Special rank one groups are perfect

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Abstract

We give a new proof of the fact that special (abstract) rank one groups with arbitrary unipotent subgroups of size at least 4 are perfect.

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MSC 2000: Primary 20E42; Secondary 51E42

1 Introduction

J. Tits [3] defined Moufang sets in order to axiomatize the linear algebraic groups of relative rank one. A closely related concept of so-called (abstract) rank one groups has been introduced by F. G. Timmesfeld [2].

Here a group $X$ is an (abstract) rank one group with unipotent subgroups $A$ and $B$, if $X = \langle A, B \rangle$ with $A$ and $B$ different subgroups of $X$, and (writing $A^b = b^{-1}Ab$)

for each $1 \neq a \in A$, there is an element $1 \neq b \in B$ such that $A^b = B^a$, and vice versa.

We emphasize that in contrast to Timmesfeld’s definition [2, p. 1] we do not assume that $A$ and $B$ are nilpotent. In an (abstract) rank one group $X$ with unipotent subgroups $A$ and $B$, the element $1 \neq b \in B$ with $A^b = B^a$ is uniquely determined for each $1 \neq a \in A$ (as $A \neq B$) and denoted by $b(a)$. Similarly we define $a(b)$.

We say $X$ is special, if $b(a^{-1}) = b(a)^{-1}$ for all $1 \neq a \in A$. This is equivalent with Timmesfeld’s original definition, see Timmesfeld [2, I (2.2), p. 17]. In this note, we give a new proof of the following result.

**Theorem 1.1.** Any special (abstract) rank one group with arbitrary unipotent subgroups of size at least 4 is perfect.
By [2, I (1.10), p. 13] this yields that a special (abstract) rank one group with abelian unipotent subgroups is either quasi-simple or isomorphic to $\SL_2(2)$ or $(P)\SL_2(3)$, as was conjectured by Timmesfeld [2, Remark, p. 26].

We remark that $X/Z(X)$ is the little projective group of a Moufang set and that is the point of view of T. De Medts, Y. Segev and K. Tent. In [1, Theorem 1.12] they prove that the little projective group $G$ of a special Moufang set $M(U, \tau)$ with $|U| \geq 4$ satisfies $U_\infty = [U_\infty, G_{0,\infty}]$. From this they deduce Theorem 1.1 above.

The proof of Theorem 1.1 given below is short, elementary and self-contained. It does not need a case differentiation whether $A$ is an elementary abelian 2-group or not. We show that $a_1(A \cap X') = a_2(A \cap X')$ for all $1 \neq a_1, a_2 \in A$ with $a_1a_2 \neq 1$. (Here $X' = [X, X]$ is the commutator subgroup of $X$.)

## 2 The proof of Theorem 1.1

Let $X$ be a special (abstract) rank one group with unipotent subgroups $A$ and $B$. For each $1 \neq a \in A$, we set $n(a) := ab(a)^{-1}a$. Then $B^{n(a)} = A$. As $b(a)^{-1} = A^{n(a)} = B$. Thus $n(a)n(a') \in H := N_X(A) \cap N_X(B)$, for all $1 \neq a, a' \in A$. For $1 \neq a \in A$, we have

$$B^{a^{-1}n(a)} = B^{a}. \quad (1)$$

**Lemma 2.1.** We have $B^{a_1a_2n(a_2^{-1})a_2} = A^{b(a_1)b(a_2)}$, for all $1 \neq a_1, a_2 \in A$.

**Proof.** By the definition of $n(a_2^{-1})$ we have $B^{a_1a_2n(a_2^{-1})a_2} = B^{a_1b(a_2^{-1})^{-1}}$. As $X$ is special, the left hand side of the claim equals $A^{b(a_1)b(a_2)}$. □

**Lemma 2.2.** We have $a_1 \in a_2(A \cap X')$, for all $1 \neq a_1, a_2 \in A$ with $a_1a_2 \neq 1$.

**Proof.** Let $1 \neq a_1, a_2 \in A$ with $a_1a_2 \neq 1$. We set $h := n(a_1a_2)n(a_2^{-1}) \in H$. By Lemma 2.1 and (1) we have $R := A^{b(a_1)b(a_2)} = B^{a_2^{-1}a_1^{-1}ha_2}$. As $a_2^{-1}a_1^{-1}ha_2 = ha_1^{-1}[a_1^{-1}, a_2][a_2^{-1}a_1^{-1}a_2, h][h, a_2]$ and $B^b = B$, we obtain that $R = B^{a_2^{-1}a_0}$ with $a_0 \in A \cap X'$.

Note that $b(a_1)b(a_2) = b(a_2)b(a_3)$, where $1 \neq a_3 = a(b_3) \in A$ with $1 \neq \neq a_3 = b(a_2^{-1}b(a_1)b(a_2) \in B$. Necessarily $a_2a_3 \neq 1$. Otherwise Lemma 2.1 implies that $R = A^{b(a_1)b(a_2)} = A^{b(a_2)b(a_3)} = B^{a_2^{-1}a_3} = A$, a contradiction as $R = B^a$ with $a \in A$. As above we have $R = A^{b(a_2)b(a_3)} = B^{a_2^{-1}a_4}$ with $a_4 \in A \cap X'$. As $N_A(B) = 1$, we obtain $a_1^{-1}(A \cap X') = a_2^{-1}(A \cap X')$. Thus the claim holds. □
When \( A \cap X' = 1 \), then Lemma 2.2 implies that \( A \subseteq \{1, a, a^{-1}\} \), where \( 1 \neq a \in A \); i.e., \( |A| \leq 3 \). Thus for \( |A| \geq 4 \), we may choose \( 1 \neq a \in A \cap X' \). By Lemma 2.2, we obtain \( A \subseteq a(A \cap X') \cup \{1, a^{-1}\} \subseteq A \cap X' \), as desired.

References


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