isomorphism.

Let $w_1 + w_2, w'_1 + w'_2 \in \Lambda$ with $w_1, w'_1 \in W_1$ and $\pi_1(w_1) = \pi_1(w'_1)$. Then $\pi_1(w_1 - w'_1) = 0$, and therefore $w_1 - w'_1 \in K_1(\Lambda)$. Similarly $w_2 - w'_2 \in K_2(\Lambda)$ and $\pi_2(w_2) = \pi_2(w'_2)$ showing $\Theta$ is well defined. It is an isomorphism, because the map $\theta$, defined by reversing the roles of 1 and 2, is an inverse map. □

We now show that $Q$ is a complete invariant of the action of $L$ on $B_e\text{GL}(V)^0$.

**Theorem 2.4.** Let $\Lambda, \Lambda'$ be $O$-lattices. Then $\Lambda$ and $\Lambda'$ are in the same $GL(W_1) \times GL(W_2)$ orbit if and only if $Q(\Lambda) = Q(\Lambda')$.

**Proof.** The class $Q(\Lambda)$ is a $GL(W_1) \times GL(W_2)$-invariant since each factor of $GL(W_i)$ commutes with the projection map $p_i$. We must show that if $Q(\Lambda) = Q(\Lambda')$ then there is a $g \in GL(W_1) \times GL(W_2)$ so that $\Lambda = g\Lambda'$.

We will need $g_1 \in GL(W_1)$ and $g_2 \in GL(W_2)$ so that $g_1P_i(\Lambda') = P_i(\Lambda)$ and $g_iK_i(\Lambda') = K_i(\Lambda)$ for $i = 1, 2$. There are certainly $g_i \in GL(W_i)$ so that
$g_i P_i(\Lambda') = P_i(\Lambda)$. Then we may assume $K_i(\Lambda), K_i(\Lambda') \subset P_i(\Lambda)$. Since $Q(\Lambda) = Q(\Lambda')$ we know by the elementary divisor theorem there are bases

$$B_i = \{e_1, \ldots, e_{n_i}\} \text{ and } B_i' = \{e'_1, \ldots, e'_{n_i}\}$$

of $P_i(\Lambda)$ so that $K_i(\Lambda)$ written in terms of $B_i$ has the same elementary divisors as $K_i(\Lambda')$ written in terms of $B_i'$. Let $h_i \in \text{GL}(P_i(\Lambda))$ be the linear transformation which takes the basis $B_i$ to $B_i'$. Then $h_i g_i \in \text{GL}(W_i)$ has the desired properties.

So we may replace $\Lambda'$ with $\Lambda'' = (h_1 g_1, h_2 g_2) \Lambda'$. Let $\Theta$ be the map from Proposition 2.3 associated to $\Lambda$, and $\Theta''$ associated to $\Lambda''$.

We claim $\Lambda = \Lambda''$ if and only if $\Theta = \Theta''$. To prove this we show that one can reconstruct $\Lambda$ from $\Theta$ (which implicitly encodes $Q(\Lambda)$ as the domain and range of the map), by taking

$$\Lambda_\Theta = \{w_1 + w_2 \mid w_1 \in P_i(\Lambda) \text{ and } \Theta(\pi_1(w_1)) = \pi_2(w_2)\} \quad (8)$$

First, we show $\Lambda \subset \Lambda_\Theta$. Let $w = w_1 + w_2 \in \Lambda$, then by definition of $\Theta$ we have $\Theta(\pi_1(w_1)) = \pi_2(w_2)$. And so $v \in \Lambda_\Theta$. We now show $\Lambda_\Theta \subset \Lambda$. Let $w_1 + w_2 \in \Lambda_\Theta$. Then $w_1 \in P_i(\Lambda)$ so there is a $w'_2 \in P_2(\Lambda)$ so that $w_1 + w'_2 \in \Lambda \subset \Lambda_\Theta$. Then $0 + (w_2 - w'_2) \in \Lambda_\Theta$. So $\pi_2(w_2 - w'_2) = 0$ which implies $w_2 - w'_2 \in K_2(\Lambda) \subset \Lambda$. Hence $w_1 + w_2 = (w_1 + w'_2) + (w_2 - w'_2) \in \Lambda$ as desired.

To complete the theorem, we will show there is an element $g \in \text{stab}(P_2(\Lambda)) \cap \text{stab}(K_2(\Lambda))$ which takes $\Theta''$ to $\Theta$. There is an $\bar{h} \in \text{GL}(P_2(\Lambda)/K_2(\Lambda))$ so that $(1, \bar{h})\Theta'' = \Theta$. By Lemma 2.1 there is a pullback $h$ of $\bar{h}$ to $h \in \text{stab}(P_2(\Lambda)) \cap \text{stab}(K_2(\Lambda)) \subset \text{GL}(W_2)$ then $(1, h)\Lambda'' = \Lambda$.  

Now let $[\Lambda] \in B_e\text{GL}(V)^0$, and $c \in F^\times$. Since $Q(\Lambda) = Q(c\Lambda)$ we will abuse notation and write $Q([\Lambda]) = Q(\Lambda)$.

**Corollary 2.5.** $Q([\Lambda])$ is a complete invariant of the action of $\text{GL}(W_1) \times \text{GL}(W_2)$ on the space of vertices in $B_e(V)^0$.

### 2.3. Orbit representatives

We now give a set representatives of each orbit. We first do this in the case when $V$ is 2-dimensional, and then use this case to determine representatives for higher dimensions.

#### 2.3.1. $\dim(V) = 2$

Let $V$ be a two-dimensional vector space over $F$, with decomposition $V = W_1 \oplus W_2$. Assume that $W_i$ is spanned by the vector $e_i$. We then define the following
class of lattices:
\[ \Lambda^k = \text{span}_O(\omega^k e_1, e_1 + e_2). \] (9)

**Proposition 2.6.** \( Q([\Lambda^k]) \cong O/P^k. \)

**Proof.** We have \( P_1(\Lambda^k) = \langle e_1 \rangle \) and \( K_1(\Lambda^k) = \langle \pi^k e_1 \rangle. \) Therefore \( Q(\Lambda) \cong O/P^k. \) \( \square \)

**Corollary 2.7.** \( \{[\Lambda^k]\}_{k=0}^{\infty} \) is a complete set of representatives for the action of \( \text{GL}(W_1) \times \text{GL}(W_2) \) on \( B_e \text{GL}(V)^0. \)

**Proof.** Let \( [\Lambda] \in B_e \text{GL}(V)^0. \) Then \( Q([\Lambda]) \cong O/P^k \) for some \( k \in \mathbb{N}. \) By Theorem 2.4, \( [\Lambda] \) is in the orbit of \( \Lambda^k. \) \( \square \)

### 2.3.2. General \( V \)

We now describe representatives when \( V \) is \( n \)-dimensional. We may assume that \( \dim W_i = n_i \) and \( n_1 \le n_2. \) Choose a basis \( \{e_1, \ldots, e_{n_1}\} \) of \( W_1 \) and \( \{f_1, \ldots, f_{n_2}\} \) of \( W_2, \) and let \( Y_i = \text{span}_P(e_i, f_i), \) for \( 1 \le i \le n_1. \) Let \( \alpha = (\alpha_i) \in \mathbb{N}^{n_1}. \) Let \( [\Lambda^\alpha] \in B_e \text{GL}(Y_i) \) defined as in equation (9) with respect to the basis \( \{e_i, f_i\}. \) This allows us to define the following class of lattices:
\[ \Lambda^\alpha = \bigoplus_{i=1}^{n_1} A^\alpha_i \bigoplus_{i=n_1+1}^{n_2} O f_i \] (10)

**Proposition 2.8.** Let \( A^n = \{\alpha = (\alpha_i) \in \mathbb{N}^n \mid \alpha_i \ge \alpha_{i+1}\}. \) Then \( [\Lambda^\alpha]_{\alpha \in A^n} \) is a complete set of representatives of the orbits of \( \text{GL}(W_1) \times \text{GL}(W_2) \) acting on \( B_e \text{GL}(V)^0. \)

**Proof.** By the elementary divisor theorem \( Q_1(\Lambda) \) decomposes into a direct sum of \( O \)-modules as follows: \( Q_1([\Lambda]) \cong O^r \bigoplus_{i=1}^{n_1} O/P^\alpha_i \) where \( \alpha_i \in \mathbb{N} \) and \( r \in \mathbb{N}. \) However, \( r = 0 \) since both \( P_1(\Lambda) \) and \( K_1(\Lambda) \) are rank \( n_1. \) We may assume \( \alpha_i \ge \alpha_{i+1}. \) Then by Theorem 2.4, \( [\Lambda] \) is in the same orbit as \( [\Lambda^\alpha]. \) \( \square \)

### 2.3.3. Double cosets

The description of orbits is equivalent to the space of double cosets \( L \backslash \text{GL}(V)/K, \) where \( K \) is the stabilizer of a vertex in \( B_e \text{GL}(V). \) We now give an explicit description of a set of double coset representatives.

The Levi subgroup \( L \) is associated to a parabolic subgroup \( P \) with a decomposition \( P = LN, \) where \( N \) is the unipotent radical of \( P. \) The Iwasawa decomposition shows that \( \text{GL}(V) = PK, \) and so we may choose the double coset representatives of \( L \backslash \text{GL}(V)/K \) to be in \( N. \)
We use the basis for $V$ of the previous section to identify $GL(V)$ with $GL_n(F)$. We will also let $K = Z(GL_n(F))GL_n(O)$. Then $N \cong M_{n_1 \times n_2}(F)$, the $n_1 \times n_2$ matrices embedded in $GL_n(F)$ as follows:

$$u : M_{n_1 \times n_2}(F) \rightarrow N$$

$$B \mapsto \begin{pmatrix} I_{n_1} & B \\ 0 & I_{n_2} \end{pmatrix}.$$  

Let $\alpha \in A^{n_1}$ and define $m^\alpha \in M_{n_1 \times n_2}$ as follows:

$$[m^\alpha]_{ij} = \begin{cases} \varpi^{-\alpha_i} & \text{if } i = j \in \{1, \ldots, n_1\}, \\ 0 & \text{else}. \end{cases}$$  \hspace{1cm} (11)

Now let $n^\alpha = u(m^\alpha)$. Then we have the following proposition.

**Proposition 2.9.** We may write $GL_n(F)$ as a disjoint union

$$GL_n(F) = \coprod_{\alpha \in A^{n_1}} Ln^\alpha K.$$

**Proof.** Let $\alpha \in A^{n_1}$, and define $l^\alpha$ to be the linear transformation that sends $e_i$ to $e_i$ and $f_i$ to $\varpi^{-\alpha_i}f_i$ for $1 \leq i \leq n_1$, and $f_j$ to $f_j$ for $n_1 + 1 \leq j \leq n_2$. Note that $l^\alpha \in L$.

Let $\Lambda = \text{span}_O(e_1, \ldots, e_{n_1}, f_1, \ldots, f_{n_2})$, and notice that $K$ stabilizes $[\Lambda]$. Furthermore, we have $l^\alpha n^\alpha(\Lambda) = \Lambda^\alpha$. \hfill \Box

This double coset decomposition is in no way canonical, although it has some nice properties. All the $n^\alpha$ are supported on the span of root groups $U^i i + n_2$ for $1 \leq i \leq n_1$, with the roots taken with respect to the diagonal torus. In fact, these root group form a set of maximally mutually orthogonal root groups in $N$. Any such set of root groups can be a support of coset representatives. This can easily be seen by having $W_n$, the Weyl groups of $GL_n$, act on the $n^\alpha$. This leads to the following conjecture for more general groups.

**Conjecture 2.10.** Let $G$ be a reductive group over $F$ and $P$ a parabolic subgroup with $P = LN$, and assume $N$ is abelian. Let $K$ be a maximal open, bounded subgroup of $G$. Then there is a discrete subset $N' \subset N$ and a maximally mutually orthogonal set of root groups $U^\alpha < N$ so that:

1. each $n \in N'$ is supported in the group generated by the $U^\alpha$;
2. $G = \coprod_{n \in N'} LnZ(G)K$. 
2.3.4. Stabilizers

We now wish to compute stabilizers for each orbit so that we may realize the orbits as homogeneous spaces. For spherical buildings knowing the stabilizers plays a role in representation theory, for instance \cite{3}. For Euclidean buildings this may have applications to understanding cuspidal representations.

Fix a $\Lambda$ and let $S_i = \text{stab}(P_i(\Lambda)) \cap \text{stab}(K_i(\Lambda))$. Furthermore, let

$$T_i = \{ I + A \mid A \in \text{End}(W_i) \text{ and } A(P_i(\Lambda)) \subset K_i(\Lambda) \} \cap S_i. \tag{12}$$

Then $T_i \triangleleft S_i$ and $S_i/T_i \cong \text{GL}(Q_i(\Lambda))$ by Lemma 2.1. Let

$$S_\Lambda = \{ (h_1, \Theta^*_\Lambda(h_1)) \mid h_1 \in \text{GL}(Q_1(\Lambda)) \} \subset (\text{GL}(Q_1(\Lambda)) \times \text{GL}(Q_2(\Lambda)) \tag{13}$$

where $\Theta^*_\Lambda$ is the isomorphism induced on $\text{GL}(Q_1(\Lambda))$ from the isomorphism $\Theta_\Lambda: Q_1(\Lambda) \to Q_2(\Lambda)$ defined in equation (7). Finally, let $S_\Lambda$ be the pullback of $S_\Lambda$ in $S_1 \times S_2$.

**Proposition 2.11.** $S_\Lambda = \text{stab}_L(\Lambda)$.

**Proof.** Let $(A_1, A_2) \in S_\Lambda$ with $A_i \in \text{GL}(W_i)$, and let $\Lambda' = (A_1, A_2) \cdot \Lambda$. Then because $A_i \in S_i$ we have $P_i(\Lambda) = P_i(\Lambda')$ and $K_i(\Lambda) = K_i(\Lambda')$. We now wish to show $\Theta_\Lambda = \Theta_{\Lambda'}$. By the proof of Theorem 2.4 this will show that $\Lambda = \Lambda'$. Let $B_i$ be the image of $A_i$ in $S_i/T_i \cong \text{GL}(Q_i(\Lambda))$, and let $v \in Q_1(\Lambda)$ and $v' = B_1^{-1}v$. Then

$$\Theta_{\Lambda'}(v) = \Theta_{\Lambda'}(B_1 B_1^{-1}v) \tag{14}$$

$$= B_2 \Theta_{\Lambda}(B_1^{-1}v) \tag{15}$$

$$= \Theta^*_\Lambda(B_1) \Theta_{\Lambda}(B_1^{-1}v) \tag{16}$$

$$= \Theta_{\Lambda}(B_1 \Theta^{-1}_{\Lambda}(\Theta_{\Lambda}(B_1^{-1}v))) \tag{17}$$

$$= \Theta_{\Lambda}(v). \tag{18}$$

Line (15) follows from the action of $(A_1, A_2)$ on $\Lambda$, line (16) comes from the fact that $B_2 = \Theta^*_\Lambda(B_1)$, and line (17) is the definition of the induced map $\Theta^*_\Lambda$.

This proves $S_\Lambda \subset \text{stab}_L(\Lambda)$. We now prove the other direction. Assume $(A_1, A_2) \in \text{stab}_L(\Lambda)$, then $A_i \in S_i$. The calculation above shows that the projection of $A_2$ in $\text{GL}(Q_2(\Lambda))$ has to equal the image of $A_1$ in $\text{GL}(Q_2(\Lambda))$ under $\Theta^*_\Lambda$, proving the result. \hfill $\Box$

We end this section by giving an explicit description of $S_\Lambda^\alpha$, the stabilizers of our orbit representatives. Let $\alpha \in A^n$, then we define $\Lambda^\alpha$ as in section 2.3. By
this definition $P_1(\Lambda^\alpha) = \text{span}_O\langle e_1, \ldots, e_{n_1} \rangle$ and $P_2(\Lambda^\alpha) = \text{span}_O\langle f_1, \ldots, f_{n_2} \rangle$. Also,

$$K_1(\Lambda^\alpha) = \text{span}_O\langle \omega^{\alpha_1} e_1, \ldots, \omega^{\alpha_{n_1}} e_{n_1} \rangle,$$

and

$$K_2(\Lambda^\alpha) = \text{span}_O\langle \omega^{\alpha_1} f_1, \ldots, \omega^{\alpha_{n_2}} f_{n_2} \rangle,$$

where $\alpha_j = 0$ if $j > n_1$. Then $S_i$ looks like

$$S_i = \begin{pmatrix} p_{\beta_{11}} & p_{\beta_{12}} & p_{\beta_{13}} & \cdots & p_{\beta_{1n_1}} \\ p_{\beta_{21}} & p_{\beta_{22}} & p_{\beta_{23}} & \cdots & p_{\beta_{2n_1}} \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ p_{\beta_{n_11}} & p_{\beta_{n_12}} & p_{\beta_{n_13}} & \cdots & p_{\beta_{n_in_1}} \end{pmatrix} \cap \text{GL}_{n_i}(O)$$

(19)

where $\beta_{ij} = \max(0, \alpha_i - \alpha_j)$.

Also, $T_i$ looks like

$$T_i = \begin{pmatrix} u_{\alpha_1} & p_{\alpha_1} & p_{\alpha_1} & \cdots & p_{\alpha_1} \\ p_{\alpha_2} & u_{\alpha_1} & p_{\alpha_2} & \cdots & p_{\alpha_2} \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ p_{\alpha_n} & p_{\alpha_n} & p_{\alpha_n} & \cdots & u_{\alpha_n} \end{pmatrix}$$

(20)

where $u_k = 1 + p_k$ if $k \geq 1$ and $u_0 = O^\times$.

The other component to Proposition 2.11 has to do with the map $\Theta_{\Lambda}$. For $\Lambda^\alpha$ there is a life of this map $\Theta_{\Lambda} : P_1(\Lambda^\alpha) \to P_2(\Lambda^\alpha)$ which is independent of $\alpha$, and is given by $\Theta_{\Lambda}(e_i) = f_i$ for $1 \leq i \leq n_1$. So by Theorem 2.11, $S_{\Lambda^\alpha}$ is the product of the group

$$\left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{n_2-n_1} \end{pmatrix} \mid A \in S_1 \right\}$$

(21)

with the group $T_1 \times T_2$ (embedded block diagonally into $\text{GL}_{n_1+n_2}(F)$).

3. Geometric interpretation of $Q$

3.1. Distance between orbits

The main result of section 2.2 gives an invariant $Q$ of the action of $L = \text{GL}(W_1) \times \text{GL}(W_2)$ acting on $B_0 \text{GL}(W_1 \oplus W_2)^0$. In this section we give a geometric interpretation of this invariant in terms of a distance between orbits.
By Proposition 2.8 we may identify the space of orbits $L \setminus B_v \text{GL}(V)$ with $A^{n_1}$. We define a function called the orbital distance as follows:

$$d_O : A^{n_1} \times A^{n_1} \to \mathbb{N}$$

$$(\alpha, \beta) \mapsto \max_{i=1, \ldots, n_1} |\alpha_i - \beta_i|.$$  \hspace{1cm} (22)

The main result of this section is that the name “orbital distance” is justified; that is, $d_O$ is actually the minimum distance between two orbits as measured in the 1-skeleton of the building $B_v \text{GL}(V)$.

For simplicity if $[\Lambda] \in B_v(V)$ then let $L[\Lambda]$ denote the orbit of $[\Lambda]$ under $L$.

**Proposition 3.1.** Let $[\Lambda_1], [\Lambda_2] \in B_v \text{GL}(V)$ be incident, then

$$d_O(L[\Lambda_1], L[\Lambda_2]) \leq 1.$$

*Proof.* Let $[\Lambda_1], [\Lambda_2]$ be two incident vertices with $\varpi \Lambda_1 \subset \Lambda_2 \subset \Lambda_1$. Let $L[\Lambda_1]$ be identified with $\alpha \in A^{n_1}$ and $L[\Lambda_2]$ with $\beta \in A^{n_1}$. We have

$$\varpi P_i(\Lambda_1) \subset P_i(\Lambda_2) \subset P_i(\Lambda_1),$$

$$\varpi K_i(\Lambda_1) \subset K_i(\Lambda_2) \subset K_i(\Lambda_1).$$  \hspace{1cm} (23) (24)

There are two extreme cases. First $P_1(\Lambda_2) = P_1(\Lambda_1)$ and $K_1(\Lambda_2) = \varpi K_1(\Lambda_1)$. In this case $\beta_i = \alpha_i + 1$ for all $i \in \{1, \ldots, n_1\}$.

In the second case $P_1(\Lambda_2) = \varpi P_1(\Lambda_1)$, and $K_1(\Lambda_2) = K_1(\Lambda_1) \cap \varpi P_1(\Lambda_1) \supset \varpi K_1(\Lambda_1)$. In this case $\alpha_i = \beta_i + 1$ or $\alpha_i = \beta_i$.

The above argument shows that no matter what $P_1(\Lambda_2)$ and $K_1(\Lambda_2)$ are we have $|\alpha_i - \beta_i| \leq 1$ as desired. \hfill \Box

Proposition 3.1 shows that if two incident vertices are in different orbits, then their $L$-orbits have orbital distance 1. To show $d_O$ is actually the proposed metric we need to show if two orbits have orbital distance 1, then there are incident representatives of each orbit. The following technical lemma proves this.

**Lemma 3.2.** Let $[\Lambda_1], [\Lambda_2] \in B_v \text{GL}(V)$. Assume $d_O(L[\Lambda_1], L[\Lambda_2]) = k > 0$. Then there is an $[\Lambda_3] \in B_v \text{GL}(V)$ incident to $[\Lambda_2]$ so that $d_O(L[\Lambda_1], L[\Lambda_3]) = k - 1$.

*Proof.* Let $[\Lambda_1], [\Lambda_2]$ be as in the statement of the lemma. Since we are working in $L$-orbits, and $L$ preserves distance in $B_v \text{GL}(V)$ we may choose any representatives for $[\Lambda_1]$ and $[\Lambda_2]$ that we like. In particular if $L[\Lambda_1], L[\Lambda_2]$ are identified with $\alpha, \beta \in A^{n_1}$ respectively, we may take for our representatives $\Lambda^\alpha, \Lambda^\beta$ respectively, as defined in Proposition 2.8.
Recall that if $W_1$ has basis $\{e_i\}_{i=1}^{n_1}$ and $W_2$ has basis $\{f_i\}_{i=1}^{n_2}$ then $\Lambda^\alpha = \bigoplus_{i=1}^{n_1} \Lambda^\alpha_i \bigoplus_{i=n_1+1}^{n_2} \mathcal{O}_f_i$ where $\Lambda^\alpha_i = \langle \varpi^\alpha, e_i, e_i + f_i \rangle$.

To find a $[\Lambda_3]$ with the desired property we need to show there exists $\gamma \in A^{n_1}$ so that $d_O(\alpha, \gamma) = k - 1$ and $d_O(\beta, \gamma) = 1$. To do this, we define $\gamma = (\gamma_i)$ where

$$\gamma_i = \begin{cases} 
\beta_i + 1 & \text{if } \alpha_i - \beta_i = k, \\
\beta_i - 1 & \text{if } \beta_i - \alpha_i = k, \\
\beta_i & \text{else.}
\end{cases}$$

Let $S = \{i \mid \beta_i - \alpha_i = k\}$. We now define $\Lambda_3$ as follows:

$$\Lambda_3 = \bigoplus_{i \in S} \varpi \Lambda^\gamma_i \quad \bigoplus_{i \in \{1, \ldots, n_1\} \setminus S} \Lambda^\gamma_i \quad \bigoplus_{i=n_1+1}^{n_2} \mathcal{O}_f_i. \quad (25)$$

By construction $d_O(L[\Lambda^\alpha], L[\Lambda_3]) = k - 1$. So all we need to show is that $[\Lambda^\beta]$ and $[\Lambda_3]$ are incident. This follows from the two-dimensional case and the fact that

$$\Lambda^k \supset \Lambda^{k+1} \supset \varpi \Lambda^k \quad (26)$$

and that

$$\Lambda^k \supset \varpi \Lambda^{k-1} \supset \varpi \Lambda^k. \quad (27)$$

Together Proposition 3.1 and Lemma 3.2 give us the following theorem.

**Theorem 3.3.** Let $[\Lambda_1], [\Lambda_2] \in B_\epsilon GL(V)^0$. Then $d_O(L[\Lambda_1], L[\Lambda_2])$ is the minimal distance between any two representatives of the orbits as measured in the 1-skeleton of $B_\epsilon GL(V)^0$.

Theorem 3.3 gives a complete combinatorial description of the geometry of the orbit space $L B_\epsilon GL(V)^0$. Figure 1 on the next page is the quotient space for $L B_\epsilon GL(V)$ when $V$ is 4-dimensional and $n_1 = n_2 = 2$.

### 3.2. Distance to $\mathcal{A}_{\mathcal{F}_1 \cup \mathcal{F}_2}$ in $B_\epsilon (GL(W_1 \oplus W_2))$

There is an important special case of Theorem 3.3. The orbit for which $Q(\Lambda) = 0$ is distinguished. In this section we give both a description of this orbit, as well as another description of the distance from a given point to this orbit.

Recall from section 1 that an apartment $\mathcal{A}_\mathcal{F}$ is specified by a frame $\mathcal{F}$ in $W_1 \oplus W_2$. Denote by Frame($V$) the set of all frames in a vector space $V$. We will be interested in the following collection of apartments:

$$\mathcal{A}_{W_1 \oplus W_2} = \bigcup_{\mathcal{F}_1 \in \text{Frame}(W_1)} \mathcal{A}_{\mathcal{F}_1 \cup \mathcal{F}_2}. \quad (28)$$
Figure 1: The quotient space for $L\backslash B_eGL(V)$ ($\dim V = 4$, $n_1 = n_2 = 2$).

**Proposition 3.4.** $\mathcal{A}_{W_1 \oplus W_2}$ is a subbuilding of $B_eGL(V)$.

**Proof.** Since $\mathcal{A}_{W_1 \oplus W_2}$ is a union of apartments from an actual building all that needs to be shown is that any two chambers $C_1, C_2 \in \mathcal{A}_{W_1 \oplus W_2}$ are in a common apartment. Let $\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset \varnothing \Lambda_1$ be a chain of $\mathcal{O}$-lattices corresponding to a chamber $C \in \mathcal{A}_{W_1 \oplus W_2}$, and $M_1 \supset M_2 \supset \cdots \supset M_n \supset \varnothing M_1$ a chain of lattices corresponding to a chamber $D \in \mathcal{A}_{W_1 \oplus W_2}$. Since each $[\Lambda_i] \in \mathcal{A}_{W_1 \oplus W_2}$ we can write $\Lambda_i = \Lambda_i^1 \oplus \Lambda_i^2$ with $[\Lambda_i^1] \in B_e(GL(W_j))$. Similarly for the $M_i$. The $\{[\Lambda_i^1]\}_{i=1}^{n}, \{[M_i^1]\}_{i=1}^{n}$ specify facets $C_j, D_j \in B_e(GL(W_j))$ since $\Lambda_i^1 \supset \Lambda_i^2 \supset \varnothing \Lambda_i^1$ (it will be the case that some of the $\Lambda_i^1 = \Lambda_{i+1}^1$ but this will not matter), and similarly for the $M_i^1$. Then there are common apartments $A_j \subset B_eGL(W_j)$ which contain $C_j$ and $D_j$. Since each $A_j$ is specified by a frame $F_j$ in $W_j$ the apartment specified by $F_1 \cup F_2$, contains the chambers $C$ and $D$. □

Now let $[\Lambda] \in B_eGL(V)^0$. We define a function on $B_eGL(V)^0$ as follows:

$$d_p: B_e(GL(W_1 \oplus W_2))^0 \to \mathbb{N}$$

$$[\Lambda] \mapsto \log_p [\text{Ann}(Q(\Lambda))].$$

(29)

Here $\text{Ann}(Q(\Lambda)) = \{x \in \mathcal{O} \mid xQ(\Lambda) = 0\}$ is the annihilator of $Q(\Lambda)$ in $\mathcal{O}$. The $p$ subscript is because it turns out $d_p$ is distance it takes to project $[\Lambda]$ onto
$A_{W_1⊕W_2}$. This follows from the fact $A_{W_1⊕W_2}$ is the orbit where $Q(Λ) = 0$. We have the following theorem.

**Theorem 3.5.** Let $[Λ] ∈ B_e \text{GL}(V)^0$ then $d_p([Λ]) = d_O(L[Λ], \overline{A_{W_1⊕W_2}})$.

**Proof.** $d_O(L[Λ], \overline{A_{W_1⊕W_2}}) = d_O(L[Λ], L[Λ^{(0)})], where Q(Λ^{(0)}) = 0$. If $L[Λ]$ is the orbit associated $α ∈ A_{n_1}$ then $d_O(L[Λ], L[Λ^{(0)})] = max(α_i)$ for $1 ≤ i ≤ n_1$ and $α_i ∈ α$, but this is the same as $d_p([Λ])$. □

In the special case when $n_1 = n_2 = 1$, $A_{W_1⊕W_2}$ is just an apartment of $B_e \text{GL}(V)^0$. Then $d_p$ is just measuring the distance of a given point to a fixed apartment. This suggests that one may be able to find the distance of a vertex to a fixed apartment by studying the action of a maximal split torus on the building.

**References**


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