Parallelisms of quadric sets

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Abstract

In this article, it is shown that every flock of a hyperbolic quadric H and every flock of a quadratic cone C in $\mathsf{PG}(3, K)$, for K a field, is in a transitive parallelism of H or C, respectively. Furthermore, it is shown it is possible to have parallelisms of quadratic cones by maximal partial flocks. The theory of parallelisms of quadratic cones is generalized to analogous results for parallelisms of α-cones.

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1. Introduction

Let K be a field and consider a flock of a quadratic cone, an elliptic quadric or a hyperbolic quadric in $\mathsf{PG}(3, K)$. This article considers whether there are "parallelisms" of these quadric sets, which in each case means a set of mutually disjoint flocks, whose union is a complete cover of the set of conics of plane intersections of the quadric set in question.

Consider first a "hyperbolic parallelism" as a union of mutually disjoint hyperbolic flocks, whose union is the set of all conics that are sections of a hyperbolic quadric.

When K is isomorphic to $\mathsf{GF}(q)$, there are $(q^4 - 1)/(q - 1)$ planes in $\mathsf{PG}(3, q)$, and there are $2(q+1)$ lines that lie in the hyperbolic quadric. Each of these lines lies in $q+1$ planes, none of which can be associated with a hyperbolic flock and each plane that contains a line of a ruling also contains a line of the opposite ruling. Hence, there are $q^3+q^2+q+1-(q+1)^2 = q(q^2-1)$ planes that intersect a hyperbolic quadric in a conic. Since a hyperbolic flock contains $q + 1$ planes, we would need $q(q - 1)$ hyperbolic flocks in a finite hyperbolic parallelism.

Consider the putative associated translation planes. The Thas/Bader–Lunardon Classification Theorem [9, 2] gives a complete classification of the corresponding spreads, and it turns out that all are nearfield spreads. Of course, we have the Desarguesian spread, and the regular nearfield spreads, but there are three irregular nearfield spreads that appear on the list of possibilities as well. These three irregular nearfield planes are of orders 11^2, 23^2 and 59^2 and they admit homology groups of order $q-1$. This fact was independently discovered by Bader [1] and Johnson [7], and for orders 11^2 and 23^2 by Baker and Ebert [3].

1.1. The Thas/Bader–Lunardon Theorem

Theorem 1.1. *A flock of a hyperbolic quadric in* $\mathsf{PG}(3,q)$ *is either*

(1) *linear, corresponding to the Desarguesian affine plane,*

(2) *a Thas flock, corresponding to the regular nearfield planes, or*

(3) *a Bader–Baker–Ebert–Johnson flock of order* p^2*, corresponding to the irregular nearfield planes of orders* 11^2*,* 23^2 *or* 59^2*.*

As noted by Bonisoli [5], using the Thas/Bader–Lunardon Theorem, it is possible to see that every flock of a finite hyperbolic quadric lies in a transitive parallelism, as every sharply 1-transitive set constructing an associated nearfield translation plane is a coset of a sharply 1-transitive group. Furthermore, Bonisoli also noted that if K is a field that admits a quadratic extension, there are parallelisms of the hyperbolic quadric in $\mathsf{PG}(3,K)$, whose corresponding translation planes are Pappian.

So, the question in the infinite or general case is whether there are non-nearfield translation planes corresponding to flocks of hyperbolic quadrics (in fact, there are, see Johnson [7]) but the bigger question is whether all flocks lead to parallelisms as they do in the finite case.

Now consider an elliptic quadric in $\mathsf{PG}(3,q)$. Since there are $1+q+q^2+q^3$ planes and there are exactly $1+q^2$ tangent planes, there are q^3+q conics of plane intersection, and an elliptic flock requires $q-1$ conics. So, there cannot be finite parallelisms of elliptic quadrics. However, Betten and Riesinger [4], have developed the concept of a covering of the elliptic quadric $\mathcal{Q}$ by a set of 2-secants. A “generalized line star” is a set $\mathcal{S}$ of lines of $\mathsf{PG}(3,K)$ that each intersect $\mathcal{Q}$ in exactly two points, such that the non-interior points of $\mathcal{Q}$ are covered by the set $\mathcal{S}$. Using the duality $\perp$ induced on $\mathsf{PG}(3,K)$ by the quadric, every 2-secant ℓ maps to an exterior line $\ell^\perp$ to $\mathcal{Q}$. Now take the set of planes containing $\ell^\perp$, which defines a linear elliptic flock. Using the Klein mapping, we obtain a Pappian spread Σ corresponding to $\mathcal{Q}$ and the linear flock corresponds to the

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Pappian spread obtained from Σ by the replacement of a set of mutually disjoint reguli that covers Σ with the exception of two components. Furthermore, Betten and Riesinger show that generalized line stars produce regular parallelisms of $\mathsf{PG}(3,K)$, which are coverings of the line set of $\mathsf{PG}(3,K)$ by Pappian (regular) spreads. Now choose any conic $\mathcal{C}$ of intersection of $\mathcal{Q}$ and let $\pi_{\mathcal{C}}$ denote the corresponding plane containing $\mathcal{C}$ and form $\pi_{\mathcal{C}}^{\perp} = P_{\mathcal{C}}$. Since $P_{\mathcal{C}}$ is exterior to $\mathcal{Q}$, there is a unique 2-secant $\ell_{P_{\mathcal{C}}}$ containing $P_{\mathcal{C}}$ of the generalized line star. Then $\ell_{P_{\mathcal{C}}}^{\perp} \subset \pi_{\mathcal{C}}$ so that $\mathcal{C}$ corresponds to one of the lines of a spread of the parallelism. All of this is noted in Betten and Riesinger [4], but not using the language of parallelisms, assuming that the characteristic of K is not 2. Also, note that the existence of Pappian spreads in $\mathsf{PG}(3,K)$, require the existence of a quadratic extension F of K. We restate this theorem using our language.

Theorem 1.2 (Betten and Riesinger [4]). *Let K be field of characteristic not 2 that admits a quadratic extension F. Every regular parallelism of $\mathsf{PG}(3,K)$ arising from a generalized line star of an elliptic quadric $\mathcal{Q}$ also produces an elliptic parallelism of $\mathcal{Q}$.*

Finally, consider a quadratic cone $\mathcal{C}$ in $\mathsf{PG}(3,K)$, for K a field. First consider K isomorphic to $\mathsf{GF}(q)$. As the number of planes that contain the vertex is $1+q+q^2$, there are $1+q+q^2+q^3-(1+q+q^2) = q^3$ conics of intersection. Since a flock of a quadratic cone (covering all points with the exception of the vertex) has q planes of intersection, we would require q^2 mutually disjoint flocks to produce a parallelism.

Surprisingly, we are able to show that all hyperbolic flocks and all conical flocks of $\mathsf{PG}(3,K)$, for K a field admitting a quadratic extension F may be embedded into a set of flocks that define a parallelism. In both cases, there are groups involved that essentially imply that there is a “transitive parallelism” in either case. We also show that it is possible to have parallelisms of quadratic cones by maximal partial flocks. Finally, we will extend the theory of parallelisms of quadratic cones to α-cones.

2. Conical parallelisms

The reader is directed to Johnson [8] for background on conical flocks and hyperbolic flocks.

Let K be a field and let a quadratic cone $\mathcal{C}$ be defined by $x_0x_1 = x_2^2$ with vertex $(0,0,0,1)$.

Definition 2.1. A “parallelism” of $\mathcal{C}$ is a partition of the conics of intersection with planes not containing the vertex by conical flocks.

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While it may seem that parallelisms would be difficult to obtain, we show, in fact, that these are readily available. First we make a few remarks. A flock of a quadratic cone is a partition of the non-vertex points by a set of plane intersections and since each of these intersections is determined by a unique plane, we may view the flock as the set of these planes. We know that to each flock of a quadratic cone there is a corresponding translation plane. Hence, a parallelism of a quadratic cone can be determined by a set of spreads of the form

$$x = 0,\ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix}; u, t \in K,$$

if and only if the flock is

$$x_0 t - x_1 f(t) + x_2 g(t) + x_3 = 0 \text{ for all } t \in K,$$

when representing the cone as $x_0 x_1 = x_2^2$, with vertex $(0, 0, 0, 1)$, where initially we may assume that $f(0) = g(0) = 0$. However, we see that we have a flock covering the points $(z_2^2, 1, z_2, \delta)$, $(1, 0, 0, \rho)$ for $\delta, \rho \in K$, if and only if $tz_2^2 - f(t) + z_2 g(t)$ is bijective for all z_2. Now notice that a function

$$\phi_u \colon t \mapsto tu^2 - f(t) + ug(t)$$

is bijective if and only if

$$\phi_u^{a,b} \colon t \mapsto tu^2 - (f(t) + a) + u(g(t) + b),$$

is bijective for all $a, b \in K$. Hence, if the functions $(f(t), g(t))$, produce a conical flock where we have taken $f(0) = g(0) = 0$, then the functions $(f(t)+a, g(t)+b)$, produce a conical flock for all $a, b \in K$.

Therefore, if $\mathcal{F}$ is a flock, we consider the associated translation plane $\pi_{\mathcal{F}}$ and a parallelism is equivalent to a set of q^2 translation planes that do not share any of their regulus nets. We note that any linear mapping of the associated 4-dimensional vector space will map a conical translation plane to another conical translation plane, isomorphic to it. Thus,

$$\tau_{a,b} = \begin{bmatrix} I & \left[\begin{smallmatrix} b & a \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & I \end{bmatrix}$$

will map

$$x = 0,\ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix}; u, t \in K,$$

to

$$x = 0,\ y = x \begin{bmatrix} u + g(t) + b & f(t) + a \\ t & u \end{bmatrix}; u, t \in K,$$

so the functions $(f(t), g(t))$ defining the flock become $(f(t)+a, g(t)+b)$. We note translation planes map to translation planes under $\tau_{a,b}$, which means that flocks map to flocks under $\tau_{a,b}$.

Theorem 2.2. *Every conical flock is in a transitive parallelism.*

Proof. We represent the planes $\pi_{a,b,c}$, for $a, b, c \in K$, that do not contain the vertex in the form

$$x_0 c + x_1 b + x_2 a + x_3 = 0.$$

The planes of a conical flock are then represented in the form

$$x_0 t - x_1 f(t) + x_2 g(t) + x_3 = 0,$$

for all $t \in K$. We also have an associated conical translation plane π with spread

$$x = 0,\ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix}; u, t \in K.$$

Now consider the group

$$G = \left\langle \tau_{a,b} = \begin{bmatrix} I & \left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & I \end{bmatrix}; a, b \in K \right\rangle.$$

We note that $\pi\tau_{a,b}$ is a conical plane isomorphic to π. Therefore, there is a corresponding conical flock. Note that $\pi\tau_{a,b}$ has the following spread

$$x = 0,\ y = x \begin{bmatrix} u + g(t) + a & f(t) + b \\ t & u \end{bmatrix}; u, t \in K.$$

Clearly, none of the associated derivable nets in $\pi\tau_{a,b}$ can be equal to the derivable nets of π. Therefore, none of the planes

$$x_0 t - x_1(f(t) + b) + x_2(g(t) + a) + x_3 = 0,$$

are equal to any of the planes of the flock, and also the associated image is also a flock. Since all planes that do not contain the vertex have the form

$$x_0 c + x_1 b + x_2 a + x_3 = 0,$$

we see that we have partitioned the conics by the flocks associated with πG. Hence, each conical flock belongs to a transitive parallelism. $\square$

Remark 2.3. The group considered in the transitive parallelism is not a group of the cone. However, since we are considering flocks as a set of planes, whose intersections with the cone partition the non-vertex points, the flocks as sets of planes can map to other flocks and therefore conics of intersection map to conics of intersection, but not necessarily of the same quadratic cone.

3. Parallelisms of hyperbolic quadrics

The reader is again directed to Johnson [8] for background on the translation planes corresponding to flocks of hyperbolic quadrics.

The feature that connects flocks of hyperbolic quadrics with the associated spreads in $\mathsf{PG}(3,q)$ is that they all admit a regulus-inducing affine homology group of order $q-1$. More generally, over an arbitrary field K, the associated spreads admit a regulus-inducing affine homology group that fixes two components of some regulus and acts regularly on the remaining components.

Let F be a flock of the hyperbolic quadric $x_0x_3 = x_1x_2$ in $\mathsf{PG}(3,K)$, whose points are represented by homogeneous coordinates (x_0,x_1,x_2,x_3) where K is a field. Then the set of planes which contain the conics in F may be represented as follows:

$$\begin{aligned} \rho\colon & x_1 = x_2, \\ \pi_t\colon & x_0 - tx_1 + f(t)x_2 - g(t)x_3 = 0 \end{aligned}$$

for all t in K where f and g are functions of K such that f is bijective. We first point out that there is a natural collineation group of the hyperbolic quadric

$$G = \left\langle \tau_{a,b} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 1 & a \end{bmatrix}; a \neq 0, b \in K \right\rangle.$$

Note that a point (x_0,x_1,x_2,x_3) maps to $(x_0, x_1a + x_0b, x_2, x_3a + x_2b)$ so if the point is on the hyperbolic quadric then $x_0x_3 = x_1x_2$, and the image point is on the hyperbolic quadric if and only if $x_0(x_3a + x_2b) = (x_1a + x_0b)x_2$, which is clearly valid. In the finite case, we note that G is a group of order $q(q-1)$, which if the hyperbolic quadric is considered a regulus net, would fix a Baer subplane of this regulus net pointwise. Therefore, $\tau_{a,b}$ will map the flock F onto another flock $F\tau_{a,b}$ and $\{F\tau_{a,b}; \tau_{a,b} \in G\}$ is a parallelism of the hyperbolic quadric if and only if F and $F\tau_{a,b}$ share no plane. Now the Baer subplane Σ_0 fixed pointwise by G is a ruling line of the hyperbolic quadric and hence each plane of the flock intersects Σ_0 in exactly one point. Now consider any plane η of F and assume that η contains the point P of Σ_0, then the image of η also contains the point P, which means that $\eta\tau_{a,b}$ cannot belong to F, unless $\eta\tau_{a,b} = \eta$.

The plane $\rho\tau_{a,b}$ is generated as follows: $\langle(1,b,0,0),(0,a,1,a),(0,0,0,1)\rangle$, which is clearly not π_t, for any t, and is ρ if and only if $a = 1$ and $b = 0$, so that $\tau_{1,0}$ is the identity mapping. A basis for π_t is

$$\{(-f(t),0,1,0),(t,1,0,0),(g(t),0,0,1)\},$$

which maps under $\tau_{a,b}$ to

$$\{(-f(t),-f(t)b,1,b),(t,tb+a,0,0),(g(t),g(t)b,0,a)\}.$$

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If $\pi_t\tau_{a,b}$ is ρ then b cannot be 0, which implies that $g(t) = 0$. Now consider the associated spread

$$y = x\begin{bmatrix} f(t)u & g(t)u \\ u & tu \end{bmatrix}, \; y = x\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}, x = 0, \text{ for all } t, v, u \neq 0 \in K.$$

If $g(t) = 0$, then the difference

$$\begin{bmatrix} f(t) & 0 \\ 1 & t \end{bmatrix} - \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} f(t) - t & 0 \\ 1 & 0 \end{bmatrix},$$

which cannot be the case.

Finally, assume that $\pi_t\tau_{a,b}$ is π_s and by the above note, we may assume that $s = t$. Hence, we obtain the following requirements:

$$\begin{aligned} -f(t) - sf(t)b + f(s) - g(s)b &= 0, \\ t - s(tb + a) &= 0, \\ g(t) - sg(t)b - g(s)a &= 0. \end{aligned}$$

If $tb + a = 0$ then $t = 0$ but then $(t, tb + a, 0, 0)$ is the zero vector. Hence, $s = t/(tb + a) = t$, so that $tb + a = 1$. Therefore, $f(t)(tb) = g(t)b$. If $b \neq 0$ then $f(t)t - g(t) = 0$, a contradiction to the fact that $\left[\begin{smallmatrix} f(t) & g(t) \\ 1 & t \end{smallmatrix}\right]$ is non-singular. Hence, $\{F\tau_{a,b}; \tau_{a,b} \in G\}$ is a parallelism in the finite case and (minimally) is a partial parallelism in the infinite case. However, given a ruling line ℓ, which is fixed by G, but not pointwise, the group G acts transitively on the non-fixed points of ℓ (actually, doubly transitively). Hence, each point of each ruling line of the hyperbolic quadric is covered by $\{F\tau_{a,b}; \tau_{a,b} \in G\}$.

Choose two conics C_1 and C_2 and assume that C_1 is the conic of intersection of a plane of F and assume without loss of generality that C_1 and C_2 share point P_1 on Σ_0. Let Q_1 and R_1 be points on ruling lines ℓ_1 and ℓ_2 of C_1 and let Q_2 and R_2 be points on lines ℓ_1 and ℓ_2 of C_2, where P_1, Q_1, R_1 uniquely defines C_1 and P_1, Q_2, R_2 uniquely defines C_2. Let $\tau \in G$ map Q_1 to Q_2 (even if $Q_1 = Q_2$). If $R_1\tau = R_2$, then τ maps C_1 to C_2. We note that $R_1\tau$ and Q_2 cannot be in the same Baer subplane, since $P_1, R_1\tau$ and Q_2 lie on a conic. Similarly, C_2 is a conic, then R_2 is not incident with P_1Q_2. Therefore, there is a collineation subgroup of G that fixes Σ_0, and fixes the point Q_2 and acts transitively on the points on $\ell_1 - \{Q_2, \ell_1 \cap \Sigma_0\}$. This means that τ' may be considered a Baer collineation that fixes the 1-dimensional subspace generated by P_1, fixes Q_2 and maps $R_1\tau$ to R_2. Then $\tau\tau'$ will map C_1 to C_2. Hence, given any conic C_2, there is a conic C_1 of F (of the conics of intersection), and a group element g of G so that $C_1 g = C_2$. Hence, we obtain a parallelism in $\mathsf{PG}(3, K)$, for any field K.

Theorem 3.1. *Let F be any flock of a hyperbolic quadric in $\mathsf{PG}(3, K)$, for K a field. Then there is a transitive parallelism $\boldsymbol{P}_F = \{F\tau_{a,b}; \tau_{a,b} \in G\}$.*

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4. Parallelisms with maximal partial flocks

Is it possible to have a parallelism of maximal partial spreads? This question was asked to the second author by Arrigo Bonisoli in the context of parallelisms of the lines of $\mathsf{PG}(3,q)$ and his initial response was that the answer was no! But, ask the question more generally: Is it possible to have a parallelism of the blocks of a geometry by flocks? In particular, is it possible to have a parallelism of the conics of intersection of a quadratic cone in $\mathsf{PG}(3,K)$ by maximal partial flocks? The very surprising answer is, "yes", at least when K is an infinite field.

Consider a putative conical (partial) spread

$$x=0,\ y=x\begin{bmatrix} u & \gamma t^3 \\ t & u \end{bmatrix};t,u\in K,\ \gamma \text{ a non-square in } K.$$

The corresponding flock would require that the functions

$$\phi_u\colon t\mapsto tu^2-(f(t)=\gamma t^3)$$

are bijective for all $u\in K$, in order that we indeed obtain a flock of a quadratic cone. Assume that some function is injective but not bijective. Then we would obtain a partial flock which does not cover all of the points of the quadratic cone, so could not be considered a flock. However, the partial spread listed above is clearly maximal since otherwise there would be a 2-dimensional K-subspace $y=xT$, such that $\left[\begin{smallmatrix} u & \gamma t^3 \\ t & u \end{smallmatrix}\right]-T$ is a non-singular matrix for all $u,t\in K$, clearly a contradiction. So, the question is, when is ϕ_u injective but not bijective?

We have $tu^2-\gamma t^3=su^2-\gamma s^3$ if and only if for $u\neq 0$, we have $u^2=\gamma(t^3-s^3)/(t-s)$ if and only if $\gamma(t^2+st+s^2)=u^2$ and for $u=0$, we must have $(t/s)^3=1$, implies that $t/s=1$. We note that $u^2-\gamma t^4=0$, would imply that γ is a square. Hence, consider

$$(t^2+st+s^2)=u^2/\gamma$$

as a quadratic in t. The discriminant is $s^2-4(s^2-u^2/\gamma)=-3s^2+4u^2/\gamma$. Now let K be an ordered field and let γ be any negative element of K. Then the discriminant is negative so can never be square. Therefore, $f(t)=\gamma t^3$ is injective in any ordered field such that $v^3=1$, implies $v=1$. But $v^3=1$, and v not 1 means that $v^2+v+1=0$, which has discriminant $1-4=-3$. Hence, the question is then whether in an ordered field when the function $f(t)=\gamma t^3$ is injective is it also bijective? Let K be a subfield of the field of real numbers that does not contain all cube roots of each of its elements. Then, $t^3=\delta$, for δ in K, where $\sqrt[3]{\delta}$ is not in K, is not surjective. For example, the rational field will have this property as well as infinitely many subfields of the field of real numbers.

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Theorem 4.1. *Let K be an ordered field such that the function $t \mapsto \gamma t^3$ is injective but not bijective, where γ is a fixed negative element in K. Then there is a maximal partial flock $\mathcal{F}$ of a quadratic cone in* $\mathsf{PG}(3, K)$ *with planes*

$$tx_0 - \gamma t^3 x_1 + x_3 = 0, t \in K.$$

Now we apply the argument of Theorem 2.2 with the group

$$G = \left\langle \tau_{a,b} = \begin{bmatrix} I & \left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & I \end{bmatrix} ; a, b \in K \right\rangle,$$

to obtain a set $\mathcal{F}G$ of mutually disjoint flocks, where

$$\mathcal{F}_{a,b} : \; tx_0 - (\gamma t^3 + a)x_1 + bx_2 + x_3 = 0.$$

Since the set of all planes of intersection have the form $cx_0 + dx_1 + ex_2 + x_3 = 0$, the set of flocks clearly cover all planes that do not contain a generator line of the cone and hence cover all conics of intersection.

Definition 4.2. For a field K, assume that there are functions $f(t)$ and $g(t)$ on K so that the functions $\phi_u : t \mapsto tu^2 - f(t) + g(t)u$ are all injective but not all bijective on K. We then say that corresponding maximal partial flock is "injective and not bijective".

Thus, we have the following theorem.

Theorem 4.3. *Let $\mathcal{F}$ be an injective and not bijective maximal partial flock of a quadratic cone in* $\mathsf{PG}(3, K)$. *Then there is a transitive parallelism of the quadratic cone (conics of intersection) by maximal partial flocks.*

5. Parallelisms of α-cones

Recently in [6], the authors consider a generalization of quadratic cones, which we shall call α-cones in this article. We consider here when there are parallelisms of α-cones by α-flokki.

Let α be an automorphism of a field K and consider $\mathsf{PG}(3, K)$, represented in the form (x_0, x_1, x_2, x_3); $x_i \in K$, as homogeneous coordinates. Let an α-cone $\mathcal{C}_\alpha$ be defined by $x_0 x_1^\alpha = x_2^{\alpha+1}$, with vertex $(0, 0, 0, 1)$. The intersection of a plane not through the vertex and an α-cone is called an "α-conic". This terminology is adopted in analogy with quadratic cones, but one should be aware that α-conics are in general not even ovals.

Definition 5.1. An "α-flokki" is a partition of the points of $\mathcal{C}_\alpha$ other than the vertex by planes of intersection that intersect in α-conics. A "parallelism" of $\mathcal{C}_\alpha$ is a partition of the set of all α-conics of intersection by α-flokki.

As with flocks of quadratic cones, it is convenient to consider the α-flokki as the set of planes rather than the intersections of these planes with the α-cone.

While it may seem that parallelisms would be difficult to obtain, we show that, again in this case, these are readily available. Our proof basically mirrors the proof for the quadratic cone case. For simplicity, we consider here only the finite case and assume that K is isomorphic to $\mathsf{GF}(q)$. By a result of Cherowitzo and Johnson [6], to each α-flokki there is a corresponding translation plane. Hence, a parallelism of an α-cone can be determined by a set of flokki spreads of the form

$$x = 0,\ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u^\alpha \end{bmatrix}; u, t \in \mathsf{GF}(q),$$

if and only if the α-flokki is

$$x_0 t - x_1 f(t)^\alpha + x_2 g(t)^\alpha + x_3 = 0 \text{ for all } t \in K,$$

when representing the α-cone as $x_0 x_1^\alpha = x_2^{\alpha+1}$ with vertex $(0,0,0,1)$. And, again, note that we may initially assume that $f(0)$ or $g(0)$ is 0. However, this is not required for the connections. In particular, for points $(z_2^{1+\alpha}, 1, z_2, \delta)$, $(1,0,0,\tau)$, for $\delta, \tau \in K$, we see that

$$t z_2^{1+\alpha} - f(t)^\alpha + z_2 g(t)^\alpha$$

is required to be bijective for all z_2 in K and this is necessary and sufficient for the existence of an α-flokki in the finite case. We note that

$$\phi_u : t \mapsto t u^{1+\alpha} - f(t)^\alpha + u g(t)^\alpha$$

is bijective if and only if

$$\phi_u : t \mapsto t u^{1+\alpha} - (f(t) + a)^\alpha + u(g(t) + b)^\alpha$$

is bijective, for all $a, b \in K$.

Therefore, if $\mathcal{F}$ is an α-flokki, we consider the associated translation plane $\pi_\mathcal{F}$ so a parallelism is a set of q^2 flokki translation planes that do not share any of their derivable nets. We note that any linear mapping of the associated 4-dimensional vector space will map a flokki translation plane to another flokki translation plane, isomorphic to it. Thus, $\tau_{a,b} = \begin{bmatrix} I & \left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & I \end{bmatrix}$ will map

$$x = 0,\ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u^\alpha \end{bmatrix}; u, t \in K,$$

to

$$x=0,\ y=x\begin{bmatrix} u+g(t)+a & f(t)+b \\ t & u^{\alpha}\end{bmatrix};u,t\in K,$$

so the functions $(f(t),g(t))$ defining the α-flock become $(f(t)^{\alpha}+a^{\alpha},g(t)^{\alpha}+b^{\alpha})$. We note translation planes map to translation planes under $\tau_{a,b}$ which means that α-flocks map to α-flocks under $\tau_{a,b}$.

Theorem 5.2. *Every α-flokki in* $\mathsf{PG}(3,q)$ *is in a transitive parallelism.*

Proof. We represent the planes $\pi_{a,b,c}$, for $a,b,c\in K$, that do not contain the vertex in the form

$$x_0c+x_1b+x_2a+x_3=0.$$

An α-flokki is then represented in the form

$$x_0t-x_1f(t)+x_2g(t)+x_3=0,$$

for all $t\in K$. We also have an associated flokki translation plane π with spread

$$x=0,\ y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u^{\alpha}\end{bmatrix};u,t\in K.$$

Now consider the group

$$G=\left\langle \tau_{a,b}=\begin{bmatrix} I & \left[\begin{smallmatrix} a & b \\ 0 & 0\end{smallmatrix}\right] \\ 0 & I\end{bmatrix};a,b\in K\right\rangle.$$

We note that $\pi\tau_{a,b}$ is a flokki plane isomorphic to π. Therefore, there is a corresponding α-flokki. Note that $\pi\tau_{a,b}$ has the following spread

$$x=0,\ y=x\begin{bmatrix} u+g(t)+a & f(t)+b \\ t & u^{\alpha}\end{bmatrix};u,t\in K.$$

Clearly, none of the associated derivable nets in $\pi\tau_{a,b}$ can be equal to the derivable nets of π. Therefore, none of the planes

$$x_0t-x_1(f(t)^{\alpha}+b^{\alpha})+x_2(g(t)^{\alpha}+a^{\alpha})+x_3=0,$$

are equal to any of the planes of the α-flokki, and so the associated image is also an α-flokki. Since all planes that do not contain the vertex have the form

$$x_0c+x_1b+x_2a+x_3=0,$$

we see that we have partitioned the α-conics by the α-flokki associated with πG. Hence, each α-flokki belongs to a transitive parallelism. $\square$

Remark 5.3. The group considered in the transitive parallelism is not a group of the α-cone. However, since we are considering α-flokki as a set of planes, whose intersections with the cone partition the non-vertex points, the α-flokki as sets of planes can map to other α-flokki and therefore α-conics of intersection map to α-conics of intersection of an equivalent α-cone.

Remark 5.4. Since the connection with associated α-flokki translation planes is not used in the proof, we note that any α-flokki of $\mathsf{PG}(3,K)$, for K any field, lies in a transitive parallelism by the same argument.

5.1. Parallelisms of α-cones by maximal partial α-flokki

Consider again a putative α-flokki as set of planes of an α-cone;

$$\rho_t\colon x_0t - x_1f(t)^\alpha + x_2g(t)^\alpha + x_3 = 0 \text{ for all } t \in K.$$

We have seen that a maximal partial α-flokki is obtained if the functions $\phi_u\colon t \mapsto u^{\alpha+1}t - f(t)^\alpha + ug(t)^\alpha$ are injective and not all bijective.

Definition 5.5. For a field K, assume that there are functions $f(t)$ and $g(t)$ on K so that the functions $\phi_u\colon t \mapsto tu^{\alpha+1} - f(t)^\alpha + g(t)^\alpha u$ all injective but not all bijective on K. We then say that corresponding maximal partial flock is "injective and not bijective".

Returning to the idea of a parallelism of an α-cone, we see that it might be possible to use maximal partial α-flokki instead of α-flokki. Considering the associated maximal partial α-flokki planes, we still may use the group

$$G = \left\langle \tau_{a,b} = \begin{bmatrix} I & \left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & I \end{bmatrix} ; a, b \in K \right\rangle .$$

If π denotes the maximal partial α-flokki spread, we note that $\pi\tau_{a,b}$ is a maximal partial α-flokki spread isomorphic to π. Therefore, there is a corresponding maximal partial α-flokki. Again, simply note that $\pi\tau_{a,b}$ has the following spread

$$x = 0,\ y = x \begin{bmatrix} u + g(t) + a & f(t) + b \\ t & u^\alpha \end{bmatrix} ; u, t \in K.$$

Clearly, none of the associated derivable partial spreads in $\pi\tau_{a,b}$ can be equal to the derivable spreads of π. Therefore, none of the planes

$$x_0t - x_1(f(t)^\alpha + b^\alpha) + x_2(g(t)^\alpha + a^\alpha) + x_3 = 0, \tag{1}$$

are equal to any of those of the original maximal partial α-flokki. Consider the set of all planes

$$x_0t + x_1u + x_2v + x_2 = 0,\ \text{for all } t, u, v \in K$$

not containing the vertex. Clearly, these planes are all covered by the planes of (1) and we have an "α-parallelism". Thus, we have the following theorem.

Theorem 5.6. *Let $\mathcal{F}$ be an injective and not bijective maximal partial α-flokki of a quadratic cone in $\mathsf{PG}(3, K)$. Then there is a transitive parallelism of the α-cone (α-conics of intersection) by maximal partial α-flokki.*

The following example is similar to the one presented when $\alpha = 1$ and is valid for α-flokki.

Theorem 5.7. *Let K be any ordered field and α an automorphism of K. Then*

$$x = 0,\ y = x \begin{bmatrix} u & -t^{3\alpha^{-1}} \\ t & u^{\alpha} \end{bmatrix}$$

is a maximal partial α-flokki spread.

There are subfields K of the field of real numbers for which the maximal partial spread is not a spread.

Finally, we list the following open problems.

Problem 1. *Find parallelisms of elliptic or hyperbolic quadrics in $\mathsf{PG}(3, K)$, by maximal partial flocks.*

Problem 2. *Extend the theory of flocks and parallelisms of quadratic cones to Laguerre planes and/or to α-flocks (flocks of translation oval cones) or to oval-flocks.*

Problem 3. *Show that there are parallelisms of elliptic quadrics of characteristic 2.*

Problem 4. *Are there parallelisms of quadratic cones or of hyperbolic quadrics that are not transitive?*

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