Innovations in Incidence Geometry
Volume 12 (2011), Pages 49–60
ISSN 1781-6475

Minimal fields of definition for simplicial arrangements in the real projective plane

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Abstract

For each simplicial arrangement in the real projective plane of the catalogue of Grünbaum [4], we determine the minimal extension of the rationals over which there exists a realization of its incidence structure. For the infinite families we use the symmetries of the incidence. For the sporadic arrangements we give an algorithm that uses Gröbner bases.

Keywords: simplicial arrangement, oriented matroid, field of definition
MSC 2010: 52C35; 20F55

1 Introduction

Recently I. Heckenberger and the author have classified the so-called finite Weyl groupoids of rank three [3]. A *Weyl groupoid* is a generalization of the Weyl group, see [2] for an introduction. It was B. Mühlherr who noticed that the root systems of rank three Weyl groupoids yield simplicial arrangements in the real projective plane. Thanks to the catalogue of B. Grünbaum [4], it was easy to identify them: It turned out that 53 of the 67 sporadic arrangements in the large component of his Hasse diagram come from Weyl groupoids. This is motivation enough to investigate simplicial arrangements from this new viewpoint, especially since it is still an open question whether the catalogue of Grünbaum is complete.

To the Weyl groupoids are associated certain *root systems*. With respect to the simple roots, the coefficients of the roots are rational integers. The Weyl groups are obtained as a special case, but for example the Coxeter group of type H_3 is not included in this setting. One reason is that there is no arrangement over the rationals with the same incidence structure as the arrangement of type H_3. Thus as a first step, it is important to understand which number fields are required to “realize” the incidence structure of an arrangement.

In this note, we develop a technique to compute these fields of definition. For the infinite series, we use the symmetry of the incidence structure to deduce that the solutions are in fact unique up to projectivity. The known sporadic arrangements are dealt by an algorithm that uses Gröbner bases to obtain enough restrictions to determine the field extension.

The infinite families $\mathcal{R}(1)$, $\mathcal{R}(2)$ require the following fields of definition (see Theorem 3.6 and Corollary 3.7 for details):

$$\mathbb{Q}(\zeta)\cap\mathbb{R} \quad \text{for } \zeta \text{ a root of unity.}$$

The known sporadic arrangements all have a realization over one of (see Theorem 4.1 for details):

$$\mathbb{Q}, \quad \mathbb{Q}(\sqrt{2}), \quad \mathbb{Q}(\sqrt{3}), \quad \mathbb{Q}(\sqrt{5}) \quad \text{or} \quad \mathbb{Q}[X]/(X^3-3X+25).$$

This note is organized as follows. We start with a section in which we prove that any simplicial arrangement has a realization over an algebraic number field. In the following section we compute the fields of definition for the infinite series. In the last section we treat the sporadic arrangements.

Acknowledgment. I wish to thank J. Maslowski and G. Malle for helpful discussions.

2 Algebraic realizations

We first recall some definitions (cfr. [5, 1.2, 5.1]).

Definition 2.1. Let K be a field and V a finite dimensional vector space over K. A *projective arrangement* $(\mathcal{A},V)$ is a finite set of projective hyperplanes in $\mathbb{P}(V)$. Let $L(\mathcal{A})$ be the set of all nonempty intersections of elements of $\mathcal{A}$. If $K\subseteq\mathbb{R}$ and every component of the complement of $\bigcup_{H\in\mathcal{A}} H$ in $V\otimes_K\mathbb{R}$ is an open simplicial cone, then we call $\mathcal{A}$ a *simplicial arrangement*.

Throughout this note, all simplicial arrangements will be in the projective plane over a subfield of $\mathbb{R}$, i.e. $K\subseteq\mathbb{R}$, $V=K^3$. We write "(,)" for the usual inner product on $\mathbb{R}^3$.

Definition 2.2. An *incidence structure* is a triple (P,L,I) where P is a finite set of "points", L is a finite set of "lines" and $I\subseteq P\times L$ is the *incidence relation*.

Definition 2.3. Given an incidence structure I, we call a *realization of I over K* an arrangement $(\mathcal{A},K^3)$ such that the poset $L(\mathcal{A})$ (with respect to inclusion) is given exactly by I. Conversely, from an arrangement $\mathcal{A}$ we obtain an incidence structure from the poset $L(\mathcal{A})$.

We will say that $K\subset\mathbb{R}$ is a *field of definition* of the incidence I of $\mathcal{A}$ if I has a realization over K and K is contained in all fields over which I can be realized.

Proposition 2.4. *If I is the incidence of an arrangement in $\mathbb{P}(\mathbb{R}^3)$, then there exists a finite field extension K of $\mathbb{Q}$ such that $K \subset \mathbb{R}$ and such that I admits a realization over K.*
The same holds for simplicial arrangements: There exists a finite field extension $K \subset \mathbb{R}$ of $\mathbb{Q}$ such that I admits a simplicial realization over K for which the same triples of lines give open simplicial cones.

Proof. Let $v_1, \dots, v_n \in \mathbb{R}^3$ be normal vectors of the n planes of an arrangement. If $g_{i,j} \neq 0$ is an element of the intersection of $v_i^\perp$ and $v_j^\perp$, then

$$(g_{i,j}, v_i) = 0 = (g_{i,j}, v_j), \tag{2.1}$$

and without loss of generality, we may assume

$$(g_{i,j}, g_{i,j}) = 1. \tag{2.2}$$

For each k such that $g_{i,j}$ is not on plane k we have

$$(g_{i,j}, v_k) x_{i,j,k} = 1 \tag{2.3}$$

for some $x_{i,j,k} \in \mathbb{R}$, and if $g_{i,j}$ is in the intersection of $v_k^\perp$ and $v_l^\perp$, then we may assume

$$g_{i,j} = g_{k,l}. \tag{2.4}$$

Equations (2.1), (2.2), (2.3), (2.4) define an ideal $\mathfrak{I}$ in the polynomial ring over $\mathbb{Q}$ with the coordinates of all $v_i, g_{i,j}$ and $x_{i,j,k}$ as indeterminates. If the corresponding variety (over $\mathbb{R}$) is non-empty, then we have at least one algebraic solution:

Assume first that $\mathfrak{I}$ has dimension 0. Then clearly all points on the variety $\mathcal{V}(\mathfrak{I})$ have algebraic coordinates, so if $\mathcal{V}(\mathfrak{I}) \neq \emptyset$ then we also have an algebraic solution. Now assume that the dimension of $\mathfrak{I}$ is greater than 0. Since $\mathcal{V}(\mathfrak{I}) \neq \emptyset$, there exists a hyperplane which has non trivial intersection with $\mathcal{V}(\mathfrak{I})$. But $\mathbb{Q}$ is dense in $\mathbb{R}$, so in the space of hyperplanes that meet $\mathcal{V}(\mathfrak{I})$ there also exists one defined by a rational form. Adding this to the ideal we get an ideal of dimension $\dim(\mathfrak{I}) - 1$; by induction we obtain an algebraic solution.

For the case of simplicial arrangements it remains to translate the fact that we need a triangulation. Each triple (i, j, l) of planes that yields an open simplicial cone gives a set of inequalities:

Write a v_k with respect to the basis (v_i, v_j, v_l). Notice that for this we need to include the generator $\det(v_i, v_j, v_l)x = 1$ for some new variable x to our ideal, otherwise (v_i, v_j, v_l) is not a basis. The base change takes place over our polynomial ring, since x is the inverse of the determinant.

Now view all vectors with respect to the basis (v_i, v_j, v_l):

$$v_k = a_{k,i} v_i + a_{k,j} v_j + a_{k,l} v_l$$

for $k = 1, \dots, n$. Then for all $k = 1, \dots, n$ either $a_{k,i}, a_{k,j}, a_{k,l} \geq 0$ or $a_{k,i}, a_{k,j}, a_{k,l} \leq 0$ if and only if the open simplicial cone

$$C = \{v \in \mathbb{R}^3 \mid (v, v_i) > 0, (v, v_j) > 0, (v, v_k) > 0\}$$

given by the planes $v_i^\perp, v_j^\perp, v_l^\perp$ intersects trivially with any hyperplane of $\mathcal{A}$ (cfr. [1, Lemma 2.2]): Assume first that w is on plane k, i.e. $(w, v_k) = 0$, and that the coordinates of v_k are all greater or equal to 0. Then $(w, a_{k,i}v_i) + (w, a_{k,j}v_j) + (w, a_{k,l}v_l) = 0$, thus either $w = 0$ or two of the inner products have different signs, say $(w, v_i)(w, v_j) < 0$, so $w \notin C$. The same holds for the case in which all coordinates of v_k are less or equal to 0. Conversely, assume that $(w, v_k) = 0$ implies $w \notin C$. Denote $v_i^\vee, v_j^\vee, v_l^\vee$ a dual basis to v_i, v_j, v_l. If without loss of generality $a_{k,i}a_{k,j} < 0$, then for $w = a_{k,j}v_i^\vee - a_{k,i}v_j^\vee$ we have either $w \in C$ or $-w \in C$ although $(w, v_k) = 0$.

Thus for each open simplicial cone we obtain a set of inequalities. But each inequality $f \geq g$ becomes an equality $f = g + x^2$ by introducing a new variable x. The same argument as for arbitrary arrangements gives: Either there is no solution, or there exists an algebraic solution. □

3 The infinite families

There are three known infinite families of simplicial arrangements in $\mathbb{P}(\mathbb{R}^3)$. They are denoted by $\mathcal{R}(0)$, $\mathcal{R}(1)$, $\mathcal{R}(2)$ in [4]. Family $\mathcal{R}(0)$ consists of *near pencils*. It is clear from definition that all near pencils may be realized over $\mathbb{Q}$, hence we will ignore them in this note.

Family $\mathcal{R}(1)$ consists of the following arrangements: Starting with a regular convex n-gon in the Euclidean plane, the arrangement $\mathcal{A}(2n, 1)$ is obtained by taking the n lines determined by the sides of the n-gon together with the n lines of mirror symmetry of that n-gon (see Figure 1 for an example).

Finally for $n = 4m + 1$, from $\mathcal{A}(4m, 1)$ one obtains a new arrangement by adjoining the "line at infinity". These are the arrangements in the family $\mathcal{R}(2)$.

Definition 3.1. We will need the arrangement $\mathcal{A}(2n, 1)$ more explicitly, so here is a more precise definition. We write I_n for its incidence structure. Let $\zeta = \exp(2\pi \mathrm{i}/n)$. We choose our labels for the points in such a way that 1 is the point in the center and $2, \dots, n+1$ are the vertices of the n-gon in counterclockwise ordering: $p_1 := (0 : 0 : 1)$ and the n-gon has vertices

$$p_{i+2} := (\mathrm{Re}(\zeta^i) : \mathrm{Im}(\zeta^i) : 1)$$

for $i = 0, \dots, n-1$. For the next lemma, we need some information about the incidence: Consider Figure 2 as a part of the n-gon and identify $p_1, \dots, p_5$ with w, u, v, x, y. Using symmetries of the n-gon one can check that the incidences between the points p, q, r, s and the lines of the figure do not depend on n.

Lemma 3.2. *A realization of the incidence structure I_n is uniquely determined by the choice of points $p_1, \dots, p_5$ in $\mathbb{P}(\mathbb{R}^3)$, where p_i corresponds to the intersection point labeled by i.*

Proof. First observe that the complete image is given by the points labeled $1, \dots, n+1$. We show that if the points labeled $1, r, \dots, r+3$ for $1 < r \leq n-4$ are given, then one can

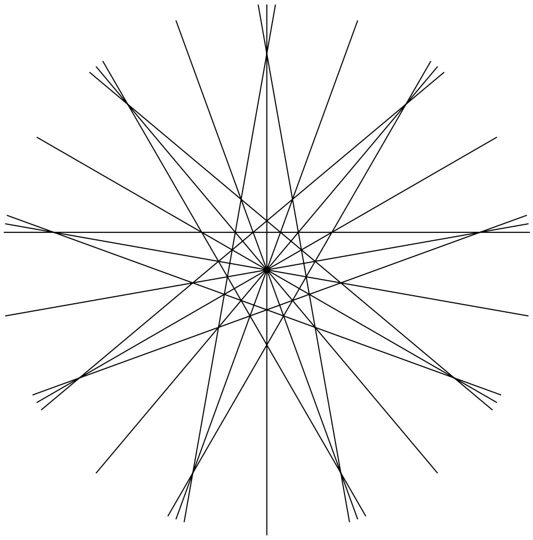

Figure 1: The arrangement $\mathcal{A}(18, 1)$

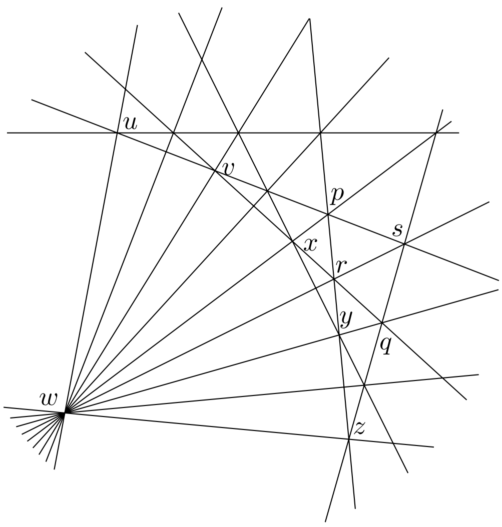

Figure 2: Proof of Lemma 3.2

construct the point $r+4$. Identify the points $p_1, p_r, \dots, p_{r+3}$ with w, u, v, x, y in Figure 2. If we write $\iota(a,b,c,d)$ for the intersection point of the lines (a,b) and (c,d), then

$$p = \iota(u,v,x,w), \quad q = \iota(v,x,y,w),$$

$$r = \iota(v,x,p,y), \quad s = \iota(u,v,w,r).$$

Thus the next point p_{r+4} is $z = \iota(p,y,s,q)$. By induction we obtain the claim. □

Definition 3.3. Call a tuple $(p_1,\dots,p_5)$ of points in $\mathbb{P}(\mathbb{R}^3)$ a *solution* for I_n, if the construction of Lemma 3.2 leads to the incidence structure I_n. We write

$$N\colon \mathbb{P}(\mathbb{R}^3)^5 \to \mathbb{P}(\mathbb{R}^3), \quad (w,u,v,x,y) \mapsto z,$$

i.e. $N(w,u,v,x,y) = z$ is the next point according to Lemma 3.2 or Figure 2.

Remark 3.4. If $\zeta = \exp(2\pi \mathrm{i} k/n)$ for some k with $\gcd(k,n) = 1$, then the points $p_1 = (0:0:1)$ and

$$p_{i+1} = (\mathrm{Re}(\zeta^i) : \mathrm{Im}(\zeta^i) : 1)$$

for $i = 1,\dots,4$ are a solution for I_n.

Lemma 3.5. *Given a realization of I_n, the points labeled $1, r, r+1, r+2$ are in general position for $1 < r < n-1$.*

Proof. If three different points with labels in $\{1, r, \dots, r+2\}$ were not in general position, then they would lie on a common line. □

Theorem 3.6. *Let $n \in \mathbb{N}$, $n > 2$ and $K \subseteq \mathbb{R}$ a field. If $p_1,\dots,p_m \in \mathbb{P}(K^3)$ are a realization of I_n, then $\mathbb{Q}(\mathrm{Re}(\zeta)) = \mathbb{Q}(\zeta) \cap \mathbb{R} \subseteq K$, where $\zeta = \exp(2\pi\mathrm{i}/n)$. Moreover, up to projectivity there exists only one realization of I_n.*

Proof. Assume that $p_1,\dots,p_5$ are a solution to the indicence structure I_n. Applying a projectivity, we may assume without loss of generality that $p_1 = (0:0:1)$, $p_2 = (1:0:1)$, $p_3 = (0:1:1)$, $p_4 = (1:1:1)$ and that $p_5 = (x:y:z)$ is an indeterminate point. Since $p_1,\dots,p_4$ have rational coordinates, this really is a projectivity on $\mathbb{P}(K^3)$.

We first construct a map $D\colon \mathbb{P}(K^3) \to \mathbb{P}(K^3)$ in the following way: Let $p_6 = N(p_1,\dots,p_5)$ be the unique next point given by Lemma 3.2. Consider the projectivity π given by (notice that the points are in general position by Lemma 3.5)

$$p_1 \mapsto p_1, \quad p_3 \mapsto p_2, \quad p_4 \mapsto p_3, \quad p_5 \mapsto p_4.$$

Since p_1, p_3, p_4, p_5, p_6 are a solution to the incidence structure, the points p_1, p_2, p_3, p_4, $\pi(p_6)$ will be a solution as well. Hence we have now two different possible choices for a fifth point: p_5 and $\pi(p_6)$ which we will denote $D(p_5) := \pi(p_6)$.

We now compute $D(p_5)$ for $p_5 = (x:y:z)$. We certainly have $x \neq 0$ because otherwise $p_5 = (0:y:z) \in \langle (0:0:1), (0:1:1) \rangle$ which contradicts I_n. One computes

$$p_6 = \big((x-y+z)(x+y-z) : xy - y^2 + 2yz - z^2 : xz - 2y^2 + 4yz - 2z^2\big)$$

and $\pi(p_6) = D(p_5)$ is

$$\left(x : \frac{(x-y+z)(x+y-z)}{x} : \frac{x^2 - xy + xz - y^2 + 2yz - z^2}{x}\right).$$

Evaluating these functions it turns out that $D(D(p_5)) = D(p_5)$ i.e. $D^2 = D$. This means that $D(p_5)$ is a solution fixed by D and we thus obtain a projectivity ψ such that $p_1, p_2, \psi(p_2) = p_3, \psi^2(p_2) = p_4, \psi^3(p_2) = D(p_5)$ is a solution and $\psi(p_1) = p_1$, $\psi^n(p_2) = p_2$.

Let φ be a linear map $\mathbb{R}^3 \to \mathbb{R}^3$ with $\psi = \mathbb{P}(\varphi)$. Since p_1, p_2, $\psi(p_2)$, $\psi^2(p_2)$ are in general position and are fixed by ψ^n, we have $\varphi^n = \lambda\mathrm{id}$ for some $\lambda \in \mathbb{R}^\times$. Choose an $\varepsilon \in \mathbb{C}$ with $\varepsilon^n = 1/\lambda$; we may assume $\varepsilon \in \mathbb{R}$ if n is odd or $\lambda > 0$, and $\varepsilon \in \mathrm{i}\mathbb{R}$ otherwise.

But then $\varepsilon\varphi = \mathrm{diag}(\xi, \eta, 1)$ with respect to some basis (b_1, b_2, b_3) and for suitable $\xi, \eta \in \mathbb{C}$ with $\xi^k = \eta^m = 1$, $n = \mathrm{lcm}(k, m)$. The constant term in the minimal polynomial of $\varphi|_{\langle b_1, b_2\rangle}$ is $\xi\eta\varepsilon^{-2}$ and has to be real, thus $\xi = \pm\eta^{-1}$. In both cases, there exists a basis $(\tilde{b}_1, \tilde{b}_2, b_3)$ such that $(\varepsilon\varphi)|_{\langle\tilde{b}_1, \tilde{b}_2\rangle}$ is a rotation of order n. Thus for a solution fixed by D, there exists a projectivity π'

$$q_1 \mapsto p_1, \quad q_2 \mapsto p_2, \quad q_3 \mapsto p_3, \quad q_4 \mapsto p_4,$$

where q_1, q_2, q_3, q_4, q_5 is our prefered solution from Remark 3.4 for $\zeta = \exp(2\pi\mathrm{i}/n)$, and such that $\pi(p_6) = \pi'(q_5)$, explicitly:

$$\pi'(q_5) = \left(\zeta + \zeta^{-1} : \frac{\zeta^4 + \zeta^2 + 1}{\zeta(\zeta^2+1)} : \frac{\zeta^4 - \zeta^3 + \zeta^2 - \zeta + 1}{\zeta(\zeta^2+1)}\right).$$

(Notice that we only need to know that $\zeta(\zeta^2 + 1) \neq 0$ to compute this expression, the relation $\zeta^n = 1$ is not used.) Using $D(p_5) = \pi'(q_5)$ we obtain

$$x = \zeta + \zeta^{-1}, \quad y = z + 1,$$

thus at least $\mathbb{Q}(\mathrm{Re}(\zeta)) \subseteq K$. For the uniqueness consider the projectivity (reflection) σ given by

$$p_1 \mapsto p_1, \quad p_5 \mapsto p_2, \quad p_4 \mapsto p_3, \quad p_3 \mapsto p_4.$$

This maps p_2 to

$$\sigma(p_2) = (x(z-y) : yz - y^2 : x^2 - xy - y^2 + yz),$$

again a new solution different from p_5. One computes

$$D(\sigma(p_2)) = \left((y-z)^2 : (z-x)(x-2y+z) : -x^2 + xy + xz + y^2 - 3yz + z^2\right)$$

which is a solution fixed by D and thus equal to $\pi'(q_5)$. Collecting these relations yields a 0-dimensional ideal in $\mathbb{R}[x, y, z]$ with exactly one solution, $(x : y : z) = \pi'(q_5)$, so in fact $D = \mathrm{id}$. □

Corollary 3.7. *Let I be the incidence structure of $\mathcal{A}(4m+1, 1)$, an arrangement of family $\mathcal{R}(2)$. If $\mathcal{A}$ is a realization of I over a field $K \subseteq \mathbb{R}$, then $\mathbb{Q}(\mathrm{Re}(\zeta)) = \mathbb{Q}(\zeta) \cap \mathbb{R} \subseteq K$, where $\zeta = \exp(2\pi\mathrm{i}/(4m))$. Moreover, there exists a realization of I over $\mathbb{Q}(\mathrm{Re}(\zeta))$, thus this is the field of definition.*

Proof. Since $\mathcal{A}(4m+1,1)$ is a descendant of $\mathcal{A}(4m,1)$, it follows from Theorem 3.6 that $\mathbb{Q}(\mathrm{Re}(\zeta)) = \mathbb{Q}(\zeta) \cap \mathbb{R} \subseteq K$, where $\zeta = \exp(2\pi \mathrm{i}/(4m))$. The second assertion holds by Remark 3.4 since adding the line at infinity does not enlarge the required field. □

4 The known sporadic arrangements

Theorem 4.1. *For each connected component of the Hasse diagram of sporadic arrangements [4, Figure 4] exists a well defined unique field of definition K:*

1. *The component of $\mathcal{A}(6,1)$ has $K = \mathbb{Q}$.*
2. *The component of $\mathcal{A}(16,1)$ has $K = \mathbb{Q}(\sqrt{2})$.*
3. *The component of $\mathcal{A}(24,1)$ has $K = \mathbb{Q}(\sqrt{3})$.*
4. *The component of $\mathcal{A}(10,1)$ has $K = \mathbb{Q}(\sqrt{5})$.*
5. *The component of $\mathcal{A}(15,5)$ has $K = \mathbb{Q}(x)/(x^3 - 3x + 25)$. This field is not Galois over $\mathbb{Q}$; its splitting field is $\mathbb{Q}(x)/(x^6 + 3x^5 + 5x^4 + 5x^3 + 5x^2 + 3x + 1)$ and has Galois group S_3.*

Proof. For each minimal and maximal arrangement in [4, Figure 4], we compute the fields of definition by the algorithm below. These are:

1. $\mathcal{A}(10,1)$, $\mathcal{A}(13,4)$, $\mathcal{A}(14,4)$, $\mathcal{A}(16,5)$, $\mathcal{A}(31,1)$ for the component on the left.
2. $\mathcal{A}(21,4)$, $\mathcal{A}(21,6)$, $\mathcal{A}(25,2)$, $\mathcal{A}(37,3)$ for the component in the middle.
3. $\mathcal{A}(16,1)$, $\mathcal{A}(17,8)$, $\mathcal{A}(25,5)$, $\mathcal{A}(15,5)$, $\mathcal{A}(21,7)$, $\mathcal{A}(24,1)$, $\mathcal{A}(37,2)$ for the remaining components.

Notice that for the component in the middle the field of definition is always $\mathbb{Q}$, so we do not need to consider the minimal arrangements. □

Remark 4.2. The only maximal arrangement in the large component in the middle which does not come from a Weyl groupoid is $\mathcal{A}(21,6)$. The fields of definition for the arrangements $\mathcal{A}(10,1)$, $\mathcal{A}(16,1)$, $\mathcal{A}(24,1)$ are given by Theorem 3.6.

4.1 A procedure to determine fields of definition

Definition 4.3. We will say that an incidence structure I is *generated* by points $\lambda_1, \ldots, \lambda_m$ if the set of all points is the smallest set P with

1. $\lambda_1, \ldots, \lambda_m \in P$,
2. for all $\lambda, \mu, \nu, \rho \in P$ such that $\langle \lambda, \mu \rangle$, $\langle \nu, \rho \rangle$ are lines in I we have $\langle \lambda, \mu \rangle \cap \langle \nu, \rho \rangle \in P$.

The main part of the procedure determines a set of polynomials:

Algorithm 4.4. FindRelations(I)
Computes equations which have solutions over the field of definition of an arrangement.
Input: an incidence structure I.
Output: the Gröbner basis of an ideal $\mathfrak{I}$ of relations satisfied by the coordinates of the points of I.

1. Find a small set of labels $\lambda_1, \dots, \lambda_m$ which generate the incidence structure and such that $\lambda_1, \dots, \lambda_4$ are in general position (for this use the incidence structure: if two points are on the same line and a third is not, then we know that their coordinates are linearly independent).
2. The points $p_1, \dots, p_4$ corresponding to the labels $\lambda_1, \dots, \lambda_4$ may be chosen in general position with coordinates in $\mathbb{Q}$; $p_5, \dots, p_m$ have indeterminate coordinates: $p_i = (x_i : y_i : z_i)$.
3. Over $F = \mathbb{Q}(x_5, y_5, z_5, \dots, x_m, y_m, z_m)$, compute the intersection spaces and new hyperplanes until we have them all. An intersection space is stored as $\langle v \rangle$ for some $v \in F^3$. Multiplying v by the least common multiple of the denominators of the coordinates of v if necessary, we obtain an element of $\mathbb{Q}[x_5, \dots, z_m]^3$.
4. For each pair of planes, compute the "difference" of the intersection space $\langle v \rangle$ and the space $\langle w \rangle$ computed in the last step:
   - Either they have a common non-zero entry, say $v_1 \neq 0 \neq w_1$. Then the "differences" are $v_2 w_1 - w_2 v_1$ and $v_3 w_1 - w_3 v_1$.
   - Or else $v_i w_i = 0$ for all $i = 1, 2, 3$. Then the "differences" are $v_1 - w_1$, $v_2 - w_2$, $v_3 - w_3$.

   Collect these differences in a set $R \subset \mathbb{Q}[x_5, \dots, z_m]$.
5. Compute a Gröbner basis of the ideal $\mathfrak{I}$ generated by the elements of R.
6. For each triple of points q_1, q_2, q_3 in general position (use the incidence structure), compute the determinant of the matrix with rows q_1, q_2, q_3. Collect these in a set D.
7. Let B be the basis of $\mathfrak{I}$. As long as an element f of B is divisible by an element g of D, replace f by f/g. We obtain a new set B'.
8. Compute the ideal $\mathfrak{I}'$ generated by B' and its basis B''. If $\mathfrak{I}' \neq \mathfrak{I}$ then go back to step 7 with $\mathfrak{I} \leftarrow \mathfrak{I}'$, $B \leftarrow B''$.
9. Return B''.

Remark 4.5. It appears that in practice, if I is the incidence structure of a simplicial arrangement then we always have $m = 5$ in step 1.

Remark 4.6. The elements of D are polynomials which must be different from 0. The usual technique is to add a new variable v for each $f \in D$ and the equation $fv = 1$. But here D is a very large set, so this would increase the number of variables considerably. Of course, one could also add only one variable v and consider the equation $v\prod_{f\in D} f = 1$, but this is a polynomial of very high degree.

After using this algorithm, it remains to perform the steps:

(1) Examine the resulting Gröbner basis B''. We obtain a subfield K of the field of definition.

(2) Compute a realization of the incidence structure over K. Then we are sure that K is minimal.

Ad (1). If the ideal $\langle B''\rangle$ has dimension 0 then it easy to determine the field. If the ideal has dimension > 0 then we can hope to extract some information about the field extension from B'': For instance, we can consider the cases that a given coordinate of (x_5, y_5, z_5) is 0 or not, and hence without loss of generality 0 or 1. In both cases we eliminate a variable and obtain an ideal in a smaller ring. This is sufficient to treat all simplicial arrangements of the catalogue.

Ad (2). This is the easiest part. Since we have a generating set of points given by step 1 of the algorithm, it suffices to choose a solution over K for $\lambda_1, \dots, \lambda_m$ and to check that it realizes the incidence structure.

Remark 4.7. The above algorithm is also applicable to arrangements which are not simplicial.

4.2 Some examples

4.2.1 $\mathcal{A}(10, 1)$

This is the incidence structure I for $\mathcal{A}(10, 1)$ (the sets of lines going through the points $1, \dots, 16$):

$$\{1,2,3,4,5\}, \{4,9\}, \{1,10\}, \{3,6\}, \{5,7\}, \{2,8\}, \{5,9,10\}, \{2,6,10\}, \{4,6,7\},$$
$$\{1,7,8\}, \{3,8,9\}, \{4,8,10\}, \{1,6,9\}, \{3,7,10\}, \{5,6,8\}, \{2,7,9\}.$$

Our algorithm suggests to start with the points labeled by

$$1, 8, 9, 10, 11.$$

For the first four points, we choose

$$(1:0:0), (1:1:0), (1:0:1), (1:1:1);$$

the fifth point will be $(x:y:z)$. The ideal is generated by 9 polynomials in x, y, z. It has a Gröbner basis with 2 generators: $x - 2z$, $y^2 - yz - z^2$. Now either $x \neq 0$ or $x = 0$. For $x = 0$ the only solution is $(x, y, z) = (0, 0, 0)$, hence $x \neq 0$, say $x = 2$. But then $z = 1$ and

$$f(y) = y^2 - y - 1 = 0.$$

The polynomial f is the minimal polynomial of $-\zeta_5 - \zeta_5^{-1}$, thus a field of definition should include $\sqrt{5}$. This agrees with Theorem 3.6.

4.2.2 $\mathcal{A}(15, 5)$

This is the incidence structure I for $\mathcal{A}(15, 5)$ (the sets of lines going through the points $1, \ldots, 34$)

$\{1, 2, 3, 15, 8\}$, $\{1, 14, 4\}$, $\{1, 5, 6\}$, $\{1, 7\}$, $\{1, 9\}$, $\{1, 13, 10\}$, $\{11, 1, 12\}$, $\{2, 4\}$, $\{2, 13, 6\}$, $\{11, 2, 14, 7, 10\}$, $\{12, 2, 9\}$, $\{12, 13, 3, 4, 7\}$, $\{3, 5, 10\}$, $\{11, 3, 9\}$, $\{3, 14\}$, $\{4, 5\}$, $\{4, 6, 9\}$, $\{4, 8, 10\}$, $\{11, 4, 15\}$, $\{5, 7, 8\}$, $\{14, 5, 9\}$, $\{11, 13, 5\}$, $\{12, 15, 5\}$, $\{15, 6, 7\}$, $\{11, 6, 8\}$, $\{12, 6, 10\}$, $\{14, 6\}$, $\{13, 8, 9\}$, $\{12, 14, 8\}$, $\{15, 9, 10\}$, $\{13, 14, 15\}$, $\{2, 5\}$, $\{3, 6\}$, $\{7, 9\}$.

Our algorithm suggests to start with the points labeled by

$$1, 2, 19, 22, 28.$$

Again, for the first four points we choose

$$(1 : 0 : 0), (1 : 1 : 0), (1 : 0 : 1), (1 : 1 : 1);$$

the fifth point will be $(x : y : z)$. The ideal is generated by 40 polynomials in x, y, z. It has a Gröbner basis (computed for instance with MAGMA) with 8 generators. There are 3105 determinants which may be used to reduce this basis, and 8 more come from the fact that certain points are different. After three cycles of steps 7, 8 of the above algorithm, we obtain an ideal with basis

$$x - 2y, \quad y^3 - 5y^2z + 4yz^2 - z^3.$$

Now either $x = 1$ or $x = 0$. For $x = 0$ the only solution is $(x, y, z) = (0, 0, 0)$, hence $x = 1$. But then $y = 1/2$ and

$$f(z) = z^3 - 2z^2 + 5/4z - 1/8 = 0.$$

The polynomial f is irreducible, thus a solution would at least contain a root α of f. After computing a solution over $\mathbb{Q}(\alpha)$, one checks that it is indeed simplicial.

The usual simplification algorithm on f yields the nicer polynomial $z^3 - 3z + 25$ defining the same number field (evaluate f at $x/6 + 2/3$ and multiply by 216).

4.2.3 $\mathcal{A}(37, 2)$

We omit the incidence structure I for $\mathcal{A}(37, 2)$ because it takes too much space. The ideal is generated by 489 polynomials in x, y, z. MAGMA computes a Gröbner basis with 5 generators. Since there are 971970 potential determinants, we concentrate on those given by the five starting points. Together with the polynomials coming from the differences of points we get 287 polynomials which must be different from 0. After three cycles of steps 7, 8 of the above algorithm, we obtain an ideal with basis

$$x - 5y + 5z, \quad y^2 - 6yz + 6z^2.$$

Again, if $x = 0$ then $(x, y, z) = (0, 0, 0)$, hence $x = 1$. But then

$$y - z - 1/5 = 0, \quad z^2 - 4/5z + 1/25 = 0,$$

thus the field of definition is $\mathbb{Q}(\sqrt{3})$.

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