Domesticity in projective spaces

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Abstract

Let J be a set of types of subspaces of a projective space. Then a collineation or a duality is called J-domestic if it maps no flag of type J to an opposite one. In this paper, we characterize symplectic polarities as the only dualities of projective spaces that map no chamber to an opposite one. This implies a complete characterization of all J-domestic dualities of an arbitrary projective space for all type subsets J. We also completely characterize and classify J-domestic collineations of projective spaces for all possible J.

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1. Introduction

Abramenko and Brown [1] show that every automorphism of an irreducible non-spherical building has infinite displacement. Their method also gives information about the spherical case. For instance, in the rank 2 case, every automorphism maps some chamber to a chamber at codistance one, and if the diameter of the incidence graph is even (odd), then any duality (collineation) maps some chamber to an opposite one. For projective planes, this shows that collineations behave normally, where 'normal' means that at least one chamber is mapped onto an opposite one. However, it is easily seen that also dualities of projective planes behave normally. Counterexamples to this normal behaviour

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are given in [1], attributed to the third author of this paper, and consist of symplectic polarities in projective spaces and central collineations in generalized polygons of even diameter. The goal of this paper is to classify all 'abnormal' automorphisms of projective spaces, which we will call 'domestic'.

2. Preliminaries and statement of the main result

We will write down most definitions in the general context of spherical buildings, but we will only be concerned with a specific class of buildings, namely, the projective spaces. Hence we do not define a building in full generality, but refer to the literature.

Let Ω be a spherical building, and let θ be an automorphism of Ω. We emphasize that θ need not be type-preserving. Then we call θ *domestic* if no chamber of Ω is mapped onto an opposite chamber. More in particular, for a subset J of the type set of Ω, we say that θ is *J-domestic*, if θ does not map any flag of type J onto an opposite one. The main result of [1, Section 5], also proved earlier by Leeb [2], using entirely different methods, asserts that every automorphism of any (thick) spherical building is not J-domestic, for some type subset J. Hence being not J-domestic seems to be the rule, and so it is worthwhile to look at automorphisms which are J-domestic, for some J.

We now specialize to projective spaces. In a projective space $\mathsf{PG}(n, \mathbb{K})$, a *flag* is a set of pairwise incident elements and a *chamber* is a flag of size n. An i-dimensional subspace U of $\mathsf{PG}(n, \mathbb{K})$ is *opposite* a j-dimensional subspace V of $\mathsf{PG}(n, \mathbb{K})$ if and only if $j = n-1-i$ and $U \cap V$ is empty. A flag F is called *opposite* a flag F' if every element of F is opposite an element of F' and conversely.

We will call a non-type preserving automorphism a *duality* and a type preserving automorphism a *collineation*. Let θ be a duality of a projective space $\mathsf{PG}(n, \mathbb{K})$, with $\mathbb{K}$ a skew field. Then an absolute element U is a subspace which is incident with its image U^θ. A *polarity* is a duality of order 2. A *symplectic polarity*, or *null-polarity*, is a polarity for which every point is absolute. Then necessarily $\mathbb{K}$ is a commutative field, n is odd, and θ is related to a non-degenerate alternating bilinear form.

In the present paper we will show the following result:

Theorem 2.1. *Every domestic duality of any finite-dimensonal projective space is a symplectic duality, and a collineation of an n-dimensional projective space, $n \geq 2$, is domestic if and only if this collineation fixes a subspace of dimension at least $\frac{n+1}{2}$ pointwise.*

3. Domestic dualities

In this section we will prove the following theorem.

Theorem 3.1. *Every domestic duality of a projective space is a symplectic polarity. In particular no even dimensional projective space admits domestic dualities.*

It is clear that in any one-dimensional projective space, the only domestic collineation is the identity, and for any domestic duality, all elements are absolute elements (and this can be considered as a symplectic polarity).

This is enough to get an induction started. Note that the problem only makes sense for finite dimensional projective spaces as infinite ones are never self dual.

We first prove some lemmas which are independent of the induction hypothesis.

Lemma 3.2. *Let θ be a duality of a projective space of dimension $d > 1$ with the property that every point is absolute. Then θ is a symplectic polarity.*

Proof. Suppose by way of contradiction that θ is not a polarity. Then there is some point x for which $x' = x^{\theta^2} \neq x$. Consequently also $H := x^\theta$ and $H' = x'^\theta$ are different, and we can choose a point $y \in H$, $y \notin H'$. Since lines are thick, there is a $z \in yx'$, $z \notin \{x', y\}$. Since $z \in H$, we have $z^\theta \ni H^\theta = x'$. Similarly, $x' \in y^\theta$. By assumption $z \in z^\theta$ and $y \in y^\theta$. Hence the line yz is in $z^\theta \cap y^\theta = (zy)^\theta \subseteq x'^\theta$. This contradicts $yz \not\subseteq H'$. Consequently θ is a polarity and hence a symplectic polarity as every point is an absolute one. □

Lemma 3.3. *If a line contains at least one non-absolute point, and $|\mathbb{K}| > 2$, then it contains at least two non-absolute points.*

Proof. Assume by way of contradiction that the line L contains exactly one non-absolute point x. Then x^θ intersects L in some point $y \neq x$. Since, by assumption, $y \in y^\theta$, we see that $L^\theta \cap L = \{y\}$. If $u_i \in L$, $i = 1, 2$, $x \neq u_i \neq y$, then $u_i^\theta = \langle u_i, L^\theta \rangle = \langle L, L^\theta \rangle$, implying $u_1 = u_2$, and so $|\mathbb{K}| = 2$. □

Proof of Theorem 3.1. In view of the induction procedure, we assume that for given $d > 0$ the only domestic dualities of a projective space of dimension $d' < d$ are the symplectic polarities for odd d', and we assume that θ is a domestic duality of a d-dimensional projective space Π.

In view of Lemma 3.2, it suffices to show that every point of Π is absolute. Let, by way of contradiction, x be a point which is not absolute, and let H be its image under θ. For any subspace S in H, the image $\langle S, x \rangle^\theta =: S'$ is a subspace of H, and the correspondence $S \mapsto S'$ is clearly a duality θ_H of H. Since for a subspace S of H we have that $\langle S, x\rangle$ is opposite $\langle S, x\rangle^\theta$ if and only if S is opposite $\langle S, x\rangle^\theta$ in H, it follows easily that this duality is domestic (because, by the foregoing remark, if C is a chamber in H, then it is mapped onto an opposite chamber in H if and only if the chamber $\{\langle x, S\rangle : S \in C \cup \{\emptyset\}\}$ of Π is mapped onto an opposite one). By induction d is even and θ_H is a symplectic polarity. It follows that every point of H is absolute. Now let z be any point in H. By construction of θ_H, the image $(xz)^\theta$ is equal to z^{θ_H}, and $(xz)^{\theta^2}$ is equal to the span of $\langle x, z^{\theta_H}\rangle^\theta$ and $H^\theta =: x'$. Note that $x' \notin H$. Since θ_H is symplectic, z is an absolute point and we see that $(xz)^{\theta^2} = x'z$.

Let us first assume that $x' \neq x$. Let y be the intersection of xx' and H. Since y is absolute and $y \in H$, the image y^θ contains xx'. Put $S = H \cap y^\theta$. Then for any point $u \in xx'$, the image u^θ contains S, but it can only contain u if $u = y$ (indeed, if it contains u, and $u \neq y$, then it contains xx' and hence coincides with y^θ, implying $u = y$ after all). It follows that all points of xx' except for y are non-absolute. But this now implies that all points of u^θ, for $y \neq u \in xx'$, are absolute, replacing x by u in the previous arguments. Now we pick a line not in y^θ meeting y^θ in a pre-chosen point $v \neq y$. Lemma 3.3 implies that v is absolute, or $|\mathbb{K}| = 2$. So, by Lemma 3.2, we may assume that $|\mathbb{K}| = 2$. In this case, all points of $x^\theta \cup x'^\theta$ are absolute. Let z be any point not in $x^\theta \cup x'^\theta$. Then z belongs to $y^\theta \setminus H$. Suppose moreover that $z \notin \{x, x'\}$. The line xz meets H in a point u that belongs to y^{θ_H}. Hence $y \in u^{\theta_H} \subseteq u^\theta$. Since also x' belongs to u^θ, we see that the line xx' is contained in u^θ. It follows that, since this line is not contained in x^θ, it is neither contained in z^θ. Since z^θ does contain y, we now see that z^θ does not contain x. Since z^θ also contains u, it cannot contain z, and so it is not an absolute point. We have shown that all points outside $x^\theta \cup x'^\theta$ are non-absolute. But now, interchanging the roles of x and z (and noting that the next paragraph is independent of the current one), we infer that all points of z^θ are absolute, and they cannot all be contained in $x^\theta \cup x'^\theta$, the final contradiction of this case.

Now we assume that $x' = x$. As before, we deduce that no point $u \notin H$ is absolute (taking $u \neq x$, considering the line ux and noting that $(ux)^\theta$ contains $ux \cap H$). But then all points of u^θ are absolute, for $u \notin H$. For $u \neq x$ we obtain points outside H that are absolute, contradicting what we just deduced.

So we have shown that the symplectic polarities are the only domestic dualities in projective space. This proves Theorem 3.1. $\square$

This has a few consequences. We assume that the type of an element of a projective space is its projective dimension as a projective subspace.

Corollary 3.4. *Let J be a subset of the set of types of an n-dimensional projective space, $n \geq 2$. If either J contains no even elements, or n is even, or the ground field (if defined) is nonabelian, then there is no J-domestic duality. In all other cases, symplectic dualities are the only J-domestic dualities.*

Proof. This follows from the fact that any symplectic polarity maps an even-dimensional subspace to a non-opposite subspace, and there exists a subspace of any odd dimension that is mapped onto an opposite subspace. These claims are easy to check and well known. Further, there do not exist symplectic polarities in even-dimensional projective space, and in projective spaces defined over proper skew fields. □

We can actually compute the displacement of a symplectic polarity ρ (the *displacement* of an automorphism is the maximal possible distance between a chamber and its image, see [1]). To do this, we first remark that , if U is a subspace of even dimension, then U^ρ meets U in at least one point (otherwise the permutation of the set of subspaces of U sending a subspace W to $W^\rho \cap U$ would be a symplectic polarity, contradicting the fact that U has even dimension). Hence, if the projective space is $(2n-1)$-dimensional, the image of any chamber contains at least n elements that are not opposite their image. In order to "walk" to an opposite chamber, we need at least n steps. This shows that the codistance from a chamber to its image is at least n. We now show that this minimum is reached. To that end, we consider the symplectic polarity ρ of $\mathsf{PG}(2n-1, \mathbb{K})$, with $\mathbb{K}$ a field, given by the standard alternating bilinear form

$$\sum_{i=1}^{n} X_{2i-1}Y_{2i} - X_{2i}Y_{2i-1},$$

where we introduced coordinates $(x_1, x_2, \ldots, x_{2n})$. Now we just consider the chamber C whose element of type i is given by the span of the first $i+1$ basis points (or, in other words, the set of points whose last $2n-i-1$ coordinates are zero). In dual coordinates, a straightforward computation shows that the element of type i of the image under ρ of C is given by putting the first $i+1$ coordinates equal to zero, if i is odd, and by putting the first $i-1$ coordinates equal to zero, together with the $(i+1)^{\text{st}}$ coordinate equal to zero, if i is even. Subsequently applying the coordinate change switching the $(2i-1)^{\text{st}}$ and $2i^{\text{th}}$ coordinates, for i taking the (subsequent) values $1, 2, \ldots, n$, we obtain a gallery of chambers ending in a chamber opposite C. This shows that minimal gallery codistance between a chamber and its image under a symplectic polarity in $(2n-1)$-dimensional space is equal to n.

4. J-domestic collineations

Now we consider collineations of projective spaces. Let us fix the projective space $\mathsf{PG}(n,\mathbb{K})$, with $\mathbb{K}$ any skew field. Let J be a subset of the type set. Define J to be *symmetric* if, whenever $i \in J$, then $n-i-1 \in J$. Then clearly, if J is not symmetric, then every collineation is J-domestic. Indeed, no flag of type J is in that case opposite any flag of type J. Hence, from now on, we assume that J is symmetric. We first prove two reduction lemmas. The first one reduces the question to type subsets of size 2, the second one reduces the question to single subspaces instead of pairs.

Lemma 4.1. *Let J be a symmetric subset of types for $\mathsf{PG}(n,\mathbb{K})$. Let i be the largest element of J satisfying $2i < n$. Then a collineation θ of $\mathsf{PG}(n,\mathbb{K})$ is J-domestic if and only if it is $\{i, n-i-1\}$-domestic.*

Proof. Clearly, if θ is $\{i, n-i-1\}$-domestic, then it is J-domestic. So assume that θ is J-domestic. Let i be as in the statement of the lemma. Suppose that θ is not $\{i, n-i-1\}$-domestic and let U and V be subspaces of dimension i, $n-i-1$, respectively, such that $U \subseteq V$ with $\{U, V\}$ opposite $\{U^\theta, V^\theta\}$, i.e., $U \cap V^\theta = V \cap U^\theta = \emptyset$.

Now choose in U any flag $\mathfrak{F}_{<i}$ of type $J_{<i}$, where with obvious notation, $J_{<i} = \{j \in J : j < i\}$. Let $\mathfrak{F}$ be an arbitrary extension of type J of the flag $\mathfrak{F}_{<i} \cup \{U, V\}$. Then $\mathfrak{F}$ is opposite $\mathfrak{F}^\theta$ if and only if each subspace $W \in \mathfrak{F}$ of type $j > n-i-1$ is disjoint from the unique subspace W' of $\mathfrak{F}^\theta_{<i}$ of type $n-j-1$ and each subspace $Z \in \mathfrak{F}_{<i}$ of type $j < i$ is disjoint from the unique subspace Z' of $\mathfrak{F}^\theta$ of type $n-j-1$. The latter is equivalent with saying that each subspace Y of $\mathfrak{F}$ of type $n-j-1 > n-i-1$ is disjoint from the unique subspace $Y' \in \mathfrak{F}^{\theta^{-1}}_{<i}$ of type j. So, we deduce that $\mathfrak{F}$ is opposite $\mathfrak{F}^\theta$ if, and only if, the flag $\mathfrak{F}_{>n-i-1}$ (with obvious notation) is opposite the two flags $\mathfrak{F}^\theta_{<i}$ and $\mathfrak{F}^{\theta^{-1}}_{<i}$. But one can always choose a flag opposite two given flags of the same type in any projective space. Indeed, this follows easily from the fact that we can always choose a subspace complementary to two given subspaces of the same dimension. Hence we have proved that θ is not J-domestic, a contradiction.

The lemma is proved. □

So we have reduced the situation to symmetric type sets of two elements. With a similar technique, we reduce this further. But first a definition. For $i \leq n-i-1$ we say that a collineation is *i-$*$-domestic*, if θ maps no subspace of dimension i to a disjoint subspace.

Then we have:

Lemma 4.2. *Let $i \leq n-i-1$. Then a collineation θ of $\mathsf{PG}(n,\mathbb{K})$ is $\{i,n-i-1\}$-domestic if and only if it is i-$*$-domestic.*

Proof. It is clear that, if θ is i-$*$-domestic, then it is $\{i,n-i-1\}$-domestic. Suppose now that θ is $\{i,n-i-1\}$-domestic and not i-$*$-domestic.

Then there exists some subspace U of dimension i mapped onto a subspace U^θ disjoint from U. We can now choose a subspace V through U of dimension $n-i-1$ such that V is disjoint from both U^θ and $U^{\theta^{-1}}$. That flag $\{U,V\}$ is mapped to an opposite flag, a contradiction.

The lemma is proved. □

Note that we can dualize the previous definition and lemma. We will not do this explicitly, and we will not need to use this duality.

So, in order to classify all J-domestic collineations of $\mathsf{PG}(n,\mathbb{K})$ for arbitrary J, it suffices to classify all i-$*$-domestic collineations, for all $i \leq n-i-1$. In order to do so, we may suppose that a given collineation θ is i-$*$-domestic, with $i \leq n-i-1$, but not j-$*$-domestic, for every $j < i$. We say that θ is *sharply i-$*$-domestic*.

In this setting, we can prove the following theorem. In the proof we need the following notation. For an i-dimensional subspace V of $\mathsf{PG}(n,\mathbb{K})$, $0 \leq i < n-1$, we denote by $\mathrm{Res}(V)$ the projective space the point set of which consists of the subspaces of dimension $i+1$ containing V, and, more generally, the nontrivial subspaces of $\mathrm{Res}(V)$ are the nontrivial subspaces of $\mathsf{PG}(n,\mathbb{K})$ containing V. If $V=\{v\}$ is a point, we also denote $\mathrm{Res}(\{v\})$ by $\mathrm{Res}(v)$.

Theorem 4.3. *A collineation θ of $\mathsf{PG}(n,\mathbb{K})$ is sharply i-$*$-domestic, $i \leq n-i-1$, if and only if it fixes a subspace of dimension $n-i$ pointwise, but it does not fix any subspace of larger dimension pointwise.*

Proof. First suppose that θ is sharply i-$*$-domestic. If θ would fix a subspace F of dimension $n-i+1$ pointwise, then it would be $(i-1)$-$*$-domestic, since every subspace of dimension $i-1$ has at least one point in common with F and hence cannot be mapped onto a disjoint subspace.

We now show that θ fixes some subspace of dimension $n-i$ pointwise. To that aim, let U be a subspace of dimension $i-1$ which is mapped onto a disjoint subspace U^θ. Let V be an arbitrary i-dimensional subspace containing U and not contained in $X =: \langle U, U^\theta\rangle$. Since θ is i-$*$-domestic, the subspace V^θ has at least one point v in common with V. If $V \cap V^\theta$ contained a line, then that line would meet both U and U^θ and so both V and V^θ would be contained in X, a contradiction. It is now our aim to show that v is fixed. But we prove a slightly stronger statement.

Let W be any $(i+1)$-dimensional subspace of $\mathsf{PG}(d,\mathbb{K})$ containing V and intersecting V^θ in just v. Since $(i+1)+i \leq n$, such a subspace exists. If, on the one hand, W^θ met W in at least a plane, then such a plane would intersect V^θ in a line, contradicting our hypothesis $W \cap V^\theta = \{v\}$. If, on the other hand, $W \cap W^\theta$ were equal to $\{v\}$, then any i-dimensional subspace of W not through v and not through $v^{\theta^{-1}}$ would be mapped onto a disjoint subspace, contradicting i-$*$-domesticity. So $W \cap W^\theta$ is a line L (and note that L is of course not contained in V). We now claim that θ fixes L pointwise. Indeed, suppose that some point x on L is not fixed under θ. Then consider all subspaces of dimension i through x contained in W and not containing L. It is easy to see that all these subspaces have only the point x in common. Hence the images only have x^θ in common, and if $x \neq x^\theta$ then there is at least one image, say V'^θ, that does not contain x. But the intersection $V' \cap V'^\theta$ is contained in L. Since V' meets L in x and V'^θ does not contain x, it follows that V' and V'^θ are disjoint, contradicting i-$*$-domesticity. Our claim is proved.

Now let $\{W_i : i = 1, 2, \ldots, n-i\}$ be a set of $(i+1)$-dimensional subspaces containing V, not being contained in $\langle V, V^\theta \rangle$ and spanning $\mathsf{PG}(n,\mathbb{K})$. Such a set can easily be obtained by choosing a set of $n-i$ independent (and hence generating) points in the $(n-i-1)$-dimensional projective space $\mathrm{Res}(V)$ avoiding the subspace $\langle V, V^\theta \rangle$. Let $\{L_i : i = 1, 2, \ldots, n-i\}$ be the corresponding set of pointwise fixed lines ($L_i = W_i \cap W_i^\theta$). Since all L_i contain v and are fixed pointwise, θ fixes the space Z generated by the L_i, $i = 1, 2, \ldots, n-i$, pointwise. The independence of the W_i in $\mathrm{Res}(V)$ now implies that the L_i are also independent in $\mathrm{Res}(v)$, and hence the subspace Z has dimension $n-i$, and that is what we had to prove.

Now suppose that θ fixes a subspace Z of dimension $n-i$ pointwise, but it does not fix any subspace of larger dimension pointwise. Clearly, every subspace of dimension i meets Z and so is not mapped onto a disjoint subspace. Hence θ is i-$*$-domestic. But if it were j-$*$-domestic for $j < i$, then by the foregoing, it would fix a subspace of dimension $n-j > n-i$ pointwise, a contradiction. Hence θ is sharply i-$*$-domestic and the theorem is proved. $\square$

As a consequence, we can now characterize all domestic collineations of $\mathsf{PG}(n,\mathbb{K})$.

Corollary 4.4. *A collineation θ of an n-dimensonal projective space, $n \geq 2$, is domestic if and only if θ fixes a subspace of dimension at least $\frac{n+1}{2}$ pointwise.*

Proof. For $n \geq 3$, this follows from Lemmas 4.1 and 4.2, and Theorem 4.3. For $n = 2$, every collineation is automatically point-domestic and line-domestic, so cannot be chamber-domestic (by Leeb [2]), unless it is the identity. $\square$

Concerning the maximal distance between a chamber and its image with respect to a domestic collineation, it is clear that this depends on the specific collineation. The maximum maximal distance occurs when the fixed point set is minimal, i.e., when the collineation is i-$*$-domestic, for $i \in \{\frac{n-1}{2}, \frac{n-2}{2}\}$. For n odd, the minimal codistance is in this case equal to 1, and for n even, it is equal to 3. In the other extreme, i.e., if the collineation fixes a hyperplane pointwise, then the maximal gallery distance between a chamber and its image is $2n+1$; this is codistance $\frac{n^2-3n-2}{2}$, which is rather large.

References

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