Appendix: Finite prolongation in RGD-systems

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1. Introduction

In this appendix we give the proof of Proposition 2.17 of [4]. It is a consequence of Theorem 4.5 below. The key step in our reasoning is based on a refinement of the arguments in the proof of Theorem 1.5 in [5]. The latter says that the set of chambers opposite a given chamber in a 2-spherical twin building is connected, if this condition holds in all rank 2 residues. Our refinement consists of giving a bound for the distance between two chambers in the set of opposite chambers depending on their distance in the building. In order to do this we have to strengthen the local condition on the rank two residues. This results in our somewhat technical Condition $(co)_k$ below. It is almost always satisfied and more explanations are given in the final section of this note.

2. Preliminaries

Let (W, S) be a Coxeter system and let $\ell : W \to \mathbf{N}$ be its associated length function. Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S). For two chambers $c, d \in \mathcal{C}$ we put $\ell(c, d) := \ell(\delta(c, d))$.

Definition 2.1. A *codistance* on $\mathcal{B}$ is a mapping $\delta_* : \mathcal{C} \to W$ such that the following is satisfied for all $s \in S$ and $c \in \mathcal{C}$ where $w := \delta_*(c)$.

(CD1) If $d \in \mathcal{C}$ is s-adjacent to c, then $\delta_*(d) \in \{w, ws\}$.

(CD2) If $\ell(ws) = \ell(w) + 1$, then there exists a unique chamber d which is s-adjacent to c such that $\delta_*(d) = ws$.

Let δ_* be a codistance on $\mathcal{B}$. Then we put $\delta_*^{\mathrm{op}} := \{c \in \mathcal{C} \mid \delta_*(c) = 1_W\}$.

Lemma 2.2. *Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S), let R be a spherical residue of $\mathcal{B}$ and δ_* a codistance on $\mathcal{B}$.*

Then there exists a unique chamber $c \in R$ such that $\ell(\delta_*(c)) = \ell(\delta_*(d)) + \ell(c, d)$ for all $d \in R$.

Proof. This is [3, Proposition 6]. □

Definition 2.3. The unique chamber c of the previous lemma is called the *projection* of δ_* onto R and it is denoted by $\operatorname{proj}_R \delta_*$.

3. Condition $(lco)_k$

Definition 3.1. Let (W, S) be a spherical Coxeter system of rank 2 and $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S).

We say that $\mathcal{B}$ satisfies Condition $(co)_k$ if the following holds for each chamber $c \in \mathcal{C}$:

> If (d, e, f) is a gallery such that $\ell(c, d) \geq \ell(c, e) = \ell(c, f) - 1$, then there exists a gallery $(d = d_0, d_1, \ldots, d_m = f)$ such that $m \leq k$ and such that $\ell(c, d_i) > \ell(c, e)$ for all $1 \leq i \leq m$.

Let (W, S) be a 2-spherical Coxeter system and let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S). We say that $\mathcal{B}$ satisfies Condition $(lco)_k$ if each rank 2 residue of $\mathcal{B}$ satisfies Condition $(co)_k$.

Convention 3.2. For the rest of this section (W, S) is a 2-spherical Coxeter system and $\mathcal{B} = (\mathcal{C}, \delta)$ is a building of type (W, S) satisfying Condition $(lco)_k$. Moreover, $\delta_* : \mathcal{C} \to W$ is a codistance on $\mathcal{B}$ and for each $c \in \mathcal{C}$ we set $\ell(c) := \ell(\delta_*(c))$.

For a gallery $\gamma = (c = c_0, \ldots, c_n = d)$ we put $\mathbf{m}(\gamma) := \max\{\ell(c_i) \mid 0 \leq i \leq n\}$ and $\mathbf{n}(\gamma) := |\{0 \leq i \leq n \mid \ell(c_i) = \mathbf{m}(\gamma)\}|$.

Lemma 3.3. *Suppose that $c, d \in \delta_*^{\mathrm{op}}$ and that $\gamma = (c = c_0, \ldots, c_n = d)$ is a gallery such that $\mathbf{m}(\gamma) > 0$. Then there exists a gallery γ' from c to d of length at most $n + k$ such that $\mathbf{m}(\gamma') \leq \mathbf{m}(\gamma)$ and such that the following hold.*

(i) *If $\mathbf{n}(\gamma) > 1$ then $\mathbf{n}(\gamma') = \mathbf{n}(\gamma) - 1$;*

(ii) *if $\mathbf{n}(\gamma) = 1$, then $\mathbf{m}(\gamma') = \mathbf{m}(\gamma) - 1$.*

Proof. We can find an $0 < i < n$ such that $\mathbf{m}(\gamma) = \ell(c_i) = \ell(c_{i+1}) + 1$.

There exists a rank 2 residue R containing c_{i-1}, c_i and c_{i+1}. Let $e := \operatorname{proj}_R \delta_*$. Note that we have $\ell(c_{i-1}) \leq \ell(c_i) = \ell(c_{i+1}) + 1$. By Lemma 2.2 we have $\ell(e, c_{i-1}) \geq \ell(e, c_i) = \ell(e, c_{i+1}) - 1$. Since $\mathcal{B}$ satisfies Condition $(lco)_k$ we can find a gallery $\gamma'' = (c_{i-1} = d_0, \ldots, d_m = c_{i+1})$ in R such that $m \leq k$ and such that $\ell(e, d_j) > \ell(e, c_i)$ for all $1 \leq j \leq m$. Applying again Lemma 2.2, it follows that $\ell(d_j) < \ell(c_i)$ for all $1 \leq j \leq m$. Now we obtain the desired gallery γ' by replacing the subgallery (c_{i-1}, c_i, c_{i+1}) by γ'' and we are done. □

Lemma 3.4. *Let $c, d \in \delta_*^{\mathrm{op}}$ and $\gamma = (c = c_0, \ldots, c_n = d)$ be a gallery such that $\mathbf{m}(\gamma) > 0$. Then there exists a gallery γ' from c to d of length at most nk such that $\mathbf{m}(\gamma') = \mathbf{m}(\gamma) - 1$.*

Proof. An obvious induction on $n(\gamma)$ using the previous lemma shows that we can find a gallery γ' from c to d of length at most $n + \mathbf{n}(\gamma)k$ with $\mathbf{m}(\gamma') = \mathbf{m}(\gamma) - 1$. As $\mathbf{n}(\gamma) \leq n - 1$ the claim follows. □

Proposition 3.5. *Let $c, d \in \delta_*^{\mathrm{op}}$ and suppose that $\ell(c, d) = n$. Then there exists a gallery of length at most nk^n from c to d in δ_*^{op}.*

Proof. Let γ be a gallery from c, d of length n and let $m := \mathbf{m}(\gamma)$. Using the previous lemma one shows by induction that there is a gallery in δ_*^{op} from c to d of length at most nk^m and as $m \leq n$ the claim follows. □

In the remainder of this section U is a group of isometries of $\mathcal{B}$ which preserve the codistance δ_*. We fix a chamber $c \in \delta_*^{\mathrm{op}}$ and let H denote its stabilizer in U. Furthermore, for each $s \in S$ we denote the stabilizer in U of the s-panel P_s containing c by U_s. Finally, we put $X := \cup_{s \in S} U_s$.

Lemma 3.6. *Suppose that U_s is transitive on $\delta_*^{\mathrm{op}} \cap P_s$ for each $s \in S$. Let $u \in U$ and $1 \leq n \in \mathbf{N}$. Then $u \in X^n$ if and only if there exists a gallery from c to $u(c)$ in δ_*^{op} of length at most n. In particular, $\langle X \rangle = U$ if and only if δ_*^{op} is connected.*

Proof. The first statement follows by an obvious induction on n and the second is consequence of the first. □

4. Moufang twin buildings

Throughout this section, let (W, S) be a 2-spherical Coxeter system and let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be a twin building of type (W, S). We recall that this means that $\mathcal{B}_\epsilon = (\mathcal{C}_\epsilon, \delta_\epsilon)$ is a building of type (W, S) and that we have a twinning $\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$.

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Two chambers $x \in \mathcal{C}_+, y \in \mathcal{C}_-$ are called opposite if $\delta^*(x,y) = 1_W$ and for a chamber $c \in \mathcal{C}_\epsilon$ we denote the set of chambers in $\mathcal{C}_{-\epsilon}$ which are opposite to c by c^{op}.

The following is immediate from the definitions:

Lemma 4.1. *Let $\epsilon \in \{+,-\}$ and $c \in \mathcal{C}_\epsilon$. Then the mapping $\delta^*_c : \mathcal{C}_{-\epsilon} \to W$ is a codistance on $\mathcal{B}_{-\epsilon}$ and $c^{\mathrm{op}} = (\delta^*_c)^{\mathrm{op}}$.*

Further conventions for this section. The twin building $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ is a 2-spherical Moufang twin building and it satisfies Condition $(lco)_k$. Moreover $(c_+.c_-)$ is a pair of opposite chambers and Σ is the unique twin apartment containing them both. Moreover, we put $\delta^*_\epsilon := \delta^*_{c_\epsilon}$; hence δ^*_ϵ is a codistance on $\mathcal{B}_{-\epsilon}$ for $\epsilon \in \{+,-\}$.

We let $\Phi := \Phi(\Sigma)$ be the denote the set of roots of Σ and for $\epsilon \in \{+,-\}$ we let Φ_ϵ denote the set of roots in Φ which contain c_ϵ. Finally, we a have an RGD-sytem $(G, (U_\alpha)_{\alpha\in\Phi})$ acting on $\mathcal{B}$ such that the U_α are mapped onto the corresponding root-groups of $\mathcal{B}$.

For $\epsilon \in \{+.-\}$ we let B_ϵ denote the stabilizer of c_ϵ in G and we set $H := B_+ \cap B_-$. We put $U_\epsilon := \langle U_\alpha \mid \alpha \in \Phi_\epsilon \rangle$ and remark that U_ϵ fixes c_ϵ. For $s \in S$ we denote the stabilizer in U_ϵ of the s panel containing $c_{-\epsilon}$ by U^ϵ_s. Finally, we put $X_\epsilon = \bigcup_{s\in S} U^\epsilon_s$ for $\epsilon \in \{+,-\}$. and $X := X_+ \cup X_-$. We remark that the subgroup H normalizes the set X_ϵ for $\epsilon \in \{+,-\}$ and hence also the set X.

Lemma 4.2. *For $\epsilon \in \{+,-\}$ the following hold:*

(i) *For each $s \in S$ the stabilzer in U_ϵ of the s-panel of $c_{-\epsilon}$ coincides with U^ϵ_s;*

(ii) *the group U_ϵ is sharply transitive on the set of chambers opposite to c_ϵ;*

(iii) *we have $U_\epsilon = \langle U^\epsilon_s \mid s \in S \rangle$.*

Proof. Assertions (i) are (ii) are basic facts for arbitrary RGD-systems. Condition $(lco)_k$ implies Condition (lco) of [5] (see the final section of this appendix) and hence the set of chambers opposite c_ϵ is connected by Theorem 1.5 in that paper. Assertion (iii) follows now from Lemma 3.6. □

As a further consequence of Lemma 3.6 we obtain the following.

Proposition 4.3. *Let $b \in B_\epsilon$ and $n := \ell(c_{-\epsilon}, b(c_{-\epsilon}))$. Then $b \in X_\epsilon^{f(n)} H$ where $f(n) = nk^n$.*

Lemma 4.4. *If $g \in X^n$, then $\ell(c_+, g(c_+)) + \ell(c_-, g(c_-)) \leq n$.*

Proof. This follows by an easy induction on n. □

Theorem 4.5. *Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $n \mapsto nk^n$. Then*

$$B_+B_- \cap X^n \subseteq (U_+ \cap X_+^{f(n)})(H \cap X_+^{f(n)} X^n X_-^{f(n)})(U_- \cap X_-^{f(n)}).$$

Proof. Let $n \in \mathbf{N}$ and suppose that $g \in B_+B_- \cap X^n$. Let $b_+ \in B_+$ and $b_- \in B_-$ be such that $g = b_+b_-$ and observe that $g(c_-) = b_+(c_-)$. As $g \in X^n$ we have $\ell(c_+, g(c_+)) + \ell(c_-, g(c_-)) \leq n$ by Lemma 4.4, and therefore $\ell(c_-, b_+(c_-)) = \ell(c_-, g(c_-)) \leq n$.

As $\ell(c_-, b_+(c_-)) \leq n$ it follows from Proposition 4.3 that $b_+ \in X_+^{f(n)}H$. Now we put $d_+ := b_-(c_+)$ and observe that $b_+(d_+) = g(c_+)$ and in particular that $\ell(c_+, d_+) = \ell(b_+(c_+), b_+(d_+)) = \ell(c_+, g(c_+)) \leq n$. Applying again Proposition 4.3 we see that also $b_- \in X_-^{f(n)}H$. Since H normalizes X_ϵ for $\epsilon \in \{+,-\}$ we see that $g \in X_+^{f(n)} H X_-^{f(n)}$. For $\epsilon \in \{+,-\}$ let $x_\epsilon \in X_\epsilon^{f(n)}$ and $h \in H$ be such that $g = x_+hx_-$. Since $g \in X^n$ it follows that $h \in X_+^{f(n)} X^n X_-^{f(n)}$ which finishes the proof. □

5. Condition $(co)_k$ for spherical buildings of rank 2

Throughout this section (W, S) is a spherical Coxeter system of rank 2, the order of W is $2m \geq 4$ and $\mathcal{B} = (\mathcal{C}, \delta)$ is a thick building of type (W, S). Hence $\mathcal{B}$ is a generalized m-gon.

Definition 5.1. We say that $\mathcal{B}$ satisfies Condition (co) if the set of chambers opposite to a chamber is connected; we say that $\mathcal{B}$ satisfies Condition (dco) if the following is satisfied for each chamber $c \in \mathcal{C}$: Any two chambers opposite c can be joined by a gallery of length at most d in the set of chambers opposite to c. In other words, the diameter of c^{op} is at most d.

It is immediate that (dco) implies $(co)_{d+2m}$. Thus, in order to check, whether $\mathcal{B}$ satisfies Condition $(co)_k$ for some k it is sufficient to show that there is a natural number d such that $\mathcal{B}$ satisfies Condition (dco).

Lemma 5.2. *The following hold.*

(i) *If $m = 2$, then $\mathcal{B}$ satisfies Condition $(2co)$;*

(ii) *if $m = 3$, then $\mathcal{B}$ satisfies Condition $(4co)$;*

(iii) *if $m = 4$ and $\mathcal{B}$ is not the building $B_2(2)$, then $\mathcal{B}$ satisfies Condition $(8co)$.*

Proof. Assertions (i) and (ii) are straightforward. Assertion (iii) can be deduced from the arguments given in the proof of [6, Proposition 1.7.15]. □

For $m > 4$ there are free constructions by Abramenko ([1, Proposition 9 in Section II.2]) which show that one cannot expect similar results without further assumptions. His construction could probably even be modified to construct polygons satifying Condition (co) without satisfying (dco) for any $d \in \mathbf{N}$. Since we are only interested in twin buildings associated with RGD-systems, all rank 2 residues are Moufang. Thus, in view of the lemma above, we are left with the Moufang hexagons and Moufang octagons. In [2] it is shown that they satisfy Condition (co) in almost all cases. A natural strategy to establish (dco) for the Moufang hexagons and octagons is to analyze the proofs of Condition (co) for those polygons given in [2]. P. Abramenko and H. Van Maldeghem are convinced that this is indeed possible. They expect that Moufang hexagons with Condition (co) satisfy Condition $(12co)$; apparently the octagons are a bit more complicated. The author thanks them for providing these informations about this question.

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