Special rank one groups are perfect

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Abstract

We give a new proof of the fact that special (abstract) rank one groups with arbitrary unipotent subgroups of size at least 4 are perfect.

`Keywords:` special (abstract) rank one group, Moufang set

`MSC 2000:` Primary 20E42; Secondary 51E42

1. Introduction

J. Tits [3] defined Moufang sets in order to axiomatize the linear algebraic groups of relative rank one. A closely related concept of so-called (abstract) rank one groups has been introduced by F. G. Timmesfeld [2].

Here a group X is an (abstract) rank one group with unipotent subgroups A and B, if $X = \langle A, B\rangle$ with A and B different subgroups of X, and (writing $A^b = b^{-1}Ab$)

> for each $1 \neq a \in A$, there is an element $1 \neq b \in B$ such that $A^b = B^a$, and vice versa.

We emphasize that in contrast to Timmesfeld's definition [2, p. 1] we do not assume that A and B are nilpotent. In an (abstract) rank one group X with unipotent subgroups A and B, the element $1 \neq b \in B$ with $A^b = B^a$ is uniquely determined for each $1 \neq a \in A$ (as $A \neq B$) and denoted by $b(a)$. Similary we define $a(b)$.

We say X is special, if $b(a^{-1}) = b(a)^{-1}$ for all $1 \neq a \in A$. This is equivalent with Timmesfeld's original definition, see Timmesfeld [2, I (2.2), p. 17]. In this note, we give a new proof of the following result.

Theorem 1.1. *Any special (abstract) rank one group with arbitrary unipotent subgroups of size at least 4 is perfect.*

By [2, I (1.10), p. 13] this yields that a special (abstract) rank one group with abelian unipotent subgroups is either quasi-simple or isomorphic to $\mathrm{SL}_2(2)$ or $(\mathrm{P})\mathrm{SL}_2(3)$, as was conjectured by Timmesfeld [2, Remark, p. 26].

We remark that $X/\mathrm{Z}(X)$ is the little projective group of a Moufang set and that is the point of view of T. De Medts, Y. Segev and K. Tent. In [1, Theorem 1.12] they prove that the little projective group G of a special Moufang set $\mathbb{M}(U,\tau)$ with $|U| \geq 4$ satisfies $U_\infty = [U_\infty, G_{0,\infty}]$. From this they deduce Theorem 1.1 above.

The proof of Theorem 1.1 given below is short, elementary and self-contained. It does not need a case differentiation whether A is an elementary abelian 2-group or not. We show that $a_1(A \cap X') = a_2(A \cap X')$ for all $1 \neq a_1, a_2 \in A$ with $a_1 a_2 \neq 1$. (Here $X' = [X, X]$ is the commutator subgroup of X.)

2. The proof of Theorem 1.1

Let X be a special (abstract) rank one group with unipotent subgroups A and B. For each $1 \neq a \in A$, we set $n(a) := ab(a)^{-1}a$. Then $B^{n(a)} = A$. As $b(a)^{-1} = b(a^{-1})$, also $A^{n(a)} = B$. Thus $n(a)n(a') \in H := N_X(A) \cap N_X(B)$, for all $1 \neq a, a' \in A$. For $1 \neq a \in A$, we have

$$B^{a^{-1}n(a)} = B^a. \tag{1}$$

Lemma 2.1. *We have $B^{a_1 a_2 n(a_2^{-1}) a_2} = A^{b(a_1)b(a_2)}$, for all $1 \neq a_1, a_2 \in A$.*

Proof. By the definition of $n(a_2^{-1})$ we have $B^{a_1 a_2 n(a_2^{-1}) a_2} = B^{a_1 b(a_2^{-1})^{-1}}$. As X is special, the left hand side of the claim equals $A^{b(a_1)b(a_2)}$. □

Lemma 2.2. *We have $a_1 \in a_2(A \cap X')$, for all $1 \neq a_1, a_2 \in A$ with $a_1 a_2 \neq 1$.*

Proof. Let $1 \neq a_1, a_2 \in A$ with $a_1 a_2 \neq 1$. We set $h := n(a_1 a_2) n(a_2^{-1}) \in H$. By Lemma 2.1 and (1) we have $R := A^{b(a_1)b(a_2)} = B^{a_2^{-1} a_1^{-1} h a_2}$. As

$$a_2^{-1} a_1^{-1} h a_2 = h a_1^{-1} [a_1^{-1}, a_2][a_2^{-1} a_1^{-1} a_2, h][h, a_2]$$

and $B^h = B$, we obtain that $R = B^{a_1^{-1} a_0}$ with $a_0 \in A \cap X'$.

Note that $b(a_1)b(a_2) = b(a_2)b(a_3)$, where $1 \neq a_3 = a(b_3) \in A$ with $1 \neq b_3 = b(a_2)^{-1} b(a_1) b(a_2) \in B$. Necessarily $a_2 a_3 \neq 1$. Otherwise Lemma 2.1 implies that $R = A^{b(a_1)b(a_2)} = A^{b(a_2)b(a_3)} = B^{n(a_3^{-1})a_3} = A$, a contradiction as $R = B^a$ with $a \in A$. As above we have $R = A^{b(a_2)b(a_3)} = B^{a_2^{-1} a_4}$ with $a_4 \in A \cap X'$. As $N_A(B) = 1$, we obtain $a_1^{-1}(A \cap X') = a_2^{-1}(A \cap X')$. Thus the claim holds. □

When $A \cap X' = 1$, then Lemma 2.2 implies that $A \subseteq \{1, a, a^{-1}\}$, where $1 \neq a \in A$; i.e., $|A| \leq 3$. Thus for $|A| \geq 4$, we may choose $1 \neq a \in A \cap X'$. By Lemma 2.2, we obtain $A \subseteq a(A \cap X') \cup \{1, a^{-1}\} \subseteq A \cap X'$, as desired.

References

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