



On topological twin buildings and topological split Kac–Moody groups

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Abstract

We prove that a two-spherical split Kac–Moody group over a local field naturally provides a topological twin building in the sense of [27]. This existence result and the local-to-global principle for twin building topologies combined with the theory of Moufang foundations as introduced and studied by Mühlherr, Ronan, and Tits allows one to immediately obtain a classification of two-spherical split Moufang topological twin buildings whose underlying Coxeter diagram contains no loop and no isolated vertices.

1 Introduction

The objective of topological geometry is to study (incidence) geometries, whose underlying sets are equipped with a topology with respect to which the natural geometric operations are continuous. Among the most prominent examples are the compact projective planes, i.e., projective planes whose point and line sets are compact Hausdorff spaces such that the maps that assign to two distinct points the unique line joining them and to two distinct lines the unique point incident to both are continuous maps. A detailed account on connected compact projective planes can be found in [38].

The gate property of Tits buildings ([1, Section 4.9]) and the resulting projection maps between opposite panels generalize these geometric operations of joining points and intersecting lines in projective planes, thus leading to the concept of compact generalized polygons and, more generally, topological (spherical) buildings; see [5, 26]. Various subclasses of these topological geometries have been classified, see for example [5, 15, 16, 38].

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An important extension of the class of spherical buildings is given by the class of twin buildings [45]. By definition, a twin building consists of a pair of (possibly non-spherical) buildings together with a codistance function (or twinning), which allows one to define the notion of opposition and (co-)projections from one half of the twin building to the other, extending the corresponding notions in the spherical case. While spherical twin buildings are just another way of looking at spherical buildings ([45, Proposition 1]), in the non-spherical case it is a very rigid property for a building to be part of a twin building. The existence of co-projections in twin buildings opens the door for the development of a theory of topological twin buildings, by requiring co-projections to be continuous. In order to be able to develop a rich theory, it seems necessary to maintain some compactness assumption, for instance compactness of the panels. In the spherical case this assumption is equivalent to compactness of the whole building, whereas in the non-spherical case it is not.

An axiomatic definition of topological twin buildings along the lines just described was first given by Kramer in [27], where he provides a geometric underpinning for the proof of Bott periodicity in [29], which was based on a somewhat ad hoc notion of a topological BN-pair. While the article [27] describes in detail an explicit model of the twin building associated with a loop group, the theory is developed in an abstract way which is independent of the existence of such a model and hence does not rely on the underlying twin building being affine.

At the time of writing of [27] examples of non-discrete, non-affine, non-spherical topological twin buildings were not well-understood—although in principle available through work by Kac and Peterson [23] and by Tits [44]. During the last decade the theory of Kac–Moody groups (Rémy [34]) and their group topologies (Glöckner, Hartnick, Köhl [13]) has been developed to a point where many such examples can be described without problem.

The purpose of the present article is twofold: on the one hand, to show that the examples of twin buildings associated with two-spherical split Kac–Moody group over local fields are indeed topological twin buildings in the sense of Kramer; on the other hand, to revisit and extend the general theory in light of these new examples up to the point necessary to obtain partial classification results. An important feature of our presentation is that we treat the connected and the totally disconnected case as well as the characteristic 0 and the positive characteristic case largely simultaneously.

The idea to use group topologies on certain Kac–Moody groups to construct new examples of topological twin buildings was already suggested in the first author’s master thesis [19], where also a group-theoretic criterion for twin building topologies based on the theory of RGD systems is stated (see [19, Sec-

tion 4.2]). However, carrying out this program—even in the case of complex Kac–Moody groups—became only possible after some rather technical insights concerning direct limit topologies had been obtained in [13]. Using a minor variation of the results from [13] it is straight-forward to carry out the program suggested in [19] for Kac–Moody groups over any local field of characteristic 0, and to associate with every such group a topological twin building. Finally, using additional techniques from [34] one can extend the result to local fields of positive characteristics. One thereby arrives at the following result:

Theorem 1. *Let G be a two-spherical simply connected split Kac–Moody group over a local field and let τ_{KP} be the Kac–Peterson topology on G . Then the associated twin building endowed with the quotient topology is a strong topological twin building.*

If the local field equals the field of real or of complex numbers, then G is connected, otherwise totally disconnected.

Theorem 1 provides a rich supply of topological twin buildings; for definitions we refer to Section 3.1 and to Definition 3.21. In the course of the proof of Theorem 1 we will explicitly construct and study the *Kac–Peterson topology*; we refer to Definition 7.8 and Remark 7.12 for its definition, to [24, Section 4G] for its original appearance and to [28] (where it is called the *analytic topology*) and to [13] for subsequent discussions in the literature.

Following the latter, we will provide a universal characterization of the Kac–Peterson topology in the general case. We emphasize that in case of a non-spherical Kac–Moody group G the Kac–Peterson topology is k_ω , but not locally compact and not metrizable; cf. Proposition 7.21 and Remark 7.28. In case of a spherical Kac–Moody group, the Kac–Peterson topology coincides with the Lie group topology; cf. Corollary 7.16 and Remark 7.24. Furthermore, the subspace topology induced on bounded subgroups turns these into algebraic Lie groups; cf. Corollary 7.30.

The hypothesis in Theorem 1 that the Kac–Moody group be two-spherical stems from Proposition 7.31. That result has been announced in [24, Section 4G] without the requirement that the Kac–Moody group be two-spherical. However, to the best of our knowledge there is no published proof of that statement available in the literature. Our own proof, which is based on a combinatorial argument by Mühlherr given in the appendix, unfortunately requires two-sphericity.

It is natural to ask whether a classification of topological twin buildings is possible under some natural conditions, mimicking known classification results in the spherical case. We will provide such a classification result in the following restricted setting: Let us call a topological twin building *k-split*, if all its rank two

residues are *compact Moufang polygons* and its rank one residues (considered as topological Moufang sets) are *projective lines over k* . We say that a k -split topological twin building is *of tree type* if the underlying Coxeter diagram is a tree. In particular, such twin buildings are two-spherical and Moufang.

Given a k -split topological twin building of tree type we can associate a Dynkin diagram in the usual way, using the fact that rank two residues are either generalized triangles or quadrangles or hexagons. Given a topological twin building Δ we denote by $\mathcal{D}(\Delta)$ the associated Dynkin diagram. With this notation understood our classification result can be formulated as follows:

Theorem 2. *Let k be a local field. Then the map $[\Delta] \mapsto [\mathcal{D}(\Delta)]$ induces a bijection between isomorphism classes of k -split topological twin buildings of tree type and isomorphism classes of simply connected simple $\{3, 4, 6\}$ -labelled graphs, where edges labelled 4 or 6 are directed.*

Note that the classification of abstract k -split twin buildings over fields is based on the theory of abstract foundations as developed by Mühlherr [30], Mühlherr–Ronan [32], Ronan–Tits [36]. The uniqueness part of our classification result is based on a topological formulation of this theory (cf. Theorem 4.6), whereas the existence part is a direct consequence of Theorem 1.

Using the concept of normal coverings of Coxeter diagrams from [30] it is actually possible to extend Theorem 2 to k -split topological twin buildings of arbitrary type, but we refrain from doing so in the present article. We simply note that for arbitrary diagrams the Moufang foundation will not uniquely determine the twin building, so the precise classification statement becomes necessarily more complicated (cf. also the discussion on page 32 before Theorem 4.6).

This article is organized as follows: In Section 2 we recall basic definitions concerning twin buildings and RGD systems. We then establish a couple of basic combinatorial properties of twin buildings. The main original result of that section (which stems from the first author’s master thesis [19]) is Theorem 2.22, where we provide an explicit formula for co-projections in the twin building associated with an RGD system in terms of group data. In Section 3 we introduce topological twin buildings and develop their basic point-set topological properties. Our main contribution in that section is the local-to-global result Theorem 3.15, which was previously only known in the spherical case. Our proof is based on the concept of Bott–Samelson desingularizations of Schubert varieties taken from [27]. Section 4 applies this local-to-global result to establish the uniqueness part of Theorem 2 (cf. Corollary 4.10). We then digress in Section 5 to re-visit [25, 26, 27], [19, Theorem 3.3.10] and to discuss the topology of connected topological twin buildings. The main result here is

Theorem 5.3, which is a topological version of the Solomon–Tits theorem. The remainder of the article is then devoted to the explicit construction of topological twin buildings. In Section 6 we develop the theory of topological groups with RGD system. In particular, we provide a list of conditions on the topology of such a group, which guarantee that the associated twin building becomes a topological twin building when equipped with the quotient topology (see Theorem 6.7). In Section 7 this result is applied to the case of split Kac–Moody groups over a local field. We use the Kac–Peterson topology on these groups in order to establish Theorem 1 and the existence part of Theorem 2. We close with some remarks concerning closure relations in Kac–Moody symmetric spaces.

Acknowledgements. The authors thank Peter Abramenko, Helge Glöckner, Guntram Hainke, Aloysius Helminck, Max Horn, Linus Kramer, Rupert McCallum, Bernhard Mühlherr, Andrei Rapinchuk, Nils Rosehr, Markus Stroppel and Stefan Witzel for several valuable comments, questions and discussions concerning the topic of the present article. Moreover, they are indebted to Rupert McCallum and to an anonymous referee for a thorough proof-reading that resulted in many extremely helpful comments, suggestions and corrections. Furthermore, they are indebted to Bernhard Mühlherr for providing the combinatorial argument in the appendix.

The first author was partially supported by ERC grant agreement No. 306706 *Ergodic Group Theory*. He also acknowledges the hospitality of Institut Mittag-Leffler during part of the revision of the article. The second and third authors gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft through funding of the research proposals GR 2077/5 and GR 2077/7.

2 Twin buildings and RGD systems

2.1 Twin buildings and their combinatorics

Buildings can be studied from the point of view of simplicial complexes (as done in [42]) or, equivalently, from the point of view of chamber systems (as introduced in [43]). The book [1] is a comprehensive introduction into the theory of buildings that explains both concepts in detail, also including the theory of twin buildings.

In the present article we will study twin building topologies using the chamber system approach to buildings. Throughout this article we reserve the letters (W, S) to denote a Coxeter system, which is always assumed to be of finite rank $|S|$. We then denote by \leq the associated Bruhat order and by $l = l_S$ the associated length function on W . Given $J \subset S$ we denote by W_J the subgroup of W

generated by J .

Definition 2.1. Let (W, S) be a Coxeter system. A *building* of type (W, S) is a pair (Δ, δ) consisting of a set of chambers Δ together with a distance function $\delta : \Delta \times \Delta \rightarrow W$ satisfying the following axioms, where $x, y \in \Delta$ and $\delta(x, y) = w$:

- (Bu1) $w = 1$ if and only if $x = y$,
- (Bu2) if $z \in \Delta$ such that $\delta(y, z) = s \in S$, then $\delta(x, z) \in \{ws, w\}$. If additionally $l(ws) > l(w)$, then $\delta(x, z) = ws$.
- (Bu3) If $s \in S$, there exists $z \in \Delta$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

A building is called *spherical* if W is finite. If Δ is spherical, then $c, d \in \Delta$ are called *opposite*, if $\delta(c, d) = w_0$, where w_0 denotes the longest element of (W, S) .

For every $c \in \Delta$ and every subset $S' \subseteq S$ we define the S' -*residue* $R_{S'}(c)$ to be

$$R_{S'}(c) := \{d \in \Delta \mid \delta(c, d) \in W_{S'} = \langle s \mid s \in S' \rangle\};$$

the collection of all S' -residues in Δ will be denoted $\text{Res}_{S'}(\Delta)$.

Remark 2.2. Using the above definition, a building of rank one is simply a set without further structure. In order to be able to develop a meaningful theory for such buildings, one has to require additional properties, such as the existence of a prescribed rank one group of automorphisms; cf. [10, 41]. In the present article we will not deal with this situation and therefore only study buildings whose Coxeter systems/Dynkin diagrams do not admit isolated points.

Lemma 2.3 ([1, Lemma 5.16 and Corollary 5.30]). *Any residue of a building Δ is again a building. For any $S' \subseteq S$ the elements of $\text{Res}_{S'}(\Delta)$ partition Δ .*

The building Δ will be called k -*spherical* if all residues of rank $\leq k$ are spherical buildings. Every building is one-spherical, and a building of type (W, S) is $|S|$ -spherical if and only if it is spherical. For our purposes the class of two-spherical buildings will play a key role.

The role residues of rank or co-rank one play is a particularly important one. Those of rank one, i.e., the elements of $\text{Pan}_s(\Delta) := \text{Res}_{\{s\}}(\Delta)$ are called s -*panels*; as a convention, we write $P_s(c)$ instead of $R_{\{s\}}(c)$. The residues of co-rank one, i.e., the elements of $\mathcal{V}_s := \text{Res}_{S \setminus \{s\}}(\Delta)$ are called s -*vertices*. There is a canonical embedding

$$\begin{aligned} \iota : \Delta &\hookrightarrow \prod_{s \in S} \mathcal{V}_s \\ c &\mapsto (R_{S \setminus \{s\}}(c))_{s \in S}, \end{aligned}$$

which, from the simplicial complexes point of view on buildings, simply maps a maximal simplex onto the tuple consisting of its vertices.

A building is *thin*, if each panel contains exactly two elements, and *thick*, if each panel contains at least three elements. For a given chamber c and a residue R there exists a unique chamber $d \in R$ such that

$$l(\delta(c, d)) = \min\{l(\delta(c, x)) \mid x \in R\},$$

see [1, Proposition 5.34]. This chamber d is called the *projection* of c onto R and is denoted by $\text{proj}_R(c)$.

Example 2.4. Let (W, S) be a Coxeter system. Then $\Delta := W$ and

$$\delta : \Delta \times \Delta \rightarrow W : (x, y) \mapsto x^{-1}y$$

yields a (thin) building of type (W, S) , denoted by $\Delta(W, S)$. For any three chambers $x, y, z \in \Delta$ one has $\delta(x, z) = x^{-1}z = x^{-1}yy^{-1}z = \delta(x, y)\delta(y, z)$; see also [1, Lemma 5.55]. Any thin building of type (W, S) is isometric to $\Delta(W, S)$, cf. [1, Exercise 4.12].

Let Δ be a building of type (W, S) . A subset of Δ which is isometric to $\Delta(W, S)$ is called an *apartment* of Δ .

Definition 2.5. A *twin building* of type (W, S) is a triple $((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ consisting of two buildings (Δ_+, δ_+) and (Δ_-, δ_-) of type (W, S) and a *codistance* function $\delta^* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \rightarrow W$ subject to the following conditions, where $x \in \Delta_\pm, y \in \Delta_\mp$ and $\delta^*(x, y) = w$:

(Tw1) $\delta^*(y, x) = w^{-1}$,

(Tw2) if $z \in \Delta_\mp$ such that $\delta_\mp(y, z) = s \in S$, and $l(ws) < l(w)$, then $\delta^*(x, z) = ws$, and

(Tw3) if $s \in S$, then there exists $z \in \Delta_\mp$ such that $\delta_\mp(y, z) = s$ and $\delta^*(x, z) = ws$.

A twin building is called *spherical*, resp. *k-spherical* if both of its halves have the corresponding property.

Morphisms of twin buildings are defined as follows:

Definition 2.6. For $j \in \{1, 2\}$ let $\Delta^{(j)} = ((\Delta_+^{(j)}, \delta_+^{(j)}), (\Delta_-^{(j)}, \delta_-^{(j)}), \delta^{*(j)})$ be twin buildings such that the type $(W^{(1)}, S^{(1)})$ is a sub-Coxeter system of the type $(W^{(2)}, S^{(2)})$. A *morphism* $\varphi : \Delta^{(1)} \rightarrow \Delta^{(2)}$ is an isometry

$$\varphi : \Delta_+^{(1)} \cup \Delta_-^{(1)} \rightarrow \Delta_+^{(2)} \cup \Delta_-^{(2)},$$

i.e., a map that preserves distances and codistances.

Example 2.7. Let (W, S) be any Coxeter system and let $\Delta_{\pm} := W$ and δ_{\pm} as in Example 2.4. Moreover, define $\delta^* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \rightarrow W$ by $\delta^*(v, w) := v^{-1}w$. Then $((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ is a thin twin building, and any thin twin building of type (W, S) is isometric to this twin building, cf. [1, Exercise 5.164]. In this case the distance and the codistance function are related by the formula

$$\delta^*(x, z) = x^{-1}z = x^{-1}yy^{-1}z = \delta^*(x, y)\delta_{\mp}(y, z), \quad x \in \Delta_{\pm}, y, z \in \Delta_{\mp};$$

see also [1, Lemma 5.173(4)].

A subset of a twin building isomorphic to a thin twin building is called a *twin apartment*. Twin apartments can be described efficiently via the notion of opposition. Here two chambers $c \in \Delta_{\pm}, d \in \Delta_{\mp}$ are called *opposite*, if $\delta^*(c, d) = 1$. Every pair of opposite chambers is contained in a unique twin apartment by [1, Proposition 5.179(1)]. Conversely, a pair of apartments (Σ_+, Σ_-) forms a twin apartment if and only if each chamber in Σ_{\pm} is opposite to exactly one chamber of Σ_{\mp} ; cf. [1, Proposition 5.173(5)].

The notion of opposition can be extended to residues by calling two residues *opposite*, if they have the same type and contain a pair of opposite chambers. Any pair of opposite residues is a twin building with respect to the restrictions of the distance and co-distance functions, cf. [1, Exercise 5.166].

Given a spherical residue $R \subseteq \Delta_{\pm}$ and a chamber $c \in \Delta_{\mp}$, there exists a unique chamber $d \in R$ such that $\delta^*(c, d)$ is maximal in the set $\delta^*(c, R)$ with respect to the Bruhat order, cf. [1, Lemma 5.149]. This chamber d is called the *co-projection* of c onto R and is denoted by $\text{proj}_R^*(c)$.

Given a chamber $c \in \Delta_{\pm}$ and an element $w \in W$ we denote by

$$\begin{aligned} E_w(c) &:= \{d \in \Delta_{\pm} \mid \delta_{\varepsilon}(c, d) = w \in W\}, \\ E_w^*(c) &:= \{d \in \Delta_{\mp} \mid \delta^*(c, d) = w \in W\} \end{aligned}$$

the *Schubert cell*, respectively, *co-Schubert cell* of radius w and centre c . The sets $E_{\leq w}(c), E_{< w}(c), E_{\leq w}^*(c), E_{< w}^*(c)$ are defined accordingly. $E_{\leq w}(c)$ and $E_{\leq w}^*(c)$ are called *Schubert varieties*, respectively, *co-Schubert varieties*. Moreover, we define

$$\Delta_w := \{(c, d) \in (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \mid \delta^*(c, d) = w\}.$$

The following combinatorial observations concerning twin buildings are quite useful.

Lemma 2.8. *Let Δ be a twin building, let $s, t \in S$ be distinct, let $c_1, c_2 \in \Delta_+$ such that $\delta_+(c_1, c_2) = s$, and let $d \in \Delta_-$ such that $\{(c_1, d), (c_2, d)\} \subset \Delta_1$. Then $\text{proj}_{P_t(d)}^*(c_1) = \text{proj}_{P_t(d)}^*(c_2)$.*

Proof. Let $a_1 := \text{proj}_{P_t(d)}^*(c_1)$. Then $\delta^*(c_1, a_1) = t$, whence $\delta^*(c_2, a_1) \in \{t, st\}$ by [1, Lemma 5.139]. On the other hand, $\delta^*(c_2, d) = 1$ and, thus, $\delta^*(c_2, a_1) \in \{1, t\}$ by [1, Lemma 5.139]. We conclude $\delta^*(c_2, a_1) = t$ and so $\text{proj}_{P_t(d)}^*(c_1) = a_1 = \text{proj}_{P_t(d)}^*(c_2)$. \square

Lemma 2.9 ([1, Lemma 5.156]). *Let Δ be a thick twin building. Then for every pair $c_1, c_2 \in \Delta_{\pm}$ there exists $d \in \Delta_{\mp}$ such that $\{(c_1, d), (c_2, d)\} \subset \Delta_1$.*

Let $c \in \Delta_{\pm}$ be a chamber and let Σ be a twin apartment of Δ containing c . Then the map $\rho = \rho_{c, \Sigma} : \Delta \rightarrow \Sigma$ which fixes c pointwise and maps every twin apartment containing c isometrically onto Σ is called the *retraction* onto Σ centred at c .

Since every two chambers are contained in a common twin apartment ([1, Proposition 5.179(3)]), the retraction ρ preserves distances from c . Moreover, ρ is distance-decreasing, i.e., $\delta(\rho(d), \rho(e)) \leq \delta(d, e)$ for any two chambers $d, e \in \Delta$, where δ is to be interpreted as δ_+ , δ_- or δ^* , whichever one makes sense.

Lemma 2.10 ([1, Lemma 5.140(1)]). *Let $c \in \Delta_{\pm}$, $d, e \in \Delta_{\mp}$ be chambers, let $\delta^*(c, d) = w$, and let $\delta_{\mp}(d, e) = v$. Then $\delta^*(c, e) = wv'$, where v' is a subexpression of v .*

Proof. Let $\rho = \rho_{c, \Sigma}$ be the retraction map onto some twin apartment Σ containing c . Then $\delta^*(c, \rho(d)) = \delta^*(c, d) = w$ as ρ preserves distances from c . Since ρ is distance-decreasing, one has $\delta_{\mp}(\rho(d), \rho(e)) \leq v$. We conclude

$$\begin{aligned} \delta^*(c, e) &= \delta^*(c, \rho(e)) \\ &\stackrel{2.7}{=} \delta^*(c, \rho(d))\delta_{\mp}(\rho(d), \rho(e)) \in \{wv' \mid v' \leq v\}. \end{aligned} \quad \square$$

Lemma 2.11. *Let Δ be a thick twin building, let $1 \neq w = s_1 \cdots s_k \in W$ be reduced, and let $c_{\pm} \in \Delta_{\pm}$ be opposite chambers. Then there exists a chamber $d \in \Delta_-$ with $\delta^*(c_+, d) = 1$ and $\delta^*(E_w^*(c_-), d) = \{s_k\}$.*

Proof. By the definition of co-projections (see above or [1, Lemma 5.149]) there is a unique chamber $a \in P_{s_1}(c_-)$ such that $\delta^*(c_+, a) = s_1$. Since Δ is thick, there exists $a_1 \in P_{s_1}(c_-) \setminus \{c_-, a\}$. Then $\delta^*(c_+, a_1) = 1$ and, by axiom (Tw2), for all $x \in E_w^*(c_-)$ one has $\delta^*(a_1, x) = s_2 \cdots s_k$. By induction we obtain a gallery a_1, \dots, a_{k-1} such that $\delta^*(c_+, a_i) = 1$ and such that for all $x \in E_w^*(c_-)$ one has $\delta^*(a_i, x) = s_{i+1} \cdots s_k$. Thus the chamber $d := a_{k-1}$ has the desired properties. \square

2.2 Twin buildings from RGD systems

A group G acts by isometries on a twin building $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ if it acts on each half and preserves the distances and the codistance. A twin building is called *homogeneous* if it admits a group action by isometries which is transitive on each half.

In this section we describe a class of homogeneous twin buildings using group theory. For the necessary background information on reflection groups and their associated root systems we refer to [1, Sections 1.5, 3.4] or to [22]. For more details on RGD systems we strongly recommend to consult [1, Chapters 7, 8] or [8].

Definition 2.12. Let G be a group and let $\{U_\alpha\}_{\alpha \in \Phi}$ be a family of subgroups of G , indexed by some root system Φ of type (W, S) , let Φ^+ be a subset of positive roots, and let T be a subgroup of G . The triple $(G, \{U_\alpha\}_{\alpha \in \Phi}, T)$ is called an *RGD system* of type (W, S) if it satisfies the following assertions.

(RGD0) For each root $\alpha \in \Phi$, one has $U_\alpha \neq \{1\}$.

(RGD1) For each prenilpotent pair $\{\alpha, \beta\} \subseteq \Phi$ of distinct roots, one has

$$[U_\alpha, U_\beta] \subseteq \langle U_\gamma \mid \gamma \in]\alpha, \beta[\rangle.$$

(Cf. [1, Sections 8.5.2, 8.5.3] for a definition of a prenilpotent pair, the “closed” interval $[\alpha, \beta]$ and the “open” interval $] \alpha, \beta [$.)

(RGD2) For each $s \in S$ there exists a function $\mu_s : U_{\alpha_s} \setminus \{1\} \rightarrow G$ such that for all $u \in U_{\alpha_s} \setminus \{1\}$ and $\alpha \in \Phi$ one has $\mu_s(u) \in U_{-\alpha_s} u U_{-\alpha_s}$ and $\mu_s(u) U_\alpha \mu_s(u)^{-1} = U_{s(\alpha)}$.

(RGD3) For each $s \in S$ one has $U_{-\alpha_s} \not\subseteq U_+ := \langle U_\alpha \mid \alpha \in \Phi^+ \rangle$.

(RGD4) $G = T \cdot \langle U_\alpha \mid \alpha \in \Phi \rangle$.

(RGD5) The group T normalises every U_α .

The tuple $(\{U_\alpha\}_{\alpha \in \Phi}, T)$ is called a *root group datum*, the U_α are called the *root subgroups*, and the $G_\alpha := \langle U_{\pm\alpha} \rangle$ are called the *rank one subgroups*.

Occasionally, for pairwise distinct simple roots $\alpha_1, \dots, \alpha_r$, we use the notation $G_{\alpha_1, \dots, \alpha_r}$ for the group generated by $G_{\alpha_1} \cup \dots \cup G_{\alpha_r}$. These groups are then referred to as *fundamental rank r subgroups*.

A root group datum $(\{U_\alpha\}_{\alpha \in \Phi}, T)$ is called *\mathbb{F} -locally split* if T is abelian and if there is a field \mathbb{F} such that $G_\alpha \cong (\text{P})\text{SL}_2(\mathbb{F})$ and $\{U_\alpha, U_{-\alpha}\}$ is isomorphic to

the canonical root group datum of $(\text{P})\text{SL}_2(\mathbb{F})$. The RGD system is called *centred* if G is generated by its root subgroups, i.e., if $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$.

Root group data give rise to BN-pairs in the sense of the following definition:

Definition 2.13. Let G be a group and let B, N be subgroups of G . The pair (B, N) is called a *BN-pair* for G , if G is generated by B and N , the intersection $T := B \cap N$ is normal in N , and the quotient group $W := N/T$ admits a set of generating involutions S such that

(BN1) for all $w \in W$ and $s \in S$ one has $wBs \subseteq BwsB \cup BwB$, and

(BN2) $sBs \not\subseteq B$ for each $s \in S$.

Two BN-pairs (B_+, N) and (B_-, N) of the same group G satisfying $B_+ \cap N = B_- \cap N$ yield a *twin BN-pair* (B_+, B_-, N) , if the following additional assertions hold:

(TBN1) for $\varepsilon \in \{+, -\}$ and all $w \in W, s \in S$ such that $l(sw) < l(w)$, one has $B_\varepsilon s B_\varepsilon w B_{-\varepsilon} = B_\varepsilon s w B_{-\varepsilon}$, and

(TBN2) for each $s \in S$ one has $B_+ s \cap B_- = \emptyset$.

If B, N is a BN-pair for G and S is as above then the quadruple (G, B, N, S) is called a *Tits system* with *Weyl group* W . The notion of a *twin Tits system* (G, B_+, B_-, N, S) is defined accordingly. We remark that the pair (W, S) is a Coxeter system; cf. [1, Theorem 6.56(1)].

A group G with a BN-pair admits a *Bruhat decomposition* $G = \bigsqcup_{w \in W} BwB$, cf. [1, Theorems 6.17 and 6.56(1)], and a group G with a twin BN-pair admits a *Birkhoff decomposition* $G = \bigsqcup_{w \in W} B_\varepsilon w B_{-\varepsilon}$, cf. [1, Proposition 6.81]. The groups B_+, B_- and their conjugates are called *Borel subgroups*.

Important examples arise from root group data:

Proposition 2.14 ([1, Theorem 8.80]). *Let G be a group with a root group datum $(\{U_\alpha\}_{\alpha \in \Phi}, T)$ of type (W, S) and for each $s \in S$ let $\mu_s : U_{\alpha_s} \setminus \{1\} \rightarrow U_{-\alpha_s} U_{\alpha_s} U_{-\alpha_s}$ be the map provided by (RGD2). Then the groups*

$$\begin{aligned} N &:= T \cdot \langle \mu_s(u) \mid u \in U_\alpha \setminus \{1\}, s \in S \rangle, \\ B_+ &:= T \cdot U_+, \\ B_- &:= T \cdot U_- \end{aligned}$$

yield a twin BN-pair (B_+, B_-, N) of the group G .

Definition 2.15. Let G be a group with a root group datum $(\{U_\alpha\}_{\alpha \in \Phi}, T)$ of type (W, S) , let (B_+, B_-, N) be the associated twin BN-pair and let $\Pi \subset \Phi$ be a set of simple roots. We say that the root group datum has the *finite prolongation property* if the following holds: For every finite sequence $(\alpha_1, \dots, \alpha_n) \in (\Pi \cup -\Pi)^n$ there exists a finite prolongation $(\alpha_1, \dots, \alpha_N) \in (\Pi \cup -\Pi)^N$ such that

$$\begin{aligned} & B_+ B_- \cap (U_{\alpha_1} \cdots U_{\alpha_n}) \\ & \subset (U_+ \cap (U_{\alpha_1} \cdots U_{\alpha_N})) \cdot (T \cap (U_{\alpha_1} \cdots U_{\alpha_N})) \cdot (U_- \cap (U_{\alpha_1} \cdots U_{\alpha_N})). \end{aligned}$$

For a twin Tits system (G, B_+, B_-, N, S) with Weyl group W define $\Delta_\pm := G/B_\pm$. Given $gB_\pm, hB_\pm \in \Delta_\pm$ using the Bruhat decomposition let

$$\delta_\pm(gB_\pm, hB_\pm) := w \in W \quad \text{if and only if} \quad B_\pm g^{-1} h B_\pm = B_\pm w B_\pm.$$

Similarly using the Birkhoff decomposition instead, given $gB_\pm \in \Delta_\pm$ and $hB_\mp \in \Delta_\mp$ let

$$\delta^*(gB_\pm, hB_\mp) := w \in W \quad \text{if and only if} \quad B_\pm g^{-1} h B_\mp = B_\pm w B_\mp.$$

Then $((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ is a twin building of type (W, S) , see [1, Theorem 6.56 and Definition 6.82].

Definition 2.16. The above twin building is denoted

$$\Delta(G, B_+, B_-, N, S) := ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$$

and is referred to as the twin building *associated with* the twin Tits system (G, B_+, B_-, N, S) . Moreover, if the twin Tits system arises from an RGD system $(\{U_\alpha\}_{\alpha \in \Phi}, T)$, then the associated twin building will also be denoted by $\Delta(G, \{U_\alpha\}_{\alpha \in \Phi}, T)$.

The following observation is due to Bernhard Mühlherr. It relies on the property (dco) for Moufang polygons, cf. [31, Definition 5.1]: For $d \in \mathbb{N}$, a Moufang polygon admits property (dco) if, given an arbitrary chamber c , any pair of chambers opposite c can be joined by a gallery of length at most d consisting of chambers opposite c only.

Proposition 2.17. *Every two-spherical root group datum for which there exists $d \in \mathbb{N}$ such that each residue of rank two of the associated twin building satisfies (dco) has the finite prolongation property.*

For infinite fields \mathbb{F} , this in particular applies to any \mathbb{F} -locally split two-spherical root group datum without generalized octagons as residues.

Proof. The first statement is [31, Theorem 4.5]; it suffices to define $\alpha_{n+1}, \dots, \alpha_N$ in such a way that it contains any sequence of length $n(2(d+16)^n + 1)$ of positive or negative simple roots, cf. [31, Section 5].

The second statement follows from [31, Lemma 5.2] and the comment thereafter. \square

2.3 A formula for co-projections onto panels

Let G be a group with root group datum $(\{U_\alpha\}_{\alpha \in \Phi}, T)$ and let

$$\Delta := \Delta(G, \{U_\alpha\}_{\alpha \in \Phi}, T) = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$$

be the associated twin building. The goal of this section is to derive a formula for co-projections (cf. [1, Lemma 5.149]) onto panels of Δ in terms of the group structure of G .

Lemma 2.18. *Let B_+, B_- be the opposite Borel subgroups provided by Proposition 2.14, let c_- be the chamber of Δ_- fixed by B_- , let $c \in \Delta_+$ be a chamber, and let $\delta^*(c_-, c) = w$. Then $B_- \cdot c = B_- w B_+$. In particular, $B_- \cdot c$ is represented by a unique double coset of the Birkhoff decomposition and every such double coset corresponds to a B_- -orbit.*

Proof. The first statement is evident, as $\delta^*(B_-, gB_+) = \delta^*(c_-, c) = w$ if and only if $B_- \cdot c = B_- g B_+ = B_- w B_+$. By [1, Lemma 6.70] the group B_- acts transitively on the chambers at codistance w from c_- , which implies the second statement. \square

Lemma 2.19. *For each $w \in W$ one has $w^{-1} B_+ w B_- \subseteq B_+ B_-$.*

Proof. We proceed by an induction on $l(w)$. As the case $l(w) = 0$ is trivial, we may assume $l(w) > 0$. Then there exist $s \in S$, $w' \in W$ such that $w = sw'$ and $l(w) = l(w') + 1$. For $x \in B_+ w B_-$ we have $sx \in B_+ s B_+ w B_- \stackrel{\text{(TBN1)}}{=} B_+ s w B_- = B_+ w' B_-$. Therefore, by induction, $w^{-1}x = w'^{-1}sx \in w'^{-1} B_+ w' B_- \subseteq B_+ B_-$. \square

Remark 2.20. The multiplication map

$$m : U_+ \times T \times U_- \rightarrow B_+ B_- : (u_+, t, u_-) \mapsto u_+ t u_-$$

is bijective by [1, Section 8.8]. Therefore also

$$\begin{aligned} \psi : B_- &\hookrightarrow B_+ B_- \rightarrow U_+ \backslash B_+ B_- \\ b_- &\mapsto U_+ b_- \end{aligned}$$

is a bijection and allows one to define a map

$$\begin{aligned}\pi: B_+B_- &\rightarrow B_- \\ x &\mapsto \psi^{-1}(U_+x).\end{aligned}$$

Finally, for $w \in W$, Lemma 2.19 allows one to define

$$\begin{aligned}\rho_w: B_+wB_- &\rightarrow B_- \\ x &\mapsto \pi(w^{-1}x).\end{aligned}$$

Proposition 2.21. *Let $x \in B_+wB_-$. Then $x \in B_+w\rho_w(x)$.*

Proof. By the Birkhoff decomposition of G there exist $u_\varepsilon \in U_\varepsilon$, $w \in W$, $t \in T$ such that $x = u_+wtu_-$. Since $w^{-1}u_+w \in U_+U_-$, there exist $u_\varepsilon^1 \in U_\varepsilon \cap w^{-1}U_+w$ such that $w^{-1}u_+w = u_+^1u_-^1$, whence $x = ww^{-1}u_+wtu_- = wu_+^1u_-^1tu_-$. Thus $w^{-1}x = u_+^1u_-^1tu_-$, so $\rho_w(x) = u_-^1tu_-$, and therefore $x = wu_+^1\rho_w(x)$. As $u_+^1 \in U_+ \cap w^{-1}U_+w$, there exists $u_2 \in U_+$ such that $u_+^1 = w^{-1}u_2w$. We conclude $x = wu_+^1\rho_w(x) = u_2w\rho_w(x) \in B_+w\rho_w(x)$. \square

We can now establish an explicit formula for co-projections onto panels:

Theorem 2.22 ([19, Theorem 4.3.5]). *Let $(G, \{U_\alpha\}_{\alpha \in \Phi})$ be an RGD system, let Δ be the associated twin building, let (W, S) be the associated Weyl group, let $c_+ = gB_+ \in \Delta_+$, let $c_- = hB_- \in \Delta_-$, let $\delta^*(c_+, c_-) = w \in W$, and let $s \in S$ such that $l(ws) > l(w)$. Then*

$$\text{proj}_{P_s(c_-)}^*(c_+) = h\rho_w(g^{-1}h)^{-1}sB_-.$$

Proof. One needs to prove that

$$\delta^*(gB_+, h\rho_w(g^{-1}h)^{-1}sB_-) = ws$$

and

$$\delta_-(hB_-, h\rho_w(g^{-1}h)^{-1}sB_-) = s.$$

Since $\delta^*(gB_+, hB_-) = w$, we have $g^{-1}h \in B_+wB_-$. Proposition 2.21 allows us to conclude that there exists $b_+ \in B_+$ such that $g^{-1}h = b_+w\rho_w(g^{-1}h)$, whence $g^{-1} = b_+w\rho_w(g^{-1}h)h^{-1}$. Therefore

$$g^{-1}h\rho_w(g^{-1}h)^{-1}s = b_+w\rho_w(g^{-1}h)h^{-1}h\rho_w(g^{-1}h)^{-1}s = b_+ws \in B_+wsB_-,$$

which shows that $\delta^*(gB_+, h\rho_w(g^{-1}h)^{-1}sB_-) = ws$. Similarly,

$$h^{-1}h\rho_w(g^{-1}h)^{-1}s = \rho_w(g^{-1}h)^{-1}s \in B_-sB_-,$$

and so $\delta_-(hB_-, h\rho_w(g^{-1}h)^{-1}sB_-) = s$. \square

Corollary 2.23 ([19, Corollary 4.3.6]). *Let $(G, \{U_\alpha\}_{\alpha \in \Phi})$ be an RGD system, let Δ be the associated twin building, let (W, S) be the associated Weyl group, let $c_+ = gB_+ \in \Delta_+$, and let $c_- = hB_- \in \Delta_-$. If $\delta^*(c_+, c_-) = 1 \in W$, then for each $s \in S$ one has*

$$\text{proj}_{P_s(c_-)}^*(c_+) = h\pi(g^{-1}h)^{-1}sB_-.$$

Proof. This follows from Theorem 2.22, as $\rho_1 = \pi$. \square

In Sections 3 and 7 we will use these projection formulae in order to derive the continuity of co-projections; in that context the following observation will become important:

Lemma 2.24. *Let τ be a group topology on G and equip T , U_\pm and B_+B_- with the subspace topologies. Assume that the continuous bijection $m : U_+ \times T \times U_- \rightarrow B_+B_- : (u_+, t, u_-) \mapsto u_+tu_-$ is open, i.e., a homeomorphism. Then the map ρ_w introduced in Remark 2.20 is continuous for every $w \in W$.*

Proof. If m is open, also ψ is open and, therefore, π and ρ_w are continuous; cf. Remark 2.20. \square

3 Topological twin buildings

3.1 Axioms for topological twin buildings

Throughout this section let $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ be a thick twin building of type (W, S) . In order to avoid pathologies we will always assume that the Coxeter diagram of (W, S) has no isolated vertices. By a *topology* τ on Δ we will always mean a pair of topologies τ_\pm on Δ_\pm . Given such topologies we equip the set $\Delta_+ \cup \Delta_-$ with the direct sum topology, i.e. Δ_+ and Δ_- are clopen subsets of $\Delta_+ \cup \Delta_-$.

We recall that a space X is the *direct limit* of subspaces X_i , denoted $X = \lim_{\rightarrow} X_i$, if (a) $X = \bigcup_i X_i$ and (b) $U \subset X$ is open if and only if $U \cap X_i$ is open for all i .

Definition 3.1. Let Δ be a thick twin building (of type (W, S) without isolated vertices in the Coxeter diagram) and τ a topology on Δ . Then the pair (Δ, τ) is called a *topological twin building* if it satisfies the following axioms:

(TTB1) τ is a Hausdorff topology.

(TTB2) For each $s \in S$ and each $c \in \Delta_{\pm}$ the map

$$\begin{aligned} E_1^*(c) &\rightarrow \Delta_+ \cup \Delta_- \\ d &\mapsto \text{proj}_{P_s(c)}^*(d) \end{aligned}$$

is continuous.

(TTB3) There exist chambers $c_{\pm} \in \Delta_{\pm}$ such that

$$\Delta_{\pm} = \lim_{\rightarrow} E_{\leq w}(c_{\pm}).$$

(TTB4) For each $s \in S$ there exists a compact panel $P \in \text{Pan}_s(\Delta_{\pm})$.

A *morphism* of topological twin buildings is a morphism of the underlying twin buildings that, additionally, is continuous with respect to the twin building topologies.

Our definition of a topological twin building is chosen in such a way that it assumes only a minimal set of axioms which we need to develop a non-trivial theory. Our definition is slightly different from Linus Kramer's original definition [27, p. 169]. Depending on the applications one has in mind one may want to add further axioms. We discuss various possible alternative axiomatizations in Section 3.5 below.

3.2 Basic point-set topology of topological twin buildings

In this section we investigate basic point-set topological properties of topological twin buildings. Initially we will not make use of the compactness assumption (TTB4); the results in this section will hold for every thick twin building $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$, which is endowed with a topology satisfying axioms (TTB1), (TTB2), and (TTB3).

Lemma 3.2. *Let $c_{\pm} \in \Delta_{\pm}$ be opposite chambers and let $w \in W \setminus \{1\}$. Then there exists an open neighbourhood U of c_+ in Δ_+ , which does not intersect $E_w^*(c_-)$.*

Proof. Fix a reduced expression $w = s_1 \cdots s_k$ with $s_j \in S$. By Lemma 2.11 there exists $d_- \in \Delta_-$ with $\delta^*(c_+, d_-) = 1$ and $\delta^*(x, d_-) = s_k$ for all $x \in E_w^*(c_-)$. For every $d' \in P_{s_k}(d_-) \setminus \{d_-, \text{proj}_{P_{s_k}(d_-)}^*(c_+)\}$ we have

$$\{c_+\} \cup E_w^*(c_-) \subset E_1^*(d').$$

By (TTB2) the restriction of $\text{proj}_{P_{s_k}(d')}^*$ defines a continuous map

$$f : E_1^*(d') \rightarrow P_{s_k}(d') = P_{s_k}(d_-).$$

Therefore the lemma follows from the fact that $P_{s_k}(d')$ is Hausdorff, (TTB1), and that $E_w^*(c_-) \subseteq f^{-1}(d_-)$. \square

Proposition 3.3. *For every $c \in \Delta_{\pm}$ the co-Schubert cell $E_1^*(c)$ is open.*

Proof. By symmetry we may assume $c \in \Delta_-$. By (TTB3) it suffices to show that for $c_0 \in \Delta_+$ the set $E_1^*(c)$ is relatively open in $E_{\leq w}(c_0)$ for all $w \in W$, i.e., any $c_+ \in E_{\leq w}(c_0) \cap E_1^*(c)$ is an interior point. By Lemma 2.10 the function $\delta^*(\cdot, c)$ takes only finitely many values on $E_{\leq w}(c_0)$, and for each non-trivial value w_j Lemma 3.2 produces an open neighbourhood U_j of c_+ in Δ_+ with $U_j \cap E_{w_j}^*(c) = \emptyset$. Then $E_{\leq w}(c_0) \cap \bigcap U_j$ is an open subset of $E_{\leq w}(c_0)$ containing c_+ and contained in $E_1^*(c)$, i.e., c_+ is an interior point of $E_{\leq w}(c_0) \cap E_1^*(c)$. \square

As a first application we deduce:

Lemma 3.4. *Let $J \subset S$. Then every J -residue in Δ_{\pm} is closed.*

Proof. We will prove that for a J -residue $R \subset \Delta_+$ the set $\Delta_+ \setminus R$ is open by showing that an arbitrary $c \in \Delta_+ \setminus R$ is an interior point. Let $d \in R$ so that $\delta_+(d, c) \notin \langle J \rangle$, let (Σ_+, Σ_-) be a twin apartment that contains both c and d , and denote by e the unique chamber in $\Sigma_- \cap E_1^*(c)$. For every $f \in R$ we have

$$\delta^*(f, e) \stackrel{2.10}{\in} \langle J \rangle \delta^*(d, e) \stackrel{2.7}{=} \langle J \rangle \delta_+(d, c) \delta^*(c, e) = \langle J \rangle \delta_+(d, c),$$

whence there exists $s \in S \setminus J$ with $\delta^*(f, e) \geq s$ for all $f \in R$. This shows $c \in E_1^*(e) \subset \Delta_+ \setminus R$, i.e., by Proposition 3.3 c is an interior point of $\Delta_+ \setminus R$. \square

Given a panel $P \subset \Delta_{\pm}$ and a chamber $c \in P$ we denote by $P^{\times} = P_c^{\times}$ the pointed panel $P \setminus \{c\}$. A pointed panel is open in its ambient panel by (TTB1).

Proposition 3.5. *Let $P \subset \Delta_{\pm}$ and $Q \subset \Delta_{\mp}$ be opposite panels. Then the map $c \mapsto \text{proj}_Q^*(c)$ restricts to a homeomorphism $p_{PQ}^* : P \rightarrow Q$.*

Proof. By [1, Proposition 5.152] the maps p_{PQ}^* and p_{QP}^* are mutually inverse bijections. Hence it remains only to establish their continuity. For this let $c \in P$ and $d := p_{PQ}^*(c) \in Q$ its projection. Then $P_c^{\times} \subset E_1^*(d)$, whence the restriction of p_{PQ}^* to P_c^{\times} is continuous by (TTB2). Since the open subsets $\{P_c^{\times} \mid c \in P\}$ cover P , this implies continuity of p_{PQ}^* . \square

Combining this with Lemma 2.9 we obtain:

Corollary 3.6. *Panels of the same type are pairwise homeomorphic.*

In particular, we deduce that—in the presence of axioms (TTB1), (TTB2) and (TTB3)— axiom (TTB4) is equivalent to the following, a priori stronger, axiom:

(TTB4+) For each $s \in S$ every panel $P \in \text{Pan}_s(\Delta_\pm)$ is compact.

As another application of the corollary we now define a functor type from the category of pointed topological twin buildings of a fixed type (W, S) to the category of topological spaces as follows: Given a topological twin building Δ we set

$$\text{type}(\Delta, c) := \bigcup_{s \in S} P_s(c)$$

and refer to $\text{type}(\Delta, c)$ as the *topological type* of the twin building Δ at c . Every based morphism φ of (pointed) topological twin buildings then induces a continuous map $\text{type}(\varphi)$ between the corresponding topological types by restriction. Corollary 3.6 ensures that, up to homeomorphism, $\text{type}(\Delta, c)$ does not depend on the choice of the basepoint c . The local-to-global principle established in Section 3.4 below shows that the topology of Δ is uniquely determined by its topological type.

As another application of Proposition 3.3 we show:

Proposition 3.7. *Let Δ be a twin building with a topology τ and Δ' a pair of opposite residues in Δ .*

- (i) Δ' is a twin building with respect to the restricted distance and codistance.
- (ii) If (Δ, τ) satisfies (TTB1), (TTB2), (TTB3), then so does $(\Delta', \tau|_{\Delta'})$.
- (iii) If (Δ, τ) is a topological twin building, then so is $(\Delta', \tau|_{\Delta'})$.

Proof. For (i) see e.g. [1, Exercise 5.166]. For (ii) and (iii) observe first that axioms (TTB1) and (TTB2) descend from Δ to arbitrary subsets, while (TTB3) descends to arbitrary closed subsets. By Lemma 3.4 it descends in particular to Δ' . Finally, if (Δ, τ) is a topological twin building, then it satisfies not only (TTB4) but also (TTB4+), and this property obviously descends to $(\Delta', \tau|_{\Delta'})$. \square

3.3 Gallery spaces and Bott–Samelson desingularizations

In this section we provide tools that will allow us to study the global point-set topology of topological twin buildings. Throughout this section we fix a topological twin building (Δ, τ) .

For a reduced word $s_1 \cdots s_k \in W$ define a *gallery* of type (s_1, \dots, s_k) as a tuple $(c_0, \dots, c_k) \in (\Delta_{\pm})^{k+1}$ satisfying $c_i \in \text{Pan}_{s_i}(c_{i-1})$ for $i = 1, \dots, k$. The chamber $c_0 \in \Delta_{\pm}$ is called the *initial chamber* of the gallery. The set of all galleries of type (s_1, \dots, s_k) with initial chamber c_0 is denoted by $\text{Gall}(s_1, \dots, s_k; c_0)$; it is endowed with the subspace topology induced from $(\Delta_{\pm})^{k+1}$.

The natural *projection* and *stammering maps* allow one to pass between different gallery spaces:

$$\begin{aligned} \pi_{s_1, \dots, s_k; c_0} : \text{Gall}(s_1, \dots, s_k; c_0) &\rightarrow \text{Gall}(s_1, \dots, s_{k-1}; c_0) \\ &(c_0, \dots, c_k) \mapsto (c_0, \dots, c_{k-1}), \\ s_{s_1, \dots, s_k; c_0} : \text{Gall}(s_1, \dots, s_{k-1}; c_0) &\rightarrow \text{Gall}(s_1, \dots, s_k; c_0) \\ &(c_0, \dots, c_{k-1}) \mapsto (c_0, \dots, c_{k-1}, c_{k-1}). \end{aligned}$$

The following observation of Linus Kramer’s provides a key insight into the topological structure of topological twin buildings. Recall from Corollary 3.6 that panels of the same type are pairwise homeomorphic.

Proposition 3.8 ([27, p. 170, 171]). *For every $c_0 \in \Delta_{\pm}$, the gallery space $\text{Gall}(s_1, \dots, s_k; c_0)$ is a locally trivial fibre bundle over $\text{Gall}(s_1, \dots, s_{k-1}; c_0)$ with fibre $P_{s_k}(c_0)$ via $\pi_{s_1, \dots, s_k; c_0}$. The stammering map $s_{s_1, \dots, s_k; c_0}$ defines a global section of this bundle.*

Proof. By symmetry we may assume $c_0 \in \Delta_+$. For each $e \in \Delta_-$ define

$$U_e := \{(c_0, \dots, c_{k-1}) \in \text{Gall}(s_1, \dots, s_{k-1}; c_0) \mid c_{k-1} \in E_1^*(e)\}.$$

By Proposition 3.3, the family $(U_e)_{e \in \Delta_-}$ provides an open covering of the gallery space $\text{Gall}(s_1, \dots, s_{k-1}; c_0)$. By Proposition 3.5, for each $e \in \Delta_-$ the map

$$\begin{aligned} h_e : U_e \times P_{s_k}(e) &\rightarrow \pi_{s_1, \dots, s_k; c_0}^{-1}(U_e) \\ (c_0, \dots, c_{k-1}, d) &\mapsto (c_0, \dots, c_{k-1}, \text{proj}_{P_{s_k}(c_{k-1})}^*(d)) \end{aligned} \quad (3.1)$$

is a homeomorphism, which in view of Corollary 3.6 provides the desired local trivialization. The final claim is obvious. \square

Remark 3.9. By (TTB4) and Corollary 3.6 panels are compact, thus so are the gallery spaces by Proposition 3.8. Hence for each reduced word $s_1 \cdots s_k$ the (surjective) endpoint map

$$\begin{aligned} p_{s_1, \dots, s_k; c_0} : \text{Gall}(s_1, \dots, s_k; c_0) &\rightarrow E_{\leq s_1 \cdots s_k}(c_0) \\ &(c_0, \dots, c_k) \mapsto c_k \end{aligned}$$

is a quotient map by (TTB1).

This remark implies:

Corollary 3.10. *Schubert varieties—in particular, spherical residues—are compact.*

The corollary has important consequences for the point-set topology of topological twin buildings.

Definition 3.11. A Hausdorff topological space X is called a k_ω space if there exists a countable ascending sequence $K_1 \subseteq K_2 \subseteq \dots \subseteq X$ of compact sets such that X is the direct limit of the K_n , i.e., $X = \bigcup_{n \in \mathbb{N}} K_n$ and such that $U \subseteq X$ is open if and only if $U \cap K_n$ is open in K_n for each n with respect to the subspace topology.

We refer the reader to [11] for an overview over the theory of k_ω spaces; the benefits of the theory of k_ω spaces for studying twin buildings and Kac–Moody groups are clearly visible in [13]. Key properties of k_ω spaces for the present article are:

Proposition 3.12 ([11], [13, Proposition 4.2]). *Each σ -compact locally compact Hausdorff space is k_ω . Moreover, the category of k_ω spaces is closed under taking closed subspaces, finite products, Hausdorff quotients and countable disjoint unions. Every k_ω space is paracompact, Lindelöf and normal.*

Corollary 3.13. *Each half of a topological twin building is a k_ω space, in particular paracompact, Lindelöf and normal. It is compact if and only if it is spherical.*

Proof. The first statement follows from Proposition 3.12, Corollary 3.10 and (TTB3), whereas the second statement then follows from the fact that a closed subset of a direct limit of compact spaces is compact if and only if it is already contained in one of the compact spaces. \square

In fact, if c_\pm are as in (TTB3) then the sequences $(E_{\leq w}(c_\pm))_{w \in W}$ are explicit k_ω -sequences for Δ_\pm . It then follows from the general theory of k_ω spaces that a sequence $(K_n) \subset \Delta_\pm$ is a k_ω -sequence for Δ_\pm if and only if each K_n is compact and for every $w \in W$ there exists $n \in \mathbb{N}$ such that $E_{\leq w}(c_\pm) \subset K_n$. In particular, we deduce that in the presence of axioms (TTB1), (TTB2), (TTB4), axiom (TTB3) is equivalent to the following strengthened version:

(TTB3+) For every chamber $c_\pm \in \Delta_\pm$,

$$\Delta_\pm = \varinjlim E_{\leq w}(c_\pm).$$

In other words, in the definition of a topological twin building the choices made in axioms (TTB3) and (TTB4) are inessential. Note that Corollary 3.13 is based on the interplay of axioms (TTB3) and (TTB4).

Another example for this interplay is provided by the following proposition:

Proposition 3.14. *Let (Δ, τ) be a topological twin building whose panels are discrete. Then Δ itself is discrete.*

Proof. By assumption all panels are discrete and compact, hence finite. Consequently, Schubert varieties are finite and Hausdorff, hence discrete. Therefore the proposition follows from (TTB3). \square

This proposition allows one to locally characterize discrete topological twin buildings. Such local-to-global results are the topic of the next section.

3.4 A local-to-global principle

Using the tools developed in the preceding section we derive a local-to-global principle for topological twin buildings. Recall that the type functor associates with every morphism of topological twin buildings a continuous map between the topological types, a concept which is meaningful by Corollary 3.6.

We intend to show that this property characterizes morphisms of topological twin buildings among all twin building morphisms.

More precisely, let $\Delta^{(1)}, \Delta^{(2)}$ be topological twin buildings and let $\varphi : \Delta^{(1)} \rightarrow \Delta^{(2)}$ be a morphism of the underlying abstract twin buildings. Choose $c \in \Delta_{\pm}^{(1)}$; then the map

$$\text{type}_c(\varphi) : \bigcup_{s \in S} P_s(c) \rightarrow \bigcup_{s \in S} P_s(\varphi(c))$$

between the topological types of the $\Delta^{(j)}$ can still be defined, but need not be continuous.

Now we have:

Theorem 3.15 (Local-to-global principle for twin building topologies). *Let $\Delta^{(1)}$ and $\Delta^{(2)}$ be topological twin buildings, let $\varphi : \Delta^{(1)} \rightarrow \Delta^{(2)}$ be a morphism of the underlying twin buildings, and let $c \in \Delta_{\pm}^{(1)}$. Then φ is continuous if and only if $\text{type}_c(\varphi)$ is continuous.*

In the spherical case the above local-to-global result was first proved in [3, Proposition 3.5] using a coordinatization procedure which, however, is not available in the general case. (Observe that the coordinatization given in [27] provides coordinates on Schubert cells rather than co-Schubert cells.)

Proof. Assume that $\text{type}_c(\varphi)$ is continuous. Since φ is a morphism, for arbitrary opposite panels P and Q of $\Delta^{(1)}$ one has

$$\varphi \circ p_{PQ}^* = p_{\varphi(P)\varphi(Q)}^* \circ \varphi,$$

where $p_{PQ}^* : P \rightarrow Q$ and $p_{\varphi(P)\varphi(Q)}^* : \varphi(P) \rightarrow \varphi(Q)$ are the projection homeomorphisms from Proposition 3.5. By Lemma 2.9 and the continuity of $\text{type}_c(\varphi)$ this implies that for each panel P of $\Delta^{(1)}$ the restriction $\varphi|_P$ is continuous. This shows that the statement that $\text{type}_c(\varphi)$ be continuous is in fact independent of $c \in \Delta_{\pm}$.

For an arbitrary reduced word $s_1 \cdots s_k \in W$ the morphism φ induces a map

$$\varphi_{s_1, \dots, s_k; c} : \text{Gall}(s_1, \dots, s_k; c) \rightarrow \text{Gall}(s_1, \dots, s_k; \varphi(c)).$$

We will prove by induction on k that $\varphi_{s_1, \dots, s_k; c}$ is continuous. For $k = 0$ there is nothing to show, so we immediately turn to the case $k > 0$. Then by Proposition 3.3 the sets

$$U_e := \{(c, c_1, \dots, c_{k-1}) \in \text{Gall}(s_1, \dots, s_{k-1}; c) \mid c_{k-1} \in E_1^*(e)\}$$

provide an open covering of $\text{Gall}(s_1, \dots, s_{k-1}; \varphi(c))$ (cf. the proof of Proposition 3.8). The homeomorphisms h_e and $h_{\varphi(e)}$ from (3.1) on page 19 yield a commuting diagram

$$\begin{array}{ccc} U_e \times P_{s_k}(e) & \xrightarrow{h_e} & \pi_{s_1, \dots, s_k; c}^{-1}(U_e) \\ \downarrow \varphi|_{U_e} \times \varphi|_{P_{s_k}(e)} & & \downarrow \varphi|_{\pi_{s_1, \dots, s_k; c}^{-1}(U_e)} \\ U_{\varphi(e)} \times P_{s_k}(\varphi(e)) & \xrightarrow{h_{\varphi(e)}} & \pi_{s_1, \dots, s_k; \varphi(c)}^{-1}(U_{\varphi(e)}). \end{array}$$

The left vertical arrow is continuous, because $\varphi|_{U_e}$ is continuous by the induction hypothesis and $\varphi|_{P_{s_k}(e)}$ is continuous by continuity of $\text{type}_e(\varphi)$ as shown above. Therefore also the right vertical arrow is continuous. Since the sets $\pi_{s_1, \dots, s_k; c}^{-1}(U_e)$ form an open covering of $\text{Gall}(s_1, \dots, s_k; c)$, this implies continuity of the map $\varphi_{s_1, \dots, s_k; c}$.

By Remark 3.9 the endpoint map $p_{s_1, \dots, s_k; c} : \text{Gall}(s_1, \dots, s_k; c) \rightarrow E_{\leq s_1 \cdots s_k}(c)$ is a quotient map, so that $\varphi|_{E_{\leq s_1 \cdots s_k}(c)}$ is continuous.

Therefore φ is continuous by (TTB3). \square

Corollary 3.16. *Let Δ be a topological twin building and let φ be an automorphism of the underlying twin building. If, for each type $s \in S$, there exists a panel P_s of type s such that $\varphi|_{P_s}$ is continuous, then φ is a homeomorphism.*

Proof. Continuity of φ is immediate by (the proof of) Theorem 3.15. By axioms (TTB1) and (TTB4) and Corollary 3.6, each $\varphi|_{P_s} : P_s \rightarrow \varphi(P_s)$ is a bijective quotient map, i.e., open. Hence also continuity of φ^{-1} follows from Theorem 3.15, and φ is a homeomorphism. \square

3.5 The axioms revisited

We have seen that a reasonably deep theory of twin building topologies can be developed assuming only the axioms (TTB1), (TTB2), (TTB3), (TTB4) given above. On the other hand, these axioms seem to be of fundamental importance in order to be able to develop a meaningful theory.

Axiom (TTB2) is the core axiom underlying—in an explicit or implicit¹ way—all approaches to topological geometry: The geometric operations should be continuous on reasonable domains. We have chosen the weakest possible formulation which requires projections between opposite panels to be only separately continuous in each variable. In certain situations it is advantageous to assume joint continuity of such projections, or even joint continuity for projection between chambers of fixed, but possibly non-trivial, codistance. One would then replace (TTB2) by one of the following stronger axioms:

(TTB2+) For each $s \in S$ the map

$$\begin{aligned} p_s : \Delta_1 &\rightarrow \Delta_+ \cup \Delta_- \\ (c, d) &\mapsto \text{proj}_{P_s(c)}^*(d) \end{aligned}$$

is continuous.

(TTB2++) For each $s \in S$ and $w \in W$ the map

$$\begin{aligned} p_s : \Delta_w &\rightarrow \Delta_+ \cup \Delta_- \\ (c, d) &\mapsto \text{proj}_{P_s(c)}^*(d) \end{aligned}$$

is continuous.

We prefer to use the weakest possible axiom as part of our definition and to assume these additional properties only when needed.

The Hausdorff axiom (TTB1) is standard in topological geometry.

The purpose of the compactness axiom (TTB4) is less obvious, and it seems tempting to try to develop at least the basic theory without it. However, all

¹As pointed out in [27], in the case of spherical topological building one can actually deduce these continuity properties by making sufficiently strong compactness assumptions and using an open mapping theorem. However, this approach is clearly limited to compact situations.

attempts to develop a sufficiently rich theory in order to allow meaningful applications have failed so far, even in the case of spherical buildings of rank two. We thus cannot avoid (TTB4) at this point.

Axiom (TTB3) has no counterpart in classical topological geometry and so requires some justification. Let us call a pair (Δ, τ) a *weak topological twin building* if it satisfies (TTB1), (TTB2) and (TTB4).

As we have seen above in Section 3.3, after choosing base points $c_{\pm} \in \Delta_{\pm}$ the corresponding combinatorial Schubert cells still yield a canonical topological cell structure on (Δ, τ) . It is thus natural to refine the topology τ by passing to the weak topology

$$\tau' := \{U \subset \Delta_+ \cup \Delta_- \mid U \cap (E_{\leq w}(c_+) \cup E_{\leq w}(c_-)) \in \tau|_{E_{\leq w}(c_+) \cup E_{\leq w}(c_-)}\}$$

with respect to this cell decomposition.

We refer to (Δ, τ') as the *Schubert completion* of (Δ, τ) and call two weak topological twin buildings *Schubert equivalent* if their Schubert completions coincide.

Schubert equivalent weak topological twin buildings have homeomorphic Schubert varieties, while their topology at infinity may be different; for the purposes of classification it seems unnatural to distinguish between them.

Moreover we observe the following:

Proposition 3.17. *If (Δ, τ) is a weak topological twin building, then its Schubert completion (Δ, τ') is a topological twin building.*

Proof. Firstly, since τ' refines τ , the Schubert completion (Δ, τ') inherits (TTB1). Also, (TTB3) holds by definition. As for (TTB4), assume $P \in \text{Pan}_s(\Delta_{\pm})$ is compact with respect to τ . Let $\{U_{\alpha}\} \subset \tau'|_P$ be a covering of P and choose w sufficiently large so that $P \subset E_{\leq w}(c_{\pm})$. Then $\{U_{\alpha}\} = \{U_{\alpha} \cap E_{\leq w}(c_{\pm})\} \subset \tau|_P$, whence there exists a finite subcovering, showing that $(P, \tau'|_P)$ is also compact.

It remains to establish (TTB2). For this, fix $s \in S$ and $c \in \Delta_+ \cup \Delta_-$. Then the map $\varphi_{s,c} : E_1^*(c) \rightarrow \Delta_+ \cup \Delta_-$ given by $d \mapsto \text{proj}_{P_s(c)}^*(d)$ takes values in $P_s(c)$. Given base chambers $c_{\pm} \in \Delta_{\pm}$ we can choose $w \in W$ so that $P_s(c) \subset E := E_{\leq w}(c_+) \cup E_{\leq w}(c_-)$. We then obtain for all $A \subset \Delta_+ \cup \Delta_-$,

$$\varphi_{s,c}^{-1}(A) = \varphi_{s,c}^{-1}(A \cap E). \quad (3.2)$$

Now, if $U \in \tau'$, then by definition there exists $V \in \tau$ such that $U \cap E = V \cap E$. Since $\varphi_{s,c}$ is τ -continuous we then deduce from (3.2) that

$$\varphi_{s,c}^{-1}(U) = \varphi_{s,c}^{-1}(U \cap E) = \varphi_{s,c}^{-1}(V \cap E) = \varphi_{s,c}^{-1}(V) \in \tau \subset \tau',$$

which shows τ' -continuity of $\varphi_{s,c}$ and thereby establishes (TTB2). \square

Corollary 3.18. *Every Schubert equivalence class of weak topological twin buildings has a unique representative which is a topological twin building.*

This corollary is the reason why we allow ourselves to include (TTB3) into our definition.

Remark 3.19. It is common practice in homotopy theory to replace a topology by its compactly generated counterpart. Schubert completion provides a re-topologization procedure which is similar in flavour.

However, we should warn the reader that we do not know whether this procedure preserves the homotopy type. For example, our proof of the topological Solomon–Tits theorem in Section 5.3 relies crucially on (TTB3), and it is not clear to us whether this assumption can be dropped.

Having justified the necessity of our axioms, we should also mention some axioms which we do not assume, but other authors might find desirable. The following is a non-exhaustive list of such properties; throughout we equip the spaces Res_J^\pm —and, in particular, the vertex sets \mathcal{V}_s^\pm —with the quotient topology with respect to the canonical map $\Delta_\pm \rightarrow \text{Res}_J^\pm$. We then equip the product $\prod \mathcal{V}_s^\pm$ with the product topology.

(TTB1+) The vertex sets \mathcal{V}_s^\pm , $s \in S$, are Hausdorff.

(TTB5) The set $\Delta_1 = \{(c, d) \in (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \mid \delta^*(c, d) = 1\}$ of opposite chambers is open.

(TTB6) For every $s \in S$ the canonical map $\Delta_\pm \rightarrow \mathcal{V}_s^\pm$ is open.

(TTB6+) For every $J \subset S$ the canonical map $\Delta_\pm \rightarrow \text{Res}_J^\pm$ is open.

(TTB7) The diagonal embedding (see page 6)

$$\begin{aligned} \iota : \Delta_\pm &\hookrightarrow \prod_{s \in S} \mathcal{V}_s^\pm \\ c &\mapsto (R_{S \setminus \{s\}}(c))_{s \in S} \end{aligned}$$

is open and, hence, a homeomorphism onto its image.

For example, Linus Kramer’s original definition of a topological twin building² involves the axioms (TTB1+), (TTB2+), (TTB4+) and (TTB7). The relevance of Axiom (TTB7) is that it allows one to pass freely between the simplicial complex picture and the chamber system picture of twin buildings in a topological

²We note Kramer formulates these axioms in terms of the simplicial complex approach to buildings; in view of the examples we have in mind, it appeared convenient to us to reformulate the theory in the language of chamber systems. A detailed discussion of the two approaches—the simplicial complex one and the chamber system one—can be found in [1].

context. From the simplicial point of view, it seems indeed more natural to start from a topology on the vertices and to induce a topology on chambers, rather than the other way round. On the other hand, for the theory presented here, (TTB7) is not strictly needed.

The above axioms are actually not quite independent:

Proposition 3.20. *Let τ be a topology on a twin building Δ .*

- (i) (TTB1+) implies (TTB1).
- (ii) (TTB2++) implies (TTB2+) implies (TTB2).
- (iii) (TTB6) and (TTB6+) are equivalent.
- (iv) If (Δ, τ) is a topological twin building, then (TTB6) implies that the space Res_J^\pm (and in particular the vertex spaces \mathcal{V}_s) are T1.
- (v) If (Δ, τ) is a topological twin building, then (TTB1+) and (TTB6) imply (TTB7).

Proof. (i) If (TTB1+) holds and ι denotes the continuous injection from (TTB7), then $\iota(\Delta_\pm) \subset \prod \mathcal{V}_s^\pm$ are Hausdorff. Since ι is continuous, it follows that Δ_\pm are Hausdorff as well.

(ii) and the implication (TTB6+) \implies (TTB6) in (iii) are obvious. For the converse, let $J = \{s_1, \dots, s_n\}$ and $U \subset \Delta_\pm$ open. Denote by $\pi(U)$ the image of U under the projection onto Res_J^\pm and by $\pi_j(U) \subset \text{Res}_j^\pm$ the preimage of the image of U in \mathcal{V}_{s_j} . Then (TTB6) implies that the $\pi_j(U)$ are open, whence $\pi(U) = \bigcap \pi_j(U)$ is open.

In order to show (iv), we fix $J \subset S$ and let R_1, R_2 be distinct J -residues in Δ_\pm . Since Δ_\pm is normal by Proposition 3.12 and R_1, R_2 are closed by Lemma 3.4, there exist open neighbourhoods U_1 and U_2 separating R_1 and R_2 in Δ_\pm . By (iii) we know that (TTB6+) holds, hence the images $\pi(U_1)$ and $\pi(U_2)$ in Res_J^\pm are open. Now $R_1 \in \pi(U_1)$ and $R_2 \notin \pi(U_1)$ (and vice versa for U_2), hence (iv) follows.

Concerning (v) we first observe that the map ι is continuous by definition of the quotient topology. By Corollary 3.10, the Schubert varieties $E_{\leq \omega}(c^\pm) \subset \Delta_\pm$ are compact, whence mapped homeomorphically by ι by (TTB1+). Now, using (TTB1+) and 3.12, each of the spaces \mathcal{V}_s itself is k_ω . Since in the category of k_ω spaces direct limits and finite products commute (cf. [12, Proposition 3.3], [13, Proposition 4.7]), the image of ι is the direct limit of the images of the Schubert varieties, and so the claim follows. \square

There might be further relations between the axioms. For example, we do not know whether the argument in (iv) can be refined to show that (TTB6) implies

(TTB1+). Also, we suspect that it might be possible to derive (TTB5) from the other axioms (cf. the argument in Proposition 6.6 and Theorem 6.7).

Definition 3.21. A topological twin building is called a *strong topological twin building* if it satisfies the additional axioms (TTB1+), (TTB2+), (TTB5), (TTB6) —and hence also (TTB6+) and (TTB7).

Note that in a strong topological twin buildings the residue spaces Res_J^\pm (and in particular the vertex spaces \mathcal{V}_s) are k_ω , since they are Hausdorff quotients of Δ_\pm . Moreover, by (TTB7) we can identify chambers of a strong topological twin building with the corresponding collections of vertices.

3.6 Algebraic operations

By [15, Proposition 1.1] punctured panels in a generalized polygon carry a multiplication operation, which can be defined in elementary geometric terms, i.e. by intersecting lines and connecting points. The following proposition provides an extension to the twin building case:

Proposition 3.22. *Let (Δ, τ) be a strong topological twin building, let c_\pm be opposite chambers, let $r, s \in S$ with $3 \leq m_{rs} \leq \infty$, let $0_+ := \text{proj}_{P_r(c_+)}^*(c_-)$ and $0_- := \text{proj}_{P_s(c_-)}^*(c_+)$, and let $P_r(c_+)^\times := P_r(c_+) \setminus \{c_+\}$ and $P_s(c_-)^\times := P_s(c_-) \setminus \{c_-\}$.*

For each choice $1_- \in P_s(c_-)^\times \setminus \{0_-\}$, there exists a continuous map

$$\bullet : P_r(c_+)^\times \times P_s(c_-)^\times \rightarrow P_r(c_+)^\times$$

with the following properties:

- (i) *For all $c \in \Delta^+$ we have $c \bullet 0_- = 0_+$ and $c \bullet 1_- = c$.*
- (ii) *For every $c' \in P_s(c_-)^\times \setminus \{0_-\}$ the map $c \mapsto c \bullet c'$ is a homeomorphism of $P_r(c_+)^\times$.*

In case (Δ, τ) is two-spherical, one reduces the proof of Proposition 3.22 to the situation of generalized polygons as follows: Suppose (Δ, τ) is a two-spherical topological twin building satisfying (TTB7). Then, by Corollary 3.10, the residues of rank two are compact polygons in the sense of [26], and one can use the argument presented there. Thus, in this case, in fact axioms (TTB1–4) and (TTB7) suffice in order to deduce the conclusion of Proposition 3.22.

Below we present a proof that does not assume two-sphericity, but uses axioms (TTB2+), (TTB5) and (TTB6+) in addition to axioms (TTB1–4).

The following lemma extracts the place where these axioms enter the picture:

Lemma 3.23. *For every $s \in S$ the map*

$$\begin{aligned} \Delta_{\langle s \rangle} &\rightarrow \Delta_+ \cup \Delta_- \\ \{P, c\} &\mapsto \text{proj}_P^*(c) \end{aligned}$$

is continuous, where

$$\begin{aligned} \Delta_{\langle s \rangle} := \{ &(P, c) \in \text{Pan}_s(\Delta_+) \times \Delta_- \mid \delta^*(P, c) \in \langle s \rangle\} \\ &\cup \{(c, P) \in \Delta_+ \times \text{Pan}_s(\Delta_-) \mid \delta^*(c, P) \in \langle s \rangle\}. \end{aligned}$$

Proof. The quotient map

$$(\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \rightarrow (\text{Pan}_s(\Delta_+) \times \Delta_-) \cup (\Delta_+ \times \text{Pan}_s(\Delta_-))$$

is open by (TTB6+), and so is its restriction $\Delta_1 \rightarrow \Delta_{\langle s \rangle}$ to the open subset Δ_1 by (TTB5). By (Tw3) this restricted map is surjective, so the claim follows from (TTB2+). \square

The remainder of the proof only uses (TTB2+) and the standard axioms:

Proof of Proposition 3.22. Note that $P_r(1_-)$ is opposite to $P_r(c_+)$. For $d_+ := \text{proj}_{P_s(c_+)}^*(c_-)$, the panel $P_r(d_+)$ is opposite to $P_r(1_-)$, because $1_- \neq c_- = \text{proj}_{P_s(c_-)}^*(d_+)$, i.e., 1_- is opposite to d_+ . Define

$$\begin{aligned} g : P_r(c_+)^{\times} \times P_s(c_-)^{\times} &\rightarrow \Delta_- \\ (x, y) &\mapsto \text{proj}_{P_r(y)}^* \circ \text{proj}_{P_r(d_+)}^* \circ \text{proj}_{P_r(1_-)}^*(x), \\ \bullet : P_r(c_+)^{\times} \times P_s(c_-)^{\times} &\rightarrow P_r(c_+)^{\times} \\ (x, y) &\mapsto \text{proj}_{P_r(c_+)}^* \circ \text{proj}_{P_r(y)}^* \circ \text{proj}_{P_r(d_+)}^* \circ \text{proj}_{P_r(1_-)}^*(x) \\ &= \text{proj}_{P_r(c_+)}^*(g(x, y)). \end{aligned}$$

Note that the fact $x \bullet y \in P_r(c_+)^{\times}$ requires a proof that we will provide below.

As $P_r(d_+)$ is opposite to $P_r(y)$ for all $y \in P_s(c_-)^{\times} = P_s(c_-) \setminus \{\text{proj}_{P_s(c_-)}^*(d_+)\}$, by Proposition 3.5 and Lemma 2.8 the map

$$g(\cdot, y) : P_r(c_+)^{\times} \rightarrow P_r(y) \setminus \{\text{proj}_{P_r(y)}^*(d_+)\}$$

is a homeomorphism for every $y \in P_s(c_-)^{\times}$.

The codistances between elements of $P_r(0_-)$ and of $P_r(c_+)$ lie in the set $\{s, rs, sr, rsr\}$. Since $rsr \neq s$, the panels $P_r(0_-)$ and $P_r(c_+)$ are not parallel and, thus, the chamber $x \bullet 0_- \in P_r(c_+)$ is independent of x . In fact, $x \bullet 0_-$

equals the unique element in $P_r(c_+)$ for which there exists a chamber in $P_r(0_-)$ at codistance rsr , i.e., $x \bullet 0_- = 0_+ \in P_r(c_+)^\times$ for all $x \in P_r(c_+)^\times$.

For $y \in P_s(c_-)^\times \setminus \{0_-\}$ the panels $P_r(y)$ and $P_r(c_+)$ are parallel, and so Lemma 2.8 implies

$$\begin{aligned} \text{proj}_{P_r(c_+)}^* \circ \text{proj}_{P_r(y)}^* \circ \text{proj}_{P_r(d_+)}^* \circ \text{proj}_{P_r(1_-)}^*(c_+) \\ = \text{proj}_{P_r(c_+)}^* \circ \text{proj}_{P_r(y)}^*(d_+) = c_+. \end{aligned}$$

We conclude that for $x \in P_r(c_+)^\times$ indeed $x \bullet y \in P_r(c_+)^\times$ and that $P_r(c_+)^\times \rightarrow P_r(c_+)^\times : x \mapsto x \bullet y$ is a homeomorphism.

We compute $g(x, 1_-) = \text{proj}_{P_r(1_-)}^* \circ \text{proj}_{P_r(d_+)}^* \circ \text{proj}_{P_r(1_-)}^*(x) = \text{proj}_{P_r(1_-)}^*(x)$, whence $x \bullet 1_- = \text{proj}_{P_r(c_+)}^* \circ \text{proj}_{P_r(1_-)}^*(x) = x$.

It remains to prove continuity of \bullet . Continuity of g follows immediately from (TTB2+) and Lemma 3.23. It thus remains to show that $\text{proj}_{P_r(c_+)}^*$ is continuous on $G := g(P_r(c_+)^\times \times P_s(c_-)^\times)$. To this end we claim that the following hold:

$$(\dagger) \quad \delta^*(c_+, G) = \{1, r, s\};$$

$$(\dagger\dagger) \quad e \in G \cap E_1^*(d_+) \Rightarrow \text{proj}_{P_r(c_+)}^*(e) = \text{proj}_{P_r(c_+)}^*(\text{proj}_{P_s(e)}^*(d_+)).$$

Let us first show that these claims imply the proposition: By (\dagger) the sets U_1, U_2 given by

$$U_1 := G \cap (E_1^*(c_+) \cup E_1^*(0_+)), \quad U_2 := G \cap E_1^*(d_+)$$

form an open covering of G . It is immediate from (TTB2+) that $\text{proj}_{P_r(c_+)}^*$ is continuous on U_1 and the map $e \mapsto \text{proj}_{P_r(c_+)}^*(\text{proj}_{P_s(e)}^*(d_+))$ is continuous on U_2 . Thus continuity of $\text{proj}_{P_r(c_+)}^*$ on all of G follows from $(\dagger\dagger)$, and we are left with verifying our claims. As far as (\dagger) is concerned, let $(x, y) \in P_r(c_+)^\times \times P_s(c_-)^\times$ and assume $y \neq 0_-$. Since $\delta^*(c_+, y) = 1$ and $\delta(y, g(x, y)) \in \langle r \rangle$ it then follows that $\delta^*(c_+, g(x, y)) \in \langle r \rangle$. On the other hand, if $y = 0_-$, then

$$\text{proj}_{P_r(c_+)}^*(g(x, y)) = x \bullet y = 0_+,$$

whence $g(x, y) \neq \text{proj}_{P_r(0_-)}^*(0_+)$ and $\delta^*(c_+, g(x, y)) = s$. To prove $(\dagger\dagger)$ we fix $e \in U_2$ and abbreviate $a := \text{proj}_{P_s(e)}^*(d_+)$. Since $\delta^*(e, c_+) \in \langle s \rangle$, we see that $b := \text{proj}_{P_r(c_+)}^*(e)$ is the unique element in $P_r(c_+)$ satisfying $\delta^*(b, e) \in r\langle s \rangle$. On the other hand we had assumed $\delta^*(d_+, e) = 1$, which implies $\delta^*(d_+, a) = s$. This in turn implies $\delta^*(c_+, a) = 1$, hence $\delta^*(\text{proj}_{P_r(c_+)}^*(a), a) = r$ and finally $\delta^*(\text{proj}_{P_r(c_+)}^*(a), e) \in r\langle s \rangle$, showing $\text{proj}_{P_r(c_+)}^*(a) = b$ and finishing the proof. \square

4 Classification methods

4.1 The Moufang condition

In this section we approach the classification of topological twin buildings. While abstract rank one buildings without further structure are not of any interest, various obstacles towards a complete classification become apparent in the context of rank two (twin) buildings of irreducible type.

Firstly, there is an apparent difference between twin trees and spherical rank two (twin) buildings: While the latter admit a unique twinning by [45, Proposition 1], the same pair of trees may instead admit many different twinings by [37, (1.1)].

In order to avoid problems arising from such non-uniqueness of twinings, we restrict our attention to buildings of two-spherical type (W, S) . This means that all rank two residues are assumed to be spherical or, equivalently, that there are no tree residues.

A second obstacle is that the automorphism group of a given compact projective plane (e.g., parametrized by a locally compact connected topological ternary field) may be small, so that a classification seems impossible. Thus, already in the classification of compact projective planes one has to assume some homogeneity condition; cf. [38].

In order to obtain a class of topological twin buildings amenable to classification we therefore further restrict ourselves to those that admit spherical rank two residues which satisfy the *Moufang condition*. We refer the reader to [46] for both an introduction to and a complete classification of such *Moufang polygons*.

Definition 4.1. A topological twin building is called a *Moufang topological twin building* if all its rank two residues are spherical and Moufang.

The hypotheses that all residues of rank two be spherical and Moufang allow one to apply the local-to-global machinery by Mühlherr and Ronan [32]. Note that the Moufang condition depends only on the underlying abstract twin building, hence guarantees the existence of many *abstract* automorphisms, but a priori not necessarily of any *continuous* automorphisms.

It turns out, however, that the topological automorphism group of a Moufang topological twin building is automatically highly transitive; this follows from the following argument which was pointed out to us by Linus Kramer.

Proposition 4.2. *Let Δ be a Moufang topological twin building. Then the topological automorphism group $\text{Aut}(\Delta)$ admits a subgroup with an RGD system that acts*

strongly transitively on Δ , i.e., transitively on the set of pairs of opposite chambers of Δ .

Proof. By [1, Proposition 8.19 and Theorem 8.81] there exists a group with centred RGD system which acts strongly transitively on the abstract twin building underlying Δ . We claim that this action is actually by homeomorphisms. In order to prove this it suffices to show that each of the root groups acts by homeomorphism. However, for each root group there exists a panel of each type on which it acts trivially. This yields the desired conclusion by Corollary 3.16. \square

4.2 Topological Moufang foundations

Under various conditions, Moufang twin buildings can be classified by local data, so-called *Moufang foundations*, see [30, 33, 36]. A topological analogue of foundations is provided by the following definition:

Definition 4.3. Let (W, S) be a two-spherical Coxeter system and denote by $E \subset \binom{S}{2}$ the set of edges in the Coxeter graph of (W, S) . A *topological Moufang foundation* of type (W, S) is a triple

$$\mathcal{F} = (\{\Delta_J \mid J \in E\}, \{c_J \mid J \in E\}, \{\theta_{jik} \mid \{i, j\}, \{j, k\} \in E\})$$

with the following properties:

- (TMF1) Each Δ_J is a topological Moufang polygon and $c_J \in \Delta_J$.
- (TMF2) $\theta_{ijk} : \text{Pan}_j(c_{i,j}) \rightarrow \text{Pan}_j(c_{j,k})$ is a base-point preserving homeomorphism and an isomorphism of Moufang sets.
- (TMF3) The θ_{ijk} satisfy the cocycle condition $\theta_{kjl} \circ \theta_{ijk} = \theta_{ijl}$.

Key examples of topological Moufang foundations arise from Moufang topological twin buildings:

Example 4.4. Let Δ be a Moufang topological twin building and $c \in \Delta_+ \cup \Delta_-$ a base chamber. We then obtain a topological Moufang foundation by setting $\Delta_J := \text{Res}_J(c)$, $c_J := c$ and θ_{jik} the appropriate (co-)restriction of the identity map of Δ .

The topological Moufang foundation from Example 4.4 will be denoted $\mathcal{F}(\Delta, c)$ and called the *collapse* of Δ along c . By Proposition 4.2, up to isomorphism the foundation $\mathcal{F}(\Delta, c)$ associated with a Moufang topological twin building only depends on Δ .

We thus denote its isomorphism class by $[\mathcal{F}(\Delta)]$.

Definition 4.5. A topological twin building is said to *globalize* a topological Moufang foundation \mathcal{F} if there exists $c \in \Delta_+ \cup \Delta_-$ with $\mathcal{F}(\Delta, c) \cong \mathcal{F}$; the foundation is then called *integrable*.

If one forgets about topologies in Definition 4.3 then one obtains the definition of an (abstract) Moufang foundation. Under our standing two-sphericity assumption an integrable Moufang foundation determines the ambient twin building uniquely, provided the Coxeter diagram of the underlying type (W, S) has no loops, see [33, p. 394] using [32, Theorem 1.3] and [35, Lemmas 5.1 and 5.2].

Theorem 3.15 allows one to promote this statement to the following topological version.

Theorem 4.6. *Let (W, S) be a (two-spherical) Coxeter system whose associated Coxeter diagram is a tree. Then every Moufang topological twin building of type (W, S) is uniquely determined by its topological foundation.*

We stress that we do not claim to be contributing to the solution of the problem that certain abstract Moufang foundations be integrable. We simply state that an *abstractly integrable* Moufang foundation endowed with a topology as in Definition 4.3 uniquely determines the topology on its (uniquely determined) twin building.

Proof. By the aforementioned local-to-global machinery any continuous isomorphism of foundations extends to an isomorphism of abstract twin buildings. This isomorphism is continuous, since its type is continuous (see Theorem 3.15). Applying the same argument to its inverse we see that it is a homeomorphism. \square

In the sequel we will say that a Moufang topological building is of *tree type* if it is of type (W, S) and the Coxeter diagram of (W, S) is a tree. We use a similar terminology concerning foundations. By means of Theorem 4.6 the classification of Moufang topological twin buildings of tree type is reduced to the following two problems:

- (1) Classify topological Moufang foundations of tree type.
- (2) Decide which of these foundations are integrable.

For illustration we will carry out this programme for the class of split Moufang topological twin buildings in the next section.

4.3 Split foundations and Dynkin diagrams

Throughout this section let k be a local field, i.e., a non-discrete locally compact σ -compact Hausdorff topological field. Recall that an archimedean local field

is isomorphic to either \mathbb{R} or \mathbb{C} , whereas the non-archimedean local fields of characteristic p , respectively 0, are fields of the form $\mathbb{F}_q((t))$ where \mathbb{F}_q is a finite field, respectively finite extensions of \mathbb{Q}_p for some prime p ([47, I, §3]).

Definition 4.7. A topological Moufang foundation $\mathcal{F} = (\{\Delta_J \mid J \in E\}, \{c_J \mid J \in E\}, \{\theta_{jik} \mid \{i, j\}, \{j, k\} \in E\})$ is called *k-split* if each rank one residue of each Δ_J (equipped with the induced topology) is isomorphic as a topological Moufang set to the projective line over k in its natural topology. A Moufang topological twin building is called *k-split* if some (hence any) of its foundations is *k-split*.

We remark that a foundation \mathcal{F} as above is *k-split* if and only if all the Δ_J are isomorphic as compact polygons to either

- (S-3) the compact projective plane over k ;
- (S-4) the compact generalized quadrangle associated with $\mathrm{Sp}_4(k)$, or its dual;
- (S-6) the compact generalized hexagon associated with the split algebraic group of exceptional type G_2 over k , or its dual.

If one replaces in Definition 4.7 the topological Moufang foundation by an abstract one, then one obtains the notion of an abstract *k-split* foundation. Concerning such foundation we have the following result of Mühlherr–Van Maldeghem:

Lemma 4.8 ([33, Proposition 2]). *Let*

$$\mathcal{F} = (\{\Delta_J \mid J \in E\}, \{c_J \mid J \in E\}, \{\theta_{jik} \mid \{i, j\}, \{j, k\} \in E\})$$

be an abstract k-split Moufang foundation of tree type. Then \mathcal{F} is uniquely determined by the list $\{\Delta_J \mid J \in E\}$.

As an immediate consequence of Theorem 4.6 the lemma carries over to the topological setting:

Corollary 4.9. *A k-split Moufang topological twin building Δ is uniquely determined by the collection $\{\Delta_J \mid J \in E\}$.*

In view of the lemma, we can visualize *k-split* foundations of tree type by *Dynkin trees*. Here, by a Dynkin tree we understand an edge-labelled tree, where edges are labelled 3, 4 or 6 and edges of label > 3 are given an orientation. Given a *k-split* Moufang foundation with associated list $\{\Delta_J \mid J \in E\}$ we define the associated Dynkin diagram as follows: The set of edges is the underlying index set S of the foundation, and two vertices i, j are joint by an edge if $J := (i, j) \in E$; the labeling of the edge J is 3, 4 or 6 according whether Δ_J is of type

(S-3), (S-4) or (S-6) respectively; in the latter two cases the edge is oriented towards the short root. Conversely, every Dynkin tree determines a unique topological k -split Moufang foundation. We call the Dynkin tree *topologically k -integrable* if this foundation is integrable.

We can summarize our discussion as follows:

Corollary 4.10. *Let k be a local field. Then there is a bijection between k -split Moufang topological twin buildings and topologically k -integrable Dynkin trees.*

We will see in Corollary 7.32 below that, in fact, every Dynkin tree is topologically k -integrable. For this we need to explicitly construct the corresponding Moufang topological twin building, which we will achieve in Section 7.7 using topological split Kac–Moody groups.

5 Connected topological twin buildings

5.1 A dichotomy: Connected vs. totally disconnected

In this section we will study (strong) topological twin buildings (Δ, τ) for which the halves $(\Delta_{\pm}, \tau|_{\Delta_{\pm}})$ are connected. By slight abuse of notation we call such a building (Δ, τ) a *connected (strong) topological twin building*. In a similar way we also define the notion of a *totally disconnected topological twin building*. We recall our standing assumption that there are no isolated vertices in the Coxeter diagram of the underlying type (W, S) . We say that a topological twin building is of *irreducible type* if the underlying Coxeter diagram is connected.

Our goal is to establish the following dichotomy:

Proposition 5.1. *Every two-spherical topological twin building of irreducible type is either connected or totally disconnected.*

Note that the irreducibility assumption is necessary, since the product of a connected and a totally disconnected topological twin building is neither connected nor totally disconnected.

Towards the proof of the proposition we first observe the following:

Lemma 5.2. *Let (Δ, τ) be a topological twin building.*

- (i) *If all panels are connected, then Δ is connected.*
- (ii) *If all panels are totally disconnected, then Δ is totally disconnected.*

Proof. If the panels are connected/totally disconnected, then the gallery spaces are connected/totally disconnected by Proposition 3.8. In the former case it follows immediately that the Schubert varieties are connected. In the latter case it follows that the set of non-stammering galleries of a given type is totally disconnected. Since this set is mapped homeomorphically onto the corresponding Schubert cell, it follows that each Schubert cell is totally disconnected.

For each $w \in W$ and $c \in \Delta_{\pm}$ one has

$$E_{\leq w}(c_0) = \bigcup_{c \in E_{\leq w}(c_0)} (E_w(c) \cap E_{\leq w}(c_0)).$$

The subsets $E_w(c) \cap E_{\leq w}(c_0)$ are open in $E_{\leq w}(c_0)$ and totally disconnected, hence the Schubert variety $E_{\leq w}(c_0)$ is totally disconnected itself.

The claim follows from the fact that direct limits of compact connected/totally disconnected spaces are connected/totally disconnected. \square

In view of Lemma 5.2, the proof of Proposition 5.1 follows from the corresponding statement concerning topological generalized polygons:

Lemma 5.3. *Let (Δ, τ) be an irreducible spherical topological twin building of rank two. Then either all panels are connected or all panels are totally disconnected.*

Proof. See [26, 2.2.3], [17, Proposition 6.13]. \square

We will provide many examples of connected, respectively totally disconnected Moufang topological twin buildings using split Kac–Moody groups over archimedean, respectively non-archimedean local fields in Theorem 1 on page 64.

For the rest of this section we will focus on the connected case.

5.2 Smooth topological twin buildings

We will be interested in the following subclass of connected topological twin buildings:

Definition 5.4. A connected topological twin building is called *smooth* if its panels are finite-dimensional real manifolds.

Note that, by definition, a topological twin building is smooth if and only if its rank two residues are smooth. In many situations, smoothness is automatic:

Proposition 5.5. *Every connected Moufang topological twin building is smooth.*

Proof. By Lemma 5.2 a topological twin building (Δ, τ) is connected if and only if its rank two residues are connected. A topological twin building (Δ, τ) is smooth, if its rank two residues are. Since a Moufang topological twin building is two-spherical by definition, each of its rank two residues is compact by Corollary 3.10. The claim therefore follows from the classification of flag-homogeneous connected compact polygons in [15, 16]. \square

Although we will not take advantage of this fact, let us briefly mention that the assumptions of the proposition can be substantially weakened. By [15, Theorem A] every flag-transitive compact connected polygon is Moufang. Therefore by the extension theorem from [32] one has:

Corollary 5.6. *Let (Δ, τ) be a connected two-spherical topological twin building. If the rank two residues have flag-transitive automorphism groups, then (Δ, τ) is smooth.*

We do not know any natural condition on twin trees which guarantee smoothness, hence we cannot extend Proposition 5.5 beyond the two-spherical case. An important observation concerning smooth connected strong topological twin buildings is that their panels are spheres:

Proposition 5.7 (cf. [15, Theorem 1.6], [25, Lemma 2.1], [26, Proposition 4.1.2]). *Let (W, S) be a Coxeter system without isolated points and let Δ be a smooth connected strong topological twin building of type (W, S) . Then each panel of Δ is a sphere.*

Proof. Since compact connected manifolds of positive dimension do not admit cutpoints, each punctured panel is connected. Let $P \subset \Delta_+$ be a panel of type r and let $c_+ \in P$, so that $P = P_r(c_+)$. By hypothesis there exists a type $s \in S$ such that $m_{rs} \geq 3$. For $c_- \in E_1^*(c_+)$ define $0_+ := \text{proj}_{P_r(c_+)}^*(c_-)$ and $0_- := \text{proj}_{P_s(c_-)}^*(c_+)$ and choose $1_- \in P_s(c_-)^\times \cap E_1^*(c_+)$. Proposition 3.22 provides a continuous map

$$\bullet : P_r(c_+)^\times \times P_s(c_-)^\times \rightarrow P_r(c_+)^\times.$$

Since $P_s(c_-)^\times$ is a connected manifold, there exists a continuous path $\gamma(t)$ from 1_- to 0_- . Then the map $H_t(x) := x \bullet \gamma(t)$ defines a pseudo-isotopic contraction of $P_r(c_+)^\times$ in the sense of [18, p. 186]. Therefore by [18, Theorem] there exists $n \in \mathbb{N}$ such that $P_r(c_+)^\times \cong \mathbb{R}^n$ and, thus, $P = P_r(c_+) \cong \mathbb{S}^n$. \square

Recall from Corollary 3.6 that panels of the same type are pairwise homeomorphic, so that the following definition is meaningful.

Definition 5.8. Let Δ be a smooth connected strong topological twin building. For each $s \in S$ the number $d(s) \in \mathbb{N}$ denotes the dimension of an s -panel.

Proposition 5.7 allows one to identify a CW structure on a smooth real twin building and its geometric realization.

Proposition 5.9 ([27, Proposition 7.9], cf. [26, Theorem 4.1.3]). *Let (Δ, τ) be a smooth connected strong topological twin building. Then the combinatorial Schubert decomposition of the halves of Δ_{\pm} with respect to base chambers c_{\pm} are CW decompositions. More precisely, there exists a CW structure on Δ_{\pm} with the following properties:*

- (i) *if $w \in W$ and $w = s_1 \cdots s_n$ is a reduced expression, then there exists a cell of dimension $d(w) := d(s_1) + \cdots + d(s_n)$ with attaching map*

$$\varphi_w : (D^{d(w)}, S^{d(w)-1}) \rightarrow (E_{\leq w}(c_{\pm}), E_{< w}(c_{\pm}));$$

the corresponding open cell is the Schubert cell $E_w(c_{\pm})$;

- (ii) *every cell is of the form φ_w for some $w \in W$.*

The proof of the proposition is based on the findings of Section 3.3 and the following observation.

Lemma 5.10 ([26, Lemma 6.2.12]). *Let $p : E \rightarrow B$ be an \mathbb{S}^d -fibre bundle over a CW complex B which admits a global section $s : B \rightarrow E$. Then there exists a unique CW structure on E with the following properties:*

- (i) *$s(B)$ is a subcomplex of E and $s : B \rightarrow s(B)$ is an isomorphism of CW complexes.*

- (ii) *Let B^{k-1} be the $(k-1)$ -skeleton of B and $\mu : (D^k, S^{k-1}) \rightarrow (B, B^{k-1})$ be a k -cell. Then there exists a unique $(k+d)$ -cell $\hat{\mu} : (D^{k+d}, \mathbb{S}^{k+d-1}) \rightarrow (E, E^{k+d-1})$ with*

$$\hat{\mu}(D^{k+d}) = p^{-1}(\mu(D^k)).$$

- (iii) *Every cell is either of type (i) or type (ii).*

Proof of Proposition 5.9. Applying Lemma 5.10 to the Bott–Samelson desingularization (see Proposition 3.8) yields a CW structure on each gallery space $\text{Gall}(s_1, \dots, s_k; c_{\pm})$ by induction on k , starting from the trivial CW structure of the point. Composing the attaching maps with the respective endpoint maps (Remark 3.9) one obtains a CW structure on Schubert varieties with centre c_{\pm} ; by (TTB3) these patch together to a CW structure on Δ_{\pm} . \square

Remark 5.11 (cf. [25, Theorem]). Proposition 5.9 yields severe restrictions on the possible values of the dimensions $d(s)$, because $d(w)$ has to be independent of the reduced expression for w . For instance, $d(\cdot)$ is constant on subsets of S

contained in a single W -conjugacy class, i.e., subsets of the connected components of the subgraph of the Dynkin diagram containing the simple edges only, as the dihedral group $\langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle$ admits the relation $sts = tst$.

Another application of Proposition 5.9 concerns the geometric realizations $|\Delta_{\pm}|$ of the two halves of the twin building, which are defined as follows: Let $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ be a topological twin building of type (W, S) and $n := |S| - 1$. Define the standard simplex and its faces as

$$\blacktriangle^n := \left\{ v \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} v_i = 1 \right\},$$

respectively,

$$\blacktriangle^n[j] := \left\{ v \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} v_i = 1, v_j = 0 \right\}, \quad (j = 1, \dots, n+1).$$

Then the geometric realizations $|\Delta_{\pm}|$ of Δ_{\pm} are given by the following construction: Enumerate $S = \{s_1, \dots, s_{n+1}\}$, equip $\Delta_{\pm} \times \blacktriangle^n$ with the product topology and identify the j th faces of chambers which are contained in the same s_j -panel. In the sequel we will denote by $q : \Delta_{\pm} \times \blacktriangle^n \rightarrow |\Delta_{\pm}|$ the canonical quotient maps.

If we equip \blacktriangle^n with the CW structure given by its faces, then the product CW structure on $\Delta_{\pm} \times \blacktriangle^n$ descend to the geometric realizations $|\Delta_{\pm}|$. Observe that the subsets

$$|E|_{\leq w}(c) := q(E_{\leq w}(c) \times \blacktriangle^n), |E|_{< w}(c) := q(E_{< w}(c) \times \blacktriangle^n)$$

are subcomplexes of $|\Delta_{\pm}|$ for every $w \in W$ and $c \in \Delta_{\pm}$. Since the inclusion of a CW subcomplex is a cofibration (see e.g. [39, Theorem 7.12]), we obtain:

Corollary 5.12 (cf. [29, Theorem 2.22(c)]). *For every $w \in W$ and $c \in \Delta_{\pm}$ the inclusion $|E|_{< w}(c) \hookrightarrow |E|_{\leq w}(c)$ is a cofibration.*

5.3 A topological Solomon–Tits theorem

The goal of this subsection is to establish the following topological variant of the Solomon–Tits theorem:

Theorem 5.13 ([27, Corollary 7.11]). *Let Δ be a smooth connected strong topological twin building of type (W, S) , whose Coxeter graph contains no isolated points. Then $|\Delta_{\pm}|$ is a homotopy sphere of dimension $d(w_0) + |S| - 1$ if W is finite and $w_0 \in W$ is the longest word, and contractible if W is infinite.*

We start with some preliminary observations and reductions. By [11, 13] direct limits in the category of k_ω spaces commute with finite products and with quotient maps. We deduce that for any fixed chamber $c \in \Delta_\pm$,

$$|\Delta_\pm| = \lim_{\rightarrow} |E|_{\leq w}(c).$$

If W is infinite, we even obtain

$$|\Delta_\pm| = \lim_{\rightarrow} |E|_{< w}(c). \quad (5.1)$$

The key step in the proof of Theorem 5.13 is to show that for every non-maximal $w \in W$ the quotient

$$B_w(c) := |E|_{\leq w}(c)/|E|_{< w}(c)$$

is contractible.

Let us assume this for the moment and explain how to deduce Theorem 5.13: We claim that our assumption implies that $|E|_{< w}(c)$ itself is always contractible, even if w is maximal. Indeed, for $l(w) \leq 1$ this is clear. Now let $l := l(w) > 1$ and let w_1, \dots, w_N be the maximal elements with respect to the Bruhat order subject to the condition $w_j < w$. By induction hypothesis, $|E|_{< w_j}(c)$ is contractible for every $j = 1, \dots, N$. Since $w_j < w$, the w_j are non-maximal, whence by our assumption also $|E|_{\leq w_j}(c)/|E|_{< w_j}(c)$ is contractible for every j . In view of Corollary 5.12 this implies that each of the sets $|E|_{\leq w_j}(c)$ is contractible. Since

$$|E|_{\leq w_j}(c) \cap |E|_{\leq w_l}(c) = |E|_{\leq \inf\{w_j, w_l\}}(c),$$

the same argument shows that finite intersections of the $|E|_{\leq w_j}(c)$ are contractible. Using Corollary 5.12 this implies

$$|E|_{< w}(c) = \bigcup_{j=1}^n |E|_{\leq w_j}(c) \simeq \bigvee_{j=1}^n |E|_{\leq w_j}(c) \simeq \{*\}$$

and, thus, establishes our claim.

In the infinite case we can combine our claim and (5.1) to deduce

$$|\Delta_\pm| \simeq \{*\};$$

if W is finite with longest word w_0 then another application of Corollary 5.12 yields

$$|\Delta_\pm| \cong |E|_{\leq w_0}(c)/|E|_{< w_0}(c) = B_{w_0}(c).$$

We have thus reduced the proof of Theorem 5.13 to the following lemma:

Lemma 5.14 ([27, Proposition 7.10], cf. [25, 2.10–2.15], [29, Theorem 2.16]). *Let $w \in W$. Then $B_w(c) \simeq S^{d(w)+|S|-1}$, if w is maximal; otherwise, $B_w(c)$ is contractible.*

Proof. We fix an enumeration $S = \{s_1, \dots, s_N\}$ and for $1 \leq i \leq N$ denote by

$$\blacktriangle^{N-1}[i] := \left\{ (t_1, \dots, t_N) \in \mathbb{R}^n \mid \sum_{j=0}^N t_j = 1, t_i = 0 \right\}$$

the i th face of the standard $(N-1)$ -simplex. We then denote by $q : \Delta_{\pm} \times \blacktriangle^{N-1} \rightarrow |\Delta_{\pm}|$ the quotient map given by identifying the i -th faces of s_i -adjacent chambers. Furthermore we denote by $\pi : |E|_{\leq w}(c) \rightarrow B_w(c)$ the canonical projection and set

$$p := \pi \circ q|_{E_{\leq w}(c) \times \blacktriangle^{N-1}} : E_{\leq w}(c) \times \blacktriangle^{N-1} \rightarrow B_w(c).$$

We observe that p maps all points in $E_{< w}(c) \times \blacktriangle^{N-1}$ to the basepoint $* := p(c)$ of $B_w(c)$. In particular p factors through a map

$$p_0 : E_{\leq w}(c)/E_{< w}(c) \times \blacktriangle^{N-1} \rightarrow B_w(c).$$

Since $E_{\leq w}(c) \times \blacktriangle^{N-1}$ is compact, the maps p and, consequently, p_0 are quotient maps.

CLAIM 1: If $l(ws_i) < l(w)$ then $p(d, (t_j)) = *$ for all $d \in E_w(c)$ and $(t_j) = (t_1, \dots, t_N) \in \blacktriangle^N[i]$ with $t_i = 0$.

Indeed, if $l(ws_i) < l(w)$ then there exists a reduced expression $w = r_1 \cdots r_M$ with $r_j \in S$, $r_M = s_i$. Let $(c = x_0, x_1, \dots, x_{M-1}, x_M = d)$ be a gallery of type (r_1, \dots, r_M) . Then $x_{M-1} \in E_{ws_i}(c) \subset E_{< w}(c)$. Since x_{M-1} and d share their i th face in $|E|_{\leq w}(c)$, the claim follows.

CLAIM 2: If $d \sim_i e$ for some $d \in E_w(c)$ and $e \in E_{\leq w}(c)$, then $l(ws_i) < l(w)$.

For $e \in E_{< w}(c)$ this is clear. Now assume $d, e \in E_w(c)$ and $l(ws_i) = l(w) + 1$; take a reduced expression (r_1, \dots, r_M) for w and let $(c = c_0, \dots, c_M = d)$ be a gallery of this type. Then $(c, \dots, c_{M-1}, d, e)$ is of reduced type, whence $l(\delta(c, e)) < l(w)$, contradicting the choice of e .

Now let $I^- := \{i \in \{1, \dots, N\} \mid l(ws_i) < l(w)\}$ and

$$\blacktriangle^{N-1}[I^-] := \bigcup_{i \in I^-} \blacktriangle^{N-1}[i].$$

By Claim 2 the map p_0 maps the set

$$[c] \times \blacktriangle^{n-1} \cup E_{\leq w}/E_{< w}(c) \times \blacktriangle^{N-1}[I^-]$$

to $*$ and is one-to-one on the complement of this set. By Proposition 5.9 we also have

$$E_{\leq w}(c)/E_{< w}(c) \cong S^{d(w)}.$$

Since p_0 is a covering map, we obtain

$$\begin{aligned} B_w(c) &\cong p_0(E_{\leq w}(c)/E_{< w}(c) \times \blacktriangle^{N-1}) \\ &\cong \frac{E_{\leq w}(c)/E_{< w}(c) \times \blacktriangle^{N-1}}{[c] \times \blacktriangle^{n-1} \cup E_{\leq w}/E_{< w}(c) \times \blacktriangle^{N-1}[I^-]} \\ &\cong (S^{d(w)} \times \blacktriangle^{N-1})/(\{p\} \times \blacktriangle^{N-1} \cup S^{d(w)} \times \blacktriangle^{N-1}[I^-]). \end{aligned}$$

If w is maximal, then $I^- = \{1, \dots, N\}$, whence

$$B_w(c) \cong (S^{d(w)} \times \blacktriangle^{N-1})/(\{p\} \times \blacktriangle^{N-1} \cup S^{d(w)} \times \partial\blacktriangle^{N-1}) \cong S^{d(w)+N-1}.$$

Otherwise we can find $i \in \{1, \dots, N\} \setminus I^-$; then we obtain a contracting homotopy

$$H_t : (S^{d(w)} \times \blacktriangle^{N-1})/(\{p\} \times \blacktriangle^{N-1} \cup S^{d(w)} \times \blacktriangle^{N-1}[I^-]) \circlearrowleft$$

by the formula

$$H_t([x, (t_1, \dots, t_N)]) := [x, ((1-t)t_1, \dots, t_i + t(1-t_i), \dots, (1-t)t_N)]. \quad \square$$

6 Topological RGD systems and topological twin buildings

We now turn to the problem of actually constructing (strong) topological twin buildings. We start by studying topological RGD systems.

Throughout this section let G be a topological group with associated RGD system $(\{U_\alpha\}_{\alpha \in \Phi}, T)$ and denote by $\Delta = \Delta(G, \{U_\alpha\}_{\alpha \in \Phi}, T)$ the associated twin building. We equip both halves Δ_\pm of Δ with the quotient topology induced by G . The goal of this section is to give conditions on the topology of G which guarantee that Δ is a (strong) topological twin building.

6.1 Orbit closure relations

We will first be concerned with the question concerning the openness of the big cells B_+B_- and B_-B_+ . The quotient map $G \rightarrow G/B_\pm$ allows one to relate the big cell $B_\mp B_\pm$ in the building to the big cell $B_\mp B_\pm$ in the group: $B_\mp B_\pm$ is open considered as a subset of G if and only if $B_\mp B_\pm$ is open considered as a subset of Δ_\pm . We conclude that for questions concerning the openness (and closedness) of unions of $B_\mp B_\pm$ -double cosets it is in fact irrelevant whether one uses the group or the building topology. As our first important reduction step we will establish the following result:

Lemma 6.1. *Let G , Δ , B_{\pm} as above. If Δ satisfies axioms (TTB1), (TTB2) and (TTB3) and if the panels of Δ are non-discrete, then B_+B_- and B_-B_+ are open in G .*

We will actually provide a more precise result: We will compute the closures of arbitrary B_-B_+ double cosets, culminating in Theorem 6.4 below, that contains Lemma 6.1 as a special case.

We begin with the following observation:

Proposition 6.2. *Let $c_{\mp} \in \Delta_{\mp}$, $d \in \Delta_{\pm}$, let $\delta^*(c_{\mp}, d) = w$, let B_{\mp} be the Borel subgroup associated to c_{\mp} , let B_{\pm} be a Borel subgroup opposite B_{\mp} , and let $v \in W$ such that $w \not\leq v$ in the Bruhat order. Then there exists an open neighbourhood of d in Δ_{\pm} disjoint from $B_{\mp}vB_{\pm}$.*

Proof. Let Σ be a twin apartment containing d and c_{\mp} and let $\tilde{w}^{-1} \in \text{Stab}_G(\Sigma)$ be a representative of w^{-1} that maps d to the chamber in Σ opposite c_{\mp} . Any chamber $x \in B_{\mp}vB_{\pm}$ of Δ_{\pm} satisfies $\delta^*(c_{\mp}, x) = v$, so $\delta^*(\tilde{w}^{-1}.c_{\mp}, \tilde{w}^{-1}.x) = v$. As $\delta_{\mp}(c_{\mp}, \tilde{w}^{-1}.c_{\mp}) = w^{-1}$, Lemma 2.10 allows us to conclude

$$\delta^*(c_{\mp}, \tilde{w}^{-1}.x) \in \{w_1v \mid w_1 \text{ is a subexpression of } w^{-1}\}.$$

Hence, for $X := \tilde{w}^{-1}B_{\mp}vB_{\pm}$, the hypothesis $w \not\leq v$ yields $1_W \notin \delta^*(c_{\mp}, X)$. Therefore, for each $a \in \delta^*(c_{\mp}, X)$, Lemma 3.2 provides an open neighbourhood U_a of $\tilde{w}^{-1}.d$ which intersects $E_a^*(c_{\mp})$ trivially. As $\delta^*(c_{\mp}, X)$ is finite, the set

$$U := \bigcap_{a \in \delta^*(c_{\mp}, X)} U_a$$

is an open neighbourhood of $\tilde{w}^{-1}.d$. Since $X \subseteq \bigcup_{a \in \delta^*(c_{\mp}, X)} E_a^*(c_{\mp})$, we moreover have $X \cap U = \emptyset$. Hence $\tilde{w}.U$ is an open neighbourhood of d satisfying $B_{\mp}vB_{\pm} \cap \tilde{w}.U = \emptyset$, as claimed. \square

Lemma 6.3. *Let $w \in W$, let $s \in S$, and assume that the panels of Δ are non-discrete.*

(i) *If $l(ws) > l(w)$, then the following inclusions hold:*

- (a) $\overline{B_-wB_+} \supseteq B_-wsB_+$,
- (b) $\overline{B_+wB_-} \supseteq B_+wsB_-$.

(ii) *If $l(sw) > l(w)$, then the following inclusions hold:*

- (a) $\overline{B_-wB_+} \supseteq B_-swB_+$,
- (b) $\overline{B_+wB_-} \supseteq B_+swB_-$.

Proof. (i) Let $\varepsilon \in \{+, -\}$. By Lemma 2.18 the orbits of $B_{-\varepsilon}$ on Δ_ε are given by the $B_{-\varepsilon}$ - B_ε -double cosets $B_{-\varepsilon}wB_\varepsilon$, $w \in W$. Let $c_{-\varepsilon}$ be the fundamental chamber in $\Delta_{-\varepsilon}$ and let $c \in \Delta_\varepsilon$ such that $\delta^*(c_{-\varepsilon}, c) = w$, i.e., c is a representative of the $B_{-\varepsilon}$ -orbit $B_{-\varepsilon}wB_\varepsilon$ in Δ_ε . Let $s \in S$ such that $l(ws) > l(w)$ and consider the s -panel $P_s(c)$ around c . The projection $d := \text{proj}_{P_s(c)}(c_{-\varepsilon})$ is the unique chamber of $P_s(c)$ satisfying $\delta^*(c_{-\varepsilon}, d) = ws$ and the group $\text{Stab}_{B_{-\varepsilon}}(P_s(c))$ acts transitively on the set $P_s(c) \setminus \{d\}$ of chambers distinct from d . Since $P_s(c)$ is non-discrete, it follows that d is contained in $\overline{P_s(c) \setminus \{d\}}$ and, thus, in $\overline{B_{-\varepsilon}wB_\varepsilon}$.

We conclude that, for each $s \in S$ such that $l(ws) > l(w)$, the closure of B_-wB_+ intersects the orbit $B_{-\varepsilon}wsB_\varepsilon$. Since this closure is a union of orbits, one has for all $s \in S$ with $l(ws) > l(w)$

$$\overline{B_-wB_+} \supseteq B_{-\varepsilon}wsB_\varepsilon.$$

(ii) As inversion in G is a homeomorphism, one has $\overline{B_{-\varepsilon}wB_\varepsilon} \supseteq B_{-\varepsilon}swB_\varepsilon$ if and only if $\overline{B_\varepsilon w^{-1}B_{-\varepsilon}} \supseteq B_\varepsilon w^{-1}sB_{-\varepsilon}$. Hence the inequality $l(w^{-1}s) = l(w^{-1}s^{-1}) = l(sw) > l(w) = l(w^{-1})$ allows one to immediately conclude (ii) from (i). \square

Now we can establish the following theorem, which contains Lemma 6.1 as a special case:

Theorem 6.4. *Let G be a topological group with RGD system $(\{U_\alpha\}_{\alpha \in \Phi}, T)$, let $\Delta = \Delta(G, \{U_\alpha\}_{\alpha \in \Phi}, T)$ be the associated twin building, equip both halves Δ_\pm of Δ with the quotient topology, and assume that this twin building topology satisfies axioms (TTB1), (TTB2) and (TTB3) and that panels of Δ are non-discrete. Let W be its Weyl group, let \leq be the Bruhat order of W , and let $w \in W$. Then the following hold:*

(i)

$$\overline{B_-wB_+} = \bigsqcup_{w' \geq w} B_-w'B_+.$$

(ii) *The smallest open union of B_- - B_+ -double cosets containing B_-wB_+ is*

$$\bigsqcup_{w' \leq w} B_-w'B_+,$$

which consists of finitely many B_- - B_+ -double cosets.

This theorem is essentially [24, Lemma 3.4]. A special case is [40, Theorem 23 (p. 127)].

Proof. (i) An induction using Lemma 6.3 shows that

$$\overline{B_-wB_+} \supseteq \bigsqcup_{w' \geq w} B_-w'B_+.$$

Conversely, let x be an element of the complement

$$X := \bigsqcup_{w' \not\geq w} B_-w'B_+$$

of $\bigsqcup_{w' \geq w} B_-w'B_+$. We will show that x lies in the interior of X . Let $n \in \mathbb{N}$ be sufficiently large such that $x \in \bigcup_{l(v) \leq n} B_+vB_+$. The intersection $\left(\bigsqcup_{w' \geq w} B_-w'B_+\right) \cap \left(\bigcup_{l(v) \leq n} B_+vB_+\right)$ meets finitely many B_-B_+ -double cosets. Let $A \subset W$ be a finite set such that these double cosets are given by the family $\{B_-aB_+\}_{a \in A}$. For every $a \in A$, Proposition 6.2 provides an open neighbourhood U_a of x in G disjoint from B_-aB_+ .

Then $\bigcap_{a \in A} U_a \cap \left(\bigcup_{l(v) \leq n} B_+vB_+\right)$ is open in $\bigcup_{l(v) \leq n} B_+vB_+$, contains x and intersects $\bigsqcup_{w' \geq w} B_-w'B_+$ trivially. Thus, this intersection is an open neighbourhood of x in $X \cap \left(\bigcup_{l(v) \leq n} B_+vB_+\right)$, and hence x is an interior point of $X \cap \left(\bigcup_{l(v) \leq n} B_+vB_+\right)$. As x was arbitrary, we conclude that $X \cap \left(\bigcup_{l(v) \leq n} B_+vB_+\right)$ is open in $\bigcup_{l(v) \leq n} B_+vB_+$ for each $n \in \mathbb{N}$. By axiom (TTB3) $\Delta_+ = \lim_{\rightarrow} \bigcup_{l(v) \leq n} B_+vB_+$, and so X is open in Δ_+ and, thus, in G .

(ii) Define the finite set

$$X_w := \{v \in W \mid v \not\leq w, \exists s \in S \text{ such that } sv \leq w \text{ or } vs \leq w\}.$$

Then, for any $w' \in W$, one has $w' \not\leq w$ if and only if there exists $v \in X_w$ such that $v \leq w'$. Hence, by (i),

$$\bigsqcup_{w' \leq w} B_-w'B_+ = G \setminus \bigcup_{x \in X_w} \overline{B_-xB_+}.$$

Since $\bigcup_{x \in X_w} \overline{B_-xB_+}$ is a finite union of closed sets, it is closed, and so its complement, $\bigsqcup_{w' \leq w} B_-w'B_+$, is open.

Moreover, if U is an arbitrary open union of B_-B_+ -double cosets containing B_-wB_+ , then by (i) for each $w' \leq w$ one necessarily has $B_-w'B_+ \subseteq U$. \square

Remark 6.5. Theorem 6.4 states that there exists a closed B_-B_+ -double coset if and only if there exists a maximal element of W with respect to the Bruhat order. This is the case if and only if W is spherical.

6.2 A group-theoretic criterion for twin building topologies

As before, let G be a topological group with RGD system $(\{U_\alpha\}_{\alpha \in \Phi}, T)$ and denote by Δ the associated twin building. Our explicit projection formulae (Theorem 2.22) immediately yield the following proposition:

Proposition 6.6. *If the bijective product map $m : U_+ \times T \times U_- \rightarrow B_+B_-$ is open, then Δ satisfies (TTB2). If, moreover, B_+B_- is open in G , then Δ also satisfies (TTB2+) and (TTB5).*

Proof. The set of pairs of chambers in $G/B_+ \times G/B_-$ at codistance 1 $\in W$ is equal to $\{(gB_+, hB_-) \mid g^{-1}h \in B_+B_-\}$. Given $s \in S$, Theorem 2.22 implies $\text{proj}_{P_s}^*(hB_-)(gB_+) = h\rho_w(g^{-1}h)^{-1}sB_-$. By Lemma 2.24 therefore Δ satisfies (TTB2).

If B_+B_- is open in G , then by continuity of multiplication and inversion the set $\{(g, h) \in G \times G \mid g^{-1}h \in B_+B_-\}$ is open in $G \times G$. As the canonical quotient map $G \times G \rightarrow (G \times G)/(B_+ \times B_-)$ is open, the set $\{(gB_+, hB_-) \mid g^{-1}h \in B_+B_-\}$ of opposite pairs of chambers is open in $\Delta_+ \times \Delta_- = G/B_+ \times G/B_-$, i.e. (TTB5) holds.

In this case the set $B_-B_+/B_+ \times B_+B_-/B_- \subset \Delta_+ \times \Delta_-$ is open, whence also $O := (B_-B_+/B_+ \times B_+B_-/B_-) \cap \Delta_1$ is open. Note that for $(c, d) \in O$ the chamber c is opposite the chamber $c_- := B_- \in G/B_-$, as B_-B_+ is exactly the set of chambers of Δ_+ opposite c_- . Therefore, by sharp transitivity of U_- on the chambers opposite c_- , there exists a unique $g_{(c,d)} \in U_-$ with $g_{(c,d)}(c) = c_+ := B_+ \in G/B_+$. The resulting map $\nu_O : O \rightarrow U_- : (c, d) \mapsto g_{(c,d)}$ is continuous, since the map $U_- \times T \times U_+ \rightarrow B_-B_+ : (u_-, t, u_+) \mapsto u_-tu_+$ is open by hypothesis.

Considering the restriction $p_{s|O} : O \rightarrow \Delta_+ \cup \Delta_- : (c, d) \mapsto \text{proj}_{P_s}^*(c)(d)$ of the projection map from axiom (TTB2+) we compute

$$\begin{aligned} \text{proj}_{P_s}^*(c)(d) &= g_{(c,d)}^{-1} \left(\text{proj}_{P_s}^*(g_{(c,d)}(c))(g_{(c,d)}(d)) \right) \\ &= \nu_O(c, d)^{-1} \left(\text{proj}_{P_s}^*(c_+)(\nu_O(c, d)(d)) \right). \end{aligned}$$

Therefore (TTB2) and continuity of ν_O imply continuity of $p_{s|O}$. Since, by the strong transitivity of the action of G on Δ , the set Δ_1 is covered by G -translates of the open set O , the validity of (TTB2+) follows. \square

Combining the proposition and Lemma 6.1 we obtain:

Theorem 6.7. *Let G be a topological group with root group datum $(\{U_\alpha\}_{\alpha \in \Phi}, T)$, let $\Delta = \Delta(G, \{U_\alpha\}_{\alpha \in \Phi}, T)$ be the associated twin building, and equip both halves with the quotient topology. Assume the following:*

- (i) B_{\pm} are closed.
- (ii) $G = \lim_{\rightarrow} \left(\bigcup_{l(w) \leq n} B_+ w B_+ \right) = \lim_{\rightarrow} \left(\bigcup_{l(w) \leq n} B_- w B_- \right)$.
- (iii) The multiplication map $m: U_+ \times T \times U_- \rightarrow B_+ B_-$ is open.
- (iv) Panels in Δ are compact.

Then Δ is a strong topological twin building, and the parabolic subgroups

$$P_{\pm}^J := B_{\pm} W_J B \quad (J \subset S)$$

are closed in G .

Proof. We first consider the axioms which follow immediately from our assumptions: (i) implies (TTB1), (ii) implies (TTB3) and (iv) implies (TTB4). Using (iii) and the first part of Proposition 6.6 we deduce that also (TTB2) holds. By Lemma 6.1 this implies that $B_+ B_-$ and $B_- B_+$ are open. Thus the second part of Proposition 6.6 applies and shows that also (TTB2+) and (TTB5) hold. (TTB6) follows from the general fact that given subgroups $H_1 < H_2$ of a topological group the canonical map $\pi: G/H_1 \rightarrow G/H_2$ is open, since for an open set $UH_1 \subset G/H_1$ the saturation $\pi^{-1}(\pi(UH_1)) = UH_2 = \bigcup UH_1 h_2$ is open as the union of (open) translates. Finally, residues in Δ are closed by Lemma 3.4, whence their stabilizers, i.e., parabolic subgroups are closed, which in turn implies (TTB1+). \square

Remark 6.8. For an \mathbb{F} -locally split root group datum there is an easy condition which guarantees property (iv) above. Indeed, the panels of Δ are homeomorphic to $\mathbb{P}_1(\mathbb{F})$, the building of $\mathrm{SL}_2(\mathbb{F})$. If \mathbb{F} is Hausdorff, non-discrete, σ -compact and locally compact, these are compact (cf. [38, Proposition 14.5 and Corollary 14.7]).

7 Topological split Kac–Moody groups

In this section we return to the classification problem for k -split topological twin buildings over a local field k , which we studied in Section 4.3. Our goal is to show that every Dynkin tree is topologically k -integrable.

Our starting point is the observation that every Dynkin tree is abstractly k -integrable; in fact the corresponding twin building is the twin building of the associated split Kac–Moody group. In view of this observation, our task is to topologize the twin buildings of split Kac–Moody groups over local fields. We will in fact topologize the Kac–Moody groups themselves and then apply the criterion developed in Theorem 6.7 to obtain topological twin buildings.

Along the way we develop a topological theory of Kac–Moody groups which is of independent interest. In particular, we will revisit various results and techniques developed and announced in [24].

7.1 Abstract split Kac–Moody groups

For the convenience of the reader let us start with the definition of a Kac–Moody root datum.

Definition 7.1. A *generalized Cartan matrix* is a matrix $A = (a_{ij})_{1 \leq i, j \leq n} \in \mathbb{Z}^{n \times n}$ satisfying $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Let $I = \{1, \dots, n\}$ and let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a generalized Cartan matrix. A quintuple $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i \in I}, \{h_i\}_{i \in I})$ is called a *Kac–Moody root datum* if Λ is a free \mathbb{Z} -module, each c_i is an element of Λ and each h_i is in the \mathbb{Z} -dual Λ^\vee of Λ such that for all $i, j \in I$ one has $h_i(c_j) = a_{ij}$.

Following [44, 3.6] to a Kac–Moody root datum \mathcal{D} one associates a triple $\mathcal{F} = (\mathcal{G}, \{\varphi_i\}_{i \in I}, \eta)$, where \mathcal{G} is a group functor on the category of commutative unital rings, the φ_i are maps $\mathrm{SL}_2(R) \rightarrow \mathcal{G}(R)$, and η is a natural transformation $\mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda], -) \rightarrow \mathcal{G}$ such that the following assertions hold:

- (KMG1) If \mathbb{F} is a field, then the group $\mathcal{G}(\mathbb{F})$ is generated by the images of the φ_i and $\eta(\mathbb{F})$.
- (KMG2) For all rings R the homomorphism $\eta(R) : \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda], R) \rightarrow \mathcal{G}(R)$ is injective.
- (KMG3) Given a ring R , $i \in I$ and $u \in R^\times$, one has

$$\varphi_i \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \eta \left(\lambda \mapsto u^{h_i(\lambda)} \right).$$

- (KMG4) If R is a ring, \mathbb{F} is a field and $\iota : R \rightarrow \mathbb{F}$ is a monomorphism, then $\mathcal{G}(\iota) : \mathcal{G}(R) \rightarrow \mathcal{G}(\mathbb{F})$ is a monomorphism as well.
- (KMG5) If \mathfrak{g} is the complex Kac–Moody algebra of type A , then there exists a homomorphism $\mathrm{Ad} : \mathcal{G}(\mathbb{C}) \rightarrow \mathrm{Aut}(\mathfrak{g})$ such that

$$\ker(\mathrm{Ad}) \subseteq \eta(\mathbb{C})(\mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda], \mathbb{C}))$$

and for a given $z \in \mathbb{C}$ one has

$$\begin{aligned} \mathrm{Ad} \left(\varphi_i \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \right) &= \exp(\mathrm{ad}_{ze_i}), \\ \mathrm{Ad} \left(\varphi_i \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \right) &= \exp(\mathrm{ad}_{-zf_i}); \end{aligned}$$

where $\{e_i, f_i\}$ are part of a standard \mathfrak{sl}_2 -triple for the fundamental Kac–Moody sub-Lie algebra corresponding to the simple root α_i ; furthermore, for every homomorphism $\gamma \in \text{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda], \mathbb{C})$ one has

$$\text{Ad}(\eta(\mathbb{C})(\gamma))(e_i) = \gamma(c_i) \cdot e_i, \quad \text{Ad}(\eta(\mathbb{C})(\gamma))(f_i) = \gamma(-c_i) \cdot f_i.$$

The Kac–Moody root datum \mathcal{D} is called *centred* if the following stronger version of (KMG1) is satisfied: If \mathbb{F} is a field, then the group $\mathcal{G}(\mathbb{F})$ is generated by the images of the φ_i .

For a given Kac–Moody root datum \mathcal{D} the group $G_{\mathcal{D}}(R) := \mathcal{G}(R)$ is called a *split Kac–Moody group* of type \mathcal{D} over R .

The main result of [44] states that under some non-degeneracy assumptions any functor defined on the category of fields satisfying the above axioms must coincide with \mathcal{G} .

A split Kac–Moody group over a field is an example of a group with an RGD system by the following result.

Proposition 7.2 ([34, Proposition 8.4.1], [6, Lemma 1.4]). *Let \mathbb{F} be a field, let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i \in I}, \{h_i\}_{i \in I})$ be a Kac–Moody root datum, and let $G_{\mathcal{D}}(\mathbb{F}) := \mathcal{G}(\mathbb{F})$ be the corresponding split Kac–Moody group of type \mathcal{D} over \mathbb{F} . Then $G_{\mathcal{D}}(\mathbb{F})$ admits an RGD system as follows. Let $M(A)$ be the associated Coxeter matrix of type (W, S) and choose a set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$ such that the reflection associated to α_i is $s_i \in S$. Define the set of real roots as $\Phi^{re} := W\Pi$. Given $i \in I$, let U_{α_i} and $U_{-\alpha_i}$ be the respective images of the subgroups of strictly upper, resp. strictly lower triangular matrices of the matrix group $\text{SL}_2(\mathbb{F})$ under the map φ_i , and denote by T the image of $\eta(\mathbb{F})$ in $G_{\mathcal{D}}(\mathbb{F})$.*

Then $T = \bigcap_{\alpha \in \Phi^{re}} N_{G_{\mathcal{D}}(\mathbb{F})}(U_{\alpha})$, $W \cong N_{G_{\mathcal{D}}(\mathbb{F})}(T)/T$ and $(G_{\mathcal{D}}(\mathbb{F}), \{U_{\alpha}\}_{\alpha \in \Phi^{re}}, T)$ is an RGD system. \square

In the sequel we refer to T as the *standard maximal torus* of $G_{\mathcal{D}}(\mathbb{F})$. Note that, since the matrix group SL_2 over a field is generated by its subgroups of strictly upper and strictly lower triangular matrices, the concept of being centred coincides with the one introduced in Section 2.2.

Remark 7.3. In general the action of a split Kac–Moody group $G_{\mathcal{D}}(\mathbb{F})$ on the associated twin building will not be effective; however the kernel $\mathcal{Z}_{\mathcal{D}}(\mathbb{F})$ of this action always equals the centre of $G_{\mathcal{D}}(\mathbb{F})$, which in turn is contained in T (cf. [34, Proposition 9.6.2]).

From now on we will reserve the letter G to denote a split Kac–Moody group $G_{\mathcal{D}}(\mathbb{F})$ over a field \mathbb{F} . Note that Propositions 2.14 and 7.2 in particular imply

that G admits a twin BN-pair. Using Proposition 2.14 we introduce the following additional notation.

For every $k \geq 0$ we set

$$G_k^\pm := \bigcup_{l(w) \leq k} B_\pm w B_\pm$$

and $G_k := G_k^+ \cap G_k^-$. For every k -tuple $\bar{\alpha} = (\alpha_1, \dots, \alpha_k) \in \Pi^k$ of simple roots we will denote by

$$G_{\bar{\alpha}} := G_{\alpha_1} \cdots G_{\alpha_k} \subseteq G$$

the subset of G consisting of products of the form

$$g = g_1 \cdots g_k, \quad g_j \in G_{\alpha_j}.$$

Note that as a special case we have $G_\alpha = G_{(\alpha)}$.

Similarly, we denote by $TG_{\bar{\alpha}}$ the image of $T \times G_{\bar{\alpha}}$ under the product map.

Note that the set of indices carries a natural partial order: We write $\bar{\alpha} \leq \bar{\beta}$ provided $\bar{\alpha}$ appears as an ordered subtuple of $\bar{\beta}$; in this case there is an obvious embedding $G_{\bar{\alpha}} \hookrightarrow G_{\bar{\beta}}$. We record the following inclusion relations for later use:

Proposition 7.4. *Let B_+, B_- be the standard Borel subgroups of the adjoint form of $G_{\mathcal{D}}(\mathbb{F})$, let $k \in \mathbb{N}$, and let $\bar{\alpha} = (\alpha_1, \dots, \alpha_k) \in \Pi^k$. Then $TG_{\bar{\alpha}} \subseteq G_k^+ \cap G_k^-$.*

Proof. We prove the result by induction on $|\bar{\alpha}| = k$. For $k = 0$ the claim follows from the inclusion $T \subseteq B_+ \cap B_-$, cf. Proposition 2.14. Let $k > 0$ and assume that for all $\bar{\beta}$ with $|\bar{\beta}| < |\bar{\alpha}|$ the set $TG_{\bar{\beta}}$ is contained in $G_{|\bar{\beta}|}^\pm$. Hence, for $\bar{\alpha}_0 = (\alpha_1, \dots, \alpha_{k-1})$, the induction hypothesis yields $TG_{\bar{\alpha}_0} \subseteq G_{|\bar{\alpha}_0|}^\pm$. For $\varepsilon \in \{\pm\}$ we have Bruhat decompositions $\mathrm{SL}_2(\mathbb{F}) \cong G_{\alpha_k} = B_{\alpha_k}^\varepsilon \cup B_{\alpha_k}^\varepsilon s_{\alpha_k} B_{\alpha_k}^\varepsilon$, where $B_{\alpha_k}^\varepsilon := B_\varepsilon \cap G_{\alpha_k}$. Hence

$$\begin{aligned} TG_{\bar{\alpha}} &= TG_{\bar{\alpha}_0} \cdot G_{\alpha_k} \subseteq \left(\bigcup_{l(w) \leq k-1} B_\varepsilon w B_\varepsilon \right) \cdot (B_{\alpha_k}^\varepsilon \cup B_{\alpha_k}^\varepsilon s_{\alpha_k} B_{\alpha_k}^\varepsilon) \\ &\subseteq \left(\bigcup_{l(w) \leq k-1} B_\varepsilon w B_\varepsilon \right) \cdot (B_\varepsilon \cup B_\varepsilon s_{\alpha_k} B_\varepsilon) \\ &\stackrel{\text{(TBN1)}}{\subseteq} \bigcup_{l(w) \leq k-1} B_\varepsilon w B_\varepsilon \cup B_\varepsilon w s_{\alpha_k} B_\varepsilon \\ &\subseteq \bigcup_{l(w) \leq k} B_\varepsilon w B_\varepsilon. \end{aligned}$$

The claim follows. \square

7.2 The centre and the adjoint representation

While there is a unique twin building associated with a given generalized Cartan matrix A over a given field \mathbb{F} , there exist several corresponding root group data and hence several corresponding Kac–Moody groups, like in the spherical case; an example to keep in mind are the groups $\mathrm{SL}_n(\mathbb{F})$ and $\mathrm{GL}_n(\mathbb{F})$. Their central quotients $\mathrm{PSL}_n(\mathbb{F})$ and $\mathrm{PGL}_n(\mathbb{F})$ are isomorphic for $\mathbb{F} = \mathbb{C}$ but distinct for $\mathbb{F} = \mathbb{R}$. For the study of topological buildings arising from Kac–Moody groups these differences do not matter much, but there are some subtleties.

Using the notation of Definition 7.1 and generalizing classical terminology in the spherical case, a Kac–Moody root datum \mathcal{D} will be called *simply connected* if the set $\{h_i \mid i \in I\}$ is a \mathbb{Z} -basis of Λ^\vee and *adjoint* if the set $\{c_i \mid i \in I\}$ is a \mathbb{Z} -basis of Λ . We will also refer to the corresponding Kac–Moody groups as simply connected or adjoint. For example, the algebraic group SL_n is simply connected and the algebraic group PGL_n is adjoint in this sense.

In general, the centre of an adjoint Kac–Moody group will be trivial, whereas the centre of a simply connected group is typically non-trivial. However, we observe:

Lemma 7.5. *The centre of a simply connected Kac–Moody group over an arbitrary field \mathbb{F} is finite.*

Proof. Let $G := G_{\mathcal{D}}(\mathbb{F})$ be a simply connected Kac–Moody group. Its centre is contained in any of its maximal tori by [34, Proposition 9.6.2]. Moreover, also by [34, Proposition 9.6.2], the centre of G equals the kernel of the adjoint representation. To prove the claim it is therefore enough to show that the adjoint representation induces an isogeny from a maximal torus T^{sc} onto its image.

By [34, Definition 7.1.1] the group of characters Λ of a maximal torus of a Kac–Moody group is a free abelian group of finite rank and by [34, 8.2.1] the torus functor is defined as the group functor $\mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda], -)$. By [34, 7.1.2] there exists a natural embedding $\Lambda^{\mathrm{ad}} \rightarrow \Lambda^{\mathrm{sc}}$ of the group of characters of an adjoint torus into the group of characters of a simply connected torus of the same type. This yields an injective \mathbb{Z} -algebra homomorphism $\mathbb{Z}[\Lambda^{\mathrm{ad}}] \rightarrow \mathbb{Z}[\Lambda^{\mathrm{sc}}]$, which in turn provides a surjective morphism $\mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda^{\mathrm{sc}}], -) \rightarrow \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda^{\mathrm{ad}}], -)$. As the ranks of Λ^{ad} and Λ^{sc} coincide, this implies that the kernel of this morphism is 0-dimensional, i.e., finite. \square

As before, let \mathcal{D} be a root group datum associated with a generalized Cartan matrix A , let \mathbb{F} be a field of arbitrary characteristic, let $G := G_{\mathcal{D}}(\mathbb{F})$ and let $Z := Z(G)$ the centre of G . We recall from [34, Section 7.3.1] how to construct

a faithful representation of $G/Z(G)$ starting from the complex Kac–Moody algebra \mathfrak{g} associated to the generalized Cartan matrix A , which generalizes the classical adjoint representation in the spherical case.

For this we denote by $\mathcal{U} := \mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} . For each $u \in \mathcal{U}$, let $u^{[n]} := (n!)^{-1}u^n$ and $\binom{u}{n} := (n!)^{-1} \cdot u \cdot (u-1) \cdots (u-n+1)$.

Let $Q := \sum_{\alpha \in \Pi} \mathbb{Z}\alpha$ be the free abelian group generated by the simple roots. Then, as in [34, Section 7.3.1], the algebras \mathcal{U} and \mathfrak{g} admit an abstract Q -grading by declaring e_i and f_i to be of degree α_i and $-\alpha_i$, respectively, and extending linearly.

With this notation, set \mathcal{U}_0 to be the subring of \mathcal{U} generated by the elements of degree 0 of the form $\binom{h}{n}$, where $h \in \mathfrak{h}, n \in \mathbb{N}$. Moreover, define \mathcal{U}_{α_i} and $\mathcal{U}_{-\alpha_i}$ to be the subrings $\sum_{n \in \mathbb{N}} \mathbb{Z}e_i^{[n]}$ and $\sum_{n \in \mathbb{N}} \mathbb{Z}f_i^{[n]}$, respectively. Let $\mathcal{U}_{\mathbb{Z}}$ be the subring of \mathcal{U} generated by \mathcal{U}_0 and $\{\mathcal{U}_{\pm\alpha} \mid \alpha \in \Pi\}$. Then $\mathcal{U}_{\mathbb{Z}}$ is a \mathbb{Z} -form of \mathcal{U} , i.e., the canonical map $\mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \mathcal{U}$ is a bijection, cf. [44, Section 4], [34, Proposition 7.4.3].

This construction allows one to replace the field \mathbb{C} with an arbitrary field \mathbb{F} : define $\mathcal{U}_{\mathbb{F}} := \mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}$. Let $\text{Aut}_{\text{filt}}(\mathcal{U}_{\mathbb{F}})$ be the group of \mathbb{F} -linear automorphisms of $\mathcal{U}_{\mathbb{F}}$ which preserve the above Q -grading.

The resulting adjoint action of a Kac–Moody group has the following nice properties.

Proposition 7.6 ([34, Proposition 9.5.2]). *Let $G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group over a field \mathbb{F} and let T denote its standard maximal torus. Then there exists a morphism of groups*

$$\text{Ad} : G_{\mathcal{D}}(\mathbb{F}) \rightarrow \text{Aut}_{\text{filt}}(\mathcal{U}_{\mathbb{F}})$$

which is characterized by the following axioms, where α_i is a real root, $r \in \mathbb{F}$ and $h \in T$:

- (i) $\text{Ad}(x_{\alpha_i}(r)) = \exp(\text{ad}_{e_i} \otimes r) = \sum_{n=0}^{\infty} \frac{(\text{ad}_{e_i})^n}{n!} \otimes r^n$,
- (ii) $\text{Ad}(T)$ fixes \mathcal{U}_0 ,
- (iii) $\text{Ad}(h)(e_i \otimes r) = h^*(\alpha_i^{\vee})(e_i \otimes r)$.

The kernel of this representation coincides with the centre of the group $G_{\mathcal{D}}(\mathbb{F})$. \square

In the sequel we will often tacitly identify $G/Z(G)$ with its image under the adjoint representation and thereby consider $G/Z(G)$ as a subgroup of $\text{Aut}_{\text{filt}}(\mathcal{U}_{\mathbb{F}})$.

7.3 The Kac–Peterson topology I: k_ω -property

Given a local field \mathbb{F} we are going to construct a group topology on every split Kac–Moody group over \mathbb{F} . Throughout we will imitate closely the arguments given in [13, Section 6] for unitary forms of complex Kac–Moody groups. The main difference is that we use the adjoint representation on the associative algebra $\mathcal{U}_{\mathbb{F}}$ instead of the Kac–Moody Lie algebra. This will allow us to include the case of positive characteristic as well.

Throughout we will reserve the letter \mathbb{F} to denote a local field, the letter \mathcal{D} to denote a Kac–Moody root datum, and the letter G to denote the associated Kac–Moody group $G := G_{\mathcal{D}}(\mathbb{F})$. As above, we then denote by $Z(G)$ the centre of G and identify $G/Z(G)$ with its image under the adjoint representation.

Our goal is to define a group topology on G that is induced by the topology on \mathbb{F} . We will actually provide two constructions of such a group topology on G in Definitions 7.8 and 7.19 below. In Proposition 7.21, however, we will show that these two topologies in fact coincide.

We equip the space $\mathbb{F}^{n \times n}$ of $(n \times n)$ matrices over \mathbb{F} with the product topology and obtain a Hausdorff group topology $\mathcal{O}_{\mathbb{F}}$ on the open subset $\mathrm{GL}_n(\mathbb{F})$. We also obtain Hausdorff group topologies on $\mathrm{SL}_n(\mathbb{F}) < \mathrm{GL}_n(\mathbb{F})$ and the central quotient $\mathrm{PSL}_n(\mathbb{F})$, which we denote by the same letter $\mathcal{O}_{\mathbb{F}}$. The rank one subgroups G_α of G are isomorphic to $\mathrm{SL}_2(\mathbb{F})$ or $\mathrm{PSL}_2(\mathbb{F})$ and, hence, by this definition can be considered as topological groups.

Starting from this topology on the rank one subgroups and the torus we now give our first definition of a topology, which is inspired by a construction of Kac and Peterson in the complex case [13, 23]. We start by defining topologies on the pieces $G_{\bar{\alpha}}$ introduced above in Section 7.1.

Definition 7.7. Let G be a split Kac–Moody group over a local field \mathbb{F} and let $\bar{\alpha} = (\alpha_1, \dots, \alpha_k) \in \bigcup_{l \in \mathbb{N}} \Pi^l$. Equip the torus T with its Lie group topology $\tau_{\mathbb{F}}$ and the rank one subgroups $G_{\alpha_1}, \dots, G_{\alpha_k}$ with the topology $\mathcal{O}_{\mathbb{F}}$. Then we denote by $\tau_{\bar{\alpha}}$ the quotient topology on $TG_{\bar{\alpha}}$ with respect to the surjective map

$$p_{\bar{\alpha}} : (T, \tau_{\mathbb{F}}) \times (G_{\alpha_1}, \mathcal{O}_{\mathbb{F}}) \times \cdots \times (G_{\alpha_k}, \mathcal{O}_{\mathbb{F}}) \rightarrow TG_{\bar{\alpha}}.$$

Note that the topological spaces $(G_{\bar{\alpha}}, \tau_{\bar{\alpha}})$ introduced in Definition 7.7 form a directed system with respect to the sub-tuple relation $\bar{\alpha} \leq \bar{\beta}$ defined in Section 7.1. Moreover, the set underlying G is the ascending union (i.e. direct limit) of the underlying directed system of sets.

Definition 7.8. The Kac–Peterson topology³ τ_{KP} on G is the direct limit topology with respect to the directed system $\{(TG_{\bar{\alpha}}, \tau_{\bar{\alpha}})\}_{\bar{\alpha}}$.

³see Remark 7.12 for the relation to their original definition

It is not obvious at all that this topology defines a group topology on G . The main problem one encounters in establishing continuity of the multiplication is that for general topological spaces G_i ,

$$\lim_{\rightarrow} G_i \times \lim_{\rightarrow} G_j \not\cong \lim_{\rightarrow} (G_i \times G_j).$$

However, such an exchange of limits is possible provided the pieces G_i are k_ω [13, Propositions 4.2, 4.7]. Our strategy for showing that τ_{KP} is a group topology will thus be to establish the k_ω -property for certain subsets of (G, τ_{KP}) . Once we can exchange direct limits and finite products freely, the proof becomes trivial. In order to establish the desired k_ω -property we will have to assume \mathbb{F} to be locally compact and σ -compact. Somewhat surprisingly, the hardest part in establishing the k_ω -property of τ_{KP} is to show that it is Hausdorff. For this we need to use some topological properties of the adjoint representation, which we summarize in the following proposition. Here we topologize all finite-dimensional \mathbb{F} -vector spaces $V \cong \mathbb{F}^n$ with the natural product topology with respect to the locally compact topology of \mathbb{F} and all corresponding general linear groups $\mathrm{GL}(V)$ with the topology $\mathcal{O}_{\mathbb{F}}$ defined above.

Proposition 7.9. *Let \mathbb{F} be a local field and let G be a split Kac–Moody group over \mathbb{F} . For each $v \in \mathcal{U}_{\mathbb{F}}$ there exists a family of subspaces $\{V_{\bar{\alpha}}^v\}_{\bar{\alpha}}$ of $\mathcal{U}_{\mathbb{F}}$ with the following properties:*

- (i) $\dim V_{\bar{\alpha}}^v < \infty$.
- (ii) *The image of the orbit map $TG_{\bar{\alpha}} \rightarrow \mathcal{U}_{\mathbb{F}} : g \mapsto g(v)$ is contained in $V_{\bar{\alpha}}^v$.*
- (iii) *The orbit map $(TG_{\bar{\alpha}}, \tau_{\bar{\alpha}}) \rightarrow (V_{\bar{\alpha}}^v, \mathcal{O}_{\mathbb{F}}) : g \mapsto g(v)$ is continuous.*
- (iv) *If $\bar{\alpha} \leq \bar{\beta}$ then $V_{\bar{\alpha}}^v \leq V_{\bar{\beta}}^v$.*
- (v) *$V_{\bar{\alpha}}^v$ is TG_{α_1} -invariant, where $\bar{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$.*
- (vi) *The kernel of the map $\rho_{\alpha_1} := \mathrm{Ad}|_{TG_{\alpha_1}}^{\mathrm{GL}(V_{\bar{\alpha}}^v)} : TG_{\alpha_1} \rightarrow \mathrm{GL}(V_{\bar{\alpha}}^v)$ is contained in $Z(G)$.*
- (vii) *The image of $U_{\pm\alpha_1} \subset G_{\alpha_1}$ under ρ_{α_1} is a closed subgroup of $(\mathrm{GL}(V_{\bar{\alpha}}^v), \mathcal{O}_{\mathbb{F}})$.*

Proof. By Proposition 7.6(ii) and (iii) we may disregard the finite-dimensional torus T .

Let $v_1, \dots, v_t \in \mathcal{U}_{\mathbb{F}}$ and $\alpha \in \Pi$. The Gauss algorithm/Bruhat decomposition of the fundamental rank one subgroup G_{α} of G implies that the product map $U_{\alpha} \times U_{-\alpha} \times U_{\alpha} \times U_{-\alpha} \rightarrow G_{\alpha}$ is surjective ([40, Lemma 24]). The adjoint action

of $u_\alpha(q) \in U_\alpha \subseteq G_\alpha$ on $\mathcal{U}_\mathbb{F}$ is given by $\text{Ad}(u_\alpha(q)) = \sum_{n=0}^{\infty} \left(\frac{(\text{ad}_{e_i})^n}{n!} \otimes q^n \right)$, see Proposition 7.6. Hence, by the Bruhat decomposition, the vector space

$$\sum_{i=1}^t \sum_{k,l,m,n \in \mathbb{N}} \left\langle \left(\frac{(\text{ad}_{e_\alpha})^k}{k!} \otimes 1 \right) \left(\frac{(\text{ad}_{f_\alpha})^l}{l!} \otimes 1 \right) \left(\frac{(\text{ad}_{e_\alpha})^m}{m!} \otimes 1 \right) \left(\frac{(\text{ad}_{f_\alpha})^n}{n!} \otimes 1 \right) v_i \right\rangle_{\mathbb{F}}$$

contains $\sum_{i=1}^t \langle G_\alpha \cdot v_i \rangle_{\mathbb{F}}$. By construction this vector space is $\text{Ad}|_{G_\alpha}$ -invariant. From the local nilpotency of ad_{e_α} and ad_{f_α} , we may conclude that the above sum is finite and hence this vector space has finite dimension.

Using bases, this argument shows that each finite-dimensional subspace of $\mathcal{U}_\mathbb{F}$ is contained in a G_α -invariant finite-dimensional subspace of $\mathcal{U}_\mathbb{F}$.

It is therefore immediate from Proposition 7.6 that for $v \in \mathcal{U}_\mathbb{F}$ there exists a family of subspaces $\{V_\alpha^v\}_{\bar{\alpha}}$ of $\mathcal{U}_\mathbb{F}$ such that (i), (ii), (iv), (v) hold.

Given a simple root α the corresponding rank one subgroup G_α falls in one of three classes and we will choose $g_\alpha \in G_\alpha$ accordingly: If $G_\alpha \cong \text{PSL}_2(\mathbb{F})$, then we let g_α be an arbitrary non-trivial element. If $G_\alpha \cong \text{SL}_2(\mathbb{F})$ and -1 is not in the centre of G , then we choose $g_\alpha := -1$. Finally, if $G_\alpha \cong \text{SL}_2(\mathbb{F})$ and -1 is contained in the centre of G , then we choose g_α different from ± 1 . In any case there exists $v_\alpha \in \mathcal{U}_\mathbb{F}$ with $\text{Ad}(g_\alpha)(v_\alpha) \neq v_\alpha$. In the first two cases, the orbit-stabilizer formula and the fact that g_α is contained in any non-trivial normal subgroup of G_α imply that $G_\alpha \rightarrow \mathcal{U}_\mathbb{F} : g \mapsto g(v_\alpha)$ is injective. Therefore, for any G_α -invariant subspace U of $\mathcal{U}_\mathbb{F}$ containing v_α , the group homomorphism $\rho : G_\alpha \rightarrow \text{GL}(U)$ induced by the adjoint action is injective. In the third case, a similar argument shows that the kernel is contained in the centre of G .

We have shown that for $v \in \mathcal{U}_\mathbb{F}$ there exists a family of subspaces $\{V_\alpha^v\}_{\bar{\alpha}}$ of $\mathcal{U}_\mathbb{F}$ such that (i), (ii), (iv), (v), (vi) hold.

It remains to show that such a family also satisfies (iii) and (vii).

Applying Borel–Tits [4] to G_{α_1} and the Zariski closure of its image under ρ_{α_1} in $\text{GL}(V_\alpha^v)$ we conclude that $\rho_{\alpha_1} : G_{\alpha_1} \rightarrow \text{GL}(V_\alpha^v)$ is algebraic and, thus, $\rho_{\alpha_1} : (G_{\alpha_1}, \tau_{\alpha_1}) \rightarrow (\text{GL}(V_\alpha^v), \mathcal{O}_\mathbb{F})$ is continuous. It follows from an open mapping theorem (cf. [13, Lemma 2.1]) that the subspace topology on $\rho_{\alpha_1}(G_{\alpha_1})$ is locally compact. Therefore, by [20, Theorem II.5.11], the image $\rho_{\alpha_1}(G_{\alpha_1})$ is a closed subgroup of $(\text{GL}(V_\alpha^v), \mathcal{O}_\mathbb{F})$ and, hence, so are $\rho_{\alpha_1}(U_{\pm\alpha_1})$. This establishes (vii).

Since $\rho_{\alpha_1} : (G_{\alpha_1}, \tau_{\alpha_1}) \rightarrow (\text{GL}(V_\alpha^v), \mathcal{O}_\mathbb{F})$ is continuous, so is the orbit map $(G_{\alpha_1}, \tau_{\alpha_1}) \rightarrow (V_\alpha^v, \mathcal{O}_\mathbb{F}) : g \mapsto g(v)$, i.e., we have shown (iii) for each 1-tuple $\bar{\alpha}$. Decompose an arbitrary tuple $\bar{\alpha}$ of length more than 1 as $(\alpha_1, \bar{\beta})$. By induction the orbit map $(G_{\bar{\beta}}, \tau_{\bar{\beta}}) \rightarrow (V_{\bar{\beta}}^v, \mathcal{O}_\mathbb{F}) : g \mapsto g(v)$ is continuous and, hence, so is $\varphi : (G_{\bar{\beta}}, \tau_{\bar{\beta}}) \rightarrow (V_\alpha^v, \mathcal{O}_\mathbb{F}) : g \mapsto g(v)$ by (iv). Since the action $\varepsilon : (\text{GL}(V_\alpha^v), \mathcal{O}_\mathbb{F}) \times$

$(V_{\bar{\alpha}}^v, \mathcal{O}_{\mathbb{F}}) \rightarrow (V_{\bar{\alpha}}^v, \mathcal{O}_{\mathbb{F}})$ is continuous, it follows that

$$\varepsilon \circ (\rho_{\alpha_1} \times \varphi) : (G_{\alpha_1}, \tau_{\alpha_1}) \times (G_{\bar{\beta}}, \tau_{\bar{\beta}}) \rightarrow (V_{\bar{\alpha}}^v, \mathcal{O}_{\mathbb{F}}) : (x_{\alpha_1}, x_{\bar{\beta}}) \rightarrow x_{\alpha_1} x_{\bar{\beta}}(v)$$

is continuous. As $G_{\bar{\alpha}}$ carries the quotient topology $\tau_{\bar{\alpha}}$, this means that also

$$(G_{\bar{\alpha}}, \tau_{\bar{\alpha}}) \rightarrow (V_{\bar{\alpha}}^v, \mathcal{O}_{\mathbb{F}}) : x \mapsto x(v)$$

is continuous, proving (iii). \square

Now we can deduce:

Proposition 7.10. *Let \mathbb{F} be a local field and let $G = G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group over \mathbb{F} . Then the Kac–Peterson topology is a k_{ω} group topology on G , which is moreover independent of the choice of the system Π of simple roots.*

Proof. To simplify the argument, we first assume that G has trivial centre. In a second step we will then remove this assumption by an easy embedding argument.

Thus let G be centre-free and fix a k -tuple $\bar{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of simple roots. Our first claim is that $(TG_{\bar{\alpha}}, \tau_{\alpha})$ is Hausdorff. For this, let $g \neq h \in G_{\bar{\alpha}}$. Since G is centre-free there then exists $v \in \mathcal{U}_{\mathbb{F}}$ such that $g(v) \neq h(v)$. By Proposition 7.9(iii) there exists a finite-dimensional subspace $V_{\bar{\alpha}}^v$ of $\mathcal{U}_{\mathbb{F}}$ that yields a continuous orbit map $f : (TG_{\bar{\alpha}}, \tau_{\alpha}) \rightarrow (V_{\bar{\alpha}}^v, \mathcal{O}_{\mathbb{F}}) : x \mapsto x(v)$. Taking preimages under f of suitable open neighbourhoods of $g(v)$, resp. $h(v)$ provides disjoint open neighbourhoods of g and h . Hence $G_{\bar{\alpha}}$ is Hausdorff. It then follows from [13, Proposition 4.2(d)] that the spaces $(TG_{\bar{\alpha}}, \tau_{\alpha})$ are k_{ω} . Moreover, the space (G, τ_{KP}) is T1 as a direct limit of Hausdorff spaces.

Since the multiplication map $TG_{(\alpha_1, \dots, \alpha_k)} \times TG_{(\beta_1, \dots, \beta_m)} \rightarrow TG_{(\alpha_1, \dots, \beta_m)}$ is continuous and these pieces are k_{ω} , it follows from [13, Propositions 4.2, 4.7] that multiplication on G is continuous with respect to τ_{KP} . A similar argument shows that also inversion is continuous, whence (G, τ_{KP}) is a topological group. As a T1 topological group it is in fact Hausdorff, hence k_{ω} as a Hausdorff direct limit of k_{ω} -spaces.

The independence of the topology τ_{KP} of the choice of Π now follows from the fact that up to conjugation there is a unique choice ([34, Theorem 10.4.2]) and the fact that conjugation in a topological group is a homeomorphism. This finishes the proof in the case where G is centre-free.

For the general case, we observe that the above proof goes through provided we are able to show that the pieces $(TG_{\bar{\alpha}}, \tau_{\alpha})$ are Hausdorff. For this, in turn, it suffices to embed them continuously into a Hausdorff topological space. We will use an embedding into a centre-free Kac–Moody group of larger rank, which generalizes the embedding of $\mathrm{SL}_n(\mathbb{F})$ into $\mathrm{PGL}_{n+1}(\mathbb{F})$.

Given a simply connected Kac–Moody group $G_{\mathcal{D}}(\mathbb{F})$ with simple roots Π , then, for any non-empty subset $\Sigma \subseteq \Pi$, the group $\langle U_{\alpha}, U_{-\alpha} \mid \alpha \in \Sigma \rangle$ again is a simply connected Kac–Moody group. If Π is irreducible and Σ is a proper subset of Π , then the centre of $\langle U_{\alpha}, U_{-\alpha} \mid \alpha \in \Sigma \rangle$ intersects the centre of $G_{\mathcal{D}}(\mathbb{F})$ trivially. Therefore, by [34, Proposition 9.6.2], the corresponding centre-free adjoint group $G_{\mathcal{D}^{\text{ad}}}(\mathbb{F})$ contains the simply connected Kac–Moody group $\langle U_{\alpha}, U_{-\alpha} \mid \alpha \in \Sigma \rangle$ as a subgroup. This argument shows that each simply connected Kac–Moody group can be embedded into an adjoint Kac–Moody group whose type is obtained by adding one further vertex to each connected component of the diagram. \square

The following result is now an immediate consequence of Proposition 7.4:

Corollary 7.11. *Let \mathbb{F} be a local field, and let G be a split Kac–Moody group over \mathbb{F} . Then, with notation as in Proposition 7.4, (G, τ_{KP}) is the direct limit of each one of the following three directed systems:*

$$\bigcup_{k \in \mathbb{N}} (G_k^+, \tau_{KP}|_{G_k^+}), \quad \bigcup_{k \in \mathbb{N}} (G_k^-, \tau_{KP}|_{G_k^-}), \quad \bigcup_{k \in \mathbb{N}} (G_k^+ \cap G_k^-, \tau_{KP}|_{G_k^+ \cap G_k^-}).$$

Remark 7.12. If \mathcal{D} is a *centred* Kac–Moody root datum and $G = \mathcal{G}_{\mathcal{D}}(\mathbb{F})$, then the torus can be recovered from the root groups. Therefore, the Kac–Peterson topology coincides with the final topology with respect to the directed system $(G_{\bar{\alpha}}, \tau_{\bar{\alpha}}|_{G_{\bar{\alpha}}})_{\bar{\alpha}}$. In this case, our definition of the Kac–Peterson topology is equivalent to the one given in 7.8 is equivalent to the one given in [24, Section 4G] using parametrizations of the root groups as follows.

Let \mathbb{F} be a local field and $G = \mathcal{G}_{\mathcal{D}}(\mathbb{F})$ a centred Kac–Moody group. For each simple root α , choose a parametrization $x_{\pm\alpha}: \mathbb{F} \rightarrow U_{\pm\alpha}$ of the root groups. For each finite sequence of positive or negative simple roots $\bar{\beta} = (\beta_1, \dots, \beta_k)$ denote by

$$x_{\bar{\beta}}: \mathbb{F}^k \rightarrow G: (t_1, \dots, t_k) \mapsto x_{\beta_1}(t_1) \cdots x_{\beta_k}(t_k)$$

the composition of the chosen parametrizations with the product map of G , and let $U_{\bar{\beta}}$ denote the image of $x_{\bar{\beta}}$. As, by the Gauss algorithm/Bruhat decomposition, for each simple root α one has $G_{\alpha} = U_{\alpha}U_{-\alpha}U_{\alpha}U_{-\alpha}$, the final topology on G with respect to the maps $x_{\bar{\beta}}$ coincides with the Kac–Peterson topology.

In the *non-centred* case, one additionally has to prescribe the topology of the torus. To this end, one classically realizes an n -dimensional split \mathbb{F} -torus with group of characters $\Lambda \cong \mathbb{Z}^n$ as the affine variety

$$\{(a_1, b_1, \dots, a_n, b_n) \in \mathbb{F}^{2n} \mid \forall 1 \leq i \leq n: a_i b_i = 1\}.$$

The parametrization in order to obtain $\text{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda], \mathbb{F})$ is given by the map that sends the a_i to the free abelian generators of Λ and the b_i to their inverses.

An important tool in the study of the Kac–Peterson topology, which goes back to the original work of Kac and Peterson (see [24, §2]), are weakly regular functions in the sense of the following definition:

Definition 7.13. A function $f : G \rightarrow \mathbb{F}$ is called *weakly regular*, if $f \circ x_{\bar{\beta}} : \mathbb{F}^k \rightarrow \mathbb{F}$ is a polynomial function for all $\bar{\beta} \in (\Pi \cup -\Pi)^k$ and all $k \in \mathbb{N}$ and $f|_T$ is a regular function in the usual sense.

Note that for centred G the last condition is automatic; thus we recover the original definition from [24, §2] in this case. The link between weakly regular functions and the Kac–Peterson topology is provided by the following lemma:

Lemma 7.14. *Every weakly regular function is continuous with respect to the Kac–Peterson topology.*

Proof. Let $A \subset \mathbb{F}$ be a closed subset and $f : G \rightarrow \mathbb{F}$ be a weakly regular function. Then for each $\bar{\beta} = (\beta_1, \dots, \beta_k) \in (\Pi \cup -\Pi)^k$ the preimage $(f \circ x_{\bar{\beta}})^{-1}(A)$ is closed with respect to the Hausdorff topology on \mathbb{F}^k , because polynomial functions are continuous. As this set equals the preimage under $x_{\bar{\beta}}$ of $f^{-1}(A)$, its image in $U_{\beta_1} U_{\beta_2} \cdots U_{\beta_k}$ under $x_{\bar{\beta}}$ is closed. This shows that $f^{-1}(A)$ is closed, as τ_{KP} equals the direct limit topology. \square

7.4 Topology of spherical subgroups

We retain the notation of the preceding section. In particular, \mathbb{F} denotes a local field and G a split Kac–Moody group over \mathbb{F} endowed with the Kac–Peterson topology τ_{KP} . We also denote by Δ_{\pm} the two halves of the associated twin building. Before we can continue our study of this topology we need to identify various closed subgroups.

We start with the following observation:

Proposition 7.15. *The subgroups $B_{\pm} < G$ are closed with respect to τ_{KP} .*

Proof. Observe that for $v \in \mathcal{U}_{\mathbb{F}}$ and $v^* \in (\mathcal{U}_{\mathbb{F}})^*$, the vector space dual of $\mathcal{U}_{\mathbb{F}}$, the map $f_{v,v^*} : G \rightarrow \mathbb{F} : g \mapsto v^*(g(v))$ is a weakly regular function, because $\text{ad}_{e_{\alpha}}$ and $\text{ad}_{f_{\alpha}}$ are locally nilpotent (cf. the proof of Proposition 7.9). Denote by $\mathcal{U}_{\mathbb{F}}^{\geq 0}$ the subspace of $\mathcal{U}_{\mathbb{F}}$ consisting of the non-negative vectors with respect to the Q -grading (cf. Section 7.2). Then, for each $g \in G \setminus B_+$ there exists a $v \in \mathcal{U}_{\mathbb{F}}^{\geq 0}$ such that $g(v) \notin \mathcal{U}_{\mathbb{F}}^{\geq 0}$. This means that there exists $v^* \in \text{Ann}(\mathcal{U}_{\mathbb{F}}^{\geq 0})$, the annihilator of $\mathcal{U}_{\mathbb{F}}^{\geq 0}$ in $(\mathcal{U}_{\mathbb{F}})^*$, such that $f_{v,v^*}(g) \neq 0$. We conclude that B_+ is the set of common zeros of the family of weakly regular functions $(f_{v,v^*})_{v \in \mathcal{U}_{\mathbb{F}}^{\geq 0}, v^* \in \text{Ann}(\mathcal{U}_{\mathbb{F}}^{\geq 0})}$. It then follows from Lemma 7.14 that B_+ is closed with respect to τ_{KP} . \square

The proposition implies in particular that the halves $\Delta_{\pm} = G/B_{\pm}$ are Hausdorff when equipped with the quotient topology with respect to τ_{KP} . We will see in the next corollary that this has massive consequences for the topology of *spherical* subgroups, i.e., fundamental subgroups of the form $G_{\alpha_1, \dots, \alpha_r} = \langle G_{\alpha_1}, \dots, G_{\alpha_r} \rangle$ for spherical subsets $\{\alpha_1, \dots, \alpha_r\}$ of the Coxeter system (W, S) , and their conjugates.

Note that in particular all rank one subgroups are spherical. Any spherical subgroup carries a unique semisimple Lie group topology $\mathcal{O}_{\mathbb{F}}$ over the ground field \mathbb{F} .

Corollary 7.16. *Let \mathbb{F} be a local field and let G be a split Kac–Moody group over \mathbb{F} . Equip the halves Δ^{\pm} of the associated twin building with the quotient topology with respect to the Kac–Peterson topology. Then:*

- (i) Δ^{\pm} are k_{ω} -spaces.
- (ii) Panels in Δ^{\pm} —and, more generally, spherical residues— are compact.
- (iii) For every real root α the restriction of τ_{KP} to G_{α} coincides with $\mathcal{O}_{\mathbb{F}}$.
- (iv) If H is a spherical subgroup, then the restriction of τ_{KP} to H coincides with its Lie group topology.
- (v) Spherical subgroups are closed.

Proof. (i) This is immediate by Proposition 7.10, Proposition 7.15 and [13, Proposition 4.2(d)].

(ii) The (continuous) action of the group $(G_{\alpha}, \mathcal{O}_{\mathbb{F}})$ on the twin building preserves the panel P_{α} . Denote by $B_{\alpha} < G_{\alpha}$ the point stabilizer of a basepoint in P_{α} , so that we obtain a continuous bijection between G_{α}/B_{α} and P_{α} . The former is compact (see Remark 6.8) and the latter is Hausdorff by (i), whence the latter is compact as it is a Hausdorff quotient of a compact space. The same argument works for spherical residues.

(iii) Denote by τ_{co} the compact-open topology on G_{α} with respect to the action on P_{α} . Then we have continuous maps

$$(G_{\alpha}, \mathcal{O}_{\mathbb{F}}) \rightarrow (G_{\alpha}, \tau_{KP}) \rightarrow (G_{\alpha}, \tau_{co}).$$

However, we have $\mathcal{O}_{\mathbb{F}} = \tau_{co}$, so in fact all three topologies coincide.

(iv) Since the Lie group topology is precisely the compact-open topology with respect to the action on the corresponding spherical residue, the same argument as in (iii) applies.

- (v) By (iii) and (iv) the subgroups in question are locally compact, hence must be closed by [20, Theorem II.5.11]. \square

From this in turn we deduce:

Corollary 7.17. (i) For every root α the root group U_α is closed with respect to τ_{KP} .

(ii) For every root α the canonical map $\mathbb{F} \rightarrow (U_\alpha, \tau_{KP}|_{U_\alpha})$ is a homeomorphism.

(iii) T is closed with respect to τ_{KP} .

Proof. (i) and (ii) are immediate since $U_\alpha < G_\alpha$ and $\tau_{KP}|_{G_\alpha}$ is the Lie group topology. (iii) then follows from (i) and $T = \bigcap_{\alpha \in \Phi^{re}} N_G(U_\alpha)$, cf. Proposition 7.2. \square

Getting the analogous statement of (ii) for the torus is slightly more involved:

Proposition 7.18. Let G be a centre-free Kac–Moody group, i.e., a subgroup of an adjoint Kac–Moody group, or a central quotient of a simply connected Kac–Moody group. Then the map $(T, \tau_{\mathbb{F}}) \rightarrow (T, \tau_{KP}|_T)$ is a homeomorphism. In particular, if G is centred, then there exists a finite group F such that $(T, \tau_{KP}|_T) \cong (\mathbb{F}^\times)^n / F$.

Proof. Consider first the case that G is centre-free. By definition $(T, \tau_{KP}|_T) = \lim_{\rightarrow} (T, \tau_{\bar{\alpha}})$. It thus suffices to show that the continuous maps $(T, \tau_{\mathbb{F}}) \rightarrow (T, \tau_{\bar{\alpha}})$ are open. Now Proposition 7.6 yields a finite-dimensional vector space $V_{\bar{\alpha}}^{\mathbb{F}}$ and a homomorphism $(T, \tau_{\bar{\alpha}}) \rightarrow \mathrm{GL}(V_{\bar{\alpha}}^{\mathbb{F}})$. Since G is assumed adjoint, this homomorphism is actually injective. It remains to show only that the map $(T, \tau_{\mathbb{F}}) \rightarrow (T, \tau_{\bar{\alpha}}) \rightarrow \mathrm{GL}(V_{\bar{\alpha}}^{\mathbb{F}})$ is a homeomorphism onto its image.

By the explicit formulae, the map in question is algebraic, whence continuous and has closed and, consequently, locally compact image. By the open mapping theorem, this map is therefore open. This finishes the proof in the centre-free case.

Now we consider the second case, where G is assumed to be a central quotient of a simply connected Kac–Moody group. The same argument as in the first case shows that τ_{KP} and $\tau_{\mathbb{F}}$ coincide on the quotient $\mathrm{Ad}(T)$. By Lemma 7.5, this is a finite quotient, hence the proposition follows from standard topological extension theory. \square

7.5 Kac–Peterson topology II: Universality

At this point we have assembled enough information about the Kac–Peterson topology to characterize it in terms of a universal property.

Definition 7.19. Let \mathbb{F} be a local field and let G be a split Kac–Moody group over \mathbb{F} . Then the *universal topology* τ on G is defined to be the final group topology with respect to the maps

$$\varphi_\alpha : \mathrm{SL}_2(\mathbb{F}) \rightarrow G, \quad \alpha \in \Phi^{re}, \quad \eta(\mathbb{F}) : \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda], \mathbb{F}) \rightarrow G,$$

where $\mathrm{SL}_2(\mathbb{F})$ and $\mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda], \mathbb{F}) \cong (\mathbb{F}^\times)^{\mathrm{rk}(\Lambda)}$ are equipped with their Lie group topologies.

Note that, as before, one obtains the same universal topology if one considers simple roots $\alpha \in \Pi$ only.

Lemma 7.20 (cf. [13, Lemma 6.2]). *Let \mathbb{F} be a local field, let G be a split Kac–Moody group, and let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a basis of simple roots of Φ^{re} . Then the universal topology on G is the final group topology with respect to the maps $(\varphi_{\alpha_i})_{1 \leq i \leq n}$ and $\eta(\mathbb{F})$.*

Proof. It suffices to observe that for every real root α there exists $w \in W$ and $\alpha_i \in \Pi$ such that $\alpha = w.\alpha_i$, whence for any representative \tilde{w} of w in G , one has $G_\alpha = G_{w.\alpha_i} = \tilde{w}G_{\alpha_i}\tilde{w}^{-1}$. \square

Again the universal topology can be defined for general topological fields, but it is unclear to us whether it has any good properties in general; we do not even know whether it is Hausdorff. However, over local fields we can show the following:

Proposition 7.21. *Assume \mathbb{F} is a local field and that G is centre-free or a central quotient of a simply connected Kac–Moody group. Then the universal topology and the Kac–Peterson topology coincide. In particular, (G, τ) is Hausdorff and k_ω .*

Proof. Since the inclusion maps $G_\alpha \rightarrow (G, \tau_{KP})$ are continuous, we obtain a continuous map $(G, \tau) \rightarrow (G, \tau_{KP})$. It remains to show that this map is open. In view of Corollary 7.16(iii), the topologies coincide on each G_α . Since multiplication is continuous, the map $(G_{\bar{\alpha}}, \tau_{\bar{\alpha}}) \rightarrow (G_{\bar{\alpha}}, \tau|_{G_{\bar{\alpha}}})$ is continuous. On the other hand the map

$$(G_{\bar{\alpha}}, \tau|_{G_{\bar{\alpha}}}) \rightarrow (G_{\bar{\alpha}}, \tau_{KP}|_{G_{\bar{\alpha}}}) = (G_{\bar{\alpha}}, \tau_{\bar{\alpha}})$$

is continuous as the restriction of a continuous map. Altogether we have shown that τ and τ_{KP} coincide on each $G_{\bar{\alpha}}$. It then follows that they coincide globally and the subgroup of G generated by the root subgroups. The claim therefore follows from Proposition 7.18. \square

In the two-spherical case we can reformulate the universal property of the Kac–Peterson topology in the form of an amalgamation result that generalizes [13, Theorem 6.20].

Theorem 7.22 (Topological Curtis–Tits Theorem). *Let \mathbb{F} be a local field and let G be a two-spherical simply connected split Kac–Moody group. Let Φ^{re} be the set of real roots and let Π be a basis of simple roots for Φ^{re} . For $\alpha, \beta \in \Pi$, set $G_\alpha := \varphi_\alpha(\mathrm{SL}_2(\mathbb{F}))$ and $G_{\alpha\beta} := \langle G_\alpha \cup G_\beta \rangle$. Moreover, let $\iota_{\alpha\beta}: G_\alpha \hookrightarrow G_{\alpha\beta}$ be the canonical inclusion morphisms.*

Then the group (G, τ_{KP}) is a universal enveloping group of the amalgam

$$\{G_\alpha, G_{\alpha\beta}; \iota_{\alpha\beta}\}$$

in the categories of

- (i) *abstract groups,*
- (ii) *Hausdorff topological groups and*
- (iii) *k_ω groups.*

Proof. (i) This is the main result of [2].

(ii) By Lemma 7.20 and Proposition 7.21 the group (G, τ_{KP}) is the direct limit of the amalgam $\{G_\alpha, G_{\alpha\beta}; \iota_{\alpha\beta}\}$ in the category of topological groups. Since τ_{KP} is Hausdorff by Proposition 7.21 the claim follows.

(iii) By (ii), the claim follows from [13, Corollary 5.10]. \square

In Theorem 7.22 the hypothesis of simple connectedness is important, since otherwise the torus will not be the universal enveloping group of the amalgam of the tori of ranks one and two. For their central quotients it is, of course, possible to derive a compact presentation as well by incorporating the finite group of toral relations manually (cf. Lemma 7.5).

We obtain:

Corollary 7.23. *Central quotients of simply connected two-spherical split Kac–Moody groups over local fields are compactly presented in the sense of [9, Definition 3.1].*

Remark 7.24. We have seen in Corollary 7.16 that in the spherical case the Kac–Peterson topology coincides with the Lie group topology. Thus Theorem 7.22 applies in particular to semisimple Lie groups over local fields. We emphasize once more that in the non-spherical case it is not locally compact and, thus, not metrizable, cf. Remark 7.28; see also [13, Example 6.14] for a detailed discussion of group topologies on affine Kac–Moody groups.

7.6 Topology of bounded and some non-spherical subgroups

We now return to the study of the topology of subgroups of G . Corollary 7.16 implies that (G, τ_{KP}) induces the natural topology on the obvious finite-dimensional pieces of G , the spherical subgroups. In this section we intend to understand the topologies induced on some infinite-dimensional subgroups and on the less obvious finite-dimensional pieces, the bounded subgroups.

Our first goal is to understand the groups U_{\pm} . The key idea is to consider them as colimits of finite-dimensional pieces. According to [8, Section 5.3, Theorem and Corollary] the group U_+ is the colimit $U_+ = \lim_{\rightarrow} U_w$ (in the category of abstract groups) with respect to the Bruhat order of the groups $U_w = U_+ \cap wB_-w^{-1}$, $w \in W$. Moreover, by [8, Section 5.5, Lemma], for each reduced $r_{\alpha_1} \cdots r_{\alpha_n} = w \in W$ for simple roots $\alpha_i \in \Pi$ with $\beta_1 := \alpha_1$ and $\beta_i := r_{\alpha_1} \cdots r_{\alpha_{i-1}} \alpha_i$ for $2 \leq i \leq n$, the multiplication map

$$m : U_{\beta_1} \times \cdots \times U_{\beta_n} \rightarrow G : (x_1, \dots, x_n) \mapsto x_1 \cdots x_n.$$

is a bijection onto U_w .

We warn the reader that the above product U_w is not to be confused with the products $U_{\bar{\alpha}}$ studied before. Unlike the α_i , the β_i are assumed to be all positive and distinct and not necessarily simple.

Lemma 7.25. *If U_{β_i} and U_w are equipped with the respective restrictions of τ_{KP} and the product $U_{\beta_1} \times \cdots \times U_{\beta_n}$ is equipped with the product topology, then m is a homeomorphism onto its image U_w .*

Proof. We are going to establish the lemma by induction on n , the case $n = 1$ being trivial. Now let $n \geq 2$, assume that the claim holds for all products of length at most $n - 1$, and consider $U_w = U_{\beta_1} \cdots U_{\beta_n}$. We may assume that G is irreducible. Then there exists a torus element $t \in T$ whose conjugation preserves the root group U_{β_1} and contracts all other root groups U_{β_i} . We deduce that if $x_i \in U_{\beta_i}$ and $x = x_1 \cdots x_n$, then

$$\lim_{n \rightarrow \infty} g^{-1} t^n g x g^{-1} t^{-n} g = x_1.$$

This shows that there is a continuous map

$$\varphi_1 : X \rightarrow U_{\beta_1} \times (U_{\beta_2} \cdots U_{\beta_n}) : x \mapsto (x_1, x_1^{-1} x).$$

By induction hypothesis, applied to $gU_{\beta_2} \cdots U_{\beta_n} g^{-1}$ for some $g \in N_G(T)$ with $gU_{\beta_2} g^{-1}$ a root subgroup for a simple root, we thus find a continuous map

$$\varphi : X \rightarrow U_{\beta_1} \times \cdots \times U_{\beta_n},$$

which is inverse to m . The claim follows. \square

There is a slight variant of the lemma, which incorporates the torus:

Lemma 7.26. *If T , U_{β_i} and TU_w are equipped with the respective restrictions of τ_{KP} and the product $T \times U_{\beta_1} \times \cdots \times U_{\beta_n}$ is equipped with the product topology, then the multiplication map*

$$T \times U_{\beta_1} \times \cdots \times U_{\beta_n} \rightarrow G : (t, x_1, \dots, x_n) \mapsto tx_1 \cdots x_n.$$

is a homeomorphism onto its image TU_w .

Proof. Since $T \cap U_+ = \{1\}$ and T is self-centralizing the same argument applies. \square

Lemma 7.26 and its obvious variant for the groups $U_{-w} = U_- \cap wB_+w^{-1}$, $w \in W$, involving negative roots allow us to describe the subspace topologies induced on B_{\pm} by τ_{KP} as direct limit topologies.

Proposition 7.27. (i) $(B_{\pm}, \tau_{KP}|_{B_{\pm}})$ is a colimit $B_{\pm} = \lim_{\rightarrow} TU_{\pm w} \cong T \times U_{\pm}$ in the category of Hausdorff topological groups.

(ii) The multiplication map $T \times U_{\pm} \rightarrow B_{\pm}$ is open.

(iii) U_{\pm} are closed in B_{\pm} , hence in G .

Proof. For (i) we need only to observe that each $U_{\pm w}$ is contained in some G_n^{\mp} and that each element in the intersection $B_{\pm} \cap G_n^{\mp}$ can be written as a product of elements contained in T and in $U_{\pm w}$ with $l(w) \leq n$ and apply Corollary 7.11. Now (ii) and (iii) are immediate from (i). \square

Remark 7.28. We conclude from Lemma 7.25 and Proposition 7.27 that in the non-spherical case (G, τ_{KP}) cannot be locally compact. Indeed, in this case the closed subgroup U_+ would be locally compact, which is absurd as it contains an infinite-dimensional vector space, e.g. the direct limit $\lim_{\rightarrow} U_{w_i}$ along an infinite chain $w_1 < w_2 < w_3 < \cdots$ with respect to the Bruhat order. Proposition 7.21 and [11, §21] therefore imply that (G, τ_{KP}) is not metrizable and, in particular, not Polish.

We conclude this section with an analysis of the topology of some further subgroups of G . The following is a straightforward generalization of Proposition 7.15.

Proposition 7.29. *Parabolic subgroups are closed.*

Proof. The proof is identical to the one given in Proposition 7.15. Let P be a parabolic subgroup of positive sign and denote by $\mathcal{U}_{\mathbb{F}}^P$ the subspace of $\mathcal{U}_{\mathbb{F}}$ spanned by the P -orbit of the subspace $\mathcal{U}_{\mathbb{F}}^{\geq 0}$ of the non-negative vectors with respect to the Q -grading. Then, for each $g \in G \setminus P$ there exists a $v \in \mathcal{U}_{\mathbb{F}}^P$ such that $g(v) \notin \mathcal{U}_{\mathbb{F}}^P$. This means that there exists $v^* \in \text{Ann}(\mathcal{U}_{\mathbb{F}}^P)$ such that $f_{v,v^*}(g) \neq 0$. We conclude that P is the set of common zeros of the family of weakly regular functions $(f_{v,v^*})_{v \in \mathcal{U}_{\mathbb{F}}^P, v^* \in \text{Ann}(\mathcal{U}_{\mathbb{F}}^P)}$. It then follows from Lemma 7.14 that P is closed with respect to τ_{KP} . \square

Corollary 7.30. *Bounded subgroups, i.e., intersections of spherical parabolics of opposite signs, are algebraic Lie groups. Its Levi decomposition is a semi-direct product of closed subgroups.*

Proof. The fact that bounded subgroups are algebraic follows from [34, Section 6.2], as does the existence of a Levi decomposition. A Levi factor is a spherical subgroup and, hence, a (closed) Lie group by Corollary 7.16. A bounded subgroup is closed by Proposition 7.29. Its unipotent radical is closed by [7, Lemma 3.3, Proposition 3.6] plus Propositions 7.27 and 7.29.

If M is a maximal bounded subgroup, then it is either equal to its own Levi subgroup (and there is nothing to prove), or by [7, Lemma 3.2, Lemma 3.3 and its proof, Theorem 4.1] its unipotent radical equals a finite product of real root subgroups, in fact a conjugate of some U_w , and hence is homeomorphic to a finite-dimensional \mathbb{F} -vector space by Lemma 7.25.

In order to prove the claim for M , it remains to show that it is a topological product of one of its Levi factors L and its unipotent radical U . Multiplication $L \times U \rightarrow M$ is clearly continuous. In order to see that it is open it suffices to show that the continuous map $L \rightarrow M \rightarrow M/U \cong L$ is open (where the latter is endowed with the quotient topology). For this we observe that U can, as in the proof of Lemma 7.25, be contracted by a torus element of M that centralizes L . (We again may assume that G is irreducible.)

A non-maximal bounded subgroup is contained in a maximal one, and so the Lie group and topological semi-direct product structure is inherited. \square

7.7 The topological twin building of a split Kac–Moody group

The goal of this section is to establish the first main result of this paper:

Theorem 1. *Let G be a two-spherical simply connected split Kac–Moody group over a local field and let τ_{KP} be the Kac–Peterson topology on G . Then the associated twin building endowed with the quotient topology is a strong topological twin building (cf. Definition 3.21).*

If the local field equals the field of real or of complex numbers, then G is connected, otherwise totally disconnected.

The key observation is the following:

Proposition 7.31. *Let \mathbb{F} be a local field and let G be a two-spherical simply connected split Kac–Moody group over \mathbb{F} endowed with the Kac–Peterson topology. Then the map $m : U_+ \times T \times U_- \rightarrow B_+B_- : (u_+, t, u_-) \mapsto u_+tu_-$ is open.*

This result has been announced in [24, Theorem 4], actually without the restriction to the two-spherical case. This more general version would in fact allow one to remove the requirement that G be two-spherical in the statement of Theorem 1.

Proof of Proposition 7.31. Let $\alpha_1, \alpha_2, \dots$ be a cofinal sequence with respect to the sub-tuple relation. Proposition 2.17 implies that for each $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ with

$$B_+B_- \cap U_{\alpha'} \subseteq (U_+ \cap U_{\alpha'}) (T \cap U_{\alpha'}) (U_- \cap U_{\alpha'}),$$

where $\alpha' = (\alpha_1, \dots, \alpha_n)$ and $\alpha'' = (\alpha_1, \dots, \alpha_m)$. Conversely, for each $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ with

$$(U_+ \cap U_{\alpha''}) (T \cap U_{\alpha''}) (U_- \cap U_{\alpha''}) \subseteq B_+B_- \cap U_{\alpha'}.$$

We conclude that

$$B_+B_- = \varinjlim (U_+ \cap U_{\alpha'}) (T \cap U_{\alpha'}) (U_- \cap U_{\alpha'}).$$

Moreover, since $U_{\pm} = \varinjlim U_{\pm} \cap U_{\beta}$ and $T = \varinjlim T \cap U_{\alpha}$, by [13, Proposition 4.7] one has

$$U_+ \times T \times U_- = \varinjlim ((U_+ \cap U_{\alpha}) \times (T \cap U_{\alpha}) \times (U_- \cap U_{\alpha})).$$

As each

$$\begin{aligned} m_{\alpha} : (U_+ \cap U_{\alpha}) \times (T \cap U_{\alpha}) \times (U_- \cap U_{\alpha}) &\rightarrow (U_+ \cap U_{\alpha}) (T \cap U_{\alpha}) (U_- \cap U_{\alpha}) \\ (u_+, t, u_-) &\mapsto u_+tu_- \end{aligned}$$

is open, this implies that $m : U_+ \times T \times U_- \rightarrow B_+B_- : (u_+, t, u_-) \mapsto u_+tu_-$ is open. \square

Now Theorem 1 is immediate:

Proof of Theorem 1. Concerning the first part of Theorem 1 it suffices to establish the hypotheses of Theorem 6.7. Condition (iii) is precisely Proposition 7.31. Since we have already established Condition (i) in Proposition 7.15, Condition (ii) in Corollary 7.11 and Condition (iv) in Corollary 7.16, the first part of Theorem 1 follows.

Concerning the second part we first note that the underlying roots groups and the torus are connected if and only if k is archimedean. In the archimedean case it then follows that the pieces $TG_{\bar{\alpha}}$, and hence their limit G are connected. In the non-archimedean case we see from Proposition 7.27 and Proposition 7.31 that the open subset B_+B_- is totally disconnected. This implies that G itself is totally disconnected, finishing the proof. \square

Note that the underlying topological foundation of the topological twin building of G is precisely the corresponding topological k -split foundation. This shows:

Corollary 7.32. *Let k be a local field. Then every Dynkin tree is topologically k -integrable.*

Proof. Let \mathcal{F} be a topological k -split Moufang foundation corresponding to a given Dynkin tree. By the main result of [30] there exists a (two-spherical) split Kac–Moody group $G = G_{\mathcal{D}}(k)$ such that the associated twin building globalizes the abstract foundation underlying \mathcal{F} . Equip G with the Kac–Petersen topology and its twin building with the associated quotient topology. In view of Theorem 1 this is a topological twin building. Since the topology on the root subgroups U_{α} is the standard one by Corollary 7.16(iii), this topological twin building realizes the given topological foundation. \square

A combination of Corollary 7.32 and Corollary 4.10 now yields the second main result of the present article:

Theorem 2. *Let k be a local field. The maps $[\Delta] \mapsto [\mathcal{D}(\Delta)]$ induces a bijection between isomorphism classes of k -split topological twin buildings of tree type and isomorphism classes of simply connected simple $\{3, 4, 6\}$ -labelled graphs, where edges labelled 4 or 6 are directed.*

7.8 Kac–Moody symmetric spaces

We conclude with a couple of observations related to Kac–Moody symmetric spaces using the theory of flips introduced in [14] and [21].

Lemma 7.33. *Let \mathbb{F} be a field, let \mathcal{D} be a centred Kac–Moody root datum, let $G_{\mathcal{D}}(\mathbb{F})$ be the corresponding split Kac–Moody group, let θ be a quasi-flip of the Kac–Moody group such that $\theta(B_+) = B_-$, let $\tau_{\theta}: G_{\mathcal{D}}(\mathbb{F}) \rightarrow G_{\mathcal{D}}(\mathbb{F}) : g \mapsto \theta(g)^{-1}g$ be the corresponding Lang map, let W be its Weyl group, let $w \in W$, and let $x \in G_{\mathcal{D}}(\mathbb{F})$. Then $\delta^*(\theta(xB_+), xB_+) = w$ if and only if $x \in \tau_{\theta}^{-1}(B_-wB_+)$.*

Proof. One has the following chain of equivalences:

$$\begin{aligned} \delta^*(\theta(xB_+), xB_+) = w &\iff B_- \theta(x)^{-1} x B_+ = B_- w B_+ \\ &\iff \theta(x)^{-1} x \in B_- w B_+ \\ &\iff \tau_{\theta}(x) \in B_- w B_+ \\ &\iff x \in \tau_{\theta}^{-1}(B_- w B_+). \quad \square \end{aligned}$$

Theorem 7.34. *Let \mathbb{F} be a local field, let G be the adjoint form of a simply connected split Kac–Moody group endowed with the Kac–Peterson topology τ_{KP} , let W be its Weyl group, let θ be a continuous quasi-flip of G such that $\theta(B_+) = B_-$, and let $G_{\theta} := C_G(\theta) = \text{Fix}_G(\theta)$. Moreover, for $w \in W$, let*

$$\Delta_w := \{c \in \Delta_+ \mid \delta^*(\theta(c), c) = w\}.$$

Furthermore, let $\text{Cod}(\theta) := \{w \in W \mid \Delta_w \neq \emptyset\}$. Then the following hold:

(i) For $w \in \text{Cod}(\theta)$ one has

$$\overline{\Delta_w} = \bigcup_{w' \geq w} \Delta_{w'}.$$

(ii) For $w \in \text{Cod}(\theta)$ the smallest open G_{θ} -invariant subset of Δ_+ containing Δ_w is

$$\bigcup_{w' \leq w} \Delta_{w'}.$$

Proof. Using the quotient map $q: G \rightarrow \Delta_+ = G/B_+$, Lemma 7.33 states that for each $w \in W$ the set $q(\tau_{\theta}^{-1}(B_-wB_+))$ equals Δ_w . As θ is continuous, so is the Lang map $\tau_{\theta}: G \rightarrow G : g \mapsto \theta(g)^{-1}g$, whence all claims follow immediately from Theorem 6.4. \square

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