Special rank one groups are perfect

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Abstract

We give a new proof of the fact that special (abstract) rank one groups with arbitrary unipotent subgroups of size at least 4 are perfect.

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1. Introduction

J. Tits [3] defined Moufang sets in order to axiomatize the linear algebraic groups of relative rank one. A closely related concept of so-called (abstract) rank one groups has been introduced by F. G. Timmesfeld [2].

Here a group $X$ is an (abstract) rank one group with unipotent subgroups $A$ and $B$, if $X = \langle A, B \rangle$ with $A$ and $B$ different subgroups of $X$, and (writing $A^b = b^{-1}Ab$)

$$\text{for each } 1 \neq a \in A, \text{ there is an element } 1 \neq b \in B \text{ such that } A^b = B^a, \text{ and vice versa.}$$

We emphasize that in contrast to Timmesfeld’s definition [2, p. 1] we do not assume that $A$ and $B$ are nilpotent. In an (abstract) rank one group $X$ with unipotent subgroups $A$ and $B$, the element $1 \neq b \in B$ with $A^b = B^a$ is uniquely determined for each $1 \neq a \in A$ (as $A \neq B$) and denoted by $b(a)$. Similarly we define $a(b)$.

We say $X$ is special, if $b(a^{-1}) = b(a)^{-1}$ for all $1 \neq a \in A$. This is equivalent with Timmesfeld’s original definition, see Timmesfeld [2, I (2.2), p. 17]. In this note, we give a new proof of the following result.

Theorem 1.1. Any special (abstract) rank one group with arbitrary unipotent subgroups of size at least 4 is perfect.
By [2, I (1.10), p. 13] this yields that a special (abstract) rank one group with abelian unipotent subgroups is either quasi-simple or isomorphic to \( \text{SL}_2(2) \) or \((\text{P})\text{SL}_2(3)\), as was conjectured by Timmesfeld [2, Remark, p. 26].

We remark that \( X/Z(X) \) is the little projective group of a Moufang set and that is the point of view of T. De Medts, Y. Segev and K. Tent. In [1, Theorem 1.12] they prove that the little projective group \( G \) of a special Moufang set \( M(U, \tau) \) with \( |U| \geq 4 \) satisfies \( U_\infty = [U_\infty, G_0, \infty] \). From this they deduce Theorem 1.1 above.

The proof of Theorem 1.1 given below is short, elementary and self-contained. It does not need a case differentiation whether \( A \) is an elementary abelian \( 2 \)-group or not. We show that \( a_1(A \cap X') = a_2(A \cap X') \) for all \( 1 \neq a_1, a_2 \in A \) with \( a_1a_2 \neq 1 \). (Here \( X' = [X, X] \) is the commutator subgroup of \( X \).)

2. The proof of Theorem 1.1

Let \( X \) be a special (abstract) rank one group with unipotent subgroups \( A \) and \( B \). For each \( 1 \neq a \in A \), we set \( n(a) := ab(a)^{-1}a \). Then \( B^{n(a)} = A \). As \( b(a)^{-1} = b(a^{-1}) \), also \( A^{n(a)} = B \). Thus \( n(a)n(a') \in H := N_X(A) \cap N_X(B) \), for all \( 1 \neq a, a' \in A \). For \( 1 \neq a \in A \), we have

\[
B^{a^{-1}n(a)} = B^a. \tag{1}
\]

Lemma 2.1. We have \( B^{a_1a_2n(a_2^{-1})a_2} = A^{b(a_1)b(a_2)} \), for all \( 1 \neq a_1, a_2 \in A \).

Proof. By the definition of \( n(a_2^{-1}) \) we have \( B^{a_1a_2n(a_2^{-1})a_2} = B^{a_1b(a_2^{-1})} \). As \( X \) is special, the left hand side of the claim equals \( A^{b(a_1)b(a_2)} \). \hfill \Box

Lemma 2.2. We have \( a_1 \in a_2(A \cap X') \), for all \( 1 \neq a_1, a_2 \in A \) with \( a_1a_2 \neq 1 \).

Proof. Let \( 1 \neq a_1, a_2 \in A \) with \( a_1a_2 \neq 1 \). We set \( h := n(a_1a_2)n(a_2^{-1}) \in H \). By Lemma 2.1 and (1) we have \( R := A^{b(a_1)b(a_2)} = B^{a_2^{-1}a_1^{-1}ha_2} \). As

\[
a_2^{-1}a_1^{-1}ha_2 = ha_1^{-1}[a_1^{-1}, a_2][a_2^{-1}a_1^{-1}a_2, h][h, a_2]
\]

and \( B^h = B \), we obtain that \( R = B^{a_1^{-1}a_0} \) with \( a_0 \in A \cap X' \).

Note that \( b(a_1)b(a_2) = b(a_2)b(a_3) \), where 1 \( \neq a_3 = a(3) \in A \) with \( 1 \neq b_3 = b(a_2)^{-1}b(a_1)b(a_2) \in B \). Necessarily \( a_2a_3 \neq 1 \). Otherwise Lemma 2.1 implies that \( R = A^{b(a_1)b(a_2)} = A^{b(a_2)b(a_3)} = B^{a_3^{-1}a_3} = A \), a contradiction as \( R = B^a \) with \( a \in A \). As above we have \( R = A^{b(a_2)b(a_3)} = B^{a_2^{-1}a_4} \) with \( a_4 \in A \cap X' \).

As \( N_A(B) = 1 \), we obtain \( a_1^{-1}(A \cap X') = a_2^{-1}(A \cap X') \). Thus the claim holds. \hfill \Box
When $A \cap X' = 1$, then Lemma 2.2 implies that $A \subseteq \{1, a, a^{-1}\}$, where $1 \neq a \in A$; i.e., $|A| \leq 3$. Thus for $|A| \geq 4$, we may choose $1 \neq a \in A \cap X'$. By Lemma 2.2, we obtain $A \subseteq a(A \cap X') \cup \{1, a^{-1}\} \subseteq A \cap X'$, as desired.

References


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