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The twist conjecture for Coxeter groups without small triangle subgroups

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Abstract

We prove Mühlherr's twist conjecture for Coxeter systems (W, S) which have no rank 3 subsystems of type 2-3-n or 2-4-n ($n \geq 3$). In combination with known results this finishes the solution of the isomorphism problem for this class of groups. The condition on the diagram does not allow spherical rank 3 subsystems, but our result covers "most" of the even Coxeter systems. With respect to earlier contributions, we develop a geometric technique to handle rank 2 twists, in particular rotation twists which occur in the even case.

Keywords: Coxeter group, twist conjecture

MSC 2000: 20F55, 51F15

1 Introduction

The twist conjecture (see [7] for details) is motivated by the isomorphism problem for Coxeter groups. In fact, a proof of this conjecture would yield a complete solution. The conjecture has been proved for skew-angled Coxeter systems [8], for chordal Coxeter systems [9] and for twist rigid Coxeter systems [4]. The first two references use the decomposition of the Coxeter system as a graph of groups. Although this approach turns out to be very efficient for the special cases considered, it seems to be very difficult to generalize it to arbitrary diagrams. The main difficulty arises when there are local twists which do not extend to global twists. The conditions required in those papers are designed to have control over the local twists.

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In the present paper we follow a strategy which had been used for the right angled case in [6]. Although the twist conjecture hasn't been formulated when that paper was written, its validity for the right-angled case is proved there. The strategy is to introduce a distance matrix for a Coxeter generating set consisting of reflections and to show that one can reduce it by elementary twists. This works very well in the right-angled case, but it becomes considerably more complicated if there are edges with finite labels in the diagram.

In this paper we prove the conjecture for diagrams which do not have certain rank 3 diagrams, including the irreducible spherical ones. The exclusion of those diagrams is essential to avoid higher rank twists, yet our condition does not allow some other types of diagrams including $\tilde{C}_2$ and $\tilde{G}_2$, which is designed to avoid technical details which become quite involved. However, although we cover a large class of Coxeter systems here for which the twist conjecture is not proved yet, our technique certainly needs substantial improvements in order to treat the general case. Yet the methods we develop are the first to directly handle rotation twists in a geometric way, using an approach which is derived from [6]. While the skew-angled and chordal Coxeter systems allow this type of twists as well, the works in [8] and [9] avoid this using Bass–Serre theory.

Here is our main result:

Main Theorem. *Suppose that (W, S) is an irreducible non-spherical Coxeter system of finite rank greater or equal 3, such that its diagram contains no subdiagrams of type* $\bullet\overset{3}{\text{———}}\bullet\overset{n}{\text{———}}\bullet$ *for $n \geq 3$ or* $\bullet\overset{4}{\text{———}}\bullet\overset{n}{\text{———}}\bullet$ *for $n \geq 4$. If $R \subset S^W$ is an irreducible sharp-angled Coxeter generating set for W, then $R \sim_t S$.*

We will later in Section 2.1 denote this condition on the diagram as condition (E), referring to the original intention to handle even Coxeter groups. The definition of sharp-angled can be reviewed in Section 2.2, the definition of twist-equivalence $\sim_t$ can be found in Section 2.5.

Remark 1.1. It is worthwhile to mention that our result covers the skew-angled Coxeter systems for which the twist conjecture was proved in [8].

In Section 2 we fix notation and recall definitions concerning the Cayley graph of a Coxeter system and its roots and walls. Most of the properties stated are taken from [8]. We also introduce longest reflections and their properties (2.3) as well as two notions of separation (2.4) and recall the definition of twists (2.5).

In Section 3 we give a characterization for a Coxeter generating set satisfying our conditions to be geometric. This will act as a base of induction for our main theorem. In the main part of this section we show that whenever neither a reflection in R nor a longest reflection separates two other reflections in R, the set R is already geometric.

In Section 4 we prove our main theorem, distinguishing three different settings of the positions of the walls in the Cayley graph. We show that in each case we find a twist or a series of twists such that the resulting Coxeter generating set has a reduced distance matrix. To do this we first prove some properties of the distances in the Cayley graph and introduce interior separation, a stronger notion of separation taking into consideration walls of longest reflections.

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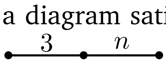
2 Preliminaries

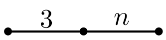
2.1 Coxeter matrices, systems, diagrams

Let I be a finite set. A *Coxeter matrix over I* is a symmetric matrix $M = (m_{ij})_{i,j\in I}$ with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ for all $i \in I$, $m_{ij} \geq 2$ for all $i \neq j \in I$.

Given a Coxeter matrix M, (W, S) is a *Coxeter system of type M* if W is a group, $S = \{s_i \mid i \in I\} \subset W$ and $\langle S \mid (s_i s_j)^{m_{ij}}, i, j \in I\rangle$ is a presentation for W. For a Coxeter matrix M the *Coxeter diagram* is the undirected graph $\Gamma = (V, E)$ with $V = I$, $E = \{\{i, j\} \mid 2 < m_{ij}\}$ and the labeling $\tau : E \to \mathbb{N}$, $\{i, j\} \mapsto m_{ij}$. The *rank* of the diagram, of the Coxeter matrix, of the Coxeter system is $|I| = |S|$. A group W is called a *Coxeter group* if there exists a subset $S \subset W$ such that (W, S) is a Coxeter system.

If W is a Coxeter group, $R \subset W$ is *universal* if $(\langle R\rangle, R)$ is a Coxeter system. A subset R is a *Coxeter generating set* if R is universal and $\langle R\rangle = W$, i.e. if (W, R) is a Coxeter system. A universal set R is *irreducible*, if there is no nontrivial partition $R = R_1 \dot\cup R_2$ such that $o(r_1 r_2) = 2$ holds for all $r_i \in R_i$, $i = 1, 2$. If R is Coxeter generating, it gives rise to a unique Coxeter matrix, justifying our notion of the *diagram of R*. For the subsets $S' \subset S$, $S' = \{s_i \mid i \in J\}$ the *special subgroups* are $W_J := \langle S'\rangle$. In this case, S' is a Coxeter generating set for W_J.

A diagram or subset $J \subset I$ is *spherical* if the generated Coxeter group W_J is finite. We say a diagram satisfies condition (E), if it does not contain subdiagrams of type  or  for $n \geq 3$.

Note that the diagram  is spherical for $2 \leq n \leq 5$.

Let R be a Coxeter generating set; for each $J \subset R$ we set

$$J^{\perp} = \{r \in R \setminus J \mid rj = jr \text{ for all } j \in J\}.$$

2.2 The Cayley graph, roots, walls, residues

Consider a Coxeter system (W, S) of type M over I. Then (C, P) with $C = W$ and $P = \{\{w, ws\} \mid w \in W, s \in S\}$ is an undirected graph. Let $\tau \colon P \to S$, $\{w, ws\} \mapsto s$ be a labeling. If for all $w \in W$, $P(w) = \{e \in P \mid w \in e\}$ the restriction $\tau|_{P(w)}$ is a bijection, then $\mathcal{C} = (C, P, \tau)$ is the *Cayley graph of* (W, S). The set C is the set of *chambers*, P the set of *panels*. We denote with $\delta \colon C \times C \to \mathbb{N}$ the distance function on the Cayley graph. For subsets $A, B \subset C$, define $\delta(A, B) = \min\{\delta(a, b) \mid a \in A, b \in B\}$. A *gallery of length* m, $\gamma = (c_0, \ldots, c_m)$, in $\mathcal{C}$ is a path of length m in (C, P), it is *minimal* if $\delta(c_0, c_m) = m$. We will sometimes identify a gallery with its set of chambers $\bigcup_{0 \leq i \leq m} \{c_i\}$.

The group W acts on the chambers of $\mathcal{C}$, denoted by $w.c = wc \in C$ for $w \in W$. Regarding this action we have $(w.p)^{\tau} = p^{\tau}$ for $p \in P$, so τ is W-invariant.

Let $r \in S^W$, $P_r = \{p \in P \mid r.p = p\}$. The graph $(C,\ P \setminus P_r)$ has two connected components (see [10, Proposition 2.6]), called the *roots associated to* r. The set $C(r) = \bigcup_{p \in P_r} p$ is the *wall of* r. For any chamber $c \in C$, $H(r, c)$ is the unique root associated to r containing c. For $A \subset C$, if A is contained in one root, $H(r, A)$ is the well-defined root associated to r containing A. If H is a root associated to r, $-H$ is the unique root associated to r not equal to H. Therefore, if $c \in C(r)$, then $-H(r, c) = H(r, r.c)$. For $r, s \in S^W$ we define $\delta(r, s) := \delta(C(r), C(s))$.

Now let $c \in C$, $J \subset I$. The set $R_J(c) := cW_J$ is called a *J-residue*. A subset $A \subset C$ is called *residue* if it is a J-residue for some $J \subset I$. A residue A is *spherical* if it is a J-residue and J is spherical. Let $s, t \in S^W$, then we will denote with $A_{s,t}$ an arbitrary maximal spherical residue of the form $R_{\{s,t\}}(c)$, i.e. a residue stabilized by $\langle s, t \rangle$. In particular, the existence of $A_{s,t}$ implies that the product st has finite order.

We will need some basic properties of roots, walls and residues. Geometric versions of these statements can be found in [8], we will recall the results we need. The following Lemma is a well known fact, for more details see [1].

Lemma 2.1 ([8, Lemma 2.3]). *A subgroup $U \leq W$ is finite if and only if it stabilizes a spherical residue.*

Lemma 2.2 ([8, Lemma 2.6]). *Let $U \leq W$ be finite, $\langle U, \{s\} \rangle$ be infinite for an $s \in S^W$. Then every spherical residue stabilized by U is contained in the same unique root associated to s.*

In the situation of the previous lemma, the notation of $H(s, U)$ for the root containing all spherical residues stabilized by U is justified whenever U is finite, $\langle s, U\rangle$ is infinite. In particular, we will write $H(s,t) := H(s, \langle t\rangle)$ if $o(st) = \infty$. Note that since $C(t)$ consists of chambers included in $\{t\}$-residues, we have $H(s,t) = H(s, C(t))$.

Remark 2.3. For convenience with our notation, we write $x^w = wxw^{-1}$ for $x \in W$ for the action of W on W by conjugation.

Lemma 2.4 ([8, Lemma 3.1])**.** (a) *W acts on the set of walls and on the set of roots associated to $r \in S^W$. Let $w \in W$, then $w.C(r) = C(r^w)$. If H_r is a root associated to r, then $w.H_r$ is a root associated to r^w.*

(b) *A root H associated to an element $r \in S^W$ is convex.*

Let $U \leq W$. A subset $F \subset C$ is a *fundamental domain for U* if $C = \dot{\bigcup}_{u\in U} u.F$. Let $s,t \in S^W$ and let H_s, H_t be roots associated to s,t. The set $\{H_s, H_t\}$ is a *geometric pair* if $H_s \cap H_t$ is a fundamental domain for $\langle s,t\rangle$. Consider a set Φ of roots, it is 2*-geometric* if each pair of roots in Φ is geometric, and *geometric* if it is 2-geometric and $\bigcap_{H\in\Phi} H \neq \emptyset$. A pair $\{H_s, H_t\}$ is *weakly geometric* if $\{H_s, H_t\}$ or $\{-H_s, -H_t\}$ is a geometric pair. A set Φ of roots is *weakly* 2*-geometric* if each pair of roots is weakly geometric. The set $R \subset S^W$ is geometric (2-geometric, weakly 2-geometric) if there exists a set $\Phi(R)$ of roots associated to the elements in R, such that $\Phi(R)$ is geometric (2-geometric, weakly 2-geometric). The set $R \subset S^W$ is *sharp-angled* if all $\{s,t\} \subset R$ are geometric. We note that if R is geometric with geometric set of roots $\Phi(R)$, then $F := \bigcap_{H\in\Phi(R)} H$ is a fundamental domain for $\langle R\rangle$ and $C(r) \cap F \neq \emptyset$ for all $r \in R$.

The following is a summary of [8, Lemma 4.3, 4.4, 4.5]; we will make constant use of these statements.

Lemma 2.5. *Let $R \subset S^W$ be a sharp-angled Coxeter generating set, $s,t \in R$. Then:*

(a) *If $o(st) = 2$, then $\{H_s, H_t\}$ is a geometric pair for all roots H_s, H_t associated to s,t.*

(b) *If $2 < o(st) < \infty$ and H_s is a root associated to s, there is a unique root H_t associated to t such that $\{H_s, H_t\}$ is a geometric pair. Then $\{-H_s, -H_t\}$ is a geometric pair as well, $\{\pm H_s, \mp H_t\}$ is not geometric.*

(c) *If st has infinite order, there exist unique roots H_s, H_t associated to s,t such that $\{H_s, H_t\}$ is a geometric pair. Then $-H_s \subset H_t$, $-H_t \subset H_s$ and $-H_s \cap -H_t = \emptyset$.*

We will denote the intersections of the geometric pairs in part (b) and (c) of the previous lemma as the *standard fundamental domains*. Note that if st has infinite order, the standard fundamental domain is uniquely determined, if $2 < o(st) < \infty$, there are two standard fundamental domains $F := H_s \cap H_t$ and $-F := -H_s \cap -H_t$ for a geometric pair $\{H_s, H_t\}$ and $w_{\{s,t\}}.F = -F$ holds for the longest element $w_{\{s,t\}}$ in $\langle s, t\rangle$ (see Section 2.3 for details).

A generalization of part (c) of the previous lemma is the following:

Lemma 2.6. *If R is universal, irreducible and non-spherical such that R is geometric, then the geometric set of roots $\Phi(R)$ is unique.*

Proof. This follows directly from Lemma 2.5 if two elements in R have infinite order. If R is 2-spherical, we can make use of Proposition 7.2 in [3]. This yields that if $R \setminus \{r\}$ is spherical for an $r \in R$, such a geometric set is unique. Now we can consider the smallest irreducible non-spherical set $\bar{R} \subset R$ such that $\bar{R} \setminus \{r\}$ is spherical for some $r \in \bar{R}$. For $\bar{R}$ we already have a unique geometric set of roots, therefore the geometric set of roots for R is unique. □

For the reader's convenience, we will also repeat a useful property in skew-angled Coxeter systems:

Lemma 2.7 ([8, Lemma 6.3])**.** *Let R be universal, $r, s, t \in R$ pairwise non-commuting elements. Then the product $rsrt$ has infinite order.*

2.3 Longest reflections and their basic properties

Consider a sharp-angled Coxeter generating set $R \subset S^W$ and a subset $J = \{s, t\} \subset R$ with $2 < o(st) < \infty$. We have a length function on W with respect to the generating set R and denote with w_J the longest element in $\langle J\rangle$. Define the *longest reflections* s_t, t_s in $\langle J\rangle$ as the elements of $\langle J\rangle \cap S^W$ of maximal length. If $o(st)$ is even, we define s_t to be the longest reflection commuting with s, t_s to be the longest reflection commuting with t. In this case we have $s_t = w_J s$, $t_s = w_J t$. If $o(st)$ is odd, we simply have $s_t = t_s = w_J$.

Remark 2.8. Since the reflections s_t, t_s are associated to the highest roots, the notion of a *highest reflection* for s_t and t_s is suggesting itself. We decided to denote them longest reflections, referring to the length function in W with respect to the Coxeter generating set R.

Also note that, given two Coxeter generating sets R, R' both containing J, the length functions on $\langle J\rangle$ with respect to R and with respect to R' are equal.

We need the following properties of s_t, t_s:

Lemma 2.9. *Let $R \subset S^W$ be a sharp-angled universal set satisfying* (E). *Let $J = \{s,t\} \subset R$, $2 < o(st) < \infty$. For all $u \in R \setminus J$ such that $J \cup \{u\}$ is irreducible $o(us_t) = o(ut_s) = \infty$ holds.*

Proof. This is a conclusion from [3, Corollary 9.5 and Lemma 9.8]. If the diagram of $\{s,t,u\}$ is a tree, the statement follows from Corollary 9.5. If it is not a tree, this is Lemma 9.8. □

Remark 2.10. Note that (E) is critical for Lemma 2.9 to hold. In fact, (E) is the weakest assumption one can make on a sharp-angled universal set R, such that Lemma 2.9 still holds.

Proposition 2.11. *Consider R as in Lemma* 2.9. *Let $J = \{s,t\}$ with $2 < o(st) < \infty$, $u \in R \setminus J$. The sets $\{u, s_t\}$, $\{u, t_s\}$ are sharp-angled.*

Proof. If $J \cup \{u\}$ is irreducible, then $o(us_t) = \infty = o(ut_s)$ holds by Lemma 2.9. Thus we can consider the sets of roots $\{H(u,s_t), H(s_t,u)\}$ and $\{H(u,t_s), H(t_s,u)\}$ associated to the sets $\{u,s_t\}$, $\{u,t_s\}$. Define $F = H(u,s_t) \cap H(s_t,u) \neq \emptyset$. Let $x \in \langle u, s_t\rangle$, then $F \cap x.F = \emptyset$ for $x \neq 1_W$ and $\bigcup_{x\in\langle u,s_t\rangle} x.F = C$ hold. The set $\{u, s_t\}$ is geometric, the same holds for $\{u, t_s\}$.

If $J \cup \{u\}$ is reducible, this implies $o(us) = o(ut) = 2$. But then $o(ut_s) = o(us_t) = 2$ holds, let H_{s_t}, H_u arbitrary roots associated to s_t, u. For $F = H_{s_t} \cap H_u$ we have $u.F = H_{s_t} \cap -H_u$, $s_t.F = -H_{s_t} \cap H_u$, $us_t.F = -H_{s_t} \cap -H_u$. So $\bigcup_{x\in\langle u,s_t\rangle} x.F = C$ and $x.F \cap F = \emptyset$ for all $x \in \langle u, s_t\rangle$, $x \neq 1_W$. The set $\{u, s_t\}$ is geometric, the same holds for $\{u, t_s\}$. □

Lemma 2.12. *Consider R as in Lemma* 2.9. *Let $J = \{s,t\} \subset R$, $2 < o(st) < \infty$. Consider $u = u_0, \ldots, u_k = v \in R \setminus (J \cup J^\perp)$, $u_i u_{i+1}$ having finite order for $i = 0, \ldots k-1$. The roots $H(s_t,u), H(s_t,v)$ are well-defined and equal.*

Proof. Because of Lemma 2.9, $o(us_t) = o(vs_t) = \infty$ holds and $H(s_t,u), H(s_t,v)$ are well-defined. Furthermore $s_t u_i$ has infinite order for $i = 0, \ldots, k$. Assume $H(s_t,u) \neq H(s_t,v)$, then using [8, Lemma 4.6] we obtain a reflection u_j such that the product $s_t u_j$ has finite order, a contradiction. □

In the beginning of Section 4 we will state further properties on the order of products of longest reflections.

2.4 Separating reflections and interiors

We extend the notion of separation used in [6] for right-angled Coxeter systems to arbitrary sharp-angled Coxeter generating sets $R \subset S^W$. Consider a sharp-angled subset $\{s,u,v\} \subset S^W$. We define $s \in [u,v]$ and say *s separates u and v* if

$o(uv) = \infty, o(su), o(sv) > 2$ and all roots H_s associated to s satisfy the following condition: Let $\{H_u, H_v\}$ be the unique geometric set of roots associated to u, v, if $\{H_u, H_s\}$ is geometric, then $\{H_v, -H_s\}$ is geometric. In other words: uv has infinite order and the set $\{s, u, v\}$ is not geometric.

We will also need a slightly sharper notion of separation. We say that s *separates* u *and* v *reducibly*, $s \in_r [u, v]$, if $s \in [u, v]$ and $\delta(u^s, v) < \delta(u, v)$.

Remark 2.13. We will show later in Lemma 4.8, that the property $\delta(v^u, w) < \delta(v, w)$ is sufficient for $u \in_r [v, w]$.

We define for a sharp-angled Coxeter generating set $R \subset S^W$ the *interior of* R to be the set $R^\circ := \{r \in R \mid \exists s, t \in R : r \in_r [s, t]\}$. Define

$$R_2 := \{s_t \in R^W \mid s, t \in R, 2 < o(st) < \infty\}$$

to be the set of longest reflections. Due to Proposition 2.11 the sets $\{s_t, u, v\}$ are sharp-angled for all $u, v \in R \setminus \{s, t\}$, thus we can define the *interior of* R_2 to be the set $R_2^\circ := \{s_t \in R_2 \mid \exists u, v \in R \setminus \{s, t\} : \ s_t \in [u, v]\}$.

2.5 Twists

For a Coxeter generating set R and $J, K, L \subset R$ satisfying

1. J is irreducible spherical,
2. $o(kl) = \infty$ for all $k \in K, l \in L$,
3. $R = J \dot\cup J^\perp \dot\cup K \dot\cup L$,

we say the pair (J, L) is R-*admissible*. For an R-admissible pair (J, L) define $T_{(J,L)}(R) := J \dot\cup J^\perp \dot\cup K \dot\cup L^{w_J}$, called *the twist of* R *by* J.

Remark 2.14. If R is Coxeter generating, $T_{(J,L)}(R)$ is a Coxeter generating set as well. See [2] for basic properties of twists as well as for a proof that $T_{(J,L)}(R)$ is indeed Coxeter generating. Our condition (E) implies for admissible pairs (J, L) that J either consists of one element or generates a finite dihedral group. We will use the fact that in the case of $|J| = 1$ the diagram of $T_{(J,L)}(R)$ coincides with the diagram of R, the same holds in the case $J = \{s, t\}$, $o(st)$ even. It is easy to see, that if R is sharp-angled, $T_{(J,L)}(R)$ is sharp-angled as well.

Two Coxeter generating sets R, $\bar{R}$ are *twist-equivalent*, $R \sim_t \bar{R}$, if there exists a series of Coxeter generating sets $R = R_0, \dots, R_m = \bar{R}$, such that R_{i+1} is a twist of R_i by some $J \subset R_i$ for $i = 0, \dots, m-1$. The relation $\sim_t$ is an equivalence relation on the set of sharp-angled Coxeter generating sets (cf. [2, Chapter 4]).

As we can interpret twists as operations on the diagram of a Coxeter group, we will need that condition (E) is preserved by twists:

Lemma 2.15. *Suppose R, R' are Coxeter generating sets for W and (J, L) is an R-admissible pair such that $R' = T_{(J,L)}(R)$. Then R satisfies* (E) *if and only if R' satisfies* (E).

Proof. Assume R satisfies (E), consider an admissible pair (J, L). Following the above remark, the diagram of R' is the same as the diagram of R if $|J| = 1$ or $J = \{s, t\}$ with $o(st)$ even. The only remaining case is $J = \{s, t\}$ and $o(st)$ odd. Let $R = J \cup J^{\perp} \cup L \cup K$, and assume the diagram of R' contains one of the rank 3 diagrams in question, say $U = \{r_1, r_2, r_3\}$. The set U cannot contain elements from both K and L, since their product has infinite order. The diagram of $J \cup J^{\perp} \cup L^{w_J} \subset R'$ is the same as the one of $J \cup J^{\perp} \cup L_J \subset R$, $J \cup J^{\perp}$ being w_J invariant. Therefore R' satisfies (E), since R satisfies (E). By symmetry R' satisfying (E) implies that R satisfies (E), which completes our proof. □

3 A characterization of geometric sets

In this section we will characterize geometric sets using the distances between reflections. For this purpose we will introduce the distance matrix of a Coxeter generating set, as already used in [6]. In particular we will show that R is already conjugate to S if no element in R or no longest reflection in any rank 2 group separates any two fundamental reflections.

Definition 3.1. Say we have a Coxeter generating set $R = \{r_i \mid i \in I\}$ for a finite I. Define the distance matrix $D_1(R) = (\delta(r_i, r_j))_{i,j \in I}$. For two Coxeter generating sets $R = \{r_i \mid i \in I\}$, $S = \{s_i \mid i \in I\}$ of same rank $|I|$ we say $D_1(R) < D_1(S)$ if there is a permutation $\sigma\colon (i, j) \mapsto (i', j')$ in $\mathrm{Sym}(I \times I)$ such that $\delta(r_{i'}, r_{j'}) \leq \delta(s_i, s_j)$ for all $i, j \in I$, and $\delta(r_{i'}, r_{j'}) < \delta(s_i, s_j)$ for at least one pair (i, j).

We can use the distance matrix to characterize if a Coxeter generating set is conjugate to S by adapting Lemma 2.8 from [6]:

Theorem 3.2. *Suppose $R \subset S^W$ is a Coxeter generating set which is sharp-angled, irreducible and non-spherical of finite rank at least 3. If R satisfies* (E)*, the following are equivalent:*

(a) *R is geometric;*

(b) *$R^{\circ} = \emptyset$ and $R_2^{\circ} = \emptyset$;*

(c) *R is conjugate to S;*

(d) $D_1(R) = 0$.

For the definition of R°, R_2°, see Section 2.5. Almost all of the arguments to prove this can be copied from [6], but the implication (b) ⇒ (a) does not follow immediately. As a main step in this deduction, we will prove the following proposition:

Proposition 3.3. *Let (W, S) be a Coxeter system, $R \subset S^W$ a Coxeter generating set for W such that R is irreducible, non-spherical, sharp-angled and the diagram for (W, R) satisfies condition* (E). *If $\{r \in R \mid \exists u, v \in R : \ r \in [u, v]\} = \emptyset$ and $R_2^\circ = \emptyset$, then R is conjugate to S.*

To show this, it suffices to show, under our conditions on R, the existence of a weakly 2-geometric set of roots associated to R. We can then make use of [3, Theorem 4.2]:

Theorem 3.4 ([3])**.** *Any finite, universal and weakly 2-geometric set of reflections is geometric.*

Note that the above mentioned result can also be deduced from [5], as the authors also pointed out in [3]. Yet the version cited is more applicable due to the geometric language it uses.

We will prove that trees and chord-free circuits of arbitrary length in the diagram yield geometric sets of roots if R satisfies $\{r \in R \mid \exists u, v \in R : \ r \in [u, v]\} = \emptyset$ and $R_2^\circ = \emptyset$. Furthermore, for the rest of this section assume that (W, S) is a Coxeter system and $R \subset S^W$ is a sharp-angled and universal set which satisfies $\{r \in R \mid \exists u, v \in R : \ r \in [u, v]\} = \emptyset$ and $R_2^\circ = \emptyset$.

Lemma 3.5. *Let R be irreducible and non-spherical. Assume the diagram of R is a tree. Let $r \in R$, and choose an arbitrary root H_r associated to r. Then there exists a unique weakly 2-geometric set of roots Φ associated to R such that $H_r \in \Phi$. In particular, R is geometric.*

Proof. Choose an arbitrary root H_r associated to r. Consider the distance d in the diagram. We prove the lemma by induction on $\max\{d(r, r') \mid r' \in R\}$. If the maximal distance to r is 0, we are done, since $R = \{r\}$. Assume we have proved the lemma for all R' and $r \in R'$ satisfying $\max\{d(r, r') \mid r' \in R'\} = m$ and we have a set R and an element $r \in R$ satisfying $\max\{d(r, r') \mid r' \in R\} = m + 1$. So we can find a weakly 2-geometric set of roots $\bar{\Phi}$ associated to $\bar{R} = \{\bar{r} \in R \mid d(r, \bar{r}) \leq m\}$ containing H_r, using that the diagram of $\bar{R}$ is also a tree. Consider $t \in R \setminus \bar{R}$. Since the diagram of R is a tree and $\bar{R}$ is connected, there is exactly one $\bar{t} \in \bar{R}$ such that $o(t\bar{t}) > 2$ and $o(tt') = 2$ for all other $t' \in R \setminus \{t\}$. In $\bar{\Phi}$ a root $H_{\bar{t}}$ associated to $\bar{t}$ is contained, and there exists a unique root H_t associated to t satisfying that $\{H_t, H_{\bar{t}}\}$ is a weakly geometric pair. Since $o(tr') = 2$ for all $r' \in R \setminus \{\bar{t}\}$, $\bar{\Phi} \cup \{H_t \mid t \in R \setminus \bar{R}\}$ is a weakly 2-geometric set for R containing H_r. By Theorem 3.4, R is geometric. □

Definition 3.6. A diagram of a universal set R is a *chord-free circuit of length* $m+1$, if there is an indexing $R = \{r_0, \dots r_m\}$ such that $o(r_i r_{i+1}) > 2$ for $i = 0, \dots, m-1$, $o(r_0 r_m) > 2$, and $o(r_i r_j) = 2$ for all other $i \neq j$.

Lemma 3.7. *Let R be irreducible and non-spherical. Assume the diagram of R is a chord-free circuit of rank 5 or greater and does not contain irreducible spherical rank 3 subdiagrams. Then R is geometric.*

Proof. Denote $R = \{r_0, \dots, r_m\}$ such that $o(r_i r_{i+1}) > 2$ for $i = 0, \dots, m-1$ and $o(r_m r_0) > 2$, $o(r_i r_j) = 2$ else.

The diagram of $\{r_0, \dots, r_{m-1}\}$ is a tree, yielding with Lemma 2.6 a unique geometric set of roots $\{H_0, \dots, H_{m-1}\}$, where H_i is associated to r_i. The set $\{r_{m-3}, r_{m-2}, r_{m-1}\}$ is irreducible non-spherical, thus by Lemma 2.6 the set $\{H_{m-3}, H_{m-2}, H_{m-1}\}$ of associated roots is the unique geometric set of roots and gives rise to a unique root H_m associated to r_m such that the set $\{H_{m-3}, H_{m-2}, H_{m-1}, H_m\}$ is geometric. We have to show that H_0, H_m is a geometric pair, then the set $\{H_0, \dots, H_m\}$ is geometric.

Using the fact that $\{H_{m-3}, H_{m-2}, H_{m-1}, H_m\}$ is unique geometric, by considering irreducible non-spherical sets whose diagrams are trees we get the following unique geometric sets of roots associated to the corresponding elements in R:

$$\{H_{m-2}, H_{m-1}, H_m\}, \{H_{m-2}, H_{m-1}, H_m, H_0'\}, \{H_{m-1}, H_m, H_0'\},$$
$$\{H_{m-1}, H_m, H_0', H_1'\}, \{H_m, H_0', H_1'\}, \{H_m, H_0', H_1', H_2'\}.$$

The last set is associated to $\{r_m, r_0, r_1, r_2\}$. Now $\{r_0, r_1, r_2\}$ is geometric with unique geometric set $\{H_0, H_1, H_2\}$, this shows $H_i = H_i'$ for $i = 0, 1, 2$ and in particular $\{H_0, H_m\}$ is geometric. □

Lemma 3.8. *Assume $|R| = 3$. Then R is geometric.*

Proof. R is geometric by Lemma 2.5 if it is reducible. So let R be irreducible. If $R = \{s, t, u\}$ is 2-spherical, then it is geometric by [3]. Assume $o(su) = \infty$, then there is a unique geometric pair of roots $\{H_s, H_u\}$ associated to s, u. Assume further all roots H_t associated to t satisfy that $\{H_s, H_t\}$ is geometric, $\{H_t, H_u\}$ is not geometric. Then we already have $t \in [s, u]$, contrary to our assumption on R. Thus, there must exist a root H_t such that $\{H_s, H_t, H_u\}$ is 2-geometric and thus geometric. □

For the next lemma we omit the properties $\{r \in R \mid \exists u, v \in R : \ r \in [u, v]\} = \emptyset$, $R_2^\circ = \emptyset$ on R.

Lemma 3.9. *Let $s, t, u, v \in R$. Let $2 < o(uv) < \infty$, $o(st) = \infty$, $s, t \notin \{u, v\}^\perp$. Let H_s, H_t, H_u, H_v be roots associated to s, t, u, v, such that the sets $\{H_s, H_u, H_v\}$ and $\{H_t, -H_u, -H_v\}$ are geometric. Let $F = H_u \cap H_v$, $F' = -H_u \cap -H_v$. Then*

(a) $C(s) \cap F \neq \emptyset$,

(b) $H(u_v, F) = -H(u_v, F')$,

(c) $u_v, v_u \in [s, t]$.

Proof. For (a) we have that $\{H_s, H_u, H_v\}$ is geometric, therefore $H_s \cap H_u \cap H_v \neq \emptyset$ and is a fundamental domain for $\langle s, u, v\rangle$. This fundamental domain contains chambers of $C(s)$, proving our first statement. For (b) we note that $H(u_v, F)$ is well-defined. Furthermore we have $F' = w_{\{u,v\}}.F$, which proves (b). Using $s, t \notin \{u, v\}^\perp$, we gain $o(u_v s) = \infty = o(u_v t)$, therefore (b) yields $H(u_v, s) = -H(u_v, t)$ and (c) holds. □

Lemma 3.10. *Let R be irreducible and non-spherical of rank 4 satisfying condition* (E). *Assume the diagram of R is a chord-free circuit. Then R is geometric.*

Proof. The result is clear if $R = \{s, t, u, v\}$ is 2-spherical by [3]. Say $o(su) = o(tv) = 2$, since the diagram of R is a chord-free circuit and assume further $o(st) = \infty$. If $o(uv) = \infty$ and $o(tu), o(vs)$ are finite, the diagram is a chord-free circuit in the sense of [8] and therefore geometric.

If $o(tu), o(vs)$ are infinite as well, the diagram is right-angled. In this setting, every pair $\{H_s, H_u\}$ of roots associated to s, u is geometric, and we can make the choice $H_s := H(s, t) = H(s, v)$ and $H_u := H(u, t) = H(u, v)$. The equality $H(s, t) = H(s, v)$ holds since $o(tv) = 2 < \infty$ and $H(s, t) = H(s, A_{t,v}) = H(s, v)$ for a spherical residue $A_{t,v}$ stabilized by $\langle t, v\rangle$. In the same way we can choose $H_t := H(t, s) = H(t, u)$ and $H_v := H(v, s) = H(v, u)$. The set $\{H_s, H_t, H_u, H_v\}$ is 2-geometric by construction. The intersection $H_s \cap H_t \cap H_u \cap H_v \cap A_{t,v}$ is not empty, since $A_{t,v} \subset H_s \cap H_u$, furthermore $A_{t,v} \cap H_t \cap H_v \neq \emptyset$, and $\{H_s, H_t, H_u, H_v\}$ is geometric, thus R is geometric.

If $o(tu)$ is finite, $o(vs)$ infinite, consider v, s instead of s, t and t, u instead of u, v. So we can assume $o(st) = \infty > o(uv)$.

In this case we have a unique geometric pair of roots $\{H_s, H_t\}$ associated to s, t. We denote H_u, H_v the unique roots associated to u, v such that $\{H_s, H_t, H_u\}$

is geometric and $\{H_s, H_t, H_v\}$ is geometric. Assume $\{H_u, H_v\}$ is not a geometric pair. Then $\{H_t, H_u, -H_v\}$ is 2-geometric as well as $\{H_s, -H_u, H_v\}$. If both sets are geometric, we can use Lemma 3.9 and have $u_v, v_u \in [s,t]$, in contradiction to $R_2^\circ = \emptyset$. If both sets are 2-geometric but not geometric, the sets $\{-H_s, -H_v, H_u\}$, $\{-H_t, H_v, -H_u\}$ are each geometric, and the same argument holds.

So assume $\{H_s, H_v, -H_u\}$ and $\{-H_t, H_v, -H_u\}$ are geometric. This is a contradiction if $o(tu) = \infty$, since then $-H_u$ cannot be part of a geometric pair with any root associated to t, $\{H_t, H_u\}$ is the only geometric pair associated to these roots. So we assume $o(tu) < \infty$. Spherical rank 2 residues stabilized by $\langle s,u\rangle$ (say $A_{s,u}$), $\langle u,v\rangle$ (say $A_{u,v}$) are contained in $H(t_u, A_{u,v}) = H(t_u, v) = H(t_u, s) = H(t_u, A_{s,u})$, else $t_u \in [s,v]$. Now $A_{s,u}, A_{u,v}$ have nonempty intersection with both roots associated to u, therefore both residues have nonempty intersection with one of the standard fundamental domains for the $\langle t,u\rangle$ action. The reflection t_u separates the two fundamental domains for this action. If $A_{s,u}, A_{u,v}$ are separated by t, such that $H(t, A_{s,u}) = -H(t, A_{u,v})$, they have nonempty intersection with different fundamental domains and $t_u \in [s,v]$ holds. So we have $H_t = H(t, A_{s,u}) = H(t, A_{u,v})$. This contradicts the fact that $\{-H_t, H_v, -H_u\}$ is geometric, because this requires $A_{u,v} \subset -H_t$.

Therefore $\Phi := \{H_s, H_t, H_u, H_v\}$ is a geometric set. □

Proof of Proposition 3.3. Choose an arbitrary reflection $r \in R$ and a root H_r associated to r. Consider an arbitrary reflection $s \in R\setminus\{r\}$ and a path connecting them in the diagram, say $\gamma_s = (r = r_0, \ldots, r_m = s)$. We define roots H_{r_i} associated to r_i, $i > 0$, inductively such that $\{H_{r_i}, H_{r_{i-1}}\}$ is a weakly geometric pair. The root $H_{r_m} = H_s$ associated to s then does not depend on the choice of the path γ_s, since all trees and chord-free circuits are geometric.

Since R is irreducible, we find for every $s \in R$ a path γ_s connecting r and s yielding a root H_s associated to s. The set $\Phi := \{H_s \mid s \in R\}$ is well-defined and weakly 2-geometric, therefore R is geometric due to Theorem 3.4. □

The proven statement will allow us to characterize a geometric set by considering the sets $\{r \in R \mid \exists u, v \in R : \ r \in [u,v]\}$ and R_2°.

To complete the proof of the implication (b) ⇒ (a) in Theorem 3.2 we will prove the useful property that $\{r \in R \mid \exists u, v \in R : \ r \in [u,v]\} = \emptyset$ if and only if $R^\circ = \emptyset$.

Definition 3.11. Let $\gamma = (c_0, \ldots, c_m)$ be a gallery in (C,P). We say γ *crosses* $r \in S^W$, if there is an index $0 \leq i < m$ such that $H(r, c_i) = -H(r, c_{i+1})$. In this situation, $\{c_i, c_{i+1}\}$ is a panel in P_r. It is easy to see that a minimal gallery crossing r crosses r only once.

Lemma 3.12. *Let $r, s, t \in S^W$. If a minimal gallery connecting $C(s)$ to $C(t)$ crosses r, then $\delta(s,t) > \delta(s,t^r)$.*

Proof. Let $\gamma = (c_0, \dots, c_m)$ be this minimal gallery with $c_0 \in C(s), c_m \in C(t)$, i the index such that $H(r, c_i) = -H(r, c_{i+1})$. Then $c_i = r.c_{i+1}$ and $\gamma = (c_0, \dots, c_i = r.c_{i+1}, \dots, r.c_m)$ is a gallery of shorter length connecting $C(s)$ to $C(t^r)$. □

Lemma 3.13. *Let $R' = \{r_0, \dots, r_m\} \subset S^W$ and let $\{H_0, \dots, H_m\}$ be a set of roots associated to the elements in R'. Assume $D = \bigcap_{i=0}^m H_i \neq \emptyset$. If $\gamma = (c_0, \dots, c_m)$ is a gallery satisfying $c_0 \notin D, c_m \in D$, then γ crosses one element in R'.*

Proof. Assume not, then $H(r_i, c_0) = H(r_i, c_m)$ for $i = 0, \dots, r_m$. Since $H_i = H(r_i, c_m)$ for $i = 0, \dots, m$, this yields $c_0 \in \bigcap_{i=0}^m H_i = D$, contradicting our assumptions. □

Lemma 3.14. *Suppose we have a universal, sharp-angled set $R \subset S^W$, $\{r, s, t\} \subset R$ and $r \in [s, t]$. Then:*

(a) *If $o(rs) = \infty = o(rt)$, then $\delta(s,t) > \delta(s,t^r)$.*

(b) *If $o(rs) < \infty > o(rt)$, then $\delta(s,t) > \delta(s,t^r)$.*

(c) *If $o(rs) < \infty = o(rt)$, then $\delta(s,t) > \delta(s,t^r)$ or $\delta(r,t) > \delta(r,t^s)$.*

Proof. Assertion (a) is obvious, since a minimal gallery connecting s, t crosses r.

For (b), if a minimal gallery emanating from s to t crosses r, we are done. Else we can say that the minimal gallery $\gamma = (c_0, \dots, c_m)$ with $c_0 \in C(s)$, $c_m \in C(t)$ is included in a root H_r associated to r. We set $\{H_r, H_s\}$ the geometric pair associated to r, s such that $\gamma \subset H_s$ and H_t the root associated to t such that $\{H_s, H_t\}$, $\{-H_r, H_t\}$ are geometric pairs. W.l.o.g. we can assume $\gamma \subset H_r \cap H_s$, else exchange s and t.

We have $c_m \in H_t$. Denote $t' = t^r$ and let $H_{t'} = H(t', c_m)$. Then the pair $\{r, t'\}$ is geometric. Let $H'_{t'}$ denote the root associated to t' such that $H'_{t'} = H(t', s)$. This is well-defined since $o(st') = \infty$, using Lemma 2.7. Since $H'_{t'} \cap H_r$ is a fundamental domain for $\langle r, t' \rangle = \langle r, t \rangle$ and $C(s) \subset H'_{t'}$, we can show $H'_{t'} = -H_{t'}$. Assume $H'_{t'} = H_{t'}$, then $c_m \in C(t) \cap H'_{t'} \cap H_r = \emptyset$, a contradiction. So we find an index $i < m$ satisfying $c_i \in H'_{t'}$, $c_{i+1} \in H_{t'}$ and $\delta(s, t^r) < \delta(s,t)$.

For (c), let $\{H_r, H_s\}$, $\{-H_r, -H_s\}$ be the geometric pairs of roots associated to r, s. Consider a minimal gallery $\gamma_s = (c_0, \dots, c_m)$ from $C(t) \ni c_0$ to $C(s) \ni c_m$. If it crosses r, then $\delta(s,t) > \delta(s,t^r)$. So assume γ_s does not cross r. Furthermore we can assume that a minimal gallery $\gamma_r = (c'_0, \dots, c'_k)$ from $C(t)$ to $C(r)$ does not cross s, else $\delta(r,t) > \delta(r,t^s)$, as required. So we have $\delta(s,t) = m$, $\delta(r,t) = k$ and we can assume $k \leq m$. Then the gallery $r.\gamma_r = (r.c'_0, \ldots, r.c'_k)$ connects $C(t^r)$ to $C(r)$. If $r.\gamma_r$ crosses s, we are done since we find $\delta(t^r, s) < k \leq m = \delta(s,t)$. Assume it does not cross s, then $C(t^r) \subset F$ for a fundamental domain $F = H_r \cap H_s$ and a geometric pair $\{H_r, H_s\}$ associated to r, s, since $r.c'_0 \in F$. But then $\{t^r, r, s\}$ are geometric, and $\delta(t,s) = \delta((t^r)^r, s) > \delta(t^r, s)$ holds. □

Corollary 3.15. *Suppose we have $R, \{r,s,t\} \subset R$ as in Lemma 3.14, $r \in [s,t]$. If $o(rs) < \infty > o(rt)$ or $o(rs) = \infty = o(rt)$, then $r \in_r [s,t]$. If $o(rs) < \infty = o(rt)$, $r \in_r [s,t]$ or $s \in_r [r,t]$.*

Proof. This is immediate from Lemma 3.14 and the definition of reducible separation. □

Proof of Theorem 3.2. Assertion (a) implies the existence of a fundamental domain $\{c\}$ for the W-action on $\mathcal{C}$, $c \in C(r)$ for all $r \in R$. This shows (a) $\Rightarrow$ (d) and (a) $\Rightarrow$ (c), the latter since c corresponds to an element $w \in W$ and $R = S^w = wSw^{-1}$. The implication (b) $\Rightarrow$ (a) follows from Proposition 3.3, since $R^\circ = \emptyset \Leftrightarrow \{r \in R \mid \exists s,t \in R : r \in [s,t]\} = \emptyset$ by Corollary 3.15. If (c) holds, R is geometric since S is geometric, so (c) $\Rightarrow$ (a).

We show (d) $\Rightarrow$ (b): Assume we have $r \in R^\circ$, then there exist $s,t \in R$ such that $r \in [s,t]$, $\delta(s^r, t) < \delta(s,t) = 0$, a contradiction. The same argument holds if $r' \in [s,t]$ for an $r' \in R_2^\circ$.

We proved (a) $\Leftrightarrow$ (c); (a) $\Rightarrow$ (d) $\Rightarrow$ (b) $\Rightarrow$ (a), thus the proposition holds. □

4 *J*-reductions

Throughout this section we will prove our main theorem using a reduction of the distance matrix of R. The proof consists of the distinction of three cases, dependent on the sets R_2° and R°. These cases are described below. Recall from Section 2.4 the definitions of $R_2 = \{s_t \in R^W \mid s,t \in R, 2 < o(st) < \infty\}$, $R_2^\circ = \{s_t \in R_2 \mid \exists u,v \in R \setminus \{s,t\} : \ s_t \in [u,v]\}$.

In this section we will always assume that $R \subset S^W$ is sharp-angled, universal, irreducible and non-spherical of finite rank at least 3. We will also assume that the diagram of R satisfies condition (E).

4.1 Proof of the main theorem

Proof of the Main Theorem. Let (W,S) be a Coxeter system, S irreducible non-spherical satisfying (E), and let R be an irreducible, sharp-angled Coxeter generating set. By considering the Cayley graph $\mathcal{C}$ of (W, R), we can switch the roles of R and S. In [4, Corollary A.4] it is proved that if R is sharp-angled in the Cayley graph of S, S is also sharp-angled in the Cayley graph of R. Thus we can assume we have an arbitrary Coxeter system (W, S) and an irreducible sharp-angled Coxeter generating set R satisfying (E).

We prove the theorem by induction on the entries in $D_1(R)$. If $D_1(R) = 0$, R is conjugate to S by Theorem 3.2.

So assume $D_1(R) > 0$. Then by Theorem 3.2 $R^\circ \neq \emptyset$ or $R_2^\circ \neq \emptyset$.

Case 1: If $R_2^\circ \neq \emptyset$, we will construct a sharp-angled Coxeter generating set $\bar{R}$ in Section 4.4, resulting from R by a series of twists. We will show in Propositions 4.18 or 4.22 that $\bar{R}$ satisfies $D_1(\bar{R}) < D_1(R)$.

Case 2: Assume $R_2^\circ = \emptyset$ and there exist $s, t \in R$ such that $o(st)$ is even, $H_s \cap H_t = F$ is a standard fundamental domain for $\langle s, t\rangle$ and $C(r) \subset H(s_t, F) \cap -H(t_s, F)$ for all $r \in R \setminus (\{s, t\} \cup \{s, t\}^\perp)$. In Section 4.6 we will construct a sharp-angled Coxeter generating set $\bar{R}$, and we will show in Proposition 4.26 that $D_1(\bar{R}) < D_1(R)$ holds.

Case 3: Assume $R_2^\circ = \emptyset$ and there do not exist $s, t \in R$ as in Case 2. Then we construct a sharp-angled Coxeter generating set $\bar{R}$ in Section 4.5, which again satisfies $D_1(\bar{R}) < D_1(R)$, this will be shown in Proposition 4.25.

In every case we can find a Coxeter generating set $\bar{R}$, twist-equivalent to R and satisfying $D_1(\bar{R}) < D_1(R)$. Furthermore $\bar{R}$ satisfies condition (E) by Lemma 2.15. Using the induction hypothesis now implies that $\bar{R}$ is already twist equivalent to S, this proves our theorem. □

Before we can continue to prove the three mentioned cases in the proof in Sections 4.4 to 4.6, we will need some more properties of longest reflections. We will also need a more precise understanding of the Case (c) in Lemma 3.14, dependent on the order of the product rs. These will be stated in Section 4.2.

We will then in Section 4.3 introduce the notion of interior separation, a notion stronger than reducible separation. This concept is useful for handling the cases occurring in the proof of the main theorem.

Remark 4.1. A note on the figures, which will occasionally be used in this section to illustrate some of the geometric ideas behind the technical proofs: We will depict the Cayley graph as a circle in the style of the Poincaré disc model for the hyperbolic plane, even though in general the Cayley graph does not result from a tessellation of a hyperbolic space.

We will always depict reflections of the Coxeter generating set we are currently considering as solid lines, attached to the boundary of the circle. If the context requires a certain root to be chosen, we will emphasize the corresponding half space with short solid lines emanating from the reflection line. Two lines intersecting means the product of the corresponding reflections having finite order, and infinite order otherwise. Conjugates of reflections will be represented by dashed lines, we will use this in particular for reflections resulting from the application of a twist. We will mark the transition caused by a twist as a dotted arrow.

Note that the figures' sole purpose is to give a geometric intuition to the methods we use, they are not part of our proofs.

4.2 Longest reflections and reducible separation

In the following part we will prove further properties of longest reflections and their products. We need this in particular for Lemma 4.4, which is necessary to handle rotation twists. Furthermore, we will give criteria for when separation implies reducible separation, based on the results in Lemma 3.14.

Lemma 4.2. *Assume $s_t, u_v \in R_2$. If $s_t u_v$ has finite order greater than 2, then $v = t$ or $u_v = t_s$. In particular $2 < o(s_t u_t) < \infty$ implies $o(st), o(tu) \in \{3, 4\}$.*

Proof. Assume $\{s,t\}$ and $\{u,v\}$ are disjoint, $\{s,t\} \not\subset \{u,v\}^\perp$. Using [3, Table 1, p. 529], computing the product of the longest roots shows the following facts.

First, we have $o(s_t u_v) = \infty$ if two of $o(su), o(tu), o(sv), o(tv)$ are greater or equal to 3, the diagram is not a tree.

Second, if only one of the above mentioned orders is ≥ 3, assume $o(tu) \geq 3$, and the diagram is a tree, then due to condition (E) our Lemma holds as well.

So the sets $\{s,t\}$, $\{u,v\}$ are not disjoint, assume we have the set $\{s,t,u\}$. We will calculate the orders of the longest reflections. If two of the orders of st, tu, su are ≥ 5, then all longest reflections in different rank 2 sets have infinite order, in particular we can assume that the diagram is not a tree. Furthermore, if $o(st), o(tu)$ are both odd, there is nothing to show, since $s_t = t_s$. So assume one of the orders, say $o(st)$, is even. A calculation shows $t_s t_u$, $t_s u_t$ have infinite order, showing our lemma.

The last assertion follows from condition (E), if one of the orders, say $o(st)$, is less than 5, (E) implies $o(su) \geq 3$. Then a calculation yields $o(s_t u_t) = \infty$ whenever $o(ut) \geq 5$. □

Lemma 4.3. *Let $\{s,t\} \neq \{u,v\}$ and $s_t, u_v \in R_2$. Then $\{s_t, u_v\}$ is geometric, with geometric pair $\{H(s_t, A_{u,v}), H(u_v, A_{s,t})\}$.*

Proof. If $o(s_t u_v) = \infty$ or s_t,u_v commute, the statement clearly holds. So let $2 < o(s_t u_v) < \infty$, by Lemma 4.2 we know $t = v$. By Lemma 4.2 the order is infinite in case $o(st) \geq 5$ or $o(tu) \geq 5$, so we can assume $o(st), o(tu)$ being 3 or 4.

Then $s_t = tst$ and $u_t = tut$. Now $\{s, u\}$ is geometric with geometric pair of roots $\{H(s, A_{t,u}), H(u, A_{s,t})\}$. Thus the pair $\{H(tst, t.A_{t,u}), H(tut, t.A_{s,t})\}$ is geometric as well. Using that $A_{t,u}, A_{s,t}$ are stabilized by t proves the lemma. □

Lemma 4.4. *Assume $R_2^\circ \neq \emptyset$. Then we can find $s, t \in R$ with $s_t \in R_2^\circ$ such that either st has odd order or such that st has even order and there exists a root H associated to s_t satisfying:*

Whenever $x_y \in R_2^\circ$ with $|\{x, y\} \cap \{s, t\}| = 1$, then $A_{x,y} \subset H$.

Proof. Assume for $s_t \in R_2^\circ$ the setup of $o(st)$ even and H a root associated to s_t such that $A_{x,y} \subset -H$, $A_{x',y'} \subset H$ for $x_y, x'_{y'} \in R_2^\circ$ and $|\{x, y\} \cap \{s, t\}| = 1 = |\{x', y'\} \cap \{s, t\}|$. If s_t does not satisfy one of these criteria, the lemma already holds.

Construct a maximal sequence $(r_0, \ldots, r_m)$ with $r_i \in R_2$ (not necessarily in R_2°) such that:

(1) If $r_i = u_v$, then $o(uv)$ is even and $u_v \in R_2^\circ$ or $v_u \in R_2^\circ$ hold, for $i = 0, \ldots, m$;

(2) if $r_i = u_v, r_{i+1} = u'_{v'}$, then $|\{u, v\} \cap \{u', v'\}| = 1$ for $i = 0, \ldots, m-1$;

(3) if $r_{i+1} = u_v$, then $r_i u_v, r_i v_u$ have infinite order for $i = 0, \ldots, m-1$;

(4) for $0 \leq i < m$, $H(r_i, A_{i-1}) = -H(r_i, A_{i+1})$.

Here A_i is a spherical residue stabilized by $\langle u, v\rangle$ for $r_i = u_v$ and $A_{-1} := A_{x',y'}$. We build the sequence such that $r_0 = s_t$ or $r_0 = t_s$, thus the sequence is nonempty. The conditions (3) and (4) imply $H(r_i, r_j)$ are defined and equal for all $j < i$. Therefore, $C(r_{i+1}) \subset \bigcap_{j=1}^{i} H(r_j, r_{j+1})$ and in particular this sequence is finite, since R_2 is finite.

So assume $r_m = u_v$, and w.l.o.g. we can assume $u_v \in R_2^\circ$. Set $H = H(u_v, r_j)$ for $j < m$. Now assume we have $a_b \in R_2^\circ$ with $|\{a, b\} \cap \{u, v\}| = 1$ and $A_{a,b} \subset -H$.

If $o(ab)$ is odd, we are done, so let $o(ab)$ be even. In case $o(u_v a_b) = \infty = o(u_v b_a)$ the sequence $(r_0, \ldots, r_m, r_{m+1} = a_b)$ satisfies (1)–(4), contradicting the maximality of the sequence. In case one of $o(u_v a_b)$ or $o(u_v b_a)$ is finite, by Lemma 4.2 the products $v_u a_b$ and $v_u b_a$ have infinite order. For the sequence $(r_0, \ldots, r_{m-1}, r'_m = v_u, a_b)$ the statements (1), (2) and (3) hold by definition, for (4) we already have $H(u_v, A_j) = -H(u_v, A_{a,b})$ for all $j < m$. The reflections u_v, v_u both separate the fundamental domains $F := H_u \cap H_v$ and $F' := -H_u \cap -H_v$ for one choice of a geometric pair $\{H_u, H_v\}$ associated to u, v. Since a_b intersects either u or v by (2) and the same holds for r_{m-1}, a_b and r_{m-1} each have nonempty intersection with one of the fundamental domains F or F'. Since u_v separates A_j and $A_{a,b}$ these fundamental domains are different, therefore $H(v_u, A_j) = -H(v_u, A_{a,b})$ holds as well. This contradicts the maximality of the sequence $(r_0, \ldots, r_{m-1}, u_v)$, proving the existence of a longest reflection $u_v \in R_2^\circ$ and an associated root H such that $A_{a,b} \subset H$ whenever we have $a_b \in R_2^\circ$ with $|\{a, b\} \cap \{u, v\}| = 1$. □

Lemma 4.5. *Let $r, s, t \in R$, $r \in [s, t]$, $o(rs)$ even and finite, $o(rt) = \infty$. If $C(t)$ is not contained in a fundamental domain $H_1 \cap H_2$ for roots associated to r_s, s_r, every minimal gallery connecting $C(t)$ and $C(s)$ or $C(t)$ and $C(r)$ is contained in $H(r_s, t) \cap H(s_r, t)$.*

Proof. Assume $\gamma = (c_0, \ldots, c_m)$ is a minimal gallery connecting $C(t) \ni c_0$ to $C(r) \ni c_m$ crossing r_s. Then $\gamma' = (c_0, \ldots, c_i = r_s.c_{i+1}, \ldots, r_s.c_m)$ for some index i is a gallery of length less than m connecting $C(t)$ and $C(r)$. So assume γ crosses s_r. The root $H(r, t)$ contains a fundamental domain $H_1 \cap H_2$ for roots associated to r_s, s_r not containing $C(t)$. So there is an index i such that $c_i \notin H_1 \cap H_2$, $c_{i+1} \in H_1 \cap H_2$. Since $C(r)$ has no chambers in this fundamental domain, there is an index $j > i$ satisfying $c_j \in H_1 \cap H_2, c_{j+1} \notin H_1 \cap H_2$. But γ cannot cross s_r twice, so by Lemma 3.13 it crosses r_s, and we are done. □

Lemma 4.6. *Assume Case* (c) *of Lemma* 3.14 *and rs having even order. Denote with H_1, H_2 roots associated to r_s, s_r such that $H_1 \cap H_2 =: F$ is a fundamental domain for $\langle s, r\rangle$ and such that $H(s, F) = H(s, t)$, $H(r, F) = H(r, t)$. Then:*

(a) *If $C(t) \subset F$, then $\delta(r, t^s) < \delta(r, t)$ and $\delta(s, t^r) < \delta(s, t)$.*

(b) *If $C(t) \subset -H_1 \cap H_2$, then $\delta(r, t^s) < \delta(r, t)$.*

(c) *If $C(t) \subset H_1 \cap -H_2$, then $\delta(s, t^r) < \delta(s, t)$.*

Proof. For (a), consider a minimal gallery $\gamma_r = (c_0, \ldots, c_m)$ emanating from t to r. We have $C(s) \cap F = \emptyset = C(r) \cap F$. Thus, γ must cross r_s or s_r by Lemma 3.13 and cannot cross r_s, since $r_s r = r r_s$ holds. Since $C(r) \cap F = \emptyset$, there is an index i satisfying $c_i \in H_2, c_{i+1} \in -H_2$. Consider the gallery $s.\gamma_r = (s.c_0, \ldots, s.c_i, s.c_{i+1}, \ldots, s.c_m)$, which is a minimal gallery connecting t^s and r^s. So $r^s = r^{s_r}$ holds, yielding a gallery $\gamma' = (s.c_0, \ldots, s.c_i = s_r s.c_{i+1}, \ldots, s_r s.c_m)$ of length less than m connecting sts to r. The same holds for a gallery emanating from t to s.

In the case of (b) a minimal gallery connecting t and r crosses s, since $C(r) \subset -H(s,t) \cup H_1$ holds and a minimal gallery cannot cross r_s. The same holds in the case of (c). □

Lemma 4.7. *Assume Case* (c) *of Lemma* 3.14 *and* rs *having odd order. If* $\delta(t,r) \leq \delta(t,s)$*, then* $\delta(s,t^r) < \delta(s,t)$ *and* $r \in_r [s,t]$ *holds.*

Proof. Let γ be a minimal gallery connecting $C(s)$ to $C(t)$. The result is immediate, if γ crosses r. So assume γ does not cross r. Then $r.\gamma$ is a gallery connecting $C(t^r)$ to $C(s^r)$. Now we can have the situation of $C(t^r)$ being in a standard fundamental domain $H_r \cap H_s$ for $\{r,s\}$ and thus $r.\gamma$ crosses s by Lemma 3.13. Otherwise $C(t^r)$ is not contained in such a fundamental domain. A minimal gallery γ' connecting $C(r)$ and $C(t)$ cannot cross s due to our assumption $\delta(t,r) \leq \delta(t,s)$. We conclude that $r.\gamma'$ emanates from $C(r)$ to $C(t^r)$, crosses s and the fact $\delta(t,r) \leq \delta(t,s)$ gives rise to a gallery of length less than $\delta(t,r)$ connecting $C(s)$ and $C(t^r)$, as required. □

Lemma 4.8. *Let* $r,s,t \in R$*. Then* $\delta(s,t^r) < \delta(s,t)$ *holds if and only if* $r \in_r [s,t]$.

Proof. If $r \in_r [s,t]$, $\delta(s,t^r) < \delta(s,t)$ holds by definition. So assume $\delta(s,t^r) < \delta(s,t)$. Since $\delta(s,t) > \delta(s,t^r) \geq 0$, $o(st) = \infty$. Furthermore we can suppose $o(rs), o(rt) > 2$, else $\delta(s,t) = \delta(s,t^r)$. Consider the geometric pair of roots $\{H_s, H_t\}$ associated to s,t. Assume there is a root H_r such that $\{H_r, H_s\}$, $\{H_r, H_t\}$ is geometric, then the triple $\{H_r, H_s, H_t\}$ is already geometric. Let $t' = t^r$, then $o(t's) = \infty$ and $\{H(t',s), -H_r\}$ is a geometric pair. Now $H(s,t) = H(s,t')$ holds, and therefore we have $r \in [s,t']$. Now we have $\delta(s,t') < \delta(s,t'^r)$, so by Corollary 3.15 we are in the situation $o(rs) < \infty$, $o(rt) = o(rt') = \infty$, else $\delta(s,t') > \delta(s,t'^r)$ holds. Let $F = H_s \cap H_t$ be the fundamental domain of $\langle r,s\rangle$ containing $C(t)$, then $C(t') \subset r.F$. Now $F \cup r.F \subset H(r_s,F) \cap H(s_r,F)$ and $\delta(r,t') < \delta(s,t')$ holds. By Lemma 4.6 and Lemma 4.7 we have $\delta(s,t') > \delta(s,t'^r) = \delta(s,t)$, a contradiction. □

4.3 Interior separation

Definition 4.9. Let $r,s,t \in R$, $T \subset R_2$. For $c \in C$, set $D(T,c) := \bigcap_{t\in T} H(t,c)$. We have $D(T,c) = C$ for $T = \emptyset$ and arbitrary $c \in C$. For $D := D(T,c)$ define $C_D(u) := C(u) \cap D$ for $u \in S^W$.

We say $r \in_D [s,t]$ if $H(r, C_D(s)), H(r, C_D(t))$ are well-defined (i.e. $C_D(s)$, $C_D(t)$ are not empty and contained in a unique root associated to r) and they satisfy $H(r, C_D(s)) = -H(r, C_D(t))$.

Note that since D is convex, it contains any gallery from $C_D(s)$ to $C_D(t)$. Since those are on two different sides of r if $r \in_D [s,t]$, such a gallery crosses r, and $C(r) \cap D \neq \emptyset$. If on the other hand a minimal gallery from $C_D(s)$ to $C_D(t)$ in D crosses r and those roots are well-defined, $r \in_D [s,t]$ holds.

Example 4.10. If $r \in [s,t]$, with $o(rs) = \infty = o(rt)$ we have $r \in_C [s,t]$. Furthermore, if for $x \in R_2$ we have a $c \in C$ such that $H(x,c)$ has nonempty intersection with $C(r), C(s), C(t)$, in particular if x commutes with r,s,t, for $D = H(x,c)$ we have $r \in_D [s,t]$.

Now let $r \in [s,t]$ such that $o(r,s) < \infty = o(rt)$. Define $D = H(r_s,t)$. Then either $r \in_D [s,t]$ or $s \in_D [r,t]$ holds.

We will give a criterion on D for the roots $H(r, C_D(s))$ to be well-defined.

Lemma 4.11. *Let $r,s \in R$, $T \subset R_2, c \in C(r), D = D(T,c)$ such that $C(s) \cap D \neq \emptyset$. If rs has infinite order, then $H(r, C_D(s))$ exists. If $2 < o(rs) < \infty$, $H(r, C_D(s))$ exists if $r_s \in T$ or $s_r \in T$.*

Proof. If rs has infinite order, $H(r, C(s))$ exists and coincides with $H(r, C_D(s))$ since $C_D(s) \subset C(s)$.

Now let $2 < o(rs) < \infty$. If $r_s \in T$ or $s_r \in T$, then assume there are chambers c, d in $C_D(s) \subset C(s)$ with the property $H(r,c) = -H(r,d)$. We can assume that c, d are contained in opposite fundamental domains for the action of $\langle r,s \rangle$, eventually considering $s.c$ or $s.d$ instead of c or d. So $H(r_s,c) = -H(r_s,c')$, a contradiction. Thus, $C_D(s)$ is contained in a unique root associated to r. $\square$

Lemma 4.12. *Let $r,s,t \in R$, $D = D(T,c)$ such that $r \in_D [s,t]$. If $o(rs) < \infty$, then $r_s \in T$ or $s_r \in T$.*

Proof. Since $C_D(s)$ is well-defined, $C_D(s) \subset H(t,d)$ for some $d \in C$ and all $t \in T$, furthermore $C_D(s) \cap -H(r, C_D(s)) = \emptyset$ holds. Then there exists a $u \in T$ with the property $H(u, C_D(s)) = -H(u, c_s)$ for all $c_s \in C(s) \cap -H(r, C_D(s))$. The product su therefore has finite order. The product ru has finite order as well, assume not, then $C_D(s) \subset H(u,r)$, since $C(r) \cap D \neq \emptyset$, but $H(u,r)$ does not satisfy $H(u, C_D(s)) = -H(u, c_s)$ for all $c_s \in C(s) \cap -H(r, C_D(s))$. If $ru = ur$, $su = us$ both hold, both roots associated to u contain chambers in $C(s) \cap -H(r, C_D(s))$. So at least one of the orders must be greater than 2.

If $o(ru) > 2$, $u \in \{r_x, x_r\}$ for some $x \in R$ and since $2 < o(rs) < \infty$ we get $o(su) = \infty$ except for the case $u \in \{r_s, s_r\}$. If $o(su) > 2$, the same argument holds, yielding $r_s \in T$ or $s_r \in T$. $\square$

Lemma 4.13. *Let $r,s \in R$, $D = D(T,c)$ such that $H(r, C_D(s)), H(s, C_D(r))$ exist. Then the pair $\{H(r, C_D(s)), H(s, C_D(r))\}$ is geometric.*

Proof. The lemma is true if $o(rs)$ is infinite or 2. Otherwise let x be a reflection in $\{r_s, s_r\}$ in T, which exists by Lemma 4.12. The root $H(x, C_D(s)) = H(x, C_D(r))$ contains a unique fundamental domain for the $\langle r, s\rangle$-action on C of the form $H_r \cap H_s$ for some choice of geometric pair $\{H_r, H_s\}$ associated to r, s, which contains chambers from $C_D(s)$ and from $C_D(r)$. Therefore $H_r = H(r, C_D(s))$ and $H_s = H(s, C_D(r))$, proving the lemma. □

Lemma 4.14. *Let $r, s, t \in R$, $D = D(T, c)$ such that $H(r, C_{D'}(s))$, $H(r, C_{D'}(t))$ are defined. Let further $T' \subset T$, $D' = D(T', c)$, such that the roots $H(r, C_{D'}(s))$, $H(r, C_{D'}(t))$ are defined. Then $r \in_D [s, t] \Leftrightarrow r \in_{D'} [s, t]$.*

Proof. This results directly from $D \subset D'$ and $C_D(s) \subset C_{D'}(s)$. □

The previous lemma allows us in particular for $r \in_D [s, t]$ to retreat to the case $D = D(T, c)$ with T consisting of one element in r_s, s_r if the order $o(rs)$ is finite and one element from r_t, t_r, if $o(rt)$ is finite.

Corollary 4.15. *Let $r, s, t \in R$, $D = D(T, c)$ such that $r \in_D [s, t]$. Then $r \in [s, t]$.*

Proof. It follows from Lemma 4.13 that we get two geometric pairs of roots $\{H(r, C_D(s)), H(s, C_D(r))\}$ and $\{H(r, C_D(t)), H(t, C_D(r))\}$. It suffices to show $o(st) = \infty$, then we know $H(t, C_D(r)) = H(t, s)$ and $H(s, C_D(r)) = H(s, t)$ since a minimal gallery connecting $C_D(s)$ to $C_D(t)$ crosses r.

If both orders rs, rt are infinite, there is nothing to show. If rs has finite order, rt has infinite order, $C(s) \subset H(r, C_D(s)) \cup -H(r_s, C_D(s))$ and $C(t) \subset H(r_s, C_D(s)) \cap -H(r, C_D(s))$ hold. Thus there can be no spherical residue stabilized by $\langle s, t\rangle$. Assume both orders rs, rt are finite. Let $u \in \{r_s, s_r\} \cap T$, $u' \in \{r_t, t_r\} \cap T$, then

$$C(s) \subset (H(r, C_D(s)) \cap H(u, t)) \cup (-H(r, C_D(s)) \cap -H(u, t)),$$
$$C(t) \subset (H(r, C_D(t)) \cap H(u', s)) \cup (-H(r, C_D(t)) \cap -H(u', s))$$

hold, a spherical residue $A_{s,t}$ stabilized by $\langle s, t\rangle$ is in the intersection of the two, which is

$$(H(r, C_D(s)) \cap H(u, t) \cap -H(u', s)) \cup (H(r, C_D(t)) \cap -H(u, t) \cap H(u', s)),$$

the union being disjoint. But $(H(r, C_D(s)) \cap H(u, t) \cap -H(u', s))$ contains no panels stabilized by s, $(H(r, C_D(t) \cap -H(u, t) \cap H(u', s))$ contains no panels stabilized by t, thus such a spherical residue cannot exist, proving $o(st) = \infty$. □

Lemma 4.16. *Let $D = D(T, c)$, $r, s, t \in R$. Then $r \in_D [s, t]$ implies $r \in_r [s, t]$.*

Proof. Let $r \in_D [s,t]$. This implies $C_D(s), C_D(t)$ are nonempty and in different unique roots associated to r.

We know $r \in [s,t]$ by Corollary 4.15. We have to show $\delta(s,t) > \delta(s,t^r)$ by Lemma 4.8. This results directly from Lemma 3.14 for $o(rs), o(rt)$ both infinite or both finite.

In the case $o(sr)$ even, $o(rt)$ infinite we have $r_s \in T$ or $s_r \in T$. The lemma holds since minimal galleries between r, s, t never cross both longest reflections in the even case by Lemma 4.5. We have yet to deal with the following case: $o(sr) < \infty$ odd, $o(rt) = \infty$. If a minimal gallery γ between $C(s), C(t)$ crosses r, meaning $\gamma \subset H(r_s, t)$, we are done, so assume $\gamma = (c_0, \ldots, c_m)$, $c_0 \in C(s), c_m \in C(t)$ does not cross r. Then it crosses r_s. In particular, $\delta(s,t) > \delta(r,t)$ holds, else we find a gallery of length less than $\delta(r,t)$ connecting r, t. So we can use Lemma 4.7 and have $\delta(s,t^r) < \delta(s,t)$, $r \in_r [s,t]$ holds. □

4.4 $\{s,t\}$-reductions

Assume that $R_2^\circ \neq \emptyset$. This implies the rank of R being at least 4.

We consider the set $J = \{s,t\}$ with $s_t \in R_2^\circ$. Recall the statement from Lemma 4.4, that we can find $s_t \in R_2^\circ$ with $o(st)$ odd or $o(st)$ even and a root H associated to s_t such that $A_{x,y} \subset H$ whenever $x_y \in R_2^\circ$ exists with $|\{x,y\} \cap J| = 1$.

First assume st having odd order. Since $s_t \in R_2^\circ$, we find $u, v \in R$ with $s_t \in [u,v]$. We define sets L_v, K_v the following way: For $r \in R \setminus (J \cup J^\perp)$ set $r \in L_v$ if $H(s_t, r) = H(s_t, u)$, and set $r \in K_v$ if $H(s_t, r) = H(s_t, v)$. Since $o(s_t r) = \infty$ for all $r \in R \setminus (J \cup J^\perp)$, this construction yields $R = J \mathbin{\dot\cup} J^\perp \mathbin{\dot\cup} K_v \mathbin{\dot\cup} L_v$. In addition:

Lemma 4.17. *The pair (J, L_v) defined as above is an R-admissible pair.*

Proof. Since $H(s_t, l) = -H(s_t, k)$ whenever $l \in L_v$, $k \in K_v$, we have $s_t \in [l,k]$ and $o(lk) = \infty$ holds for all such l, k. □

Proposition 4.18. *Set $\bar{R} := T_{(J,L_v)}(R)$. Then $D_1(\bar{R}) < D_1(R)$.*

Proof. For $l \in L_v, k \in K_v$ a minimal gallery emanating from $C(l)$ to $C(k)$ crosses s_t, yielding a shorter gallery emanating from $C(l^{s_t})$ to $C(k)$. Thus, $\delta(l,k) > \delta(l^{s_t}, k)$ holds at least for the pair $l = u, k = v$. The relations in W yield $s^{s_t} = t$, $t^{s_t} = s$. So we have for all $l \in L_v$: $\delta(l,s) = \delta(l^{s_t}, t)$, $\delta(l,t) = \delta(l^{s_t}, s)$. Then, using a permutation mapping (l,s) to (l,t) and vice versa, we gain $D_1(\bar{R}) < D_1(R)$. □

Now assume $o(st)$ is even, there exists a root H associated to s_t such that $A_{x,y} \subset H$ whenever x_y exists with $|\{x, y\} \cap J| = 1$ by Lemma 4.4. Let H_s, H_t be roots associated to s, t such that $H_s \cap H_t =: F$ is a fundamental domain and $H = H(s_t, F)$. In case $R_2^\circ \setminus (\{s_t, t_s\} \cup \{r \in R_2^\circ \mid rs_t = s_t r\})$ is empty, choose an arbitrary geometric pair $\{H_s, H_t\}$.

We note that due to our assumptions on s_t, whenever we take $u_v \in R_2$, $|\{u, v\} \cap J| = 1$, with $A_{u,v} \subset -H$, this yields $C(r) \subset H(u_v, F) = H(u_v, A_{s,t})$ for all $r \in R \setminus (\{u, v\} \cup \{u, v\}^\perp)$, else $u_v \in [r, t]$ if $s \in \{u, v\}$ or $u_v \in [r, s]$ if $t \in \{u, v\}$ in contradiction to our assumptions on s_t.

Define $T_s = \{s_t\} \cup \{u_v \in R_2 \mid A_{u,v} \subset -H, |\{u, v\} \cap J| = 1\}$. Let $c \in w_J.F \cap C(s) \cap C(t)$ and define $D_s = D(T_s, c)$. Then for all $r \in R$ satisfying $C(r) \subset -H(s_t, F)$ the intersection $C(r) \cap D_s$ is nonempty and the roots $H(s, C_{D_s}(r))$, $H(t, C_{D_s}(r))$ are defined by Lemma 4.11.

Now we define two sets L_s, K_s satisfying $R = \{s\} \ \dot\cup\ s^\perp \ \dot\cup\ K_s \ \dot\cup\ L_s$. For $r \in R \setminus (\{s\} \cup s^\perp)$, set $r \in L_s$ if $C_{D_s}(r) \neq \emptyset$ and $s \in_{D_s} [r, t]$. Else $r \in K_s$.

Lemma 4.19. *The pair $(\{s\}, L_s)$ is an R-admissible pair.*

Proof. Consider $l \in L_s$, $k \in K_s$. If $C_{D_s}(k) \neq \emptyset$ we have $s \in_{D_s} [l, k]$. This implies $s \in_r [l, k]$ by Lemma 4.16, $o(lk) = \infty$ holds and we are done. If $C_{D_s}(k) = \emptyset$, this implies $C(k) \subset H(s_t, F)$, and $s_t \in_r [l, k]$ holds. Therefore $o(lk) = \infty$ holds in all cases and $(\{s\}, L_s)$ is an R-admissible pair. □

Set $R' := T_{(\{s\}, L_s)}(R)$. Note that since $o(lk) = \infty$ for all $l \in L_s, k \in K_s$, $o(l^s k) = \infty$, this results from Lemma 2.7, or from the fact that the diagram is not changed by a rank 1 twist. Furthermore for $l \in L_s$ we have $o(lt) = \infty$ since $s \in_r [t, l]$, so $o(l^s t) = \infty$ as well. In consequence, the pair $J \subset R'$ and the root $H(t_s, F)$ still satisfy the property $A_{x,y} \subset H(t_s, F)$ whenever x_y exists with $|\{x, y\} \cap J| = 1$.

We apply the same for t. To be exact, we define

$$T_t = \{t_s\} \cup \{u_v \in R'_2 \mid A_{u,v} \subset -H, |\{u, v\} \cap J| = 1\}$$

and set $D_t = D(T_t, c)$. Again for all $r \in R$ satisfying $C(r) \subset -H(t_s, F)$ the intersection $C(r) \cap D_t$ is nonempty and $H(s, C_{D_t}(r))$, $H(t, C_{D_t}(r))$ are defined.

Define L_t, K_t in the same manner. For $r \in R' \setminus (\{t\} \cup \{t\}^\perp)$ set $r \in L_t$ if $C_{D_t}(r) \neq \emptyset$, $r \notin L_s^s$ and $t \in_{D'} [r, s]$. Else $r \in K_t$.

Lemma 4.20. *The pair $(\{t\}, L_t)$ is an R'-admissible pair.*

Proof. The proof copies from the proof of Lemma 4.19, except for $l \in L_t$, $k \in L_s^s$. This case results from $L_t \subset K_s$, and as mentioned above $o(lk) = \infty$ follows from Lemma 2.7 or the fact that rank 1 twists preserve the diagram. □

Set $R'' := T_{(\{t\},L_t)}(R')$. Now define L_J, K_J. For $r \in R'' \setminus (J \cup J^\perp)$ set $r \in L_J$ if $C(r) \subset -H(s_t, F) \cup -H(t_s, F)$, else $r \in K_J$. Clearly (J, L_J) is an R''-admissible pair and $L_s^s \cup L_t^t \subset L_J$ holds. We then define $\bar{R} := T_{(J,L_J)}(R'')$.

Remark 4.21. For the set K_J we have

$$K_J = \{r \in R \mid C(r) \subset H(s_t, F) \cap H(t_s, F)\} \subset K_s \cap K_t.$$

Define

$$L_0 := \{r \in R \mid C_{D_s}(r) \neq \emptyset, s \notin_D [t, r]\} \cap \{r \in R' \mid C_{D_t}(r) \neq \emptyset, t \notin_{D_t} [s, r]\}.$$

For $r \in s^\perp$ one of $r \in L_0$, $r \in J^\perp$, $r \in K_t$ or $r \in L_t$ holds. If $r \in K_t \setminus K_J$, meaning $C(r) \subset -H(t_s, F)$ and $t \notin_{D_t} [s, r]$, either $r \in K_s$, and thus $r \in L_0$ holds, or $r \in L_s$. Therefore we have:

$$\begin{aligned} R &= J \,\dot\cup\, J^\perp \,\dot\cup\, L_s \,\dot\cup\, L_t \,\dot\cup\, L_0 \,\dot\cup\, K_J, \\ R' &= J \,\dot\cup\, J^\perp \,\dot\cup\, L_s^s \,\dot\cup\, L_t \,\dot\cup\, L_0 \,\dot\cup\, K_J, \\ R'' &= J \,\dot\cup\, J^\perp \,\dot\cup\, L_s^s \,\dot\cup\, L_t^t \,\dot\cup\, L_0 \,\dot\cup\, K_J, \\ \bar{R} &= J \,\dot\cup\, J^\perp \,\dot\cup\, L_s^{sw_J} \,\dot\cup\, L_t^{tw_J} \,\dot\cup\, L_0^{w_J} \,\dot\cup\, K_J \\ &= J \,\dot\cup\, J^\perp \,\dot\cup\, L_s^{s_t} \,\dot\cup\, L_t^{t_s} \,\dot\cup\, L_0^{w_J} \,\dot\cup\, K_J. \end{aligned}$$

The transition of R to $\bar{R}$ in the case of st having order 4 is shown schematically in Figure 1.

Proposition 4.22. *The Coxeter generating set $\bar{R}$ satisfies $D_1(\bar{R}) < D_1(R)$.*

Proof. Distances to elements in $J^\perp$ are preserved. The same holds for the distances from L_s to s, from L_t to t, from L_0 and K_J to J. Since $s \in_{D_s} [l, t]$ for all $l \in L_s$ and $t \in_{D_t} [l', s]$ for all $l' \in L_t$, distances from L_s to t and from L_t to s are reduced.

The sets L_s, K_J are separated by s_t, in the sense that each pair of elements is separated by s_t, so $\delta(l, k) > \delta(l^{s_t}, k)$ holds for $l \in L_s$, $k \in K_J$ by Lemma 3.12. The same argument holds for L_t, K_J, which are separated by t_s.

Assume we have $l \in L_s$, $l' \in L_0$. Then $s \in_{D_s} [l, l']$ holds and $\delta(l, l') > \delta(l^s, l') = \delta(l^{s_t}, l'^{w_J})$. The same holds for $l \in L_t$, $l' \in L_0$.

Let $l \in L_s$, $l' \in L_t$. Then consider a minimal gallery $\gamma = (c_0, \ldots, c_m)$ of length m, with $c_0 \in C(l), c_m \in C(l')$. Assume there are indices i, j such that $c_i \in H_s, c_{i+1} \in -H_s, c_j \in -H_t, c_{j+1} \in H_t$, so we assume γ crosses s and t. W.l.o.g. $i < j$. Then $\gamma' = (s.c_0, \ldots s.c_i = c_{i+1}, \ldots, c_j = t.c_{j+1}, \ldots, t.c_m)$ is a gallery of length $m - 2$ connecting $C(l^s)$ to $C({l'}^t)$. We find

$$\gamma'' := w_J.\gamma' = (s_t.c_0, \ldots, s_t.c_i = w_J.c_{i+1}, \ldots, w_J.c_j = t_s.c_{j+1}, \ldots, t_s.c_m)$$

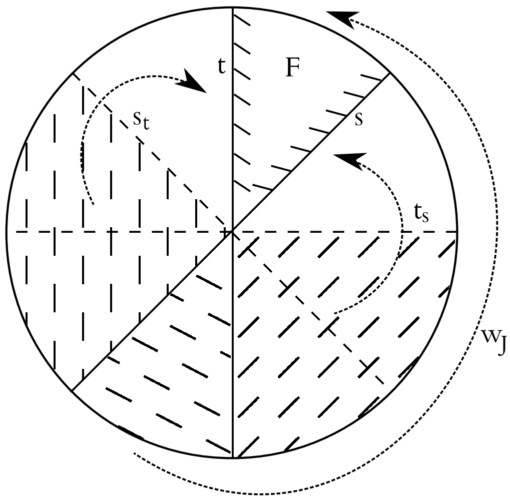


Figure 1: {s,t}-reductions in the even case

is a gallery of length $m-2$ connecting $C(l^{s_t})$ to $C(l'^{t_s})$, as required. Assume γ does not cross s. This implies $o(sl) < \infty$, l_s or $s_l \in T_s$. Denote this reflection x and set $D_x = D(\{x, s_t\}, c)$. We have $s \in_{D_s} [l, t]$ by construction of L_s. Consequently $s \in_{D_x} [l, t]$ and $s \in_{D_x} [l, l']$ since $H(s, C_{D_x}) = H(s, l')$, hereby we gain $\delta(l^s, l') < \delta(l, l')$. Due to $t \in_{D_t} [s, l']$, $t \in_{D_t} [l^s, l']$ holds as well, since we have $\delta(l, s) = 0 = \delta(l^s, s)$. This yields $H(t, C_{D_t}(s)) = H(t, C_{D_t}(l)) = H(t, C_{D_t}(l^s))$ and $\delta(l^s, l'^t) < \delta(l^s, l')$ holds, as required.

Finally consider $l \in L_0$, $k \in K_J$ and a minimal gallery $\gamma = (c_0, \dots, c_m)$, $c_0 \in C(l)$, $c_m \in C(k)$. Clearly γ crosses s_t and t_s. If it crosses s or t as well or k commutes with s or t, $w_J = ss_t = tt_s$ yields a shorter gallery emanating from l^{w_J} to k.

Now assume γ does not cross s and t and w.l.o.g. assume $\gamma \subset H_s \cap -H_t$. The other case, $\gamma \subset -H_s \cap H_t$, follows in the same manner substituting s and t. The fact $s \notin_D [l, t]$ implies $o(ls) < \infty$, else γ crosses s. If l, s commute, we are done, since γ crosses s_t, so assume $o(ls) > 2$.

In the case $o(kt) = \infty$, either $C(k) \subset H_t$, a contradiction to $\gamma \subset -H_t$, or $C(k) \subset -H_t$. In the last case $C(k) \subset H(s_t, F) \cap H(t_s, F)$ implies $C(k) \subset H_s = -H(s, C_{D_s}(l))$. For $D_x = D(\{l_s\}, c)$ we get $s \in_{D_x} [l, k]$, and $\delta(l^s, k) < \delta(l, k)$ holds.

Now let $2 < o(kt) < \infty$. Furthermore we can assume $2 < o(ks) < \infty$ since $o(ks) = \infty$ implies again $s \in_{D_x} [k, l]$. Therefore the set $\{k, s, t\}$ is geometric with geometric set of roots $\{H_k, H_s, H_t\}$, since $C(k) \subset H(s_t, F) \cap H(t_s, F)$.

The root H_k associated to k satisfies $H_k = H(k, A_{s,t}) = H(k,l)$. The pair $\{H(s, C_{D_s}(l)), H(l, C_{D_s}(s))\}$ is geometric by Lemma 4.13. For $D := D(\{l_s\}, c)$ the same holds. Since we can assume that $D_s \subset D$, we have $H(l, C_D(s)) = H(l, A_{s,t}) = H(l,k)$ and $H(s, C_D(l)) = H(s, C_D(t)) = -H_s$ yields $s \in [l,k]$. Now we can use Lemma 3.14 and have $\delta(l^s, k) < \delta(l,k)$, as required.

As a final step we need to show that there are at least two reflections whose distance is reduced in $\bar{R}$, using $s_t \in R_2^\circ$. If L_s or L_t is nonempty, the distance to t or s is reduced. So assume they are empty, then L_0 and K_J must be nonempty and as shown above for $l \in L_0$, $k \in K_J$ the inequality $\delta(l^{w_J}, k) < \delta(l,k)$ holds. □

4.5 r-reductions

We now assume that R satisfies $R_2^\circ = \emptyset$ and $R^\circ \neq \emptyset$. Throughout this section we will also assume the following condition $(*)$ on R:

Consider an arbitrary pair $s,t \in R$, $2 < o(st) < \infty$ even and $u \notin \{s,t\} \cup \{s,t\}^\perp$. Denote with $F := H_s \cap H_t$, $-F := -H_s \cap -H_t$ the standard fundamental domains for the action of $\langle s,t \rangle$. Then either $H(s_t, u) = H(s_t, F)$ and $H(t_s, u) = H(t_s, F)$ hold for all $u \in R \setminus (\{s,t\} \cup \{s,t\}^\perp)$ or $H(s_t, u) = H(s_t, -F)$ and $H(t_s, u) = H(t_s, -F)$ hold for all $u \in R \setminus (\{s,t\} \cup \{s,t\}^\perp)$. In other words, $C(u)$ is not contained in the fundamental domain generated by the geometric pair of roots associated to $\{s_t, t_s\}$.

Since we further require R_2° to be empty, then all $u \notin \{s,t\} \cup \{s,t\}^\perp$ are on the same side of s_t, t_s. Also we have $\{s,t\} \cup \{s,t\}^\perp \neq R$, since R is irreducible. So we see that such a u always exists.

Define $T := R_2$. The intersection

$$D = \bigcap_{\substack{2<o(st)<\infty,\\ u \notin \{s,t\} \cup \{s,t\}^\perp}} H(s_t, u)$$

is nonempty, and for a $c \in D$ we have $D = D(T,c)$. Furthermore, $C_D(r) \neq \emptyset$ for all $r \in R$ due to $(*)$ and $H(r', C_D(r))$ is defined for all $r, r' \in R$ with $rr' \neq r'r$.

Lemma 4.23. *If $R^\circ \neq \emptyset$, there exist $r,s,t \in R$ such that $r \in_D [s,t]$.*

Proof. The assumption $R^\circ \neq \emptyset$ yields r,s,t such that $r \in_r [s,t]$. The roots $H(r, C_D(s))$, $H(r, C_D(t))$ are well-defined, and $r \in_D [s,t]$ holds if $o(rs), o(st)$ are both infinite. If they are both finite and a minimal gallery between $C(s)$, $C(t)$ does not cross r, it is easy to see that it crosses r_s or r_t. We conclude that every gallery not crossing r_s or r_t crosses r, proving $r \in_D [s,t]$.

Consider the case $o(rs) < \infty$, $o(rt) = \infty$. Let $D' = D(\{r_s, s_r\}, c)$. If the minimal gallery γ connecting $C_{D'}(s), C_{D'}(t)$ crosses r or the minimal gallery γ' connecting $C_{D'}(r), C_{D'}(t)$ crosses s, this yields $r \in_{D'} [s,t]$ or $s \in_{D'} [r,t]$. If neither γ nor γ' cross r, s, the first chambers in γ, γ' are not contained in the fundamental domain $F = H(r, C_{D'}(s)) \cap H(s, C_{D'}(r)) \subset H(r_s, c) \cap H(s_r, c)$ by Lemma 3.13. Therefore, $\gamma \subset -H(r, C_{D'}(s))$, $\gamma' \subset -H(s, C_{D'}(r))$ and $C(t) \subset -H(r, C_{D'}(s)) \cap -H(s, C_{D'}(r)) = w_{\{r,s\}}.F$. But longest reflections separate the two standard fundamental domains, thus $w_{\{r,s\}}.F \cap H(r_s, c) = \emptyset$, a contradiction. □

Now let $r \in R^\circ$, $s, t \in R$ such that $r \in_D [s,t]$, these exist by Lemma 4.23. We find an R-admissible pair $(\{r\}, L_r)$ by defining L_r, K_r the following way. For $r' \in R \setminus (\{r\} \cup r^\perp)$ we define $r' \in L_r \Leftrightarrow C_D(r') \subset H(r, C_D(s))$ and $r' \in K_r \Leftrightarrow C_D(r') \subset H(r, C_D(t))$. This yields a partition $R = \{r\} \,\dot\cup\, r^\perp \,\dot\cup\, L_r \,\dot\cup\, K_r$.

Lemma 4.24. $(\{r\}, L_r)$ *is an* R*-admissible pair.*

Proof. Let $l \in L_r, k \in K_r$, then by construction $r \in_D [l,k]$ and $r \in_r [l,k]$ by Lemma 4.16, thus $o(lk) = \infty$. Thus the pair $(\{r\}, L_r)$ is admissible. □

Define for an $r \in R^\circ$ the set $\bar{R} = T_{(\{r\}, L_r)}(R)$. See Figure 2 for an example of the above construction, with the property $o(rs) < \infty > o(rt)$. The longest reflections here give rise to a convex set D, which can be seen in the first depiction as the space between the longest reflections s_r, t_r.

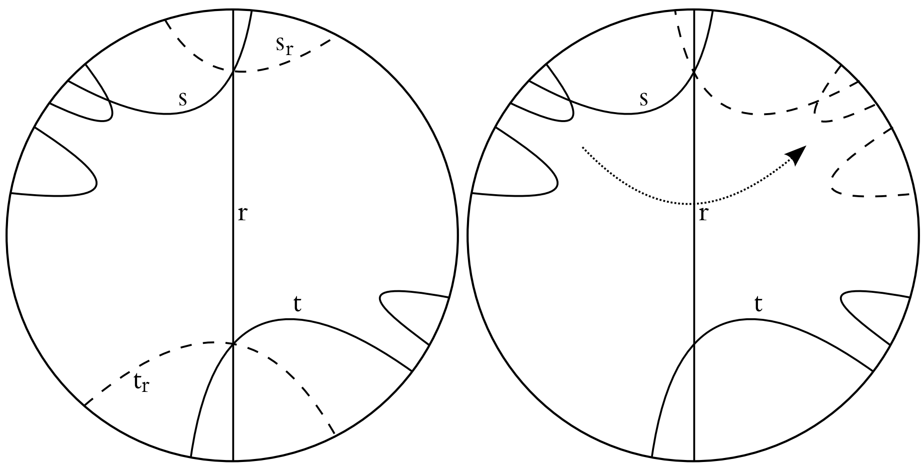


Figure 2: r-reductions using interior separation

Proposition 4.25. *The Coxeter generating set $\bar{R}$ satisfies $D_1(\bar{R}) < D_1(R)$.*

Proof. Let $l \in L_t, k \in K_t$, both sets are not empty since $s \in L_t, t \in K_t$. Then $r \in_D [l,k]$ and $r \in_r [l,k]$ by Lemma 4.16. Thus, $\delta(l^r, k) < \delta(l,k)$. Distances to $r, r^\perp$ are preserved. □

4.6 r-reductions in an exceptional case

In order to reduce distances in every case, we have yet to deal with one case.

Assume we have $R_2^\circ = \emptyset$ and $R^\circ \neq \emptyset$. If we cannot apply a reduction as constructed in Section 4.5, we can find $J = \{s,t\}$, $2 < o(st) < \infty$ even, together with a standard fundamental domain $F = H_s \cap H_t$, such that we can find an $r \in R \setminus (J \cup J^\perp)$ satisfying $C(r) \subset H(s_t, F) \cap -H(t_s, F)$. Since $R_2^\circ = \emptyset$ and $o(r's_t) = o(r't_s) = \infty$ for all $r' \in R \setminus (J \cup J^\perp)$, we have $C(r') \subset H(s_t, F) \cap -H(t_s, F)$ for all $r' \in R \setminus (J \cup J^\perp)$. In particular, if r commutes with t, it commutes with s as well.

Define $L_s = R \setminus (J \cup s^\perp)$, $K_s = \{t\}$, then $(\{s\}, L_s)$ is clearly an R-admissible pair. Let $\bar{R} = T_{(\{s\}, L_s)}$. An example of the sets R and $\bar{R}$ for a sample of reflections in L_s can be found in Figure 3.

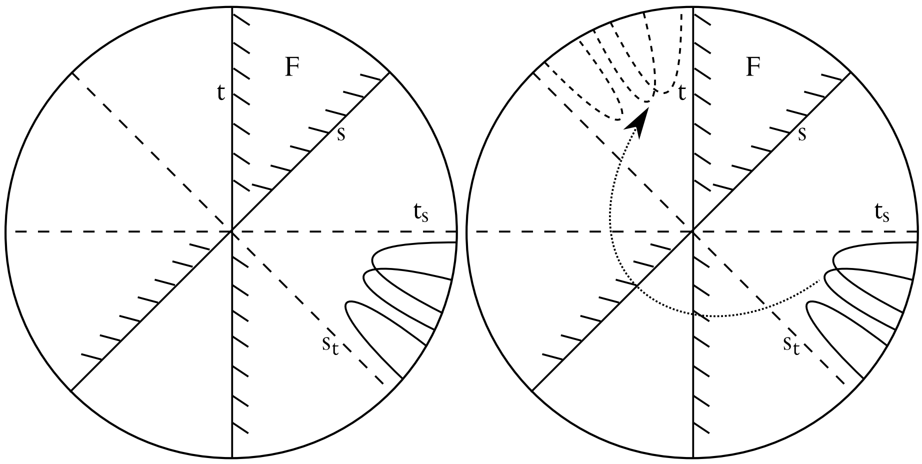


Figure 3: r-reductions in an exceptional case

Proposition 4.26. *The Coxeter generating set $\bar{R}$ satisfies $D_1(\bar{R}) < D_1(R)$.*

Proof. For $l \in L_s$ $\delta(l^s, t) < \delta(l,t)$ holds by Lemma 4.6. □

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