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On Weyl modules for the symplectic group

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Abstract

A rich information can be found in the literature on Weyl modules for $\mathrm{Sp}(2n, \mathbb{F})$, but the most important contributions to this topic mainly enlighten the algebraic side of the matter. In this paper we try a more geometric approach. In particular, our approach enables us to obtain a sufficient condition for a module as above to be uniserial and a geometric description of its composition series when our condition is satisfied. Our result can be applied to a number of cases. For instance, it implies that the module hosting the Grassmann embedding of the dual polar space associated to $\mathrm{Sp}(2n, \mathbb{F})$ is uniserial.

Keywords: symplectic groups, symplectic grasmmannians, Weyl modules.

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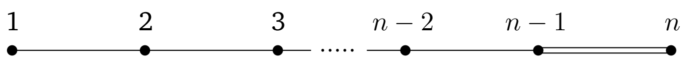
1 Introduction

Let V be a $2n$-dimensional vector space over a field $\mathbb{F}$ and, for a given non-degenerate alternating form $\alpha(.,.)$ of V, let $G \cong \mathrm{Sp}(2n, \mathbb{F})$ be the symplectic group associated with it and Δ the building associated with G. The elements of Δ of type $k = 1, 2, \ldots, n$ are the k-dimensional subspaces of V totally isotropic for the form α.



For $1 \leq k \leq n$, let $\mathcal{G}_k$ be the k-grassmannian of $\mathrm{PG}(V)$, where the k-subspaces of V are taken as points. We recall that the lines of $\mathcal{G}_k$ are the sets

$$l_{X,Y} := \{Z \mid X \subset Z \subset Y,\ \dim(Z) = k\}$$

for a $(k+1)$-subspace Y of V and a $(k-1)$-subspace X of Y. Put $W_k := \wedge^k V$ and let $\iota_k \colon \mathcal{G}_k \mapsto \mathrm{PG}(W_k)$ be the natural embedding of $\mathcal{G}_k$ in $\mathrm{PG}(W_k)$, sending a k-subspace $\langle v_1, \ldots, v_k \rangle$ of V to the 1-dimensional subspace $\langle v_1 \wedge \cdots \wedge v_k \rangle$ of W_k. Let Δ_k be the k-grassmannian of Δ, elements of Δ of type k being taken as points of Δ_k. When $1 < k < n$ the lines of Δ_k are the lines $l_{X,Y}$ of $\mathcal{G}_k$ where X and Y are totally α-isotropic, while Δ_1 and Δ_n are respectively the polar space and the dual polar space associated to Δ. In any case, Δ_k is a full subgeometry of $\mathcal{G}_k$. The embedding ι_k induces an embedding $\varepsilon_k \colon \Delta_k \mapsto \mathrm{PG}(V_k)$, called the *natural embedding* of Δ_k, where V_k is the subspace of W_k spanned by the ι_k-image of the set of points of Δ_k. We recall that $\dim(V_k) = \binom{2n}{k} - \binom{2n}{k-2}$ while $\dim(W_k) = \binom{2n}{k}$. When $\mathrm{char}(\mathbb{F}) \neq 2$ the embedding ε_k is absolutely universal. This follows from the fact that Δ_k admits the absolutely universal embedding (Kasikova and Shult [14]) and, when $\mathrm{char}(\mathbb{F}) \neq 2$, it also admits a generating set of size equal to $\dim(V_k)$ (Blok [4]).

The group G acts on V_k via ε_k. In the language of Chevalley groups, V_k is the Weyl module whose highest weight is the k-th fundamental dominant weight. We are interested in the structure of the G-module V_k.

It is well known that if $\mathrm{char}(\mathbb{F}) = 0$ then V_k is irreducible (see Steinberg [16], for instance). When $\mathrm{char}(\mathbb{F}) = p > 0$ the module V_k can be reducible. In fact V_k admits a unique maximal proper G-submodule $R(V_k)$, which we call the *radical* of the embedding ε_k, also the *radical* of V_k. The radical $R(V_k)$ can be characterized as the intersection of all hyperplanes of V_k spanned by ε_k-images of singular hyperplanes of Δ_k (see Blok [5]).

The dimension $f_k := \dim(V_k/R(V_k))$ can be explicitly computed with the help of the following recursive formula (Premet and Suprunenko [15], Baranov and Suprunenko [3, Section 2]; see also Adamovich [1, 2] and Brouwer [9]):

$$f_k = \binom{2n}{k} - \binom{2n}{k-2} - \sum_{j \in J_p(k,n)} f_j,$$

where

$$J_p(k,n) := \big\{ j \mid 0 \leq j < k,\ k-j \equiv 0 \pmod{2},\ n-j+1 \geq_p (k-j)/2 \big\}$$

and, for two integers $a \geq b \geq 0$, expressed as $a = a_0 + a_1 p + \cdots + a_r p^r$ and $b = b_0 + b_1 p + \cdots + b_s p^s$ to the base p, we write $a \geq_p b$ and say that a *contains* b *to the base* p if $s \leq r$ and for every $i = 1, \ldots, s$ either $b_i = a_i$ or $b_i = 0$. Note that $f_0 = 1$, namely $V_0 = \wedge^0 V$ is the trivial 1-dimensional G-module.

More explicitly, put $m := |J_p(k,n)|$ and let $k_1, k_2, \ldots, k_m$ be the elements of $J_p(k,n)$, listed in increasing order.

Theorem 1.1 (Premet and Suprunenko [15]). *The composition series of V_k have length $m + 1$. If $0 = S_0 \subset S_1 \subset \cdots \subset S_{m-1} \subset S_m = R(V_k) \subset V_k$ is a composition series of V_k, then there exists a permutation σ of $\{1, 2, \ldots, m\}$ such that $S_i/S_{i-1} \cong V_{k_{\sigma(i)}}/R(V_{k_{\sigma(i)}})$ for $i = 1, 2, \ldots, m$.*

Note that, in general, V_k admits more than one composition series. However, according to the Jordan–Hölder theorem, the family of irreducible sections S_i/S_{i-1} of a composition series does not depend on the choice of the series. These sections and the quotient $V_k/R(V_k)$ are the *irreducible sections* of V_k, a *section* of V_k being a quotient S'/S for two submodules $S \subset S'$ of V_k.

Even if in general V_k admits more than one composition series, the first nonzero member S_1 of a composition series of V_k does not depend on the choice of the series (see Adamovich [2]; also Baranov and Suprunenko [3, Section 2]). Hence $S_1 = S(V_k)$, where $S(V_k)$ stands for the socle of V_k. In other words, the socle $S(V_k)$ of V_k is simple. When $J_p(k) = \emptyset$ we put $S(V_k) = 0$, by convention.

Let $\mathcal{L}(V_k)$ be the lattice of submodules of V_k. A description of the isomorphism type of the lattice $\mathcal{L}(V_k)$, originally due to Adamovich [2], is offered by Baranov and Suprunenko [3, Section 2]. They define a particular ordering relation on certain finite sequences of integers depending on n, k and p, thus obtaining a poset which is proved to be isomorphic to $\mathcal{L}(V_k)$. This description as well as Theorem 1.1 have been obtained by purely algebraic methods. For instance, Theorem 1.1 has been obtained by an investigation of the weight subspaces of V_k based on the theory of representations of symmetric groups (see James [13]).

In this paper, carrying on a project laid down in Blok, Cardinali and Pasini [8] (but already implicit in Cardinali and Lunardon [10] and Blok, Cardinali and De Bruyn [6]), we try a different, more geometric approach to this matter. Our dream is to obtain Theorem 1.1 and a description of $\mathcal{L}(V_k)$ in a geometric way. We made our first steps towards this goal in [8]. In this paper we go on further.

Our investigation will exploit poles, introduced in [8]. When k is odd the group G acts fixed-point freely on $\mathrm{PG}(W_k)$ while when k is even G fixes exactly one point P_k of $\mathrm{PG}(W_k)$ (see [8, Theorem 4.1]). The point P_k is called the *pole* of G in W_k, also the pole of W_k, for short.

Lemma 1.2. *The pole P_k is contained in V_k if and only if $\dim(S(V_k)) = 1$. If $\dim(S(V_k)) = 1$ then $S(V_k) = P_k$ and k is even.*

The second claim and the 'if' part of the first claim of this lemma immediately follow from the uniqueness of P_k and the fact that P_k exists only if k is even. The 'only if' part of the first claim follows from the fact that $S(V_k)$ is simple.

The first claim of the next lemma is obvious. The second claim is a little piece of Theorem 1.1. We put it in evidence since it is the only part of Theorem 1.1 which we need in the proof of our main theorem (Theorem 1.4, to be stated below). As we will show in Section 3.2, nearly all the rest of Theorem 1.1 can be deduced from it.

Lemma 1.3. (1) *A nonnegative integer $r < k$ belongs to $J_p(k, n)$ if and only if $0 \in J_p(k-r, n-r)$, namely $k-r$ is even and $(k-r)/2 \leq_p n-r+1$.*

(2) *The module V_k admits a 1-dimensional section if and only if $0 \in J_p(k, n)$.*

Clearly, if $S(V_k) = P_k$ then P_k is a 1-dimensional section of V_k. In fact, it is the unique 1-dimensional section of V_k since, according to Theorem 1.1, no two 1-dimensional sections can occur in the same composition series of V_k. On the other hand, it can happen that V_k admits a 1-dimensional section but $P_k \not\subset V_k$ (see Remark 5.3).

We slightly change our notation by writing $W_{k,n}$, $V_{k,n}$, $\iota_{k,n}$, $\varepsilon_{k,n}$ and $P_{k,n}$ instead of W_k, V_k, ι_k, ε_k and P_k, in order to keep a record of the rank n of G and Δ in these symbols, but we refrain from extending this heavier notation further, thus keeping the symbols G, Δ, $\mathcal{G}_k$ and Δ_k with no change.

In [8] we proved that, for any given value of the difference $h = n - k$, denoted by $n(h, p)$ the smallest n for which $V_{k,n}$ is reducible, if $n = n(h, p)$ then $R(V_{k,n}) = P_{k,n}$ while if $n > n(h, p)$ then $R(V_{k,n})$ contains a submodule spanned by 'local poles'. We shall explain in a few lines what local poles are.

Given a positive integer $r < k$ with $k - r$ even, for every r-element X of Δ let $\mathcal{G}_{k,X}$ be the set of k-subspaces of V that contain X and $W^X_{k,n}$ the subspace of $W_{k,n}$ spanned by $\iota_{k,n}(\mathcal{G}_{k,X})$. Also, let $\Delta_{k,X}$ be the set of k-elements of Δ that contain X and $V^X_{k,n}$ the subspace of $V_{k,n}$ spanned by $\varepsilon_{k,n}(\Delta_{k,X})$. Let G_X be the stabilizer of X in G and let K_X be the element-wise stabilizer of $\Delta_{k,X}$. Then $G_X/K_X \cong \mathrm{Sp}(2n-2r, \mathbb{F})$ and K_X also fixes all elements of $\mathcal{G}_{k,X}$. Thus G_X/K_X also acts on $W^X_{k,n}$. Moreover $W^X_{k,n} \cong W_{k-r,n-r}$ and $V^X_{k,n} \cong V_{k-r,n-r}$ as $\mathrm{Sp}(2n-2r, \mathbb{F})$-modules (see also Proposition 2.1 of this paper). As $k - r$ is even, $W_{k-r,n-r}$ admits a pole $P_{k-r,n-r}$. Let P_X be the point of $\mathrm{PG}(W^X_{k,n})$ corresponding to $P_{k-r,n-r}$ in the isomorphism $W^X_{k,n} \cong W_{k-r,n-r}$. Then P_X is the unique fixed point of G_X in its action on $\mathrm{PG}(W^X_{k,n})$. We call P_X the *pole* of G_X in $W^X_{k,n}$, also the *local pole* of G at X.

Suppose that $P_{k-r,n-r} \subset V_{k-r,n-r}$, namely $\dim(S(V_{k-r,n-r})) = 1$. Then $P_X \subset V^X_{k,n}$ and we can consider the following subspace of $V_{k,n}$:

$$\mathcal{P}^r_{k,n} := \langle P_X \mid X \text{ is an } r\text{-element of } \Delta \rangle.$$

If $P_{k,n} \subset V_{k,n}$ we put $\mathcal{P}^0_{k,n} := P_{k,n}$. Let

$$\widetilde{J}_p(k, n) := \{r \mid 0 \leq r < k,\ \dim(S(V_{k-r,n-r})) = 1\}.$$

By Lemma 1.3, $\widetilde{J}_p(k,n) \subseteq J_p(k,n)$, with $\widetilde{J}_p(k,n) = J_p(k,n)$ if and only if, for every $r = 0, 1, \ldots, k-1$, if $0 \in J_p(k-r, n-r)$ then $\dim(S(V_{k-r,n-r})) = 1$.

In [8] we proved that $R(V_{k,n}) \supseteq \cup_{r \in \widetilde{J}_p(k,n)} \mathcal{P}^r_{k,n}$. A sharper version of this result will be given in Section 4 of this paper.

When writing [8] we believed that all of $R(V_{k,n})$ could be explained by means of the submodules $\mathcal{P}^r_{k,n}$. Considering that $V_{k,n}$ can admit a 1-dimensional section even if it does not contain $P_{k,n}$, whence $\widetilde{J}_p(k,n)$ can be smaller than $J_p(k,n)$, we now feel differently. However, as we will show in this paper, that belief is still right when $\widetilde{J}_p(k,n) = J_p(k,n)$.

In order to state the main result of this paper we need one more definition. Recall that a module is said to be uniserial when it admits exactly one composition series. Let $V_{k,n}$ be uniserial, let $0 = S_0 \subset S_1 \subset \cdots \subset S_m = R(V_{k,n}) \subset V_{k,n}$ be its unique composition series and let σ be the permutation of $\{1, 2, \ldots, m\}$ such that $S_i/S_{i-1} \cong V_{j_{\sigma(i)},n}/R(V_{j_{\sigma(i)},n})$ for $i = 1, 2, \ldots, m$. In general, σ is not the identity permutation, even if $V_{k,n}$ is uniserial. If $V_{k,n}$ is uniserial and σ is the identity permutation, then we say that $V_{k,n}$ is *plainly uniserial*.

Theorem 1.4. *Let* $\mathrm{char}(\mathbb{F}) \neq 2$. *Assume that* $\widetilde{J}_p(k,n) = J_p(k,n)$. *Then* $V_{k,n}$ *is plainly uniserial. If* $k_1 < k_2 < \cdots < k_m$ *are the elements of* $J_p(k,n)$ *and* $0 = S_0 \subset S(V_{k,n}) = S_1 \subset S_2 \subset \cdots \subset S_m = R(V_{k,n}) \subset V_{k,n}$ *is the composition series of* $V_{k,n}$*, then* $S_i = \mathcal{P}^{k_i}_{k,n}$ *for every* $i = 1, 2, \ldots, m$ *and* $\mathcal{P}^{k_i}_{k,n}$ *is a homomorphic image of* $V_{k_i,n}$*. In particular,* $S_i/S_{i-1} \cong V_{k_i,n}/R(V_{k_i,n})$.

We shall prove Theorem 1.4 in Section 5. As previously said, the second claim of Lemma 1.3 is the only part of Theorem 1.1 that we need to assume in that proof.

As we shall prove in Section 6 (Lemma 6.1), the equality $\widetilde{J}_p(k,n) = J_p(k,n)$ holds whenever $n - k < p - 1$. Moreover, by the first part of Lemma 1.3 and Lemma 2.3 of Section 2 one can see that if $n - k < p - 1$ then

$$J_p(k,n) = \{2(n+1) - k - 2p^t \mid t = 1, 2, \ldots, m\}$$

where $m = \lfloor \log_p(n + 1 - k/2) \rfloor$ (integral part of $\log_p(n + 1 - k/2)$). Note that $m = 0$ if and only if $n < p - 1 + k/2$. Clearly, $m = 0$ precisely when $J_p(k,n) = \emptyset$, namely $V_{k,n}$ is irreducible. By Theorem 1.4 and the above, we immediately obtain the following:

Corollary 1.5. *If* $\mathrm{char}(\mathbb{F}) \neq 2$ *and* $n - k < p - 1$ *then* $V_{k,n}$ *is plainly uniserial. The composition series of* $V_{k,n}$ *contains* m *non-zero proper submodules* $S_1, S_2, \ldots, S_m$ *where* $m = \lfloor \log_p(n + 1 - k/2) \rfloor$. *For* $i = 1, 2, \ldots, m$ *the module* S_i *is a homomorphic image of* $V_{k_i,n}$*, where* $k_i = 2(n+1) - k - 2p^{m+1-i}$.

In particular, the above applies to $V_{n,n}$, which hosts the Grassmann embedding of the dual polar space Δ_n.

Most likely the hypothesis $\mathrm{char}(\mathbb{F}) \neq 2$, assumed in Theorem 1.4 and inherited by Corollary 1.5, is superfluous (compare Blok, Cardinali and De Bruyn [6], where a part of the statement of Corollary 1.5 is obtained for $V_{n,n}$, but in even characteristic). We have assumed that $\mathrm{char}(\mathbb{F}) \neq 2$ mainly because, in the sequel, we will sometimes exploit the fact that when $\mathrm{char}(\mathbb{F}) \neq 2$ the natural embedding $\varepsilon_{k,n}$ is absolutely universal in order to prove that certain embeddings are homomorphic images of it, but perhaps this conclusion can also be obtained in a straightforward way, allowing $\mathrm{char}(\mathbb{F}) = 2$.

We finish this introduction by mentioning a few problems which should be solved in order to pursue our project of obtaining a complete geometric explanation of the structure of $V_{k,n}$:

1. As previously remarked, when $n - k \geq p$ it can happen that $V_{k,n}$ admits a 1-dimensional section but $\dim(S(V_{k,n})) > 1$. Find a geometric explanation of the occurrence of these sections.
2. Find a geometric proof of the second claim of Lemma 1.3.
3. Lemma 1.2 follows from the the simplicity of $S(V_{k,n})$ but this crucial property of $S(V_{k,n})$ is obtained in [3] and [2] as a by-product of the description of the lattice $\mathcal{L}(V_{k,n})$. Find a more straightforward way to prove Lemma 1.2.

2 Preliminaries

2.1 Notation and conventions

Throughout this paper V, $\alpha(.,.)$, G, Δ, Δ_k, $\mathcal{G}_k$, $\iota_{k,n}$, $\varepsilon_{k,n}$, $W_{k,n}$ and $V_{k,n}$ have the meaning stated in the introduction. The orthogonality relation with respect to α will be denoted by $\perp$.

Henceforth $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is a given basis of V, hyperbolic for the form α. For a subset $J = \{j_1, j_2, \ldots, j_s\}$ of $\{1, 2, \ldots, n\}$, where $j_1 < j_2 < \cdots < j_s$, we put $e_J := e_{j_1} \wedge \cdots \wedge e_{j_s}$ and $f_J := f_{j_1} \wedge \cdots \wedge f_{j_s}$. We also put $I = \{1, 2, \ldots, n\}$ and, for a nonnegative integer $r \leq n$, we denote by $\binom{I}{r}$ the collection of r-subsets of I. With this notation, a sum as $\sum_{J \in \binom{I}{r}} e_J \wedge f_J$ is read as follows:

$$\sum_{J \in \binom{I}{r}} e_J \wedge f_J = \sum_{1 \leq j_1 < \cdots < j_r \leq n} e_{j_1} \wedge \cdots \wedge e_{j_r} \wedge f_{j_1} \wedge \cdots \wedge f_{j_r}.$$

We will make use of a few notions from the theory of embeddings, as isomorphism and morphisms between embeddings, absolute universality, homogeneity. We are not going to recall these notions here. We presume that the reader is familiar with them. If not, we refer to [7, Section 2.2] or Kasikova and Shult [14].

As explained in the introduction, we must assume $\mathrm{char}(\mathbb{F}) \neq 2$ because we need $\varepsilon_{k,n}$ to be absolutely universal. On the other hand, if $\mathrm{char}(\mathbb{F}) = 0$ then $R(V_{k,n}) = 0$. In this case there is nothing to study. So, from now on we assume $\mathrm{char}(\mathbb{F}) = p > 2$.

2.2 Induced embeddings of residues of elements of Δ

Given an element X of Δ of type $r \in \{1, 2, \ldots, k\}$, let Δ^+_X be its upper residue, formed by the totally isotropic subspaces of V properly containing X. It is well known that Δ^+_X is isomorphic to the building of a symplectic polar space of rank $n - r$. We take $\{1, 2, \ldots, n - r\}$ as the type-set of Δ^+_X. So, an element of Δ^+_X of type i has type $i + r$ when regarded as an element of Δ. In particular, elements of Δ^+_X of type $k - r$ have type k in Δ. The $(k - r)$-grassmannian $(\Delta^+_X)_{k-r}$ of Δ^+_X is a full subgeometry of Δ_k and $\varepsilon_{k,n}$ induces an embedding of $(\Delta^+_X)_{k-r}$ in the subspace $\langle \varepsilon_{k,n}((\Delta^+_X)_{k-r}) \rangle$ of $V_{k,n}$ spanned by the $\varepsilon_{k,n}$-image of the set of points of $(\Delta^+_X)_{k-r}$. (By a little abuse, we denote that image by $\varepsilon_{k,n}((\Delta^+_X)_{k-r})$.) The embedding of $(\Delta^+_X)_{k-r}$ in $\langle \varepsilon_{k,n}((\Delta^+_X)_{k-r}) \rangle$ induced by $\varepsilon_{k,n}$ will be denoted by $\varepsilon^X_{k,n}$.

Proposition 2.1. $\varepsilon^X_{k,n} \cong \varepsilon_{k-r,n-r}$.

Proof. Without loss of generality we can assume that $X = \langle e_1, \ldots, e_r \rangle$. Therefore $X^\perp/X \cong V' := \langle e_{r+1}, \ldots, e_n, f_{r+1}, \ldots, f_n \rangle$ and the points of $(\Delta^+_X)_{k-r}$ bijectively correspond to the totally isotropic $(k - r)$-subspaces of V'. We may regard $W_{k-r,n-r}$ as the same thing as $\wedge^{k-r} V'$. There exists a unique linear mapping

$$\begin{aligned} \varphi_{e_1,\ldots,e_r} \colon W_{k-r,n-r} = \wedge^{k-r} V' &\to W_{k,n} \\ v_1 \wedge v_2 \wedge \cdots \wedge v_{k-r} &\mapsto v_1 \wedge \cdots \wedge v_{k-r} \wedge e_{\{1,\ldots,r\}} \end{aligned}$$

where $(v_1, \ldots, v_{k-r})$ stands for any independent $(k - r)$-tuple of vectors of V' and $e_{\{1,\ldots,r\}} = e_1 \wedge \cdots \wedge e_r$, as stated in Section 2.1. Clearly, $\varphi_{e_1,\ldots,e_r}$ maps $\langle \varepsilon_{k,n}((\Delta^+_X)_{k-r}) \rangle$ isomorphically onto $V_{k-r,n-r}$. It yields the desired isomorphism from $\varepsilon^X_{k,n}$ to $\varepsilon_{k-r,n-r}$. □

2.3 Radical and 1-dimensional sections of $V_{k,n}$

In the introduction, the radical $R(V_{k,n})$ of $V_{k,n}$ has been defined as the largest proper submodule of $V_{k,n}$. We have also mentioned that $R(V_{k,n})$ can be characterized as the intersection of all hyperplanes of $V_{k,n}$ spanned by $\varepsilon_{k,n}$-images of singular hyperplanes of Δ_k (Blok [5]). This characterization can be rephrased in the following way, more suited to our needs in this paper.

A non-degenerate bilinear form $\alpha_k(.,.)$ can be defined on $W_{k,n}$ such that, for any two points X and Y of Δ_k and any non-zero vectors $x \in \varepsilon_{k,n}(X), y \in \varepsilon_{k,n}(Y)$, we have $\alpha_k(x,y) = 0$ if and only if X and Y are non-opposite as elements of Δ (see [8, Section 2]). Blok's characterization of $R(V_{k,n})$ amounts to say that $R(V_{k,n}) = V_{k,n} \cap V_{k,n}^{\perp_k}$, where $\perp_k$ stands for the orthogonality relation with respect to α_k.

We shall now describe when 1-dimensional sections occur. To this end, we consider pairs (k,n) with a fixed difference $h = n-k$. It turns out that the decomposition of $V_{k,n}$ largely depends on this difference h. Let $N(h,p)$ be the smallest integer $n > h$ such that p divides $\binom{1+\lfloor (n+h)/2 \rfloor}{h+1}$. The following proposition is a corollary of the proof of Theorem 1.1 by Premet and Suprunenko [15]. A different, more geometric proof of this proposition is given in [8, Section 5], but only valid when $p-1$ does not divide h. Another proof is given by De Bruyn [11].

Proposition 2.2. *Let $h = n-k$. If $n < N(h,p)$ then $R(V_{k,n}) = 0$. If $n = N(h,p)$ then $R(V_{k,n})$ is 1-dimensional. If $n > N(h,p)$ then $\dim(R(V_{k,n})) > 1$.*

In view of the next formula we need to state a few conventions. Let $h = \sum_{j=0}^{\infty} h_j p^j$ be the expansion of h to the base p. Let e the smallest j such that $h_j < p-1$. So,

$$h = \left[(p-1) \cdot \sum_{j=0}^{e-1} p^j\right] + h_e p^e + h_{e+1} p^{e+1} + \cdots \tag{1}$$

with $0 \leq h_e < p-1$. Note that $e = 0$ is allowed in the above. In this case $h_0 < p-1$. As remarked in [8, Section 5],

$$N(h,p) = 2(p-1-h_e)p^e + h. \tag{2}$$

By claim (2) of Lemma 1.3, $V_{k,n}$ admits a 1-dimensional section if and only if $0 \in J_p(k,n)$, namely $k/2 \leq_p n+1$.

Lemma 2.3. *Let k be even. Then $k/2 \leq_p n+1$ if and only if p^e divides k and*

$$\frac{k}{2p^e} = p^{t+1} - 1 - \sum_{j=0}^{t} h_{j+e} p^j + \sum_{i=1}^{r} \left[p^{s_i+t_i+1} - \sum_{j=s_i}^{s_i+t_i} h_{j+e} p^j\right] \tag{3}$$

for an integer $t \geq -1$*, a nonnegative integer* r*, positive integers* $s_1, s_2, \ldots, s_r$ *and nonnegative integers* $t_1, t_2, \ldots, t_r$ *such that*

$$\begin{aligned}
&t + r \geq 0, \quad t + 1 < s_1, \\
&s_1 + t_1 + 1 < s_2, \quad s_2 + t_2 + 1 < s_3, \ \ldots\ , s_{r-1} + t_{r-1} + 1 < s_r, \\
&p - 1 > h_{j+e} \qquad \textit{for } j = 0, 1, \ldots, t, \\
&h_{e+s_i} \neq 0 \qquad\quad \textit{for } i = 1, 2, \ldots, r, \ \textit{and} \\
&p - 1 > h_{e+s_i+j} \ \textit{for } i = 1, 2, \ldots, r, \ j = 1, 2, \ldots, t_i.
\end{aligned}$$

We will turn to the proof of this lemma in a few lines. We make a few remarks first. If $t = -1$ then $\sum_{j=0}^{t} h_{j+e} p^j = 0$ by convention, while if $r = 0$ then $\sum_{i=1}^{r} \left[p^{s_i+t_i+1} - \sum_{j=s_i}^{s_i+t_i} h_{j+e} p^j\right] = 0$. Note that either $t \geq 0$ or $r > 0$, since $t + r \geq 0$. Note also that, if we put $t = r = 0$ in (3), then we obtain $n = N(h, p)$ as in (2).

Proof of Lemma 2.3. We only give a sketch of the proof of the 'only if' part of the lemma, leaving the rest and all details for the reader. Note that $n = k + h$. Let $k/2 = k_0 + k_1 p + k_2 p^2 + \cdots$ be the expansion of $k/2$ to the base p. So,

$$n + 1 = [2k_0 + 2k_1 p + \cdots] + [(1 + h_e)p^e + h_{e+1} p^{e+1} + h_{e+2} p^{e+2} + \cdots].$$

It follows that $k/2 \leq_p n+1$ only if $k_0 = k_1 = \cdots = k_{e-1} = 0$. Therefore, assuming that $k/2 \leq_p n+1$, we obtain that p^e divides $k/2$. Assume first that $k_e \neq 0$ and let t be the largest integer such that $k_j \neq 0$ for every $j = e, e+1, \ldots, e+t$. Then condition $k/2 \leq_p n+1$ implies that $k_j = p - h_j - 1$ for every $j = e, e+1, \ldots, e+t$. So,

$$\sum_{j=0}^{t} k_{j+e} p^{j+e} = p^{t+e+1} - p^e - \sum_{j=0}^{t} h_{j+e} p^{j+e}.$$

If $k_j = 0$ for every $j > e+t$ then $k/2p^e$ is as in (3) with $r = 0$. Suppose that $k_j \neq 0$ for some $j > t + e$ and let s_1 be the smallest integer $j > t$ such that $k_{j+e} \neq 0$. As $k_{e+t+1} = 0$ by the choice of t, we have $s_1 > t + 1$. Moreover, $k/2 \leq_p n + 1$ forces $k_{e+s_1} = p - h_{e+s_1}$. Hence $h_{e+s_1} \neq 0$, because $k_{e+s_1} \neq 0$. Let t_1 be the largest integer such that $k_j \neq 0$ for every $j = e + s_1, e + s_1 + 1, \ldots, e + s_1 + t_1$. Then $k_{e+s_1+j} = p - 1 - h_{e+s_1+j}$ for $j = 1, 2, \ldots, t_1$. For these values of j we have $h_{e+s_1+j} < p - 1$ because $k_{e+s_1+j} \neq 0$. Moreover,

$$\sum_{j=s_1}^{s_1+t_1} k_{j+e} p^{j+e} = p^{t_1+s_1+e+1} - \sum_{j=s_1}^{s_1+t_1} h_{j+e} p^{j+e}.$$

It is now clear how to go on. We end up with (3). We have assumed $k_e \neq 0$. If $k_e = 0$ then we still obtain (3) but with $t = -1$. In this case s_1 is the smallest integer j such that $k_{j+e} \neq 0$. $\square$

Proposition 2.4. *We have $0 \in J_p(k,n)$ if and only if k is even, p^e divides k and $k/2p^e$ is as in* (3) *of Lemma* 2.3.

Proof. By claim (2) of Lemma 1.3, $0 \in J_p(k,n)$ if and only if k is even and $k/2 \leq_p n+1$. Lemma 2.3 yields the conclusion. □

2.4 The basic series and the pole

For $0 \leq i \leq \lfloor k/2 \rfloor$, where $\lfloor k/2 \rfloor$ is the integral part of $k/2$, we denote by $V_{k-2i}^{(k,n)}$ the subspace of $W_{k,n}$ spanned by the vectors $\iota_{k,n}(X)$ for a k-subspace X of V with $\dim(X \cap X^{\perp}) \geq k-2i$. In particular, $V_k^{(k,n)} = V_{k,n}$. Clearly, $V_{k-2i}^{(k,n)}$ is G-invariant and $V_{k-2i}^{(k,n)} \subseteq V_{k-2j}^{(k,n)}$ for $0 \leq i \leq j \leq \lfloor k/2 \rfloor$. Note that $k - 2\lfloor k/2 \rfloor$ is equal to 0 or 1 according to whether k is even or odd. In any case, $V_{k-2\lfloor k/2 \rfloor}^{(k,n)} = W_{k,n}$. We put $V_{k+2}^{(k,n)} := 0$, by convention. The series of the G-submodules of $W_{k,n}$ defined above is called the *basic series* of G in $W_{k,n}$:

$$0 = V_{k+2}^{(k,n)} \subseteq V_k^{(k,n)} \subseteq V_{k-2}^{(k,n)} \subseteq \cdots \subseteq V_{k-2\lfloor k/2 \rfloor}^{(k,n)} = W_{k,n}.$$

When k is odd the clause $i < k/2 - 1$ is equivalent to $i < \lfloor k/2 \rfloor$. When k is even and $i = k/2 - 1$ then $V_2^{(k,n)}$ is a hyperplane of $V_0^{(k,n)} = W_{k,n}$.

Let $0 \leq i < k/2$. Given a totally singular $(k-2i)$-subspace X of V, choose a k-subspace Y of V such that $Y \cap Y^{\perp} = X$ and put

$$\varphi_i(X) := \iota_{k,n}(Y) + V_{k-2i+2}^{(k,n)} \in \mathrm{PG}\big(V_{k-2i}^{(k,n)}/V_{k-2i+2}^{(k,n)}\big).$$

Proposition 2.5. *The mapping φ_i is well defined, it is an embedding of Δ_{k-2i} in $\mathrm{PG}(V_{k-2i}^{(k,n)}/V_{k-2i+2}^{(k,n)})$ and it is isomorphic to the natural embedding $\varepsilon_{k-2i,n}$ of Δ_{k-2i}.*

(See [8, Theorem 3.5]; we warn that the universality of $\varepsilon_{k-2i,n}$ is exploited in the proof of that theorem.) As recalled in the introduction of this paper, if k is odd then G acts fixed-point-freely on $\mathrm{PG}(W_{k,n})$ while when k is even G fixes exactly one point $P_{k,n}$ of $\mathrm{PG}(W_{k,n})$, called the pole of G in $W_{k,n}$. Clearly, G stabilizes the 1-dimensional subspace $(V_2^{(k,n)})^{\perp_k}$ of $W_{k,n}$ (where $\perp_k$ is defined as in Section 2.3). Hence $P_{k,n} = (V_2^{(k,n)})^{\perp_k}$.

Proposition 2.6. $P_{k,n} = \langle v_{P_{k,n}} \rangle$ *where* $v_{P_{k,n}} = \sum_{J \in \binom{I}{k/2}} e_J \wedge f_J$.

(See [8, Lemma 4.2].) We take the vector $v_{P_{k,n}}$ as the canonical representative of $P_{k,n}$. The following is also proved in [8, Theorem 4.3]:

Proposition 2.7. *We have $v_{P_{k,n}} \in V_2^{(k,n)}$ if and only if p divides $\binom{n}{k/2}$.*

As said in the introduction, $P_{k,n}$ is contained in $V_k^{(k,n)} = V_{k,n}$ if and only if the socle $S(V_{k,n})$ of $V_{k,n}$ is 1-dimensional, namely $P_{k,n} = S(V_{k,n})$. Let i be minimal with $v_{P_{k,n}} \in V_{k-2i}^{(k,n)}$. Then $v_{P_{k,n}} \in V_{k-2i}^{(k,n)} \setminus V_{k-2i+2}^{(k,n)}$. The next proposition gives necessary conditions for this to happen.

Proposition 2.8. *Assume that $v_{P_{k,n}} \in V_{k-2i}^{(k,n)} \setminus V_{k-2i+2}^{(k,n)}$ for a nonnegative index $i < k/2$. Then:*

(1) *p divides $\binom{n}{k/2-i}$;*

(2) *$k/2 - i \leq_p n+1$;*

(3) *either p divides both $\binom{n-k+2i}{i}$ and $\binom{k/2}{k/2-i}$ or it divides neither of them.*

Proof. Let $v_{P_{k,n}} \in V_{k-2i}^{(k,n)} \setminus V_{k-2i+2}^{(k,n)}$. Then $\dim(S(V_{k-2i,n})) = 1$ by Proposition 2.5, hence p divides $\binom{n}{k/2-i}$ by Proposition 2.7 and $k/2 - i \leq_p n+1$ by claim (2) of Lemma 1.3 applied to $V_{k-2i,n}$. Claim (3) follows from [8, Proposition 4.7]. □

3 Irreducible sections

In this section we show how to exploit claim (2) of Lemma 1.3 and the information collected in Section 2 to prove that every irreducible section of $V_{k,n}$ has dimension as it can be obtained from Theorem 1.1.

3.1 A few lemmas

Let B be the Borel subgroup of G stabilizing the chamber $(\langle e_1, \ldots, e_j\rangle)_{j=1}^n$ of Δ and let U be the unipotent radical of B. For every $i = 0, 1, \ldots, \lfloor k/2 \rfloor$ we put $\widehat{e}_i = e_{\{1,2,\ldots,k-2i\}}$ and

$$\widehat{v}_i = \sum_{J \in \binom{\{k-2i+1,\ldots,n\}}{i}} e_J \wedge f_J.$$

So, $\widehat{e}_i$ corresponds to the subspace $A_i := \langle e_j \rangle_{j=1}^{k-2i}$ of V and $\langle \widehat{v}_i \rangle$ is the local pole of G at A_i, namely the pole of the group induced by G_{A_i} on $W_{A_i} := \wedge^{2i}\langle e_{k-2i+1}, \ldots, e_n, f_{k-2i+1}, \ldots, f_n\rangle \cong W_{2i,n-k+2i}$.

Lemma 3.1. *A vector of $W_{k,n}$ is fixed by U if and only if it belongs to $\langle \widehat{e}_i \wedge \widehat{v}_i \rangle_{i=0}^{\lfloor k/2 \rfloor}$. A point of $\mathrm{PG}(W_{k,n})$ is fixed by B if and only if it is equal to $\langle \widehat{e}_i \wedge \widehat{v}_i \rangle$ for some $i = 0, 1, \ldots, \lfloor k/2 \rfloor$.*

Proof. Let Θ_k be the set of ordered triples (X, Y, Z) of pairwise disjoint (and possibly empty) subsets of $I = \{1, 2, \dots, n\}$ such that $|X| + 2|Y| + |Z| = k$. Every vector $v \in W_{k,n}$ can be written in a unique way as a linear combination

$$v = \sum_{(X,Y,Z)\in\Theta_k} \lambda_{X,Y,Z} e_X \wedge (e_Y \wedge f_Y) \wedge f_Z.$$

Suppose that $U(v) = v$. This condition is equivalent to $L(U)(v) = 0$, where $L(U)$ is the nilpotent subalgebra of the Lie algebra $L(G)$ of G corresponding to U. Considering elements of $L(U)$ corresponding to long roots, it is straightforward to check that $\lambda_{X,Y,Z} = 0$ whenever $Z \neq \emptyset$. So,

$$v = \sum(\lambda_{X,Y} e_X \wedge (e_Y \wedge f_Y) \mid X \cap Y = \emptyset, |X| + 2|Y| = k)$$

where $\lambda_{X,Y} := \lambda_{X,Y,\emptyset}$. We can now consider elements of $L(U)$ corresponding to short simple roots or sums of short simple roots. Given two disjoint subsets $X, Y \subset I$ such that $|X| + 2|Y| = k$, we write $X < Y$ if every element of X is smaller than all elements of Y. Recalling that the elements of $L(U)$ map v to 0, one can see that $\lambda_{X,Y} = 0$ only if $X < Y$ and that if $X < Y, Y'$ then $\lambda_{X,Y} = \lambda_{X,Y'}$. We leave details for the reader. At this stage,

$$v = \sum\big(\lambda_{X,Y} e_X \wedge (e_Y \wedge f_Y) \mid X < Y, |X| + 2|Y| = k\big).$$

It remains to prove that X is an initial segment of I. This can be seen by considering elements of $L(U)$ corresponding to sums of short simple roots and one long root. Again, we leave details for the reader. The first claim of the lemma is proved.

Turning to the second claim, note that if $B(\langle v\rangle) = \langle v\rangle$ then $U(v) = v$. Therefore, if $\langle v\rangle$ is fixed by B, then $v \in \langle \widehat{e}_i \wedge \widehat{v}_i\rangle_{i=0}^{\lfloor k/2\rfloor}$, say $v = \sum_{i=0}^{\lfloor k/2\rfloor} \lambda_i \widehat{e}_i \wedge \widehat{v}_i$. Let now H be the Cartan subgroup of B stabilizing each of the subspaces $\langle \widehat{e}_i \wedge \widehat{v}_i\rangle$. Recall that $H \cong (\mathbb{F}^*)^n$. If $g \in H$ corresponds to $(t_1, \dots, t_n) \in (\mathbb{F}^*)^n$, then g maps v to

$$g(v) = \sum_{i=0}^{\lfloor k/2\rfloor} (t_1 \cdots t_{k-2i}) \cdot \lambda_i \widehat{e}_i \wedge \widehat{v}_i.$$

The vectors $\widehat{e}_i \wedge \widehat{v}_i$ are independent, the scalars $t_1, \dots, t_n$ are arbitrary elements of $\mathbb{F}^*$ and $\mathbb{F}^*$ contains at least two elements. It follows that $H(v) \in \langle v\rangle$ if and only if v is proportional to one of the vectors $\widehat{e}_i \wedge \widehat{v}_i$. □

Lemma 3.2. *For $J \subset I = \{1, 2, \dots, n\}$, let $j = |J|$, put $V^J = \langle e_i, f_i\rangle_{i\in I\setminus J} \cong V(2n - 2j, \mathbb{F})$ and $W^J_{k-j} = \wedge^{k-j} V^J \cong W_{k-j,n-j}$ and let $V^J_{k-j} \cong V_{k-j,n-j}$ be the subspace of W^J_{k-j} spanned by the vectors representing totally isotropic subspaces of V^J. Then $e_J \wedge V^J_{k-j} = (e_J \wedge W^J_{k-j}) \cap V_{k,n}$.*

Proof. Clearly $e_J \wedge V^J_{k-j} \subseteq (e_J \wedge W^J_{k-j}) \cap V_{k,n}$. To prove the converse, we exploit a result by De Bruyn [11]. As in the proof of Lemma 3.1, let Θ_k be the set of triples $\{X, Y, Z\}$ of (possibly empty) subsets of I such that X, Y and Z are pairwise disjoint and $|X|+2|Y|+|Z| = k$. Then $\{e_X \wedge (e_Y \wedge f_Y) \wedge f_Z\}_{\{X,Y,Z\}\in\Theta_k}$ is a basis of $W_{k,n}$. For every $l \in \{0, \dots, \lfloor\frac{k}{2}\rfloor\}$, let $\theta_{k,l} \colon W_{k,n} \to W_{k-2l,n}$, be the linear mapping defined as follows on the basis vectors $e_X \wedge (e_Y \wedge f_Y) \wedge f_Z$ of $W_{k,n}$:

$$\theta_{k,l} \colon e_X \wedge (e_Y \wedge f_Y) \wedge f_Z \ \mapsto \sum_{Y' \subset Y,\ |Y'|=|Y|-l} e_X \wedge (e_{Y'} \wedge f_{Y'}) \wedge f_Z$$

with the convention the sum is 0 when $|Y| < l$. (Note that $\theta_{k,l}$ only effects the anisotropic term $e_Y \wedge f_Y$ of the vector $e_X \wedge (e_Y \wedge f_Y) \wedge f_Z$ leaving the isotropic term $e_X \wedge f_Z$ unchanged.)

De Bruyn [11, Theorem 3.5] proves that $V_{k,n} = \cap_{i=1}^{\lfloor\frac{k}{2}\rfloor} \ker(\theta_{k,i})$. So, let $v \in (e_J \wedge W^J_{k-j}) \cap V_{k,n}$, say $v = e_J \wedge w$ for a suitable $w \in W^J_{k-j}$. As $v \in V_{k,n}$ we have $\theta_{k,i}(v) = 0$ for every $i = 1, \dots, \lfloor\frac{k}{2}\rfloor$. However, it is clear from the definition that $\theta_{k,i'}(e_J \wedge w) = e_J \wedge \theta^J_{k-j,i'}(w)$, $i' = 1, \dots, \lfloor\frac{k-j}{2}\rfloor$, where $\theta^J_{k-j,i'} \colon W^J_{k-j} \to W_{k-j-2i',n-j}$ is defined just in the same way as $\theta_{k,i}$, but on $W^J_{k-j} \cong W_{k-j,n-j}$. Since $\theta_{k,i}(v) = 0$ for every $i = 1, \dots, \lfloor\frac{k}{2}\rfloor$, we must have $\theta^J_{k-j,i'}(w) = 0$ for every $i' = 1, \dots, \lfloor\frac{k-j}{2}\rfloor$. However, $\cap_{i'=1}^{\lfloor\frac{k-j}{2}\rfloor} \ker(\theta^J_{k-j,i'}) = V^J_{k-j}$ by [11, Theorem 3.5]. Hence $w \in V^J_{k-j}$. □

Lemma 3.3. *The module $V_{k,n}/R(V_{k,n})$ is self-dual.*

Proof. With $\alpha_k(.,.)$ as in Section 2.3, let f be the linear mapping from $V_{k,n}$ to its dual $V^*_{k,n}$ sending $v \in V_{k,n}$ to the functional f_v that maps every $x \in V_{k,n}$ onto $\alpha_k(v, x)$. As $R(V_{k,n})$ is the radical of the restriction of α_k to $V_{k,n}$, the linear mapping f induces an isomorphism $\widehat{f}$ from $V_{k,n}/R(V_{k,n})$ to its dual $(V_{k,n}/R(V_{k,n}))^*$ Clearly, G commutes with $\widehat{f}$. Hence $V_{k,n}/R(V_{k,n}) \cong (V_{k,n}/R(V_{k,n}))^*$ as G-modules. □

The next lemma immediately follows from Theorem 1.1, but we prefer not to use that theorem, as far as possible. So, we shall give a more straightforward proof here.

Lemma 3.4. *If $V_{r,n}/R(V_{r,n}) \cong V_{s,n}/R(V_{s,n})$ as G-modules, then $r = s$.*

Proof. Let B and B' be the stabilizers of the chambers $(\langle e_1, \dots, e_i\rangle)_{i=1}^n$ and respectively $(\langle f_1, \dots, f_i\rangle)_{i=1}^n$ and let U and U' be their unipotent radicals. Let $V_{r,n}/R(V_{r,n}) \cong V_{s,n}/R(V_{s,n})$ and put $J_r = \{1, 2, \dots, r\}$ and $J_s = \{1, 2, \dots, s\}$.

Let f be an isomorphism from $V_{s,n}/R(V_{s,n})$ to $V_{r,n}/R(V_{r,n})$. Turning to the Lie algebra $L(G)$ of G, $V_{r,n}/R(V_{r,n})$ and $V_{s,n}/R(V_{s,n})$ are also isomorphic $L(G)$-modules. Moreover, e_{J_r} and e_{J_s} are highest weight vectors in $V_{r,n}$ and $V_{s,n}$ respectively, where the positive (negative) roots correspond to the root subgroups of U (respectively, U'). It follows that $f(e_{J_s})$ is $R(V_{r,n})$-equivalent to a weight vector of $V_{r,n}$. Similarly, $f^{-1}(e_{J_r})$ is $R(V_{s,n})$-equivalent to a weight vector of $V_{s,n}$. Therefore, if $A(U')$ is the subalgebra of the enveloping associative algebra of $L(G)$ generated by the subalgebra of $L(G)$ corresponding to U', then $f(e_{J_s}) + R(V_{r,n}) = u_1(e_{J_r}) + R(V_{r,n})$ for an element $u_1 \in A(U')$. Similarly, $f^{-1}(e_{J_r}) + R(V_{s,n}) = u_2(e_{J_s}) + R(V_{s,n})$ for an element $u_2 \in A(U)$. Hence $e_{J_r} + R(V_{r,n}) = u_2(f(e_{J_s})) + R(V_{r,n})$. It follows that $u_2u_1(e_{J_r}) + R(V_{r,n}) = e_{J_r} + R(V_{r,n})$. This can happen only if $u_2u_1 = 1$, namely $u_1 = u_2 = 1$. This forces $f(e_{J_r}) = e_{J_s}$. Let λ_r and λ_s be the fundamental dominant weights relative to the types r and s respectively and let $\mathcal{H}$ be the Cartan subalgebra of $L(G)$, relative to the choice of $(\langle e_1, \ldots, e_i\rangle)_{i=1}^n$ as the fundamental chamber. Then $h(e_{J_r}) = \lambda_r(h)e_{J_r}$ and $h(e_{J_s}) = \lambda_s(h)e_{J_s}$ for every $h \in \mathcal{H}$. However, $f(e_{J_r}) = e_{J_s}$ and f is an isomorphism of $L(G)$-modules. Therefore $\lambda_s(h)e_{J_s} = h(e_{J_s}) = h(f(e_{J_r})) = f(h(e_{J_r})) = f(\lambda_r(h)e_{J_r}) = \lambda_r(h)f(e_{J_r}) = \lambda_r(h)e_{J_s}$. Hence $\lambda_s(h) = \lambda_r(h)$ for every $h \in \mathcal{H}$. It follows that $r = s$. □

3.2 From Lemma 1.3 to Theorem 1.1

Lemma 3.5. *Every irreducible section of $V_{k,n}$ is a copy of a section $V_{r,n}/R(V_{r,n})$ for some nonnegative integer $r \leq k$.*

Proof. By induction on k. If $k = 1$ then $V_{1,n} = V(2n, \mathbb{F})$, which is irreducible. In this case there is nothing to prove. Let $k > 1$ and let S'/S be an irreducible section of $V_{k,n}$. If $S' = V_{k,n}$ then $S = R(V_{k,n})$ and we are done. So, let $S' \subset V_{k,n}$, namely $S' \subseteq R(V_{k,n})$. Then $V_{k,n} \subset T' := S'^{\perp_k} \subset T := S^{\perp_k}$. Moreover, T/T' is dually isomorphic to S'/S.

We shall now exploit the basic series of G in $W_{k,n}$. Suppose that $T \cap V^{(k,n)}_{k-2i} + V^{(k,n)}_{k-2i+2} = T' \cap V^{(k,n)}_{k-2i} + V^{(k,n)}_{k-2i+2}$ for every i. However $T \cap V^{(k,n)}_{k-2i} \supset T' \cap V^{(k,n)}_{k-2i}$ for at least one i, since $T \supset T'$ and $\cup_{i\geq 0} V^{(k,n)}_{k-2i} = W_{k,n}$. Let i be such that $T \cap V^{(k,n)}_{k-2i} \supset T' \cap V^{(k,n)}_{k-2i}$, but as small as possible. Certainly $i > 0$, since $T \cap V_{k,n} = T' \cap V_{k,n} = V_{k,n}$. Choose $x \in T \cap V^{(k,n)}_{k-2i} \setminus T' \cap V^{(k,n)}_{k-2i}$. Since by assumption $T \cap V^{(k,n)}_{k-2i} + V^{(k,n)}_{k-2i+2} = T' \cap V^{(k,n)}_{k-2i} + V^{(k,n)}_{k-2i+2}$, we have $x = x' + y$ with $x' \in T' \cap V^{(k,n)}_{k-2i}$ and $y \in V^{(k,n)}_{k-2i+2}$. So, $y = x - x' \in T \cap V^{(k,n)}_{k-2i+2}$. By the minimality of i, $T \cap V^{(k,n)}_{k-2i+2} = T' \cap V^{(k,n)}_{k-2i+2}$. Therefore $y \in T'$. It follows that $x = x' + y \in T'$, contrary to the choice of x. This contradiction shows that $T\cap V^{(k,n)}_{k-2i}+V^{(k,n)}_{k-2i+2}\supset T'\cap V^{(k,n)}_{k-2i}+V^{(k,n)}_{k-2i+2}$ for at least one $i>0$.

Let $i>0$ be such that $T\cap V^{(k,n)}_{k-2i}+V^{(k,n)}_{k-2i+2}\supset T'\cap V^{(k,n)}_{k-2i}+V^{(k,n)}_{k-2i+2}$ and consider the quotient

$$Q:=\frac{T\cap V^{(k,n)}_{k-2i}+V^{(k,n)}_{k-2i+2}}{T'\cap V^{(k,n)}_{k-2i}+V^{(k,n)}_{k-2i+2}}.$$

Then Q is a homomorphic image of T/T'. However T/T' is irreducible, since it is dually isomorphic to S'/S which is irreducible. Hence $Q\cong T/T'$. It follows that $Q\cong T/T'$ is an irreducible section of $V^{(k,n)}_{k-2i}/V^{(k,n)}_{k-2i+2}$. The latter is isomorphic to $V_{k-2i,n}$, by Proposition 2.5. Therefore $Q\cong T/T'$ is isomorphic to an irreducible section of $V_{k-2i,n}$. We can now apply our inductive hypothesis: all irreducible sections of $V_{k-2i,n}$ are isomorphic to a section $V_{r,n}/R(V_{r,n})$ for some $r<k-2i$. Hence $T/T'\cong V_{r,n}/R(V_{r,n})$, namely S'/S is dually isomorphic to $V_{r,n}/R(V_{r,n})$. However $V_{r,n}/R(V_{r,n})$ is self-dual (Lemma 3.3). So, $S'/S\cong V_{r,n}/R(V_{r,n})$. □

Lemma 3.6. *Let S'/S be an irreducible section of $V_{k,n}$. Then there exists a unique nonnegative integer $r\le k$ such that $S'/S\cong V_{r,n}/R(V_{r,n})$. Moreover, there exists a unique isomorphism of G-modules $f\colon V_{r,n}/R(V_{r,n})\to S'/S$. Let $J=\{1,2,\dots,r\}$. If $r<k$ then S' contains a vector $e_J\wedge v_J$ where $v_J\in V^J_{k-r}\cong V_{k-r,n-r}$ and $f(e_J+R(V_{r,n}))=e_J\wedge v_J+S$ (notation as in Lemma 3.2). If $r=k$ then $S'=V_{k,n}$, $S=R(V_{k,n})$ and f is the identity automorphism of $V_{k,n}/R(V_{k,n})$.*

Proof. The existence of r follows from Lemma 3.5 and its uniqueness follows from Lemma 3.4. Assume first that $r<k$ and let f be an isomorphism from $V_{r,n}/R(V_{r,n})$ to S'/S. Then f is induced by a unique homomorphism $\widehat{f}\colon V_{r,n}\to S'/S$. Let $J=\{1,2,\dots,r\}$ and let B be the Borel subgroup of G stabilizing the chamber $(\langle e_1,\dots,e_i\rangle)_{i=1}^n$. The fundamental weight vector e_J of $V_{r,n}$ is mapped by $\widehat{f}$ onto a vector $v+S$ of S'/S on which B acts as on e_J. (In particular, B stabilizes $\langle v\rangle+S$, but when saying that B acts on $v+S$ as on e_J we say more than that.) For every $X\subseteq J$ we can find a vector v_X of $\wedge^{k-|X|}\langle e_{r+1},\dots,e_n,f_1,\dots,f_n\rangle$ in such a way that $v=\sum_{X\subseteq J}e_X\wedge v_X$. The vectors v_X are linear combinations of vectors $e_K\wedge f_H$ where $K\subseteq\{r+1,\dots,n\}$, $H\subseteq\{1,\dots,n\}$ and $|K|+|H|=k-r$. For every $j\in J$, we can split v_X as $v_X=v^+_{X,j}+v^-_{X,j}$ where $v^+_{X,j}$ is a linear combination of vectors $e_K\wedge f_H$ as above with $j\in H$ while $v^-_{X,j}$ is a linear combination of vectors $e_K\wedge f_H$ with $j\notin H$.

So, $v = v_j^+ + v_j^- + w_j^+ + w_j^-$ where

$$v_j^+ = \sum_{j\in X\subseteq J} e_X \wedge v_{X,j}^+, \qquad v_j^- = \sum_{j\in X\subseteq J} e_X \wedge v_{X,j}^-,$$
$$w_j^+ = \sum_{j\notin X\subseteq J} e_X \wedge v_{X,j}^+, \qquad w_j^- = \sum_{j\notin X\subseteq J} e_X \wedge v_{X,j}^-.$$

Let b be the element of B fixing e_i and f_i for $i \neq j$ and sending e_j to te_j and f_j to $t^{-1}f_j$. Then

$$b(v) - tv \in S. \tag{4}$$

Indeed $b(e_J) = te_J$ in $V_{r,n}$ because $j \in J$ and $b(v+S) = t(v+S)$. (Recall that B acts on $v+S$ as on e_J.) On the other hand,

$$b(v) = v_j^+ + tv_j^- + t^{-1}w_j^+ + w_j^-. \tag{5}$$

By substituting (5) in (4) we obtain that $(1-t)v_j^+ + (t^{-1}-t)w_j^+ + (1-t)w_j^- \in S$. If $t \neq 1$ then

$$v_j^+ + \frac{t^{-1}-t}{1-t}w_j^+ + w_j^- \in S. \tag{6}$$

By putting $t = -1$ in (6) we obtain that

$$v_j^+ + w_j^- \in S. \tag{7}$$

Suppose that $\mathbb{F}$ contains at least four elements. Then we can assume to have chosen $t \neq 1, -1$ in (6). Hence $(t^{-1}-t)/(1-t) \neq 1$ and by comparing (6) with (7) we also obtain that $w_j^+ \in S$. Therefore $v_j^+ + w_j^+ + w_j^- \in S$ and we can assume to have chosen the representative v of $v+S$ in such a way that $v = v_j^- = \sum_{j\in X\subseteq J} e_X \wedge v_{X,j}^-$. However this holds for every $j \in J$. By considering the elements of J in some order and adjusting at every step the choice of v as explained above, we can eventually assume to have chosen v so that $v = e_J \wedge v_J$ where $v_J \in W_{k-r}^J = \wedge^{k-r}\langle e_{r+1}, \ldots, e_n, f_{r+1}, \ldots, f_n\rangle$ (notation as in Lemma 3.2). On the other hand, $e_J \wedge v_J \in S' \subseteq V_{k,n}$. By Lemma 3.2, $v_J \in V_{k-r}^J$. So, $\widehat{f}(e_J) = e_J \wedge v_J + S$ with $v_J \in V_{k-r}^J$. Note that no use is made in the above argument of the hypothesis that S'/S is irreducible. We have only exploited the existence of a surjective homomorphism of G-modules $\widehat{f}\colon V_{r,n} \to S'/S$.

Let now $\mathbb{F} = \mathbb{F}_3$. (Recall that $\mathbb{F} \neq \mathbb{F}_2$ because $\mathrm{char}(\mathbb{F}) \neq 2$ by assumption.) Then (6) and (7) only allow us to choose $v = \sum_{X\subseteq J} e_X \wedge f_{J\setminus X} \wedge v_X$ with $v_X \in W_{k-r}^J$ for every $X \subseteq J$. However, we can get out from this blind alley by the following trick. Note first that the G-invariant subspaces of $V_{k,n}$ are precisely the $L(G)$-invariant subspaces of $V_{k,n}$, where $L(G)$ is the Lie algebra of G. However, $L(G)$ bears the structure of a vector space. So, given any extension $\overline{\mathbb{F}}$ of $\mathbb{F}$ (for instance, $\overline{\mathbb{F}} = \mathbb{F}_9$), we can consider the scalar extensions $\overline{L}(G) := \overline{\mathbb{F}} \otimes L(G)$ as well as $\overline{V}_{k,n} := \overline{\mathbb{F}} \otimes V_{k,n}$. For every $a \in L(G)$, $x \in V_{k,n}$ and scalars $\alpha, \xi \in \overline{\mathbb{F}}$, we have $(\alpha \otimes a) \cdot (\xi \otimes x) = (\alpha\xi)(a \cdot x) = a \cdot (\alpha\xi x)$. Therefore the $\overline{\mathbb{F}}$-extensions of the $L(G)$-invariant subspaces of $V_{k,n}$ are $\overline{L}(G)$-invariant subspaces of $\overline{V}_{r,n}$. In this way, by replacing $\mathbb{F}$ with $\overline{\mathbb{F}}$, S with $\overline{S} := \overline{\mathbb{F}} \otimes S$, S' with $\overline{S}' := \overline{\mathbb{F}} \otimes S'$ and $\widehat{f}$ with the homomorphism $\overline{\mathbb{F}} \otimes \widehat{f} \colon \overline{V}_{r,n} \to \overline{S}'/\overline{S}$, we are led back to the case where $|\mathbb{F}| > 3$ and we are done.

So far, we have assumed $r < k$. When $r = k$ we can use the same argument as above, except that now V^J_{k-r} is the trivial (1-dimensional) G-module and $e_J \wedge v_J$ means the same as e_J. In this case, $\widehat{f}(e_J) = e_J$. As the G-orbit of e_J spans $V_{k,n}$, we obtain that $S' = V_{k,n}$. Consequently, $S = R(V_{k,n})$. As $f(g(e_J + R(V_{k,n}))) = g(f(e_J + R(V_{k,n}))) = g(e_J + S)$ for every $g \in G$, the isomorphism f is the identity. Turning back to the case of $r < k$, if there are two distinct isomorphisms f_1, f_2 from $V_{r,n}/R(V_{r,n})$ to S'/S then $f := f_2^{-1} f_1$ is an automorphism of $V_{r,n}/R(V_{r,n})$. By the above, $f = \mathrm{id}$. Hence $f_1 = f_2$. □

In the next proposition S'/S is an irreducible section of $V_{k,n}$ and $r \leq k$ is the nonnegative integer such that $S'/S \cong V_{r,n}/R(V_{r,n})$, existing and unique by Lemmas 3.5 and 3.4.

Proposition 3.7. *Let $S'/S \cong V_{r,n}/R(V_{r,n})$ be an irreducible section of $V_{k,n}$, with $S' \subset V_{k,n}$. Then $r < k$ and $r \in J_p(k, n)$.*

Proof. By Lemma 3.6, we have $r < k$ and there exists a unique isomorphism $f \colon V_{r,n}/R(V_{r,n}) \to S'/S$. Put $J = \{1, 2, \ldots, r\}$ and $v + S = f(e_J + R(V_{r,n}))$. By Lemma 3.6, we can choose $v = e_J \wedge v_J$ with $v_J \in V^J_{k-r}$.

Let $A = \langle e_1, \ldots, e_r \rangle$ and let L be the Levi complement of the unipotent radical of the stabilizer of A in G. It is well known that $L = L_1 \times L_2$ where $L_1 \cong \mathrm{GL}(r, \mathbb{F})$ and $L_2 \cong \mathrm{Sp}(2n - 2r, \mathbb{F})$. The latter group acts naturally on $V^J_{k-r} \cong V_{k-r,n-r}$ (notation as in Lemma 3.2) and fixes e_J. So, if $g \in L_2$, then $e_J \wedge (v_J - g(v_J)) = v - g(v) \in S$. Let S_J be the subspace of V^J_{k-r} formed by the vectors $x \in V^J_{k-r}$ such that $e_J \wedge x \in S$. Clearly, S_J is a submodule of V^J_{k-r}. By the above, $S'_J := \langle v_J, S_J \rangle$ is also a submodule of V^J_{k-r} and S_J has codimension 1 in S'_J. Hence the composition series of V^J_{k-r} admits a 1-dimensional section. By Lemma 1.3, $0 \in J_p(k - r, n - r)$, whence $r \in J_p(k, n)$. □

The first claim of the next proposition is a special case of Proposition 3.7. However the argument we will use to prove the second part entails a proof of the first claim. So, we prefer to regard this result as a new proposition, independent of Proposition 3.7.

Proposition 3.8. *Let $S \neq 0$ be an irreducible proper submodule of $V_{k,n}$. Then $S \cong V_{r,n}/R(V_{r,n})$ for a unique integer $r \in J_p(k,n)$. Moreover, $S(V_{k-r,n-r})$ is 1-dimensional.*

Proof. Let B be the Borel subgroup of G stabilizing the chamber $\{\langle e_1, \ldots, e_i\rangle\}_{i=1}^{n}$. By Lie's Theorem, B stabilizes a 1-dimensional subspace $\langle v\rangle$ of S. By Lemma 3.1, $\langle v\rangle = \langle \widehat{e}_i \wedge \widehat{v}_i\rangle$ for some $i = 0, 1, \ldots, \lfloor k/2\rfloor$. Without loss of generality, we may suppose $v = \widehat{e}_i \wedge \widehat{v}_i$. Let $r = k - 2i$. As $v \in S$ and $S \subset V_{k,n}$, the vector $\widehat{v}_i$, which generates the local pole of G at $A_i = \langle e_i\rangle_{i=1}^{k-2i}$, belongs to $V_{k-r,n-r}$ by Lemma 3.2. Hence $\langle \widehat{v}_i\rangle = S(V_{k-r,n-r})$. Therefore $\dim(S(V_{k-r,n-r})) = 1$.

As S is irreducible, the G-orbit of v spans S. Therefore $S \cong V_{r,n}/R(V_{r,n})$. By Lemma 3.4, r is the unique integer such that $S \cong V_{r,n}/R(V_{r,n})$. By Proposition 3.7, $r \in J_p(k,n)$. □

So far we have shown that claim (2) of Lemma 1.3 and the results collected in Section 2 are sufficient to prove that the irreducible sections of $V_{k,n}$ have the dimensions that can be obtained from Theorem 1.1. Two things remain to prove in order to get back the whole of Theorem 1.1, namely the following:

1. At most one 1-dimensional section occurs in $V_{k,n}$.
2. If $V_{k-r,n-r}$ admits a 1-dimensional section then $V_{k,n}$ admits a section isomorphic to $V_{r,n}/R(V_{r,n})$.

Claim 1 is sufficient to prove that no two distinct irreducible sections of $V_{k,n}$ can be isomorphic. Indeed let $S'/S \cong T'/T \cong V_{r,n}$ for irreducible sections S'/S and T'/T of $V_{k,n}$. Let $J = \{1, 2, \ldots, r\}$ and choose $v_J, w_J \in V_{k-r,n-r}$ so that e_J corresponds to $e_J \wedge v_J + S$ in S'/S and to $e_J \wedge w_J + T$ in T'/T. As in the proof of Proposition 3.7, let $S_J = \langle g(v_J) - v_J\rangle_{g\in G}$, $S'_J = \langle v_J, S_J\rangle$, $T_J = \langle g(w_J) - w_J\rangle_{g\in G}$ and $T'_J = \langle w_J, T_J\rangle$. Then S'_J/S_J and T'_J/T_J are 1-dimensional sections of $V_{k-r,n-r}$. By claim 1, $S'_J = T'_J$ and $S_J = T_J$. Hence $v_J = w_J$. Consequently $S' = T'$ and $S = T$.

By Lemma 1.3, claim 2 is sufficient to prove that $V_{k,n}$ admits a section isomorphic to $V_{r,n}/R(V_{r,n})$, for every $r \in J_p(k,n)$.

4 Geometric submodules of $V_{k,n}$

Put $h := n - k$, as in Section 2.3. Given a positive integer $r < k$ and an r-element X of Δ, we put $V^X_{k,n} := \langle \varepsilon_{k,n}((\Delta^+_X)_{k-r})\rangle$, where Δ^+_X, $(\Delta^+_X)_{k-r}$ and $\varepsilon_{k,n}((\Delta^+_X)_{k-r})$ have the meaning stated in Section 2.2. According to Proposition 2.1, the embedding $\varepsilon^X_{k,n} \colon (\Delta^+_X)_{k-r} \mapsto V^X_{k,n}$ is isomorphic to $\varepsilon_{k-r,n-r}$.

Suppose that $r \in \widetilde{J}_p(k,n)$, namely the socle $S(V_{k-r,n-r})$ of $V_{k-r,n-r}$ is 1-dimensional. So, by Lemma 1.2, $k-r$ is even and, for every r-element X of Δ, the local pole P_X of G at X is equal to $S(V_{k,n}^X)$. (Recall that P_X is the unique point of $\mathrm{PG}(W_{k,n}^X)$ fixed by the stabilizer G_X of X in G; see Section 1.) Then $n-r \geq N(h,p)$ by Proposition 2.2 applied to $V_{k-r,n-r}$. Hence $n > N(h,p)$. Therefore $\dim(R(V_{k,n})) > 1$ again by Proposition 2.2, but now applied to $V_{k,n}$.

As in Section 1, let $\mathcal{P}_{k,n}^r$ be the subspace of $V_{k,n}$ spanned by the 1-dimensional subspaces P_X, for X an r-element of Δ. Clearly, $\mathcal{P}_{k,n}^r$ is stabilized by G. We call $\mathcal{P}_{k,n}^r$ a *geometric submodule* of $V_{k,n}$, also geometric submodule of *type* r. Define the following map

$$\begin{aligned} \pi_{k,n}^r \colon \Delta_r &\to \mathrm{PG}(\mathcal{P}_{k,n}^r), \\ X &\mapsto P_X. \end{aligned}$$

Theorem 4.1. *The mapping $\pi_{k,n}^r$ is an embedding of Δ_r. Moreover $\pi_{k,n}^r$ is G-homogeneous and there exists a morphism from the natural embedding $\varepsilon_{r,n}$ of Δ_r to $\pi_{k,n}^r$.*

Proof. Assume $r < n$, to fix ideas. The case of $r = n$ can be dealt with in a similar way, modulo minor modifications, which we leave to the reader.

We first show that lines of Δ_r are mapped onto lines of $\mathcal{P}_{k,n}^r$. Let X_1 and X_2 be two distinct collinear points of Δ_r. They are r-dimensional totally isotropic subspaces of V. As they are assumed to be collinear, without loss of generality we may suppose that $X_1 = \langle e_1, \dots, e_{r-1}, e_r \rangle$ and $X_2 = \langle e_1, \dots, e_{r-1}, e_{r+1} \rangle$. So, a point $X_3 \neq X_1, X_2$ on the line of Δ_r through X_1 and X_2 corresponds to an r-dimensional totally isotropic subspace of the form $\langle e_1, \dots, e_{r-1}, e_r + te_{r+1} \rangle$, $t \in \mathbb{F} \setminus \{0\}$.

By Proposition 2.6 and the proof of Proposition 2.1, we obtain that $P_{X_1} = \langle v_1 \rangle$ and $P_{X_2} = \langle v_2 \rangle$, where

$$\begin{aligned} v_1 &= \sum_{J \in \binom{\{r+1, r+2, \dots, n\}}{(k-r)/2}} e_J \wedge f_J \wedge (e_1 \wedge \cdots \wedge e_{r-1} \wedge e_r), \\ v_2 &= \sum_{J \in \binom{\{r, r+2, \dots, n\}}{(k-r)/2}} e_J \wedge f_J \wedge (e_1 \wedge \cdots \wedge e_{r-1} \wedge e_{r+1}). \end{aligned}$$

In order to compute P_{X_3} we need to extend the basis $\{e_1, \dots, e_{r-1}, e_r + te_{r+1}\}$ of X_3 to a basis of $X_3^\perp$ by adding a hyperbolic basis B of a complement of X_3 in $X_3^\perp$. We make the following choice:

$$B = \{e_{r+2}, e_{r+3}, \dots, e_n, \tfrac{1}{t} f_{r+1} - f_r, f_{r+2}, f_{r+3}, \dots, f_n, e_r\}.$$

We also put $e_* := -f_r + \frac{1}{t} f_{r+1}$ and $f_* := e_r$, regarding the symbol $*$ as an additional index, in order to get a list of $n-r$ indices, namely $r+2, r+3, \ldots, n, *$. With this convention, $P_{X_3} = \langle v_3 \rangle$ where

$$v_3 = \sum_{J \in \binom{\{r+2, r+3, \ldots, n, *\}}{(k-r)/2}} e_J \wedge f_J \wedge (e_1 \wedge \cdots \wedge e_{r-1} \wedge (e_r + te_{r+1})).$$

It is now straightforward to check that $v_3 = v_1 + tv_2$. It is now clear that $\pi^r_{k,n}$ maps lines of Δ_r onto lines of $\mathrm{PG}(\mathcal{P}^r_{k,n})$.

We shall now prove that the map $\pi^r_{k,n}$ is injective. The group G permutes the fibers of $\pi^r_{k,n}$. Moreover, G acts transitively and imprimitively on the point-set of Δ_r. Therefore, either $\pi^r_{k,n}$ is injective or it maps all points of Δ_r to one single point. However, the previous discussion makes it clear that the latter case is impossible. Hence $\pi^r_{k,n}$ is injective.

So, $\pi^r_{k,n}$ is an embedding of Δ_r. As $g(P_X) = P_{g(X)}$ for every r-element X of Δ and every element g of G, the embedding $\pi^r_{k,n}$ is G-homogeneous. As the natural embedding $\varepsilon_{r,n}$ of Δ_r is absolutely universal, there exists a homomorphism of vectors spaces $\varphi\colon V_{r,n} \mapsto \mathcal{P}^r_{k,n}$ such that $\varphi\varepsilon_{r,n} = \pi^r_{k,n}$, namely φ is a morphism of embeddings from $\varepsilon_{r,n}$ to $\pi^r_{k.n}$. $\square$

Corollary 4.2. $\mathcal{P}^r_{k,n} \subseteq R(V_{k,n})$.

Proof. By Theorem 4.1, the embedding $\pi^r_{k,n}$ is a homomorphic image of $\varepsilon_{r,n}$. Hence $\mathcal{P}^r_{k,n}$ is a homomorphic image of $V_{r,n}$. However $\dim(V_{r,n}) < \dim(V_{k,n})$. Therefore $\mathcal{P}^r_{k,n} \subset V_{k,n}$. It follows that $\mathcal{P}^r_{k,n} \subseteq R(V_{k,n})$, since $\mathcal{P}^r_{k,n}$ is G-invariant. $\square$

Let now $1 \leq s < r$ and suppose that $S(V_{k-s,n-s})$ is also 1-dimensional. Thus, we can also consider the geometric submodule $\mathcal{P}^s_{k,n}$ of type s.

Lemma 4.3. $\mathcal{P}^s_{k,n} \subset \mathcal{P}^r_{k,n}$.

Proof. Let X be an element of Δ of type s and let $\mathcal{P}^r_X$ be the subspace of $V^X_{k,n}$ spanned by the set of poles P_Y for Y an r-element of Δ incident to X, namely a point of $(\Delta^+_X)_{r-s}$. Then $\mathcal{P}^r_X \subseteq R(V^X_{k,n})$, by Corollary 4.2. However $P_X = S(V^X_{k,n})$ by Lemma 1.2. Hence $P_X \subseteq \mathcal{P}^r_X$. On the other hand, $\mathcal{P}^r_X \subseteq \mathcal{P}^r_{k,n}$. Therefore $\mathcal{P}^s_{k,n} \subseteq \mathcal{P}^r_{k,n}$.

It remains to prove that $\mathcal{P}^s_{k,n}$ is properly contained in $\mathcal{P}^r_{k,n}$. Suppose to the contrary that $\mathcal{P}^s_{k,n} = \mathcal{P}^r_{k,n}$. We know by the second part of Theorem 4.1 that $\mathcal{P}^r_{k,n} \cong V_{r,n}/U_r$ and $\mathcal{P}^s_{k,n} \cong V_{s,n}/U_s$ for suitable subspaces U_r and U_s of $V_{r,n}$ and $V_{s,n}$ respectively. As both $\mathcal{P}^r_{k,n}$ and $\mathcal{P}^s_{k,n}$ are G-homogeneous, both U_r and U_s are G-invariant [7, Proposition 2.4]. Hence $U_r \subseteq R(V_{r,n})$ and $U_s \subseteq R(V_{s,n})$.

However $V_{r,n}/U_r \cong V_{s,n}/U_s$, since $\mathcal{P}^r_{k,n} = \mathcal{P}^s_{k,n}$. This forces $V_{r,n}/R(V_{r,n})$ to be a quotient of $V_{s,n}$, which is clearly impossible since $V_{r,n}/R(V_{r,n})$ is irreducible, $V_{s,n}/R(V_{s,n})$ is the unique non-trivial irreducible quotient of $V_{s,n}$ and $V_{r,n}/R(V_{r,n}) \ncong V_{s,n}/R(V_{s,n})$ because $r > s$ (Lemma 3.4). □

Corollary 4.4. *With r and s as above, the factor module $\mathcal{P}^r_{k,n}/\mathcal{P}^s_{k,n}$ admits a quotient isomorphic to $V_{r,n}/R(V_{r,n})$.*

Proof. By Theorem 4.1, $\mathcal{P}^r_{k,n} \cong V_{r,n}/X$ for a submodule X of $R(V_{r,n})$. So, $\mathcal{P}^r_{k,n}/\mathcal{P}^s_{k,n} \cong V_{r,n}/Y$ for a submodule Y of $V_{r,n}$ containing X. Lemma 4.3 implies that $Y \subset V_{r,n}$ whence $Y \subseteq R(V_{r,n})$. □

Still assuming $n \geq N(h,p)$ with $h = n-k$, let $\{r_1, r_2, \ldots, r_t\}$ be the set of integers $0 \leq r < k$ such that $\dim(S(V_{k-r,n-r})) = 1$. We assume that $r_1, \ldots, r_2$ are given in decreasing order, namely $k > r_1 > r_2 > \cdots > r_t \geq 0$. So, $r_1 = n - N(h,p)$. If $r_t = 0$ then $\dim(S(V_{k,n})) = 1$. In this case we put $\mathcal{P}^{r_t}_{k,n} := S(V_{k,n})$. In any case, we set $r_{t+1} := -1$ and $\mathcal{P}^{-1}_{k,n} := 0$.

Lemma 4.3, Corollaries 4.2 and 4.4 and Theorem 4.1 imply the following.

Theorem 4.5. *With $r_1, r_2, \ldots, r_t$ as above,*

$$0 = \mathcal{P}^{-1}_{k,n} \subset \mathcal{P}^{r_t}_{k,n} \subset \mathcal{P}^{r_{t-1}}_{k,n} \subset \cdots \subset \mathcal{P}^{r_2}_{k,n} \subset \mathcal{P}^{r_1}_{k,n} \subseteq R(V_{k,n}).$$

Moreover, for every $i = 1, 2, \ldots, t$ the factor module $\mathcal{P}^{r_i}_{k,n}/\mathcal{P}^{r_{i+1}}_{k,n}$ admits a quotient isomorphic to $V_{r_i,n}/R(V_{r_i,n})$, with the convention that $\dim(V_{r_t,n}) = 1$ and $R(V_{r_t,n}) = 0$ when $r_t = 0$.

We call $(0, \mathcal{P}^{r_t}_{k,n}, \ldots, \mathcal{P}^{r_2}_{k,n}, \mathcal{P}^{r_1}_{k,n})$ the *geometric series* of $V_{k,n}$. Clearly, all proper submodules of $V_{k,n}$ are geometric if and only if $\mathcal{P}^{r_1}_{k,n} = R(V_{k,n})$, the geometric series is a composition series and it is the unique composition series of $R(V_{k,n})$ (so, $V_{k,n}$ is plainly uniserial). However, in general, not all proper submodules of $V_{k,n}$ are geometric. It can also happen that $\mathcal{P}^{r_1}_{k,n} \subset R(V_{k,n})$. The reader can see the remark at the end of the next section for an example.

5 Proof of Theorem 1.4

Assume that $\widetilde{J}_p(k,n) = J_p(k,n)$. Put $h := n-k$ and let $N(h,p)$ be defined as in Section 2.3. Let $n \geq N(h,p)$, otherwise $R(V_{k,n}) = 0$ and there is nothing to prove.

Put $\mathbf{N}(h,n) := \{m \mid h < m \leq n \text{ and } 0 \in J_p(m-h,m)\}$. Clearly, $N(h,p)$ is the smallest member of $\mathbf{N}(h,n)$. Let $h_1, h_2, \ldots, h_t$ be the members of $\mathbf{N}(h,n)$, given in increasing order, so $h_1 = N(h,p) < h_2 < \cdots < h_{t-1} < h_t \leq n$. For $i = 1, 2, \ldots, t$ put $r_i = n - h_i$. As in the previous section, $(0, \mathcal{P}^{r_t}_{k,n}, \ldots, \mathcal{P}^{r_2}_{k,n}, \mathcal{P}^{r_1}_{k,n})$ is the geometric series of $V_{k,n}$.

Lemma 5.1. *The geometric series of $V_{k,n}$ is a composition series. In particular $\mathcal{P}^{r_1}_{k,n} = R(V_{k,n})$.*

Proof. We must prove that the geometric series $\mathcal{S} := (\mathcal{P}^{r_{t+1-i}}_{k,n})^t_{i=0}$ of $V_{k,n}$ is a composition series and its largest member $\mathcal{P}^{r_1}_{k,n}$ is equal to $R(V_{k,n})$. (Recall from Section 4 that $r_{t+1} = -1$ and $\mathcal{P}^{-1}_{k,n} = 0$). The non-zero terms of $\mathcal{S}$ bijectively correspond to the members of $\mathbf{N}(h,n)$. Let S be a proper submodule of $V_{k,n}$ such that $\mathcal{P}^{r_i}_{k,n} \subset S$ for some $i < t+1$ and $S/\mathcal{P}^{r_i}_{k,n}$ is irreducible. By Lemma 3.5, $S/\mathcal{P}^{r_i}_{k,n} \cong V_{r,n}$ for some $r < k$. By Proposition 3.7, $k - r \in \mathbf{N}(h,n)$. The proof of Proposition 3.7 also shows that S contains $e_J \wedge P_{r,n}$, where $P_{r,n}$ is the pole of G in $W_{r,n}$ and $J = \{1, 2, \ldots, k-r\}$. Consequently, S contains the G-orbit of $e_J \wedge P_{r,n}$. This orbit spans $\mathcal{P}^r_{k,n}$. Hence S contains $\mathcal{P}^r_{k,n}$. Since we have assumed that $S/\mathcal{P}^{r_i}_{k,n}$ is irreducible, either $i > 1$, $r = r_{i-1}$ and $S = \mathcal{P}^{r_{i-1}}_{k,n}$ or $r = r_j$ for some $j \geq i$. Assume the latter. The module S properly contains $\mathcal{P}^{r_i}_{k,n}$. It also contains the vector $e_J \wedge P_{r,n} = e_{J_j} \wedge P_{r_j,n}$, where $J_j = \{1, 2, \ldots, k - r_j\}$. As $j \geq i$, the vector $e_{J_j} \wedge P_{r_j,n}$ belongs to $\mathcal{P}^{r_i}_{k,n}$. So, $e_J \wedge P_{r,n} \in \mathcal{P}^{r_i}_{k,n}$ while, according to the proof of Proposition 3.7, $e_J \wedge P_{r,n} \notin \mathcal{P}^{r_i}_{k,n}$. We have reached a contradiction. Therefore $S = \mathcal{P}^{r_{i-1}}_{k,n}$.

By a similar argument, but exploiting Proposition 3.8 instead of 3.7, we can see that $\mathcal{P}^{r_t}_{k,n}$ is irreducible. Therefore $\mathcal{S}$ is a composition series and $\mathcal{P}^{r_1}_{k,n} = R(V_{k,n})$. □

The following is also implicit in the proof of the previous lemma.

Lemma 5.2. $\dim(\mathcal{P}^{r_i}_{k,n}/\mathcal{P}^{r_{i+1}}_{k,n}) \in J_p(k,n)$ *for every $i = 1, 2, \ldots, t$.*

In order to finish the proof of Theorem 1.4 it remains to prove that the geometric series of $V_{k,n}$ is the unique composition series of $V_{k,n}$. By way of contradiction, suppose it is not. Then for at least one index $i < s$ the geometric submodule $\mathcal{P}^{r_i}_{k,n}$ admits two proper submodules S_1 and S_2 such that $\mathcal{P}^{r_i}_{k,n} = S_1 + S_2$. On the other hand, $\mathcal{P}^{r_i}_{k,n}$ is a homomorphic image of $V_{r_i,n}$, by Theorem 4.1. Let $\overline{S}_1$ and $\overline{S}_2$ be the pre-images of S_1 and S_2 by the projection of $V_{r_i,n}$ onto $\mathcal{P}^{r_i}_{k,n}$. Then $V_{r_i,n} = \overline{S}_1 + \overline{S}_2$. However this is impossible. Indeed both $\overline{S}_1$ and $\overline{S}_2$ are proper submodules of $V_{r_i,n}$, whence they are both contained in $R(V_{r_i,n})$.

Remark 5.3. For $i = 1, 2, \ldots, t$, let $k_i = n - h_i$. In general $\widetilde{J}_p(k_i, n) \subset J_p(k_i, n)$. Indeed, while $0 \in J_p(k,n)$ implies $0 \in J_p(k_i, n)$, the converse is false in general. To fix ideas, suppose that $t = 2$ and $n \in \mathbf{N}(h,n)$. So, $h_1 = N(h,p)$ and $h_2 = n$

are the only members of $\mathbf{N}(h,n)$. Then $\dim(S(V_{k,n}))=1$ and $R(V_{k,n})/S(V_{k,n})$ is irreducible. Suppose that $\mathbf{N}(h_1,n)$ contains at least two members and n is one of them. If $\widetilde{J}_p(k_1,n)=J_p(k_1,n)$ then $S(V_{k_1,n})$ would be a 1-dimensional proper submodule of $R(V_{k_1,n})$. However $R(V_{k,n})$ is a quotient of $V_{k_1,n}$, say $R(V_{k,n})\cong V_{k_1,n}/S$ for a suitable submodule S of $V_{k_1,n}$. On the other hand, S is contained in $R(V_{k_1,n})$ and it contains $S(V_{k_1,n})$. So, no 1-dimensional submodule can appear in $V_{k_1,n}/S$, while $R(V_{k,n})$ does admit a 1-dimensional submodule; we have reached a contradiction. Therefore, in the considered situation, $\widetilde{J}_p(k_1,n)\subset J_p(k_1,n)$.

For a concrete example of the above situation, choose $p=3$, $h=0$ and $n=16$. Then $\mathbf{N}(h,n)=\{4,16\}$, whence $h_1=4$. We have $\mathbf{N}(h_1,n)=\{6,12,16\}$ and $R(V_{12,16})$ admits a unique composition series $0\subset S_1\subset S_2\subset R(V_{12,16})$ where $S_1=S(V_{12,16})\cong V_{4,16}/R(V_{4,16})$, $S_2/S_1\cong V_{10,16}/R(V_{10,16})$ and S_2 is a hyperplane of $R(V_{12,16})$.

6 Proof of Corollary 1.5

Put $h:=n-k$ and assume $h<p-1$. Then $e=0$ in the expansion of h to the base p (see formula (1)). As a consequence, the second sum of formula (3) of Lemma 2.3 vanishes while the first sum contains just one summand, namely $h_0=h$. Thus, by Proposition 2.4, $0\in J_p(k,n)$ if and only if $n=2(p^t-1)-h$ for a positive integer t. So, in order to show that the hypotheses of Theorem 1.4 are satisfied we only need to prove the following:

Lemma 6.1. *Let* $n=2(p^t-1)-h$. *Then* $\dim(S(V_{k,n}))=1$.

Proof. If $t=1$ then $n=N(h,p)$ by formula (2), hence $\dim(S(V_{k,n}))=1$ by Proposition 2.2. Assume $t>1$.

We first show that $v_{P_{k,n}}\in V_2^{(k,n)}$ by showing that $p\mid\binom{n}{k/2}$. To this end, we introduce the symbol $\mathrm{ord}_p(m)$. For a positive integer m, we denote by $\mathrm{ord}_p(m)$ the largest exponent f such that p^f divides m. It is well known that $\mathrm{ord}_p(m!)=\sum_{j\geq 1}\lfloor m/p^j\rfloor$, where $\lfloor m/p^j\rfloor$ is the integral part of m/p^j (see [12, Theorem 416] for instance). Therefore

$$\mathrm{ord}_p\binom{m}{r}=\sum_{j\geq 1}\left(\left\lfloor\frac{m}{p^j}\right\rfloor-\left\lfloor\frac{r}{p^j}\right\rfloor-\left\lfloor\frac{m-r}{p^j}\right\rfloor\right). \tag{8}$$

Given an integer $x\geq 0$ we denote by $|x|_p$ the remainder of its division by p.

By exploiting formula (8) and recalling that $n=2(p^t-1)-h$ and $k=n-h=2(p^t-1-h)$, it is straightforward to check that, for a nonnegative integer $x\leq k/2$,

$$p \text{ divides } \binom{n}{x} \text{ iff } |x|_p \geq p-1-h. \tag{9}$$

It follows that p divides $\binom{n}{k/2}$. Indeed $k/2 = p^t - 1 - h = p^t - p + [p-1-h]$. Since p divides $\binom{n}{k/2}$, the pole $P_{k,n} = \langle v_{P_{k,n}} \rangle$ of G in $V_{k,n}$ is contained in $V_2^{(k,n)}$, by Proposition 2.7. Let $j < k/2$ be the least nonnegative integer such that $v_{P_{k,n}} \in V_{k-2j}^{(k,n)}$, namely $v_{P_{k,n}} \in V_{k-2j}^{(k,n)} \setminus V_{k-2j+2}^{(k,n)}$. In order to prove the lemma we only must show that $j = 0$.

Let $r := 2(p-1-h)$. By (9), r is the smallest even integer such that p divides $\binom{n}{r/2}$. Hence $k - 2j \geq r$ by the minimality of r. So, $j \leq (k-r)/2 = p^t - p$. In order to align our notation to that of Proposition 2.8, we set $k - 2j = r + 2i$. So, $0 \leq i \leq p^t - p$ and the equality $j = 0$, which we want to prove, is equivalent to $i = p^t - p$.

By condition (1) of Proposition 2.8, p divides $\binom{n}{r/2+i}$. Hence $|r/2 + i|_p \geq p - 1 - h$, by equivalence (9). As $r/2 = p - 1 - h$ and $h < p - 1$, the previous inequality is equivalent to the following:

$$|i|_p \leq h. \tag{10}$$

On the other hand, condition (2) of Proposition 2.8 implies that $r/2 + i \leq_p n+1$, namely

$$p - 1 - h + i \leq_p 2(p^t - 1) - h + 1 = 2p^t - 1 - h. \tag{11}$$

In view of (10), condition (11) implies $|i|_p = 0$ and $i \leq_p p^t - p$. So,

$$i = i_1 p + i_2 p^2 + \cdots + i_{t-1} p^{t-1}$$

where $i_1, i_2, \ldots, i_{t-1} \in \{0, p-1\}$. Consider now the binomial coefficient

$$\binom{k/2}{k/2 - j} = \binom{k/2}{r/2 + i} = \binom{p^t - 1 - h}{p - 1 - h + i}.$$

By exploiting formula (8) one can see that p divides $\binom{k/2}{r/2+i}$ if and only if

$$p + i_1 p + i_2 p^2 + \cdots + i_s p^s > p^{s+1}$$

for at least one $s \in \{1, \ldots, t-1\}$. However, this is impossible, since $p + i_1 p + i_2 p^2 + \cdots + i_s p^s \leq p^{s+1}$. Therefore, p does not divide $\binom{k/2}{k/2-j}$. In view of Proposition 2.8 (3), the prime p neither divides

$$\binom{n - k + 2j}{j} = \binom{n - r - 2i}{(k-r)/2 - i} = \binom{2(p^t - p) + h - 2i}{p^t - p - i}.$$

By exploiting (8) once again, one can see that $\binom{2(p^t-p)+h-2i}{p^t-p-i}$ is prime to p if and only if

$$2p + 2i_1p + 2i_2p^2 + \cdots + 2i_sp^s > p^{s+1} + h$$

for every $s = 1, 2, \ldots, t-1$. This condition implies that $i_s \geq (p-1)/2$ for every $s = 1, 2, \ldots, t-1$. However, $i_s \in \{0, p-1\}$. Therefore, $i_s = p-1$. It follows that $i = p^t - p$. Equivalently, $j = 0$. □

Conjecture 6.2. *Let e be as in formula* (1)*. We conjecture that* $\dim(S(V_{k,n})) = 1$ *if and only if* $k = 2p^e \cdot [p^{t+1} - 1 - \sum_{j=0}^{t} h_{j+e}p^j]$ *for an integer* $t \geq -1$ *(compare Lemma* 2.3*, formula* (3)*). This would include Lemma* 6.1 *as a special case.*

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