Innovations in Incidence Geometry
Volume 13 (2013), Pages 97–105
ISSN 1781-6475

A near projective plane of order 6

Alan R. Prince

Abstract

We construct a near projective plane of order 6 containing 15 pure lines by extending the dual of the point-line geometry of $\mathsf{PG}(3,2)$.

Keywords: projective plane of order 6, spreads of $\mathsf{PG}(3,2)$, Conwell heptads

MSC 2000: 05B25, 51E30

1 Introduction

A *near projective plane of order* n is a point-line incidence structure with (n^2+n+1) points and lines, each line incident with $(n+1)$ points and each point incident with $(n+1)$ lines, in which any two distinct lines intersect in at most two points and any two distinct points are both incident with at most two lines. Any projective plane of order n is clearly a near projective plane of order n. The dual of a near projective plane of order n is also a near projective plane of order n. In a near projective plane, if the number of lines meeting a given line ℓ in $0,1,2$ points is x,y,z respectively, then counting in two different ways the number of flags (P,m), where P is a point on ℓ and m is a line through P with $m \neq \ell$, gives $y+2z = n(n+1)$. Since $x+y+z = n^2+n$, it follows that $x = z$. Thus, in a near projective plane, if the number of lines intersecting a given line in two points is λ, then the number of lines not intersecting the given line is also λ. A line with $\lambda = 0$ is called a *pure* line. If all lines are pure, then we have a projective plane.

Near projective planes are relatively easy to construct from sets which are almost difference sets. For example, a near projective plane of order 6 can be constructed by expanding $\{0,1,2,6,14,17,24\}$ modulo 43. However, such examples are cyclic and have no pure lines (unless they are projective planes).

In this paper, we are interested in near projective planes which contain a relatively large proportion of pure lines and we describe the construction of a near projective plane of order 6 with 15 pure lines.

R.H. Bruck conjectured that, in some instances, it might be possible to construct a projective plane of order $q(q+1)$ by extending the point-line geometry of $\mathsf{PG}(3,q)$ [1, 2]. Of course, some cases can be immediately ruled out by the Bruck–Ryser theorem [3], in particular the case $q=2$. In these cases, it is interesting to ask whether a near projective plane extension exists. The purpose of this paper is to consider the case $q=2$. We extend the dual of the point-line geometry of $\mathsf{PG}(3,2)$ to a near projective plane of order 6, in which the 15 initial lines are pure lines. The additional 28 lines of the extension each have $\lambda=6$.

We give an elementary description of most of the point-line incidences of the near projective plane in terms of unordered pairs and triples from the set $\{1,2,3,\dots,7\}$. However, a particular Fano plane defined on $\{1,2,3,\dots,7\}$, with the points on each line cyclically ordered, is used in the definition of the remaining point-line incidences and plays a crucial role in the structure of the near projective plane.

2 Preliminary results

We use the theory of $\mathsf{PG}(3,2)$ described in [8, Section 17.5]. Conwell [4] showed that there are eight sets of seven pairwise non-conjugate points (*heptads*) off the Klein quadric in $\mathsf{PG}(5,2)$. Any two heptads have a unique point in common and each of the 28 points off the quadric is contained in two heptads. The existence of the Conwell heptads allows an 8-set description of much of the structure of $\mathsf{PG}(3,2)$. The lines of $\mathsf{PG}(3,2)$ can be identified with the partitions of $\{1,2,3,\dots,8\}$ into two 4-sets. Since one of a pair of the complementary 4-sets must be of the form $8ijk$, the lines can also be represented by an unordered triple ijk of elements of $\{1,2,\dots,7\}$. Two distinct lines intersect if and only if the corresponding triples have precisely one element in common.

There are 30 ways to choose a set of seven triples from $\{1,2,\dots,7\}$, such as $\{123,145,167,246,257,347,356\}$, to define a Fano plane (a projective plane of order 2). This is easily checked by enumeration but also follows from the fact that the collineation group of the Fano plane has order 168 and hence index 30 in S_7. These 30 sets of triples form a single orbit under the action of S_7, but two orbits of length 15 under the action of A_7. The 15 points of $\mathsf{PG}(3,2)$ can be identified with the 15 sets of triples in one A_7-orbit and the 15 planes can be identified with the 15 sets of triples in the other orbit. Two sets of triples in the same orbit have precisely one triple in common. Two sets of triples in different orbits are either disjoint or have three triples in common: a given set of triples meets seven sets of the other orbit in three triples and is disjoint to the remaining eight. If we identify the points of $\mathsf{PG}(3,2)$ with the orbit of $\{123, 145, 167, 246, 257, 347, 356\}$, then the Fano plane F defined by $\{124,\ 235,\ 346,\ 457,\ 156,\ 267,\ 137\}$ corresponds to a plane of $\mathsf{PG}(3,2)$ (with lines identified with the triples defining F).

The 56 spreads in $\mathsf{PG}(3,2)$ can be identified with the unordered triples from $\{1,2,3,\ldots,8\}$. The spread ijk contains the lines $ijka$ (5 choices for a). Distinct spreads intersect in $2,0,1$ lines according as the corresponding triples have $0,1,2$ elements, respectively, in common. A set of spreads, which pairwise have at most one line in common, corresponds to a set of triples from $\{1,2,\ldots,8\}$, which pairwise have non-empty intersection. By the Erdös–Ko–Rado theorem [6], the maximum number of triples with this property is 21 and occurs if and only if we take all the triples containing some fixed element of $\{1,2,\ldots,8\}$. If we identify $8ij$ with ij, then the 56 spreads can be identified with the 21 unordered pairs ij from $\{1,2,\ldots,7\}$ together with the 35 unordered triples from $\{1,2,\ldots,7\}$. The spread ij contains the lines ija (5 choices for a) and the spread ijk contains the lines corresponding to ijk and the four triples disjoint to ijk. The 21 spreads of $\mathsf{PG}(3,2)$ identified with pairs ij consitute one of the 8 possible maximal sets of spreads of size 21 with the property that any two have at most one line in common. The action of the collineation group $\mathsf{PSL}(4,2)$ of $\mathsf{PG}(3,2)$ on these 8 sets of spreads explains the exceptional isomorphism $\mathsf{PSL}(4,2) \cong A_8$ (cf. Edge [5]). The spreads corresponding to the triples of any Fano plane form a parallelism of $\mathsf{PG}(3,2)$ and there are $30 \cdot 8 = 240$ possible parallelisms, corresponding to the sets of triples defining a Fano plane on each of the subsets of size 7 of $\{1,2,\ldots,8\}$ (see [8]).

3 The construction

We define a point-line incidence structure based on the set $\{1,2,\ldots,7\}$. The definition depends on the specific Fano plane

$$F = \{124,\ 235,\ 346,\ 457,\ 561,\ 672,\ 713\},$$

together with a cyclic ordering of the points on each line, given by the 3-cycles

$$(124),\ (235),\ (346),\ (457),\ (561),\ (672),\ (713).$$

The lines of the incidence structure are of three types:

(a) the Fano planes in the A_7-orbit of $\{123, 145, 167, 246, 257, 347, 356\}$;

(b) the unordered pairs ij from $\{1, 2, \dots, 7\}$;

(c) the unordered triples defining the lines of the Fano plane F.

Thus, we have a total of 43 lines: fifteen of type (a), twenty-one of type (b) and seven of type (c).

The points of the incidence structure are of three types:

(a) the unordered triples from $\{1, 2, \dots, 7\}$;

(b) the 3-cycles ordering the lines of F;

(c) the Fano plane F itself.

Thus, we have a total of 43 points: thirty-five of type (a), seven of type (b) and a single point of type (c).

Incidence is defined as follows. A line of type (a) is incident precisely with the seven points of type (a), given by the seven triples of the corresponding Fano plane. A line ij of type (b) is incident precisely with the five points of type (a) corresponding to the triples containing ij and with two points of type (b) to be defined. A line ijk of type (c) is incident precisely with the five points of type (a), given by the triple ijk and the triples disjoint to ijk, with the point of type (b) given by the 3-cycle ordering the line ijk and with the unique point of type (c). Incidence between lines and points of type (b) is defined as follows. The cyclic ordering (pqr) of the points of the line pqr of F induces the following ordering on each pair of points of the line: $(p, q), (q, r), (r, p)$. Identifying the lines of type (b) with *ordered* pairs, the line (i, j) of type (b) is incident with the point (pqr) of type (b) if and only if $i \in \{p, q, r\}$ and $j \notin \{p, q, r\}$. Thus, the line (i, j) is incident with two points of type (b), given by the 3-cycles ordering the further two lines through i, other than the line joining i and j. The point (pqr) is incident with six lines of type (b), given by the ordered pairs with first component one of p, q, r and second component *not* one of p, q, r.

As described in Section 2, the points of type (a) can be identified with the lines of $\mathsf{PG}(3, 2)$ and the lines of type (a) can be identified with the points of $\mathsf{PG}(3, 2)$. Thus, the lines and points of type (a), with the incidence defined, form an incidence structure isomorphic to the dual of the point-line geometry of $\mathsf{PG}(3, 2)$. Furthermore, the lines of type (b) and (c) can be identified with spreads of $\mathsf{PG}(3, 2)$ and the defined incidence of these lines with points of type (a) corresponds to the natural incidence of lines and spreads of $\mathsf{PG}(3, 2)$ (a line is incident with a spread if the line is contained in the spread).

The following observation on the structure of the Fano plane F is useful. If ijk is a line of F and a, b, c, d are the points not on ijk, then abc, abd, acd, bcd are *triangles* of F (sets of three points, no three collinear). Moreover, given any triangle pqr, there is a unique line not containing p, q, r. Thus, the 28

triangles of F are partioned into seven sets of four of type $\{abc, abd, acd, bcd\}$, corresponding to the lines ijk of F.

Consider the complete graph K_7 with vertices corresponding to the points of F. The cyclic ordering of the lines of F induces a direction on each edge of the K_7. The directed K_7 has a group of automorphisms of order 21 generated by the permutations (1234567) and $(124)(365)$ (the *doubling map* modulo 7). This automorphism group permutes the triangles of the directed K_7 in orbits of lengths $7, 7, 21$. The triangles in both orbits of length 7 have all vertices with out-degree 1 and in-degree 1 (in the induced directed subgraph). We call these *cyclic* triangles since an edge directed into a vertex is always followed by an edge directed out of the vertex. The triangles in the orbit of length 21 have a vertex with out-degree 2, a vertex with in-degree 2 and a vertex with out-degree 1 and in-degree 1 and are called *non-cyclic* triangles. The 35 triangles of the K_7 correspond to the 7 lines and 28 triangles of F. Thus, the triangles of F are of two types with respect to the cyclic ordering of the lines of F: the cyclic triangles and the non-cyclic triangles. Hence, the 35 points of type (a) can be further subdivided according as the associated triple is (i) a line of F, (ii) a cyclic triangle of F or (iii) a non-cyclic triangle of F. We have seven points of type (a)(i), seven points of type (a)(ii) and twenty-one points of type (a)(iii).

Lemma 3.1. *If two distinct lines intersect in more than one point, then they intersect in precisely two points. This happens in, and only in, the following cases. Two lines $(i, j), (k, l)$ of type* (b)*, where either $i = k$ or $j = l$, or a line (i, j) of type* (b) *and a line pqr of type* (c)*, where $\{i, j\}$ and $\{p, q, r\}$ are disjoint.*

Proof. This follows from a careful check of the various cases in the incidence relation. In the case of lines (i, j) and (i, l), there are precisely two points of intersection: the point ijl of type (a) and a point of type (b) given by the 3-cycle ordering the third line through i other than the two lines joining i to j and i to l. Note that ijl is a non-cyclic triangle of F since i is the first component in both ordered pairs. In the case of lines (i, j) and (k, j), there are precisely two points of intersection: the point ijk of type (a) and a point of type (b) given by the 3-cycle ordering the line joining i and k. Note that ijk is a non-cyclic triangle of F since j is the second component in both ordered pairs. In the case of a line (i, j) and a line pqr, where $\{i, j\}$ and $\{p, q, r\}$ are disjoint, there are precisely two points of intersection: the points ijk and ijl of type (a), where i, j, k, l are the points of F not on the line pqr. $\square$

The following tables summarise all the occurences of pairs of doubly intersecting lines, which correspond to submatrices J_2, the 2×2 matrix with each entry 1, in the incidence matrix of the near projective plane, as given in the proof of Lemma 3.1. The rows are labelled by lines and the columns by points.

	ijl	(ipq)
(i,j)	1	1
(i,l)	1	1

	ijk	$(ikp)^{\pm 1}$
(i,j)	1	1
(k,j)	1	1

	ijk	ijl
(i,j)	1	1
pqr	1	1

In the first table, (ipq) is the 3-cycle corresponding to the third line through i, other than the two lines joining i to j and i to l. In the second table, ikp is the line joining i and k and $(ikp)^{\pm 1}$ denotes the corresponding 3-cycle, either (ikp) or (kip). In the first table, ijl is a non-cyclic triangle, as is ijk in the second table. In the third table, one of the triangles ijk, ijl may be cyclic since, of the four triangles involving points not on the line pqr, one is cyclic and three are non-cyclic.

Theorem 3.2. *The incidence structure is a near projective plane of order* 6*. The* 15 *lines of type* (a) *are pure lines and each of the remaining* 28 *lines has* $\lambda = 6$.

Proof. The incidence structure contains 43 points and 43 lines. By definition, each line is incident with seven points. Each point of type (a) is incident with three lines of type (a) (each line of $\mathsf{PG}(3,2)$ contains three points), with three lines of type (b) (each triple contains three pairs) and with a unique line of type (c) (each line of $\mathsf{PG}(3,2)$ is in a unique spread of the parallelism of $\mathsf{PG}(3,2)$ corresponding to the lines of F). Each point of type (b) is incident with six lines of type (b) (the point (pqr) is incident with the lines corresponding to the six ordered pairs with first component p, q or r and second component *not* p, q or r) and with a unique line of type (c). The single point of type (c) is incident precisely with the seven lines of type (c). Thus, each point is incident with seven lines.

From Lemma 3.1, distinct lines in the incidence structure intersect in at most two points. From the description of the points of intersection of doubly intersecting lines, any two distinct points incident two lines cannot both be incident with any further line. Thus, any two distinct points are incident with at most two lines. Hence, the incidence structure is a near projective plane of order 6.

From the proof of Lemma 3.1, neither of two doubly intersecting lines is type (a). Thus, the 15 lines of type (a) are pure lines. Each line (i,j) of type (b) intersects four lines of type (b) in two points (two lines of type (i,l) and two lines of type (k,j)) and two lines of type (c) in two points (corresponding to the two lines of F not passing through i nor j). Each line pqr of type (c) intersects six lines of type (b) in two points, namely those corresponding to pairs of points neither of which lies on the line pqr of F. Thus, each of the 28 lines of type (b) or type (c) intersects precisely 6 other lines in two points. □

Remark 3.3. It follows from the definition of the incidence relation that distinct lines of type (b) corresponding to the ordered pairs (i, j) and (k, l) are skew if and only if i, j, k, l are distinct and either ikl is a line or kij is a line. Thus, the line (i, j) is skew to precisely four lines (k, l). In addition, line (i, j) is skew to the two lines of type (c) corresponding to the two lines of F passing through j but not i. The line ijk of type (c) is skew to the six lines of type (b) corresponding to the unique ordered pairs with i, j or k as second component determined by each of the six lines of F distinct from ijk.

Theorem 3.4. *In the dual near projective plane of order 6, there are eight pure lines, seven lines with $\lambda = 3$, twenty-one lines with $\lambda = 5$ and seven lines with $\lambda = 6$.*

Proof. From the description of the points of intersection of doubly intersecting lines in the proof of Lemma 3.1 (summarised in the tables above), distinct points of type (a) are incident with two lines if and only if they correspond to triangles of F with the same complementary line (see the third table). Thus, in the incidence matrix, each of the 28 columns corresponding to points of type (a)(ii) or type (a)(iii) has inner product 2 with precisely three other columns of these 28. In addition, each column corresponding to points of type (a)(iii) has inner product 2 with precisely two columns corresponding to points of type (b) (see the first and second tables). In fact, if abc is a non-cyclic triangle of F with edges $a \to b$, $b \to c$, $a \to c$, then column abc has inner product 2 with columns (abp) and (aqr), where p is the third point on the line ab and (aqr) is the 3-cycle ordering the third line through a, other than ab and ac. By symmetry, each column corresponding to a point of type (b) has inner product 2 with precisely six columns corresponding to points of type (a)(iii). These are all the instances of columns with inner product 2. Thus, in the dual near projective plane, the eight lines corresponding to points of type (a)(i) or (c) have $\lambda = 0$, the seven lines corresponding to points of type (a)(ii) have $\lambda = 3$, the twenty-one lines corresponding to points of type (a)(iii) have $\lambda = 5$ and the seven lines corresponding to points of type (b) have $\lambda = 6$. □

Remark 3.5. In the dual plane, two lines corresponding to points of type (a)(ii) or (a)(iii) are skew if and only if the corresponding triples are disjoint triangles of F. Thus, each of these lines is skew to three other lines of this type. In addition, the line corresponding to the point abc of type (a)(iii), where abc is a non-cyclic triangle of F with edges $a \to b$, $b \to c$, $a \to c$, is skew to the lines corresponding to the points (bcq) and (cpr), where q is the third point on the line bc and (cpr) is the 3-cycle ordering the third line through c, other than ca

and cb (p is the third point on the line ab). This describes all pairs of skew lines in the dual plane.

4 Conclusion

The investigation of the case $q = 2$ gives some insight into the general problem of extending the point-line geometry of $\mathsf{PG}(3, q)$ (or its dual) to a near projective plane of order $q(q + 1)$. It was shown in [9] (see also [10]) that a projective plane extension exists only if $\mathsf{PG}(3, q)$ contains a *covering set* of spreads (a set of spreads with the property that any given pair of skew lines is contained in a unique spread of the set). This was used to show the non-existence of a projective plane extension in the case $q = 3$ [7, 9]. A *partial covering set* of spreads in $\mathsf{PG}(3, q)$ is a set of spreads, which pairwise have at most one line in common. As noted in Section 2, the maximum size of a partial covering set of spreads in $\mathsf{PG}(3, 2)$ is 21 and hence no covering set of spreads exists (a covering set would contain 28 spreads). The construction of the near projective plane of order 6 utilises a set of 28 spreads, containing a maximum partial covering set, to define the extra lines extending the dual point-line geometry of $\mathsf{PG}(3, 2)$. This indicates the role that sets of spreads, which are close to being covering sets, might play in the general problem.

References

[1] **R.H. Bruck**, Construction problems of finite projective planes, *Combinatorial Mathematics and its Applications* (Univ. of North Carolina at Chapell Hill, 1967), University of North Carolina Press, 1969, 426–514.

[2] ______, Construction problems in finite projective space, *Finite Geometric Structures and their Applications* (A. Barlotti, Ed.), Cremonese, Rome, 1973, 107–188.

[3] **R.H. Bruck** and **H.J. Ryser**, The nonexistence of certain finite projective planes, *Canad. J. Math* **1** (1949), 88–93.

[4] **G.M. Conwell**, The 3-space $\mathsf{PG}(3, 2)$ and its group, *Ann. of Math.* **11** (1910), 60–76.

[5] **W.L. Edge**, The geometry of the linear fractional group LF(4,2), *Proc. London Math. Soc.* **4** (1954), 317–342.

[6] **P. Erdös**, **C. Ko** and **R. Rado**, Extremal problems among subsets of a set, *Q. J. Math.* **12** (1961), 313–318.

[7] **M. Hall** and **R. Roth**, On a conjecture of R.H. Bruck, *J. Combin. Theory Ser. A* **37** (1984), 22–31.

[8] **J.W.P. Hirschfeld**, *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford, 1985.

[9] **A.R. Prince**, Projective planes of order 12 and $\mathsf{PG}(3,3)$, *Discrete Math.* **208/209** (1999), 477–483.

[10] ———, Covering sets of spreads in $\mathsf{PG}(3,q)$, *Discrete Math.* **238** (2001), 131–136.

Alan R. Prince

Maxwell Institute for Mathematical Sciences, Heriot-Watt University, Edinburgh EH14 4AS, Scotland

e-mail: `a.r.prince@hw.ac.uk`