A rank 3 geometry for the O’Nan group connected with the Livingstone graph

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Abstract

We construct a rank 3 geometry $\Gamma(O’N)$ over the diagram $\begin{array}{c}
\begin{array}{ccc}
\circ & \circ & \circ \\
0 & c & 1
\end{array}
\end{array}$ whose automorphism group is the O’Nan sporadic simple group. The maximal parabolic subgroups are the Janko group $J_1$, $2 \times S_5$ and the Mathieu group $M_{11}$. Our construction is based on a convenient amalgam of known geometries of rank 2 for $J_1$ and $M_{11}$ extracted from the subgroup lattice of $O’N$.

Keywords: Diagram geometry, coset geometry, sporadic simple groups, O’Nan group, Livingstone graph

MSC 2000: 51E24, 20D08

1. Introduction

We provide a construction of a rank 3 coset geometry $\Gamma(O’N)$ for O’Nan’s sporadic simple group $O’N$ over the type set $I = \{0, 1, 2\}$. The geometry $\Gamma(O’N)$ belongs to the diagram

\[
\begin{array}{c}
\begin{array}{ccc}
\circ & \circ & \circ \\
0 & c & 1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\circ & \circ & \circ \\
5 & 8 & 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\circ & \circ & \circ \\
10 & 1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\circ & \circ & \circ \\
8 & 5 & 8
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\circ & \circ & \circ \\
1 & 0
\end{array}
\end{array}
\end{array}
\]

The construction is based on a convenient amalgam of known rank 2 coset geometries for the sporadic groups $J_1$ and $M_{11}$, and on a theorem due to Aschbacher [1]. We prove that $\Gamma(O’N)$ is flag-transitive, residually connected, residually weakly primitive, locally 2-transitive and that it satisfies the intersection property in rank 2. Finally we prove that the automorphism group of
the geometry $\Gamma(O'N)$ is $O'N$, using a study of the truncation of $\Gamma(O'N)$ on its elements of type in $\{0,1\}$.

The paper is organized as follows. In section 2 we give the basic definitions needed to understand this paper. Then we state a theorem due to Aschbacher used later to prove that the geometry $\Gamma(O'N)$ is flag-transitive. In section 3, we introduce a characterization of the Janko group $J_1$ due to Perkel [13] using the Livingstone graph. Then we construct the geometry $\Gamma(O'N)$ in section 4. In the next three sections, we study properties of $\Gamma(O'N)$ and finally, in section 8, we prove that its automorphism group $\text{Aut}(\Gamma(O'N))$ is $O'N$.

## 2. A theorem of Aschbacher

Following [4, Chapter 3, §3], a geometry is a 4-tuple $(X, *, t, I)$ with $X$ a set of elements, * a binary, reflexive and symmetric relation on $X$ called incidence, $I$ a set of types, and $t : X \to I$ a surjective map, called the type function, which assigns a type to each element of $X$. Moreover the incidence relation satisfies the following condition: given $x, y \in X$ such that $x * y$ and $t(x) = t(y)$, it follows that $x = y$. Finally, every maximal set of pairwise incident elements contains one element of each type.

In [15, 16], Tits introduces and develops the concept of a coset geometry. Given a group $G$ and a family $\mathcal{F} = \{G_i \mid i \in I\}$ of subgroups of $G$, where $I$ is a finite set, define $\Gamma(G, \mathcal{F})$ to be the pregeometry over $I$ as follows. For $i \in I$ the set of elements of type $i$ is the set of right cosets of $G_i$ in $G$. Two elements $G_ig$ and $G_jh$ are incident, and we write $G_ig * G_jh$ for that, if and only if $G_ig \cap G_jh \neq \emptyset$. Clearly, $G$ acts as a group of automorphisms of $\Gamma(G, \mathcal{F})$ under right multiplication.

Following Tits, Aschbacher [1] studied the interaction between incidence geometries and groups. He studies a useful criterion so as to determine whether a coset geometry with a string diagram is flag-transitive. In order to state and to apply easily this criterion, we introduce some notation used in [1]. Let $G$ and $\mathcal{F} = \{G_i \mid i \in I\}$ be as above. For $J \subseteq I$, let $G_J = \cap_{j \in J} G_j$. We set $S_J = \{G_j \mid j \in J\}$, $\mathcal{F}_J = \{G_j \cap \{i\} \mid i \in I - J\}$ and $\Gamma(J) = \Gamma(G_J, \mathcal{F}_J)$. Obviously, $S_J$ is a flag of type $J$ in $\Gamma(G, \mathcal{F})$. A diagram on $I$ in the sense of [1] is a tuple $\mathcal{D} = \{D_J \mid J \subseteq I, |J| = 2\}$ such that $D_J$ is a nonempty family of geometries on $J$. The graph of $\mathcal{D}$ is the undirected graph with vertex set $I$ and $i$ adjacent to $j$ if $J = \{i, j\}$ is of order 2 and some member of $D_J$ is not a generalized digon. The diagram $\mathcal{D}(G, \mathcal{F})$ of $\Gamma(G, \mathcal{F})$ is the diagram on $I$ with $\mathcal{D}(G, \mathcal{F})_J = \{\Gamma(I - J)\}$. A graph on $I$ is a string if we can order $I = \{0, 1, \ldots, n\}$ so that the edges of the graph are $\{i, i + 1\}, 0 \leq i < n$. 
Now comes the result of Aschbacher.

**Lemma 2.1** ([1, Main theorem]). Let $G$ be a group and let $\mathcal{F} = \{G_i \mid i \in I\}$ be a family of subgroups of $G$. Assume

(a) for each subset $J$ of $I$ of corank at least 2, $G_J = \langle G_{J \cup \{i\}} \mid i \in I - J \rangle$, and

(b) the connected components of the graph of $\mathcal{D}(G, \mathcal{F})$ are strings.

Then

1. $G$ is flag-transitive on $\Gamma(G, \mathcal{F})$.
2. $\Gamma(G, \mathcal{F})$ is residually connected.
3. For each $J \subseteq I$, the map

$$(G_{J \cup \{i\}})z \mapsto G_i z, \quad i \in I - J, z \in G_J$$

is an isomorphism of $\Gamma(J)$ and the residue of $S_J$.

3. The Livingstone graph

The Livingstone graph was first described in [11] as an 11-regular graph of 266 vertices on which $J_1$ acts flag-transitively. This graph motivated further work consisting of constructions and characterizations as in [3, 13, 17]. We provide the characterization of the Livingstone graph given in [13].

3.1. Perkel’s characterization

Let $\mathcal{L}$ be a connected, finite, regular undirected graph of girth 5 on a set $\Omega$ of vertices, with automorphism group $G \cong \text{Aut}(\mathcal{L})$, satisfying the following four properties:

1. the valency of $\mathcal{L}$ is 11;

2. for any $x \in \Omega$, the point stabilizer in $G$ is isomorphic to the simple group $L_2(11)$;

3. for some path $(x, y, z)$ of length 2, $x, y, z \in \Omega$ ($x \neq z$), $G_{xyz}$ fixes a pentagon (a circuit of length 5) containing $(x, y, z)$;

4. the vertices and edges fixed by an involution of $G_{xyz}$ constitute a connected subgraph of $\mathcal{L}$.

Then $\mathcal{L}$ is a distance regular graph of 266 vertices with $G \cong J_1$. 
3.2. The Livingstone graph as a coset geometry

In [3, Chapter 11, §7], Brouwer, et al. provide a construction of the Livingstone graph as a coset geometry for $J_1$ as follows. Let $F \cong J_1$. This group possesses exactly one conjugacy class of subgroups isomorphic to $L_2(11)$, and one conjugacy class of involutions. The centralizer of an involution is isomorphic to $2 \times A_5$. Now choose a subgroup $L < F$ isomorphic to $L_2(11)$ and an involution $i \in F$ such that $L_2(11) \cap C_F(i) = A_5$ (there are 11 such involutions) and set $C := C_F(i)$. Define the coset geometry $\Gamma(J_1) = \Gamma(F, \{L, C\})$ with Borel subgroup isomorphic to $A_5$. Then the underlying incidence geometry $\mathcal{L}$ is precisely the Livingstone graph. It is a regular distance transitive graph of degree 11 with $V(\mathcal{L}) = 266$ and $E(\mathcal{L}) = 1463$. The distance distribution diagram of the graph $\mathcal{L}$ is provided in Figure 1.

![Figure 1: The distance distribution diagram of $\mathcal{L}$](image)

The coset geometry $\Gamma(J_1)$ is flag-transitive, firm, residually connected, residually weakly primitive, locally 2-transitive and satisfies the intersection property. We provide the Buekenhout diagram of the dual geometry of $\Gamma(J_1)$ in Figure 2 (see Gottschalk and Leemans [6] or Leemans [10, Geometry 2.7]).

![Figure 2: The dual diagram of $\Gamma(J_1)$](image)

4. Construction of a rank 3 coset geometry for $O'N$

The maximal subgroups of $O'N$ are known (see [14, 18, 19]). This group has, among others, one conjugacy class of subgroups isomorphic to $J_1$ and two conjugacy classes of subgroups isomorphic to $M_{11}$ that are fused under the action of $\text{Aut} O'N = O'N : 2$. The residually weakly primitive and locally 2-transitive
Coset geometries for $M_{11}$ are known (see Leemans [10] and Dehon et al. [5]). Exactly one of those geometries has a Borel subgroup isomorphic to $A_5$. We call it $\Gamma(M_{11})$. Its underlying incidence structure is a complete graph on 12 vertices and it belongs to the Buekenhout diagram of Figure 3.

Since $\Gamma(M_{11})$ has a maximal parabolic subgroup isomorphic to $L_2(11)$, we see that $\Gamma(J_1)$ and $\Gamma(M_{11})$ might be residues of a residually weakly primitive and locally 2-transitive rank 3 geometry $\Gamma(O'N)$ for $O'N$.

**Theorem 4.1.** The O'Nan group contains a boolean lattice $\Psi$ of subgroups as in Figure 4, in which all inclusions are maximal except the inclusion of $2 \times S_5$ in $O'N$. There are exactly two such lattices in $O'N$ up to conjugacy. These two classes of boolean lattices are fused in $\text{Aut}(O'N)$.

**Proof.** Let $\Lambda$ denote the subgroup lattice of $O'N$ available in [9]. There exists a unique class of conjugate subgroups isomorphic to $J_1$ in $O'N = G$. This is class number 6 in $\Lambda$. Let $J_1 \cong G_0 < G$. There is exactly one conjugacy class of subgroups isomorphic to $L_2(11)$ in $O'N$. This is class number 119 in $\Lambda$. Moreover, $J_1$ possesses exactly one conjugacy class of subgroups isomorphic to $L_2(11)$ and exactly one conjugacy class of subgroups isomorphic to $2 \times A_5$ (class number 277 in $\Lambda$). Let $G_{02} < G_0$ and $G_{01} < G_0$ be isomorphic respectively
to $L_2(11)$ and $2 \times A_5$. $J_1$ has two conjugacy classes of subgroups isomorphic to $A_5$. Only one of them has subgroups that are contained simultaneously in a subgroup isomorphic to $L_2(11)$ and in a subgroup isomorphic to $2 \times A_5$. It corresponds to class number 370 in $\Lambda$. Let $A_5 \cong B = G_{01} \cap G_{02}$.

There are two conjugacy classes of subgroups isomorphic to $M_{11}$ in $O'N$: they are classes number 23 and 24 in $\Lambda$. These classes are fused in $\text{Aut}(O'N) = O'N : 2$. In $M_{11}$, there is exactly one conjugacy class of subgroups isomorphic to $L_2(11)$. Choose $M_{11} \cong G_2 < G$ in any of the two conjugacy classes (without loss of generality, say class 23), such that $G_{02} < G_2$. Now let $S_5 \cong G_{12} < G_2$ be such that $B < G_{12}$. Hence, $G_{12}$ is a subgroup of class 278 in $\Lambda$. Indeed, there are two classes of subgroups isomorphic to $2 \times S_5$ in $O'N$, but only class number 278 is in class 23 of subgroups isomorphic to $M_{11}$.

The subgroups $G_{01}$ and $G_{12}$ share a unique common minimal overgroup $G_1$. This subgroup is isomorphic to $2 \times S_5$ and belongs to class number 212 in $\Lambda$. Moreover, $G_{12}$ has exactly two other minimal overgroups. They are isomorphic to $M_{11}$ and thus they are themselves maximal subgroups of $G$. Since they do not contain $G_{01}$, we see that the subgroups $G_{01}$ and $G_{12}$ generate $G_1$ or $G$ itself. We claim that they generate $G_1$. By way of contradiction, suppose they generate $G$. Since $G_{01} \cap G_{12} = B \cong A_5$, there would exist a thin rank 2 geometry for $O'N$ with a Borel $B \cong A_5$. Consequently, $B \trianglelefteq G$, i.e. $O'N$ would have a normal subgroup isomorphic to $A_5$, a contradiction.

In conclusion, the choice of a conjugacy class of subgroups $M_{11}$ determines uniquely the boolean lattice $\Psi$ of Figure 4.

**Theorem 4.2.** The boolean lattice $\Psi$ of Figure 4 determines a coset geometry $\Gamma(O'N)$ over the Buekenhout diagram of Figure 5: $\Gamma(O'N)$ is flag-transitive, firm, residually connected, residually weakly primitive, locally 2-transitive and satisfies the intersection property in rank 2 residues as defined in Leemans [10] and Pasini [12].

![Figure 5: The diagram of $\Gamma(O'N)$](image)

**Proof.** The boolean lattice $\Psi$ of Figure 4 provides us with $G = O'N$ and a family of subgroups $F = \{G_0, G_1, G_2\}$. Observe that $G_{J} = \langle G_{J \cup \{i\}} : i \in I - J \rangle$ for
every $J \subseteq I$, $|J| \geq 2$ since all inclusions of subgroups in $\Psi$ are maximal except the inclusion $G_1 < G$. Furthermore the geometries $\Gamma_0 = (G_0, \{G_{01}, G_{02}\})$ and $\Gamma_2 = (G_2, \{G_{02}, G_{12}\})$ are the geometries $\Gamma(J_1)$ and $\Gamma(M_{11})$ and belong to the Buekenhout diagrams of Figure 2 and Figure 3 respectively, while the geometry $\Gamma_1 = (G_1, \{G_{01}, G_{12}\})$ is obviously a generalized digon. Consequently the diagram $D$ of $\Gamma(O'N) = (G, \{G_0, G_1, G_2\})$, in the sense of Aschbacher [1], is a string diagram. By Lemma 2.1, $\Gamma(O'N)$ is a flag-transitive and residually connected geometry. We deduce the Buekenhout diagram of $\Gamma(O'N)$ as it is drawn in Figure 5 by amalgamating the Buekenhout diagrams of $\Gamma_0$ and $\Gamma_2$.

Moreover $\Gamma(O'N)$ is residually weakly primitive because all inclusions of parabolic subgroups are maximal except the inclusion $G_1 < G$. Clearly, it is firm and it satisfies the intersection property in rank 2. Finally, it is locally 2-transitive as all of its rank 2 residues are.

5. The line graph of the Livingstone graph

Let $\mathcal{G} = (V, E)$ be a graph with vertex set $V$ and edge set $E$. We define the line graph of $\mathcal{G}$ as the graph $L(\mathcal{G}) = (E, X)$ which represents the adjacencies between the edges of $\mathcal{G}$ (see Harary [7, Chapter 8]). In other words, the vertex set of $L(\mathcal{G})$ is the edge set $E$ of $\mathcal{G}$; two vertices of $L(\mathcal{G})$ are joined by an edge if and only if the corresponding edges in $\mathcal{G}$ share a common vertex in $\mathcal{G}$.

We require the next characterization of line graphs in the development of section 7.

Lemma 5.1 ([7, Theorem 8.4]). A graph is a line graph if and only if its edges can be partitioned into complete subgraphs in such a way that no vertex lies in more than two of the subgraphs.

Given a line graph $L(\mathcal{G})$ and a partition $P$ as in Lemma 5.1, we recover the original graph $\mathcal{G}$ in a natural way: the vertex set of $\mathcal{G}$ is the set $P$; two vertices of $G$ are now joined by an edge if and only if the corresponding cliques share a vertex in $L(\mathcal{G})$.

Accordingly, the line graph $L(\mathcal{L})$ of the Livingstone graph $\mathcal{L}$ is a graph of 1463 vertices. Each vertex lies in exactly two cliques of 11 points and there are 266 11-cliques in $L(\mathcal{L})$. The associated incidence geometry of rank 2 is a flag-transitive geometry that belongs to the diagram of Figure 6. Let us observe that $J_1$ acts on the set of 11-cliques and on the set of vertices of $L(\mathcal{L})$ (with incidence provided by symmetrized inclusion) in the same way as it acts on the Livingstone graph.
6. Two graphs on 1463 vertices on which $J_1$ acts flag-transitively

Using MAGMA, it can be checked that there are exactly two nonisomorphic flag-transitive coset geometries $\Gamma_1(J_1)$ and $\Gamma_2(J_1)$ for $J_1$ of rank 2 with parabolic subgroups $G_0 = 2 \times A_5$ and $G_1 = D_{12}$ and with Borel subgroup isomorphic to $S_3$. They belong to the Buekenhout diagrams given in Figure 7.

![Diagram of $\Gamma_1(J_1)$](image1.png)

![Diagram of $\Gamma_2(J_1)$](image2.png)

Figure 7: Two nonisomorphic geometries with same parabolic subgroups

We readily see that $\Gamma_1(J_1)$ is the line graph of the Livingstone graph that we discussed in Section 5. The geometry $\Gamma_2(J_1)$ is a graph of degree 20 with 1463 vertices and 14630 edges.

7. A graph extension involving the line graph of the Livingstone graph

The truncation $T$ of $\Gamma(O'N)$ on its elements of type in $\{0, 1\}$ is a graph whose vertices are the elements of type 0 of $\Gamma(O'N)$ and whose edges are the elements of type 1 of $\Gamma(O'N)$. Every edge is incident to 2 vertices in $\Gamma(O'N)$ and every vertex is incident to 1463 edges as a consequence of Theorem 4.2.
The vertices of $\mathcal{T}$ correspond to the cosets of $H = J_1$ in $G = O'N$. Since $J_1$ is a maximal subgroup of $G$, the action of $G$ on the vertices of $\mathcal{T}$ is primitive. Using Magma [2], we determine that 35 orbits appear when we fix a vertex $p$. They are distributed as follows:

$$1 + 1463^3 + 5852 + 12540^2 + 21945^2 + 29260^5 + 58520^2 + 87780^{12} + 17560^7.$$ 

There are 3 orbits of size 1463, say $\Omega_1$, $\Omega_2$ and $\Omega_3$. Given $q_1 \in \Omega_1$, $q_2 \in \Omega_2$ and $q_3 \in \Omega_3$, we obtain that, up to relabelling, $\text{Stab}_G\{p, q_1\} \cong \text{Stab}_G\{p, q_2\} \cong 2 \times S_5$ and $\text{Stab}_G\{p, q_3\} \cong 4 \times A_5$. However, $\text{Stab}_G\{p, q_1\}$ and $\text{Stab}_G\{p, q_2\}$ are not conjugate in $G$. In order to distinguish two subgroups isomorphic to $2 \times S_5$ that belong to two different conjugacy classes of subgroups in $G$, we denote a subgroup of the class corresponding to the $p$-eventually determine that $p$-clique number of $\{\} \cup \{p, q_1, p, q_2\}$ | $\{\} \cup \{p, q_1, p, q_3\}$ | $\{\} \cup \{p, q_2, p, q_3\}$ of points of $\Omega$. Using $\text{Magma}$, we determine that, up to relabelling, $\text{Stab}_G\{p, q_1\} \cong \text{Stab}_G\{p, q_2\}$, if $p$-denote a subgroup of the class corresponding to the $1$-elements of $\Gamma(O'N)$ by $(2 \times S_5)_A$ (according to the construction of $\Gamma(O'N)$ in section 4, it corresponds to conjugacy class number 212 in $\Lambda$) and the other by $(2 \times S_5)_B$ (it corresponds to conjugacy class number 213 in $\Lambda$).

Let us denote by $p^\perp = (V_p, E_p)$ the induced subgraph on the neighborhood of $p$ in $\mathcal{T}$. Using Magma, we build $p^\perp$ in the following way. Start with the set $V_p$ of points of $\Omega_1$ and an empty edge set $E_p$. For each (unordered) pair of points $\{x, y\}$ of $V_p$, if $\text{Stab}_G\{x, y\} = (2 \times S_5)_A$ then add $\{x, y\}$ to the edge set $E_p$. We eventually determine that $p^\perp$ is $40$-regular and that $|E_p| = 29260 = 2 \times 14630$. The clique number of $p^\perp$ is $11$ and it occurs that $p^\perp$ has exactly $266$ such cliques. Let us denote by $E_p^1$ the set of edges of $p^\perp$ occuring in at least one $11$-clique. We check that each vertex of $p^\perp$ is in exactly $2$ cliques of $11$ points. Consequently, by Theorem 5.1, the graph $(V_p, E_p^1)$ is isomorphic to $L(L) = \Gamma_1(J_1)$. Let us now consider the set $E_p^2 = E_p - E_p^1$. The resulting graph $(V_p, E_p^2)$ is isomorphic to $\Gamma_2(J_1)$. The set of edges of $p^\perp$ is thus the disjoint union of the edge sets of $\Gamma_1(J_1)$ and $\Gamma_2(J_1)$.

Intuitively, we can think of the induced subgraph on the neighborhood of a point of $\mathcal{T}$ as a superposition of the graphs $\Gamma_1(J_1)$ and $\Gamma_2(J_1)$.

**Theorem 7.1.** The automorphism group of the graph $\mathcal{T}$ is isomorphic to $O'N$.

**Proof.** Let $T$ denote the automorphism group $\text{Aut} \mathcal{T}$ of $\mathcal{T}$. Let $p$ be a vertex of $\mathcal{T}$, let $T_p$ denote $\text{Stab}_T p$ and let $T(p)$ denote the orbit of $p$. By the orbit-stabilizer theorem, $|T| = |T_p| \times |T(p)|$. The automorphism group $P$ of $p^\perp$ is isomorphic to $J_1$ and $P \leq T_p$. By contradiction, suppose that the inclusion is strict. Let $S = \text{Stab}_T[p^\perp \cup p]$ denote the pointwise stabilizer of $p^\perp \cup p$ in $T$. Since $P \neq T_p$, we have that $S$ is nontrivial. Let $q \in p^\perp$. By the developments of sections 6 and 7, we know that $q^\perp \cong p^\perp$ and $|p^\perp \cap q^\perp| = 40$. Moreover, those 40 vertices are fixed. Using Magma, it is checked easily that if we fix pointwise a vertex of $q^\perp$ and its neighborhood in $q^\perp$, then $q^\perp$ is fixed pointwise. Therefore, if $p$ and
are pointwise fixed, all vertices at distance 2 from \( p \) are fixed as well, and, repeating the same argument, we show that all vertices are fixed. Hence \( S \) is trivial, a contradiction.

This means that fixing pointwise a vertex \( p \) and its neighborhood \( p \perp \) in \( T \) implies that the set of points of \( T \) at distance 2 from \( p \) is fixed pointwise. It follows that \( T \) itself is fixed pointwise. Consequently, \( P = T_p \) and thus \( J_1 \cong T_p \). Now we conclude:

\[
|T| = |T_p| \times |T(p)| \\
= |J_1| \times \frac{|O'N|}{|J_1|} \\
= |O'N|.
\]

By the developments of section 2, the O'Nan group acts as an automorphism group on \( T \) and thus \( G \leq T \). Since \( |O'N| = |T| \), we have \( O'N \cong G = T \). \( \Box \)

8. Automorphism group of \( \Gamma(O'N) \)

In order to prove the next theorem, we introduce the following notation. The incidence graph of \( \Gamma(O'N) \) is a tripartite graph \( \mathcal{X} = (X, *) \) with \( X = X_0 \sqcup X_1 \sqcup X_2 \) where \( X_i \) is the set of elements of \( \Gamma(O'N) \) of type \( i \), and \( * \) is the incidence relation inherited from \( \Gamma(O'N) \).

**Theorem 8.1.** The automorphism group of the geometry \( \Gamma(O'N) \) is isomorphic to \( O'N \).

**Proof.** We apply the same strategy as in the proof of Theorem 7.1. First, let us observe that \( \Xi := \text{Aut} \mathcal{X} = \text{Aut} \Gamma(O'N) \).

Let us consider a vertex \( j \) of \( X_0 \) and denote by \( j \perp \) the induced subgraph in \( \mathcal{X} \) on the neighborhood of \( j \). This graph corresponds to the incidence graph of the residue in \( \Gamma(O'N) \) of \( j \). It is readily seen that \( j \perp \) is therefore the incidence graph of the Livingstone graph whose automorphism group is \( J \cong J_1 \). Consequently, \( J \leq \Xi_j \). By way of contradiction, assume the inclusion is strict. Let \( S = \text{Stab}_\Xi[j \perp \cup j] \) denote the pointwise stabilizer of \( j \perp \cup j \) in \( \Xi \). Since \( J \neq \Xi_j \), we have that \( S \) is nontrivial. Observe that the induced subgraph \( \mathcal{X}_{01} \) in \( \mathcal{X} \) on the vertices of \( X_0 \) and \( X_1 \) is isomorphic to the incidence graph of \( T \). By Theorem 7.1, if we fix pointwise \( j \) and \( j \perp \) in \( \mathcal{X}_{01} \), then all the vertices of \( \mathcal{X}_{01} \) are fixed. It follows that the vertices of \( X_2 \) are pointwise fixed. Indeed, the truncation \( T_{12} \) of \( \Gamma(O'N) \) on its elements of type \( \{1, 2\} \) is a graph whose vertices correspond to the elements of \( X_2 \) and whose edges correspond to the elements of \( X_1 \). Since
all the edges of this graph are fixed, it follows that all of its vertices are fixed because $T_{12}$ is not a generalized digon.

We conclude by applying the orbit-stabilizer theorem:

$$|\Xi| = |\Xi_j| \times |\Xi(j)|$$
$$= |J_1| \times \frac{|O'N|}{|J_1|}$$
$$= |O'N|$$

and thus $\Xi \cong O'N$. □

9. Final remark

Buekenhout\(^1\) observed that the geometry $\Gamma(O'N)$ is a truncation of a rank 5 geometry due to Ivanov and Shpectorov [8]. This observation leads to upcoming work of Buekenhout and the author.

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