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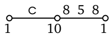
A rank 3 geometry for the O’Nan group connected with the Livingstone graph

Thomas Connor

Abstract

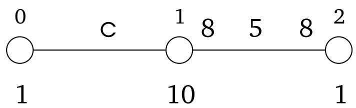
We construct a rank 3 geometry $\Gamma(O'N)$ over the diagram 

whose automorphism group is the O’Nan sporadic simple group. The maximal parabolic subgroups are the Janko group J_1, $2 \times S_5$ and the Mathieu group M_{11}. Our construction is based on a convenient amalgam of known geometries of rank 2 for J_1 and M_{11} extracted from the subgroup lattice of $O'N$.

Keywords: Diagram geometry, coset geometry, sporadic simple groups, O’Nan group, Livingstone graph
MSC 2000: 51E24, 20D08

1 Introduction

We provide a construction of a rank 3 coset geometry $\Gamma(O'N)$ for O’Nan’s sporadic simple group $O'N$ over the type set $I = \{0, 1, 2\}$. The geometry $\Gamma(O'N)$ belongs to the diagram



The construction is based on a convenient amalgam of known rank 2 coset geometries for the sporadic groups J_1 and M_{11}, and on a theorem due to Aschbacher [1]. We prove that $\Gamma(O'N)$ is flag-transitive, residually connected, residually weakly primitive, locally 2-transitive and that it satisfies the intersection property in rank 2. Finally we prove that the automorphism group of the geometry $\Gamma(O'N)$ is $O'N$, using a study of the truncation of $\Gamma(O'N)$ on its elements of type in $\{0,1\}$.

The paper is organized as follows. In section 2 we give the basic definitions needed to understand this paper. Then we state a theorem due to Aschbacher used later to prove that the geometry $\Gamma(O'N)$ is flag-transitive. In section 3, we introduce a characterization of the Janko group J_1 due to Perkel [13] using the Livingstone graph. Then we construct the geometry $\Gamma(O'N)$ in section 4. In the next three sections, we study properties of $\Gamma(O'N)$ and finally, in section 8, we prove that its automorphism group $\mathrm{Aut}(\Gamma(O'N))$ is $O'N$.

2 A theorem of Aschbacher

Following [4, Chapter 3, §3], a geometry is a 4-tuple $(X,*,t,I)$ with X a set of elements, $*$ a binary, reflexive and symmetric relation on X called incidence, I a set of types, and $t : X \to I$ a surjective map, called the type function, which assigns a type to each element of X. Moreover the incidence relation satisfies the following condition: given $x, y \in X$ such that $x*y$ and $t(x) = t(y)$, it follows that $x = y$. Finally, every maximal set of pairwise incident elements contains one element of each type.

In [15, 16], Tits introduces and develops the concept of a coset geometry. Given a group G and a family $\mathcal{F} = \{G_i \mid i \in I\}$ of subgroups of G, where I is a finite set, define $\Gamma(G,\mathcal{F})$ to be the pregeometry over I as follows. For $i \in I$ the set of elements of type i is the set of right cosets of G_i in G. Two elements G_ig and G_jh are incident, and we write $G_ig * G_jh$ for that, if and only if $G_ig \cap G_jh \neq \emptyset$. Clearly, G acts as a group of automorphisms of $\Gamma(G,\mathcal{F})$ under right multiplication.

Following Tits, Aschbacher [1] studied the interaction between incidence geometries and groups. He studies a useful criterion so as to determine whether a coset geometry with a string diagram is flag-transitive. In order to state and to apply easily this criterion, we introduce some notation used in [1]. Let G and $\mathcal{F} = \{G_i \mid i \in I\}$ be as above. For $J \subseteq I$, let $G_J = \cap_{j\in J} G_j$. We set $S_J = \{G_j \mid j \in J\}$, $\mathcal{F}_J = \{G_{J\cup\{i\}} \mid i \in I - J\}$ and $\Gamma(J) = \Gamma(G_J, \mathcal{F}_J)$. Obviously, S_J is a flag of type J in $\Gamma(G,\mathcal{F})$. A *diagram* on I in the sense of [1] is a tuple $\mathcal{D} = \{\mathcal{D}_J \mid J \subseteq I, |J| = 2\}$ such that $\mathcal{D}_J$ is a nonempty family of geometries on J. The *graph* of $\mathcal{D}$ is the undirected graph with vertex set I and i adjacent to j if $J = \{i,j\}$ is of order 2 and some member of $\mathcal{D}_J$ is not a generalized digon. The diagram $\mathcal{D}(G,\mathcal{F})$ of $\Gamma(G,\mathcal{F})$ is the diagram on I with $\mathcal{D}(G,\mathcal{F})_J = \{\Gamma(I-J)\}$. A graph on I is a *string* if we can order $I = \{0,1,\ldots,n\}$ so that the edges of the graph are $\{i, i+1\}$, $0 \le i < n$.

Now comes the result of Aschbacher.

Lemma 2.1 ([1, Main theorem]). *Let G be a group and let $\mathcal{F} = \{G_i \mid i \in I\}$ be a family of subgroups of G. Assume*

(a) *for each subset J of I of corank at least 2, $G_J = \langle G_{J\cup\{i\}} \mid i \in I - J\rangle$, and*

(b) *the connected components of the graph of $\mathcal{D}(G, \mathcal{F})$ are strings.*

Then

(1) *G is flag-transitive on $\Gamma(G, \mathcal{F})$.*

(2) *$\Gamma(G, \mathcal{F})$ is residually connected.*

(3) *For each $J \subseteq I$, the map*

$$(G_{J\cup\{i\}})z \mapsto G_i z, \quad i \in I - J, z \in G_J$$

is an isomorphism of $\Gamma(J)$ and the residue of S_J.

3 The Livingstone graph

The Livingstone graph was first described in [11] as an 11-regular graph of 266 vertices on which J_1 acts flag-transitively. This graph motivated further work consisting of constructions and characterizations as in [3, 13, 17]. We provide the characterization of the Livingstone graph given in [13].

3.1 Perkel's characterization

Let $\mathcal{L}$ be a connected, finite, regular undirected graph of girth 5 on a set Ω of vertices, with automorphism group $G \cong \mathrm{Aut}(\mathcal{L})$, satisfying the following four properties:

1. the valency of $\mathcal{L}$ is 11;
2. for any $x \in \Omega$, the point stabilizer in G is isomorphic to the simple group $L_2(11)$;
3. for some path (x, y, z) of length 2, $x, y, z \in \Omega$ $(x \neq z)$, G_{xyz} fixes a pentagon (a circuit of length 5) containing (x, y, z);
4. the vertices and edges fixed by an involution of G_{xyz} constitute a connected subgraph of $\mathcal{L}$.

Then $\mathcal{L}$ is a distance regular graph of 266 vertices with $G \cong J_1$.

3.2 The Livingstone graph as a coset geometry

In [3, Chapter 11, §7], Brouwer, *et al.* provide a construction of the Livingstone graph as a coset geometry for J_1 as follows. Let $F \cong J_1$. This group possesses exactly one conjugacy class of subgroups isomorphic to $L_2(11)$, and one conjugacy class of involutions. The centralizer of an involution is isomorphic to $2 \times A_5$. Now choose a subgroup $L < F$ isomorphic to $L_2(11)$ and an involution $i \in F$ such that $L_2(11) \cap C_F(i) = A_5$ (there are 11 such involutions) and set $C := C_F(i)$. Define the coset geometry $\Gamma(J_1) = \Gamma(F, \{L, C\})$ with Borel subgroup isomorphic to A_5. Then the underlying incidence geometry $\mathcal{L}$ is precisely the Livingstone graph. It is a regular distance transitive graph of degree 11 with $V(\mathcal{L}) = 266$ and $E(\mathcal{L}) = 1463$. The distance distribution diagram of the graph $\mathcal{L}$ is provided in Figure 1.

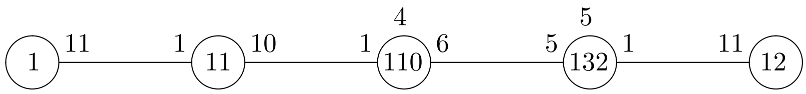


Figure 1: The distance distribution diagram of $\mathcal{L}$

The coset geometry $\Gamma(J_1)$ is flag-transitive, firm, residually connected, residually weakly primitive, locally 2-transitive and satisfies the intersection property. We provide the Buekenhout diagram of the dual geometry of $\Gamma(J_1)$ in Figure 2 (see Gottschalk and Leemans [6] or Leemans [10, Geometry 2.7]).

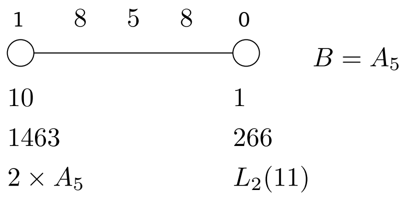


Figure 2: The dual diagram of $\Gamma(J_1)$.

4 Construction of a rank 3 coset geometry for $O'N$

The maximal subgroups of $O'N$ are known (see [14, 18, 19]). This group has, among others, one conjugacy class of subgroups isomorphic to J_1 and two conjugacy classes of subgroups isomorphic to M_{11} that are fused under the action of $\operatorname{Aut} O'N = O'N : 2$. The residually weakly primitive and locally 2-transitive coset geometries for M_{11} are known (see Leemans [10] and Dehon *et al.* [5]). Exactly one of those geometries has a Borel subgroup isomorphic to A_5. We call it $\Gamma(M_{11})$. Its underlying incidence structure is a complete graph on 12 vertices and it belongs to the Buekenhout diagram of Figure 3.

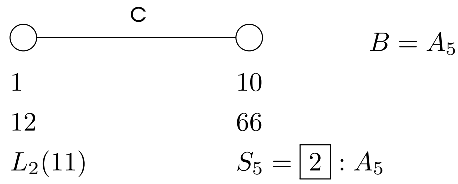


Figure 3: The diagram of $\Gamma(M_{11})$.

Since $\Gamma(M_{11})$ has a maximal parabolic subgroup isomorphic to $L_2(11)$, we see that $\Gamma(J_1)$ and $\Gamma(M_{11})$ might be residues of a residually weakly primitive and locally 2-transitive rank 3 geometry $\Gamma(O'N)$ for $O'N$.

Theorem 4.1. *The O'Nan group contains a boolean lattice* Ψ *of subgroups as in Figure 4, in which all inclusions are maximal except the inclusion of* $2 \times S_5$ *in* $O'N$. *There are exactly two such lattices in* $O'N$ *up to conjugacy. These two classes of boolean lattices are fused in* $\mathrm{Aut}(O'N)$.

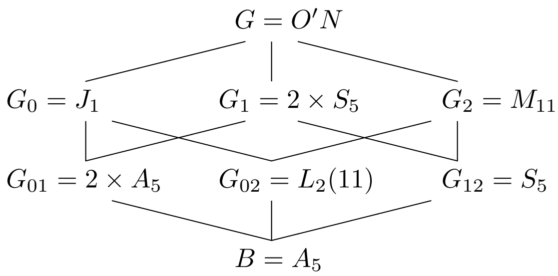


Figure 4: The Boolean lattice Ψ of $\Gamma(O'N)$.

Proof. Let Λ denote the subgroup lattice of $O'N$ available in [9]. There exists a unique class of conjugate subgroups isomorphic to J_1 in $O'N = G$. This is class number 6 in Λ. Let $J_1 \cong G_0 < G$. There is exactly one conjugacy class of subgroups isomorphic to $L_2(11)$ in $O'N$. This is class number 119 in Λ. Moreover, J_1 possesses exactly one conjugacy class of subgroups isomorphic to $L_2(11)$ and exactly one conjugacy class of subgroups isomorphic to $2 \times A_5$ (class number 277 in Λ). Let $G_{02} < G_0$ and $G_{01} < G_0$ be isomorphic respectively to $L_2(11)$ and $2 \times A_5$. J_1 has two conjugacy classes of subgroups isomorphic to A_5. Only one of them has subgroups that are contained simultaneously in a subgroup isomorphic to $L_2(11)$ and in a subgroup isomorphic to $2 \times A_5$. It corresponds to class number 370 in Λ. Let $A_5 \cong B = G_{01} \cap G_{02}$.

There are two conjugacy classes of subgroups isomorphic to M_{11} in $O'N$: they are classes number 23 and 24 in Λ. These classes are fused in $\mathrm{Aut}(O'N)$ $= O'N : 2$. In M_{11}, there is exactly one conjugacy class of subgroups isomorphic to $L_2(11)$. Choose $M_{11} \cong G_2 < G$ in any of the two conjugacy classes (without loss of generality, say class 23), such that $G_{02} < G_2$. Now let $S_5 \cong G_{12} < G_2$ be such that $B < G_{12}$. Hence, G_{12} is a subgroup of class 278 in Λ. Indeed, there are two classes of subgroups isomorphic to $2 \times S_5$ in $O'N$, but only class number 278 is in class 23 of subgroups isomorphic to M_{11}.

The subgroups G_{01} and G_{12} share a unique common minimal overgroup G_1. This subgroup is isomorphic to $2 \times S_5$ and belongs to class number 212 in Λ. Moreover, G_{12} has exactly two other minimal overgroups. They are isomorphic to M_{11} and thus they are themselves maximal subgroups of G. Since they do not contain G_{01}, we see that the subgroups G_{01} and G_{12} generate G_1 or G itself. We claim that they generate G_1. By way of contradiction, suppose they generate G. Since $G_{01} \cap G_{12} = B \cong A_5$, there would exist a thin rank 2 geometry for $O'N$ with a Borel $B \cong A_5$. Consequently, $B \trianglelefteq G$, i.e. $O'N$ would have a normal subgroup isomorphic to A_5, a contradiction.

In conclusion, the choice of a conjugacy class of subgroups M_{11} determines uniquely the boolean lattice Ψ of Figure 4. $\square$

Theorem 4.2. *The boolean lattice Ψ of Figure 4 determines a coset geometry $\Gamma(O'N)$ over the Buekenhout diagram of Figure 5: $\Gamma(O'N)$ is flag-transitive, firm, residually connected, residually weakly primitive, locally 2-transitive and satisfies the intersection property in rank 2 residues as defined in Leemans [10] and Pasini [12].*

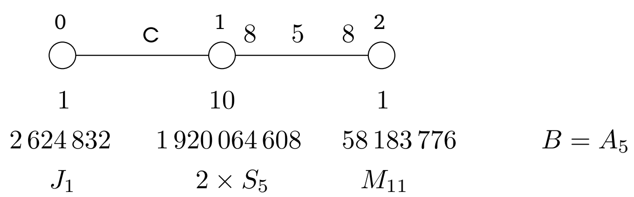


Figure 5: The diagram of $\Gamma(O'N)$

Proof. The boolean lattice Ψ of Figure 4 provides us with $G = O'N$ and a family of subgroups $\mathcal{F} = \{G_0, G_1, G_2\}$. Observe that $G_J = \langle G_{J\cup\{i\}} : i \in I - J\rangle$ for every $J \subseteq I$, $|J| \geq 2$ since all inclusions of subgroups in Ψ are maximal except the inclusion $G_1 < G$. Furthermore the geometries $\Gamma_0 = (G_0, \{G_{01}, G_{02}\})$ and $\Gamma_2 = (G_2, \{G_{02}, G_{12}\})$ are the geometries $\Gamma(J_1)$ and $\Gamma(M_{11})$ and belong to the Buekenhout diagrams of Figure 2 and Figure 3 respectively, while the geometry $\Gamma_1 = (G_1, \{G_{01}, G_{12}\})$ is obviously a generalized digon. Consequently the diagram $\mathcal{D}$ of $\Gamma(O'N) = (G, \{G_0, G_1, G_2\})$, in the sense of Aschbacher [1], is a string diagram. By Lemma 2.1, $\Gamma(O'N)$ is a flag-transitive and residually connected geometry. We deduce the Buekenhout diagram of $\Gamma(O'N)$ as it is drawn in Figure 5 by amalgamating the Buekenhout diagrams of Γ_0 and Γ_2.

Moreover $\Gamma(O'N)$ is residually weakly primitive because all inclusions of parabolic subgroups are maximal except the inclusion $G_1 < G$. Clearly, it is firm and it satisfies the intersection property in rank 2. Finally, it is locally 2-transitive as all of its rank 2 residues are. □

5 The line graph of the Livingstone graph

Let $\mathcal{G} = (V, E)$ be a graph with vertex set V and edge set E. We define the *line graph* of $\mathcal{G}$ as the graph $L(\mathcal{G}) = (E, X)$ which represents the adjacencies between the edges of $\mathcal{G}$ (see Harary [7, Chapter 8]). In other words, the vertex set of $L(\mathcal{G})$ is the edge set E of $\mathcal{G}$; two vertices of $L(\mathcal{G})$ are joined by an edge if and only if the corresponding edges in $\mathcal{G}$ share a common vertex in $\mathcal{G}$.

We require the next characterization of line graphs in the development of section 7.

Lemma 5.1 ([7, Theorem 8.4])**.** *A graph is a line graph if and only if its edges can be partitioned into complete subgraphs in such a way that no vertex lies in more than two of the subgraphs.*

Given a line graph $L(\mathcal{G})$ and a partition P as in Lemma 5.1, we recover the original graph $\mathcal{G}$ in a natural way: the vertex set of $\mathcal{G}$ is the set P; two vertices of G are now joined by an edge if and only if the corresponding cliques share a vertex in $L(\mathcal{G})$.

Accordingly, the line graph $L(\mathcal{L})$ of the Livingstone graph $\mathcal{L}$ is a graph of 1463 vertices. Each vertex lies in exactly two cliques of 11 points and there are 266 11-cliques in $L(\mathcal{L})$. The associated incidence geometry of rank 2 is a flag-transitive geometry that belongs to the diagram of Figure 6. Let us observe that J_1 acts on the set of 11-cliques and on the set of vertices of $L(\mathcal{L})$ (with incidence provided by symmetrized inclusion) in the same way as it acts on the Livingstone graph.

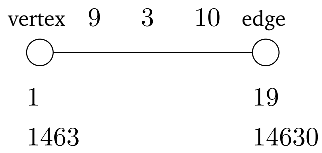


Figure 6: The diagram of $L(\mathcal{L})$.

6 Two graphs on 1463 vertices on which J_1 acts flag-transitively

Using MAGMA, it can be checked that there are exactly two nonisomorphic flag-transitive coset geometries $\Gamma_1(J_1)$ and $\Gamma_2(J_1)$ for J_1 of rank 2 with parabolic subgroups $G_0 = 2 \times A_5$ and $G_1 = D_{12}$ and with Borel subgroup isomorphic to S_3. They belong to the Buekenhout diagrams given in Figure 7.

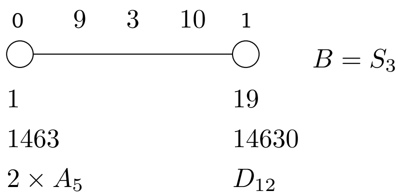


Figure 7.a: The diagram of $\Gamma_1(J_1)$.

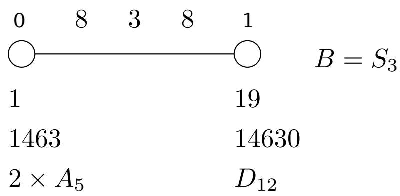


Figure 7.b: The diagram of $\Gamma_2(J_1)$.

Figure 7: Two nonisomorphic geometries with same parabolic subgroups

We readily see that $\Gamma_1(J_1)$ is the line graph of the Livingstone graph that we discussed in Section 5. The geometry $\Gamma_2(J_1)$ is a graph of degree 20 with 1463 vertices and 14630 edges.

7 A graph extension involving the line graph of the Livingstone graph

The truncation $\mathcal{T}$ of $\Gamma(O'N)$ on its elements of type in $\{0,1\}$ is a graph whose vertices are the elements of type 0 of $\Gamma(O'N)$ and whose edges are the elements of type 1 of $\Gamma(O'N)$. Every edge is incident to 2 vertices in $\Gamma(O'N)$ and every vertex is incident to 1463 edges as a consequence of Theorem 4.2.

The vertices of $\mathcal{T}$ correspond to the cosets of $H = J_1$ in $G = O'N$. Since J_1 is a maximal subgroup of G, the action of G on the vertices of $\mathcal{T}$ is primitive. Using MAGMA [2], we determine that 35 orbits appear when we fix a vertex p. They are distributed as follows:

$$1 + 1463^3 + 5852 + 12540^2 + 21945^2 + 29260^5 + 58520^2 + 87780^{12} + 175560^7.$$

There are 3 orbits of size 1463, say Ω_1, Ω_2 and Ω_3. Given $q_1 \in \Omega_1$, $q_2 \in \Omega_2$ and $q_3 \in \Omega_3$, we obtain that, up to relabelling, $\mathrm{Stab}_G\{p, q_1\} \cong \mathrm{Stab}_G\{p, q_2\} \cong 2 \times S_5$ and $\mathrm{Stab}_G\{p, q_3\} \cong 4 \times A_5$. However, $\mathrm{Stab}_G\{p, q_1\}$ and $\mathrm{Stab}_G\{p, q_2\}$ are not conjugate in G. In order to distinguish two subgroups isomorphic to $2 \times S_5$ that belong to two different conjugacy classes of subgroups in G, we denote a subgroup of the class corresponding to the 1-elements of $\Gamma(O'N)$ by $(2 \times S_5)_A$ (according to the construction of $\Gamma(O'N)$ in section 4, it corresponds to conjugacy class number 212 in Λ) and the other by $(2 \times S_5)_B$ (it corresponds to conjugacy class number 213 in Λ).

Let us denote by $p^\perp = (V_p, E_p)$ the induced subgraph on the neighborhood of p in $\mathcal{T}$. Using MAGMA, we build $p^\perp$ in the following way. Start with the set V_p of points of Ω_1 and an empty edge set E_p. For each (unordered) pair of points $\{x, y\}$ of V_p, if $\mathrm{Stab}_G\{x, y\} = (2 \times S_5)_A$ then add $\{x, y\}$ to the edge set E_p. We eventually determine that $p^\perp$ is 40-regular and that $|E_p| = 29260 = 2 \times 14630$. The clique number of $p^\perp$ is 11 and it occurs that $p^\perp$ has exactly 266 such cliques. Let us denote by E_p^1 the set of edges of $p^\perp$ occuring in at least one 11-clique. We check that each vertex of $p^\perp$ is in exactly 2 cliques of 11 points. Consequently, by Theorem 5.1, the graph (V_p, E_p^1) is isomorphic to $L(\mathcal{L}) = \Gamma_1(J_1)$. Let us now consider the set $E_p^2 = E_p - E_p^1$. The resulting graph (V_p, E_p^2) is isomorphic to $\Gamma_2(J_1)$. The set of edges of $p^\perp$ is thus the disjoint union of the edge sets of $\Gamma_1(J_1)$ and $\Gamma_2(J_1)$.

Intuitively, we can think of the induced subgraph on the neighborhood of a point of $\mathcal{T}$ as a superposition of the graphs $\Gamma_1(J_1)$ and $\Gamma_2(J_1)$.

Theorem 7.1. *The automorphism group of the graph $\mathcal{T}$ is isomorphic to $O'N$.*

Proof. Let T denote the automorphism group $\mathrm{Aut}\,\mathcal{T}$ of $\mathcal{T}$. Let p be a vertex of $\mathcal{T}$, let T_p denote $\mathrm{Stab}_T\, p$ and let $T(p)$ denote the orbit of p. By the orbit-stabilizer theorem, $|T| = |T_p| \times |T(p)|$. The automorphism group P of $p^\perp$ is isomorphic to J_1 and $P \leq T_p$. By contradiction, suppose that the inclusion is strict. Let $S = \mathrm{Stab}_T[p^\perp \cup p]$ denote the pointwise stabilizer of $p^\perp \cup p$ in T. Since $P \neq T_p$, we have that S is nontrivial. Let $q \in p^\perp$. By the developments of sections 6 and 7, we know that $q^\perp \cong p^\perp$ and $|p^\perp \cap q^\perp| = 40$. Moreover, those 40 vertices are fixed. Using MAGMA, it is checked easily that if we fix pointwise a vertex of $q^\perp$ and its neighborhood in $q^\perp$, then $q^\perp$ is fixed pointwise. Therefore, if p and

$p^\perp$ are pointwise fixed, all vertices at distance 2 from p are fixed as well, and, repeating the same argument, we show that all vertices are fixed. Hence S is trivial, a contradiction.

This means that fixing pointwise a vertex p and its neighborhood $p^\perp$ in $\mathcal{T}$ implies that the set of points of $\mathcal{T}$ at distance 2 from p is fixed pointwise. It follows that $\mathcal{T}$ itself is fixed pointwise. Consequently, $P = T_p$ and thus $J_1 \cong T_p$. Now we conclude:

$$\begin{aligned} |T| &= |T_p| \times |T(p)| \\ &= |J_1| \times \frac{|O'N|}{|J_1|} \\ &= |O'N|. \end{aligned}$$

By the developments of section 2, the O'Nan group acts as an automorphism group on $\mathcal{T}$ and thus $G \le T$. Since $|O'N| = |T|$, we have $O'N \cong G = T$. □

8 Automorphism group of $\Gamma(O'N)$

In order to prove the next theorem, we introduce the following notation. The incidence graph of $\Gamma(O'N)$ is a tripartite graph $\mathcal{X} = (X, *)$ with $X = X_0 \sqcup X_1 \sqcup X_2$ where X_i is the set of elements of $\Gamma(O'N)$ of type i, and $*$ is the incidence relation inherited from $\Gamma(O'N)$.

Theorem 8.1. *The automorphism group of the geometry $\Gamma(O'N)$ is isomorphic to $O'N$.*

Proof. We apply the same strategy as in the proof of Theorem 7.1. First, let us observe that $\Xi := \mathrm{Aut}\,\mathcal{X} = \mathrm{Aut}\,\Gamma(O'N)$.

Let us consider a vertex j of X_0 and denote by $j^\perp$ the induced subgraph in $\mathcal{X}$ on the neighborhood of j. This graph corresponds to the incidence graph of the residue in $\Gamma(O'N)$ of j. It is readily seen that $j^\perp$ is therefore the incidence graph of the Livingstone graph whose automorphism group is $J \cong J_1$. Consequently, $J \le \Xi_j$. By way of contradiction, assume the inclusion is strict. Let $S = \mathrm{Stab}_\Xi[j^\perp \cup j]$ denote the pointwise stabilizer of $j^\perp \cup j$ in Ξ. Since $J \neq \Xi_j$, we have that S is nontrivial. Observe that the induced subgraph $\mathcal{X}_{01}$ in $\mathcal{X}$ on the vertices of X_0 and X_1 is isomorphic to the incidence graph of $\mathcal{T}$. By Theorem 7.1, if we fix pointwise j and $j^\perp$ in $\mathcal{X}_{01}$, then all the vertices of $\mathcal{X}_{01}$ are fixed. It follows that the vertices of X_2 are pointwise fixed. Indeed, the truncation T_{12} of $\Gamma(O'N)$ on its elements of type $\{1, 2\}$ is a graph whose vertices correspond to the elements of X_2 and whose edges correspond to the elements of X_1. Since all the edges of this graph are fixed, it follows that all of its vertices are fixed because T_{12} is not a generalized digon.

We conclude by applying the orbit-stabilizer theorem:

$$\begin{aligned} |\Xi| &= |\Xi_j| \times |\Xi(j)| \\ &= |J_1| \times \frac{|O'N|}{|J_1|} \\ &= |O'N| \end{aligned}$$

and thus $\Xi \cong O'N$. □

9 Final remark

Buekenhout[1] observed that the geometry $\Gamma(O'N)$ is a truncation of a rank 5 geometry due to Ivanov and Shpectorov [8]. This observation leads to upcoming work of Buekenhout and the author.

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Thomas Connor

e-mail: tconnor@ulb.ac.be

BOURSIER F.R.I.A., UNIVERSITÉ LIBRE DE BRUXELLES, DÉPARTEMENT DE MATHÉMATIQUES - C.P.216, BOULEVARD DU TRIOMPHE, B-1050 BRUXELLES