Embeddings of orthogonal Grassmannians

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Abstract

In this paper I survey a number of recent results on projective and Veronesean embeddings of orthogonal Grassmannians and propose a few conjectures and problems.

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1. Organization of the paper

This paper consists four sections, besides the present one. Section 2 is a survey of basic notions on projective and Veronesean embeddings and generating sets, to be freely used in the rest of the paper. In Section 3 we recall the definition of orthogonal Grassmannians and define two embeddings for each of them, called the Grassmann and Weyl embedding, respectively. In the special case of a dual polar space of type B_n those two embeddings are Veronesean and one more Veronesean embedding can be defined, which we call the Veronese-spin embedding. In Section 4 we compare the embeddings defined in Section 3: the Veronese-spin embedding and the Weyl embedding of a dual polar space are always isomorphic while the Grassmann and Weyl embeddings are isomorphic when the underlying field has characteristic different from 2. In the case of characteristic 2 things are more complicated. Most of Section 4 is devoted to a discussion of that case. Section 5 is devoted to universality. Nearly all results discussed in Sections 4 and 5 are taken from Cardinali and Pasini [9, 10, 11]. We will omit their proofs, referring the reader to [9, 10, 11] for them, but we shall give short sketches of the proofs whenever it will be possible, so that the reader can get at least a flavor of the arguments used in them.

2. Basics on embeddings and generation

Throughout this section $\Gamma = (\mathcal{P}, \mathcal{L})$ is a point-line geometry, $\mathcal{P}$ is its set of points and $\mathcal{L}$ its set of lines. We assume that the collinearity graph of Γ is connected, that no two distinct lines of Γ meet in more than one point and every line of Γ has at least three points. The second condition is necessary for Γ to admit a projective or laxly projective embedding while the third condition is necessary for the existence of a projective or Veronesean embedding. Connectedness is a sensible requirement. Anyway, I don't like disconnected objects.

2.1. Projective embeddings

Given Γ as above, a *projective embedding* of Γ in the projective space $\mathrm{PG}(V)$ of a vector space V is an injective mapping ε from the point-set $\mathcal{P}$ of Γ to the set of points of $\mathrm{PG}(V)$ such that ε maps every line of Γ surjectively onto a line of $\mathrm{PG}(V)$ and $\varepsilon(\mathcal{P})$ spans $\mathrm{PG}(V)$.

Henceforth we will freely switch from $\mathrm{PG}(V)$ to V. In particular, we will commit the abuse of regarding V instead of $\mathrm{PG}(V)$ as the codomain of ε, thus writing $\varepsilon : \Gamma \to V$ instead of $\varepsilon : \Gamma \to \mathrm{PG}(V)$. Accordingly, if $p \in \mathcal{P}$ we regard $\varepsilon(p)$ as a 1-dimensional subspace of V and we take the dimension of V as the *dimension* $\dim(\varepsilon)$ of ε.

If $\mathbb{F}$ is the underlying division ring of V then we say that ε is *defined over* $\mathbb{F}$, also that ε is a *projective* $\mathbb{F}$*-embedding* for short. If all projective embeddings of Γ are defined over the same division ring $\mathbb{F}$ (as it is the case for all geometries to be considered in this paper), then Γ is said to be *defined over* $\mathbb{F}$.

Given two projective $\mathbb{F}$-embeddings $\varepsilon_1 : \Gamma \to V_1$ and $\varepsilon_2 : \Gamma \to V_2$, a *morphism* $f : \varepsilon_1 \to \varepsilon_2$ from ε_1 to ε_2 is a semi-linear mapping $f : V_1 \to V_2$ such that $\varepsilon_2 = f \cdot \varepsilon_1$. Note that, since $\langle \varepsilon_2(\mathcal{P}) \rangle = V_2$, the equality $\varepsilon_2 = f \cdot \varepsilon_1$ forces $f : V_1 \to V_2$ to be surjective. If f is bijective then f is called an *isomorphism*. When ε_1 and ε_2 are isomorphic we write $\varepsilon_1 \cong \varepsilon_2$. Note that, if a morphism $f : \varepsilon_1 \to \varepsilon_2$ exists then f is uniquely determined by ε_1 and ε_2 modulo scalars (see e.g. Pasini and Van Maldeghem [23, Proposition 9]; we warn that the connectedness of Γ is essential to obtain this result). If a morphism exists from ε_1 to ε_2 then we write $\varepsilon_1 \geq \varepsilon_2$ and we say that ε_2 is a *morphic image* of ε_2. When $\varepsilon_1 \geq \varepsilon_2$ but $\varepsilon_1 \not\cong \varepsilon_2$ we write $\varepsilon_1 > \varepsilon_2$.

Given an embedding $\varepsilon : \Gamma \to V$, let K be a subspace of V satisfying the following:

($Q1$) if $x, y \in \mathcal{P}$ (possibly $x = y$) then $\langle \varepsilon(x), \varepsilon(y) \rangle \cap K = 0$. In particular $K \cap \varepsilon(x) = 0$ for every point $x \in \mathcal{P}$.

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Then the function $\varepsilon/K : \Gamma \to V/K$ mapping $x \in \mathcal{P}$ to $\langle \varepsilon(x), K\rangle/K$ is an embedding of Γ in V/K and the canonical projection of V onto V/K is a morphism from ε to ε/K. We call ε/K a *quotient* of ε. We also say that K *defines a quotient* of ε. Note that $(Q1)$ implies the following:

$(Q2)$ For $p \in \mathcal{P}$ and $l \in \mathcal{L}$, if $p \notin l$ then $\varepsilon(p) \cap \langle \varepsilon(l), K\rangle = 0$.

This remark may look futile here, but in the next subsection it will appear in a different light.

If $f : V_1 \to V_2$ is a morphism from ε_1 to ε_2 then $\ker(f)$ defines a quotient of ε_1 and $\varepsilon_2 \cong \varepsilon_1/\ker(f)$. In view of this fact, we take the liberty to call ε_2 a *quotient* of ε_1 (a *proper quotient* if $\varepsilon_1 \not\cong \varepsilon_2$) thus taking the word 'quotient' as a synonym of 'morphic image'. We also call the morphism $f : \varepsilon_1 \to \varepsilon_2$ the *projection* of ε_1 onto ε_2.

Following Kasikova and Shult [20], we say that a projective embedding of Γ is *relatively universal* when it is not a proper quotient of any other projective embedding of Γ. Every projective embedding ε of Γ admits a *hull* $\tilde{\varepsilon}$, uniquely determined up to isomorphism by the following properties (Ronan [25]): $\tilde{\varepsilon}$ is a projective embedding of Γ, ε is a quotient of $\tilde{\varepsilon}$ and we have $\tilde{\varepsilon} \geq \varepsilon'$ for every projective embedding ε' of Γ such that $\varepsilon' \geq \varepsilon$. Clearly, $\tilde{\varepsilon}$ is relatively universal. A projective embedding is relatively universal if and only if it is its own hull.

The hull $\tilde{\varepsilon}$ of an embedding $\varepsilon : \Gamma \to V$ can be constructed as follows ([25]): denote by $\mathcal{F}$ the set of flags of Γ, and consider the presheaf

$$(\{V_x\}_{x\in\mathcal{P}\cup\mathcal{L}}, \{\iota_{p,l}\}_{(p,l)\in\mathcal{F}}),$$

where if $x \in \mathcal{P}$ then $V_x = \varepsilon(x)$, if $x \in \mathcal{L}$ then $V_x = \langle\varepsilon(x)\rangle = \langle \cup_{p\in x} V_p\rangle$ $(= \cup_{p\in x} V_p)$ and $\iota_{p,l}$ is the inclusion embedding of V_p in V_l, for every flag $(p,l) \in \mathcal{F}$. Let J be the subspace of $\oplus_{x\in\mathcal{P}\cup\mathcal{L}}\varepsilon(x)$ spanned by the vectors $v - \iota_{p,l}(v)$ for every flag $(p,l) \in \mathcal{F}$ and every vector $v \in \varepsilon(p)$. Put $\widetilde{V} = (\oplus_{x\in\mathcal{P}\cup\mathcal{L}}\varepsilon(x))/J$ and define the mapping $\tilde{\varepsilon} : \Gamma \to \widetilde{V}$ by the following clause: $\tilde{\varepsilon}(p) = \langle \varepsilon(p), J\rangle/J$ for every point $p \in \mathcal{P}$. Then $\tilde{\varepsilon}$ is the hull of ε.

A projective $\mathbb{F}$-embedding ε of Γ is *absolutely universal* if all projective $\mathbb{F}$-embeddings of Γ are quotients of ε. The absolutely universal projective $\mathbb{F}$-embedding of Γ, if it exists, is uniquely determined up to isomorphisms. It is the hull of all projective $\mathbb{F}$-embeddings of Γ. Obviously, it is relatively universal. Up to isomorphisms, it is the unique relatively universal $\mathbb{F}$-embedding of Γ. So, if we know that Γ admits the absolutely universal projective $\mathbb{F}$-embedding we may say that a given projective $\mathbb{F}$-embedding of Γ is or is not *universal*, dropping the adverbs 'relatively' or 'absolutely'. We refer the reader to Kasikova and Shult [20] for a very far-reaching sufficient condition for the existence of the absolutely universal projective embedding.

Given an embedding $\varepsilon : \Gamma \to V$ and an automorphism g of Γ, a *lifting* of g through ε is a semi-linear mapping $\varepsilon(g) : V \to V$ such that $\varepsilon(g) \cdot \varepsilon = \varepsilon \cdot g$. The lifting $\varepsilon(g)$ of g, if it exists, is uniquely determined modulo scalars. Clearly, it is invertible. Given a group G acting on Γ as a group of automorphisms, the embedding ε is said to be *G-homogeneous* if for every $g \in G$ the automorphism of Γ induced by g lifts through ε to a semi-linear map of V.

If ε is absolutely universal then it is $\mathrm{Aut}(\Gamma)$-homogeneous. Let ε_1 and ε_2 be projective embeddings of Γ and $f : \varepsilon_1 \to \varepsilon_2$ a morphism. Suppose that ε_1 is G-homogeneous for some $G \leq \mathrm{Aut}(\Gamma)$ and $\ker(f)$ is stabilized by G. Then ε_2 is G-homogeneous.

2.2. Veronesean embeddings

Various definitions of Veronesean embeddings have appeared in the literature, sometimes under different names. The reader may see [27] for one of them. The underlying idea of each of those definitions is that lines are mapped onto conics, or even plane arcs, but that idea can be worked out in different ways, thus obtaining different definitions. We choose the following one.

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point line geometry satisfying the assumptions made at the beginning of this section. A *Veronesean embedding* of Γ in (the projective space $\mathrm{PG}(V)$ of) a vector space V defined over a commutative division ring (namely a field) $\mathbb{F}$ is an injective mapping ε from the point-set $\mathcal{P}$ of Γ to the set of points of $\mathrm{PG}(V)$ such that ε maps every line of Γ onto a non-singular conic of $\mathrm{PG}(V)$, for every line $l \in \mathcal{L}$ the projective plane spanned by the conic $\varepsilon(l)$ intersects the set $\varepsilon(\mathcal{P})$ just in $\varepsilon(l)$, and $\varepsilon(\mathcal{P})$ spans $\mathrm{PG}(V)$.

All definitions and conventions stated for projective embeddings in Subsection 2.1 can be rephrased for Veronesean embeddings word for word. In particular, we can still construct the hull $\tilde{\varepsilon}$ of ε starting from $(\{V_x\}_{x \in \mathcal{P} \cup \mathcal{L}}, \{\iota_{p,l}\}_{(p,l) \in \mathcal{F}})$ where $V_x = \varepsilon(x)$ when $x \in \mathcal{P}$ and $V_x = \langle \varepsilon(x) \rangle = \langle \cup_{p \in x} V_x \rangle$ when $x \in \mathcal{L}$. As before, for every flag $(p, l) \in \mathcal{F}$ the mapping $\iota_{p,l}$ is the inclusion embedding of V_p in V_l. Now $\varepsilon(l) = \{\varepsilon(p)\}_{p \in l}$ is a non-singular conic of the projective plane $\mathrm{PG}(V_l)$ while in the previous subsection $\mathrm{PG}(V_l)$ is a projective line and $\varepsilon(l)$ is its set of points. However this difference has no effect on the construction of $\widetilde{V}$ (see [22], where the hulls as defined here are called *linear hulls*).

We have claimed that all what is said in Subsection 2.1 for projective embeddings can be carried to Veronesean embeddings with nearly no modification. This is true, but when dealing with quotients more precise remarks are necessary. Indeed, certain situations can occur now that are not paralleled by anything occurring for projective embeddings.

Given a Veronesean embedding $\varepsilon : \Gamma \to V$, let K be a subspace of V satisfying conditions $(Q1)$ and $(Q2)$ of Subsection 2.1. Note that now we must assume $(Q2)$ in addition to $(Q1)$, since in the present context $(Q2)$ does not follow from $(Q1)$. (Examples where $(Q1)$ holds but $(Q2)$ fails to hold are easy to construct.) By $(Q1)$, the function ε/K mapping $p \in \mathcal{P}$ to $\langle\varepsilon(p), K\rangle/K$ is an injective mapping from $\mathcal{P}$ to the point-set of $\mathrm{PG}(V/K)$. By $(Q2)$, given a point p and a line l of Γ, if $p \notin l$ then $(\varepsilon/K)(p) \cap \langle(\varepsilon/K)(l)\rangle = 0$, no matter if $\langle(\varepsilon/K)(l)\rangle$ is 2- or 3-dimensional. Moreover $\langle(\varepsilon/K)(\mathcal{P})\rangle = V/K$, since $\langle\varepsilon(\mathcal{P})\rangle = V$ by assumption. Sticking to the terminology introduced in the previous subsection, we still call ε/K a *quotient* of ε and we say that K *defines a quotient* of ε. However, when the underlying field $\mathbb{F}$ of ε has characteristic 2 the embedding ε/K might not be Veronesean, as we shall see in a few lines. Nevertheless, such non-Veronesean quotients are worth of consideration.

Note firstly that ε/K is a Veronesean embedding whenever K also satisfies the following:

$(Q3)$ $K \cap \langle\varepsilon(l)\rangle = 0$ for every line l of Γ.

Indeed, if $(Q3)$ holds then the canonical projection of V onto V/K induces an isomorphism from $\langle\varepsilon(l)\rangle$ to $\langle\varepsilon(l), K\rangle/K$.

Condition $(Q3)$ always holds when $\mathrm{char}(\mathbb{F}) \neq 2$. In this case ε/K is a Veronesean embedding.

Let $\mathrm{char}(\mathbb{F}) = 2$. Then $K \cap \langle\varepsilon(l)\rangle$ might be non-trivial. This happens precisely when $K \cap \langle\varepsilon(l)\rangle$ is the nucleus of the conic $\varepsilon(l)$ in the projective plane $\mathrm{PG}(\langle\varepsilon(l)\rangle)$. Denote by n_l the nucleus of $\varepsilon(l)$, and suppose that $K \cap \langle\varepsilon(l)\rangle = n_l$ for every line l of Γ. Assume moreover that $\mathbb{F}$ is perfect. Then every line of $\mathrm{PG}(\langle\varepsilon(l)\rangle)$ through n_l is tangent to $\varepsilon(l)$. It follows that the canonical projection of V onto V/K induces a bijection from the conic $\varepsilon(l)$ to the set of points of the projective line $\mathrm{PG}(\langle\varepsilon(l), K\rangle/K) \cong \mathrm{PG}(\langle\varepsilon(l)\rangle/n_l)$. Thus, ε/K is a projective embedding.

Still assuming that $\mathrm{char}(\mathbb{F}) = 2$ and $K \cap \langle\varepsilon(l)\rangle = n_l$ for every line l, suppose that $\mathbb{F}$ is non-perfect. In this case some of the lines of $\mathrm{PG}(\langle\varepsilon(l)\rangle)$ through n_l are exterior to $\varepsilon(l)$. The subspace $\mathrm{PG}(\langle\varepsilon(l), K\rangle/K)$ of $\mathrm{PG}(V/K)$ is still a line but ε/K induces an injective but non-surjective mapping from the set of points of l to the set of points of $\mathrm{PG}(\langle\varepsilon(l), K\rangle/K)$. Moreover, if l and m are distinct lines of Γ then $(\varepsilon/K)(l)$ and $(\varepsilon/K)(m)$ are contained in distinct lines of $\mathrm{PG}(V/K)$. We say that ε/K is a *laxly projective embedding*.

Finally, keeping the assumption that $\mathrm{char}(\mathbb{F}) = 2$, suppose that $K \cap \langle\varepsilon(l)\rangle = n_l$ for some but not all lines of Γ. In this case ε/K maps some lines of Γ into (or onto, if $\mathbb{F}$ is perfect) lines of $\mathrm{PG}(V/K)$ while other lines of Γ are mapped by ε/K onto non-singular conics of $\mathrm{PG}(V/K)$. In this case we say that ε/K is *quasi-Veronesean*. We include Veronesean, projective and laxly projective embeddings in the class of quasi-Veronesean embeddings as borderline cases.

Remark 2.1. In the literature projective embeddings as defined in Subsection 2.1 are also called *full projective embeddings* while the embeddings that we have called laxly projective are often called *lax projective embeddings*, also *lax embeddings* for short (as in [29]). This terminology is used in contexts where both full and lax embeddings must be considered. When all projective embeddings to consider are full people normally prefer to drop the word ‘full’, thus calling them just ‘projective embeddings’, as we have done here. We have replaced the adjective ‘lax’ with the adverb ‘laxly’ because, since a projective embedding as defined in this paper is never lax, the phrase ‘lax projective embedding’ would sound as an oxymoron. The phrase ‘laxly projective embedding’ sounds better.

Remark 2.2. Nothing is known on the existence of absolutely universal Veronesean embeddings except the following: if all lines of $\Gamma = (\mathcal{P}, \mathcal{L})$ have just three points then Γ admits the absolutely universal Veronesean embedding, which can be constructed as follows. Let $\widetilde{V}$ be an $\mathbb{F}_2$-vector space of dimension equal to $|\mathcal{P}|$. Then any bijection from $\mathcal{P}$ to a basis of $\widetilde{V}$ can be taken as the absolutely universal Veronesean embedding of Γ.

The previous construction is admittedly too trivial to be interesting. Perhaps, Veronesean embeddings of geometries with three points per line are devoid of interest.

2.3. Subspaces and generation

With $\Gamma = (\mathcal{P}, \mathcal{L})$ as in the previous subsections, a subset S of $\mathcal{P}$ is a *subspace* of Γ if S contains every line l of Γ for which $|l \cap S| \geq 2$. A subspace S is *proper* if $S \neq \mathcal{P}$. A proper subspace of Γ meeting every line of Γ non-trivially is called a *hyperplane* of Γ.

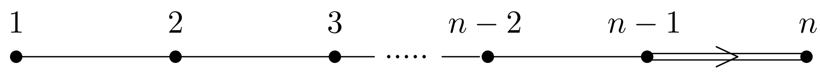
Intersections of subspaces are still subspaces. So, given a set X of points of Γ we can consider the *span* $\langle X \rangle_\Gamma$ of X in Γ, namely the smallest subspace of Γ containing X, defined as the intersection of all subspaces containing X. We say that X *generates* Γ if $\langle S \rangle_\Gamma = \mathcal{P}$. The *generating rank* $\mathrm{grk}(\Gamma)$ of Γ is the minimum size of a generating set of Γ. Clearly, if Γ admits a projective embedding then $\mathrm{grk}(\Gamma) \geq \dim(\varepsilon)$ for every projective embedding ε of Γ. In particular, if $\dim(\varepsilon) = \mathrm{grk}(\Gamma) < \infty$ then ε is relatively universal.

3. Embeddings of orthogonal Grassmannians

3.1. Grassmannians of the building of type B_n and their embeddings

For $n \geq 2$ let $V_{2n+1} \cong V(2n+1, \mathbb{F})$ be a $(2n+1)$-dimensional vector space over a field $\mathbb{F}$ and q a non-singular quadratic form of V_{2n+1} with Witt index $n \geq 2$. Let $\mathcal{B}_n$ the building of type B_n associated to the pair (V_{2n+1}, q), where the elements of type $k = 1, 2, \ldots, n$ (k-elements for short) are the k-dimensional subspaces of V_{2n+1} totally singular for q, with containment as the incidence relation. The building $\mathcal{B}_n$ is described by the following Dynkin diagram, where the integers written over the nodes of the diagram are the types:

(B_n)



For $1 \leq k \leq n$, the k-shadow of a flag F of $\mathcal{B}_n$ is the set of k-elements incident to F. The *k-Grassmannian* $\mathcal{B}_{n,k}$ of $\mathcal{B}_n$ is the point-line geometry defined as follows. The points of $\mathcal{B}_{n,k}$ are the k-elements of $\mathcal{B}_n$. When $1 < k < n$ the lines of $\mathcal{B}_{n,k}$ are the k-shadows of the flags of $\mathcal{B}_n$ of type $\{k-1, k+1\}$. The geometry $\mathcal{B}_{n,1}$ is the polar space associated to $\mathcal{B}_n$. Its lines are the 1-shadows of the 2-elements of $\mathcal{B}_n$. The geometry $\mathcal{B}_{n,n}$ is usually regarded as the dual of the polar space $\mathcal{B}_{n,1}$. Its lines are the n-shadows of the $(n-1)$-elements of $\mathcal{B}_n$. For such an element X, let l_X be its n-shadow. Then

$$l_X = \{Z \mid X \subset Z \subset X^{\perp},\ \dim(Z) = n,\ Z \text{ totally singular}\}$$

where $X^{\perp}$ is the orthogonal of X with respect to q. (Recall that X is an $(n-1)$-dimensional totally singular subspace of V_{2n+1}.) The vector space $X^{\perp}/X$ is 3-dimensional and l_X is a non-singular conic in the projective plane $\mathrm{PG}(X^{\perp}/X)$.

Grassmann embeddings. For $1 \leq k \leq n$ let $W_{2n+1,k} := \wedge^k V_{2n+1}$ and let $\varepsilon^{\mathrm{gr}}_{n,k}$ be the mapping from the set of points of $\mathcal{B}_{n,k}$ to the set of points of $\mathrm{PG}(W_{2n+1,k})$ defined by the following clause. Let X be a k-element of $\mathcal{B}_n$ and $\{v_1, \ldots, v_k\}$ a basis of X regarded as a subspace of V_{2n+1}. The 1-dimensional subspace $\langle v_1 \wedge \cdots \wedge v_k \rangle$ of $W_{2n+1,k}$ does not depend on the choice of the basis $\{v_1, \ldots, v_k\}$ of X. We put $\varepsilon^{\mathrm{gr}}_{n,k}(X) = \langle v_1 \wedge \cdots \wedge v_k \rangle$. It is easy to see (and well known) that this mapping is injective.

The mapping $\varepsilon^{\mathrm{gr}}_{n,1}$ is just the natural embedding of the polar space $\mathcal{B}_{n,1}$ in V_{2n+1}. If $1 < k < n$ then $\varepsilon^{\mathrm{gr}}_{n,k}$ is a projective embedding of $\mathcal{B}_{n,k}$ in the subspace $W^{\mathrm{gr}}_{2n+1,k} := \langle \varepsilon^{\mathrm{gr}}_{n,k}(\mathcal{B}_{n,k}) \rangle$ of $W_{2n+1,k}$ spanned by $\varepsilon^{\mathrm{gr}}_{n,k}(\mathcal{B}_{n,k})$ (where we take the liberty of using the symbol $\mathcal{B}_{n,k}$ to denote also the point-set of the geometry $\mathcal{B}_{n,k}$). We call $\varepsilon^{\mathrm{gr}}_{n,k}$ the *Grassmann embedding* of $\mathcal{B}_{n,k}$.

Let $k = n$. If X is an $(n-1)$-element of $\mathcal{B}_n$ then the set $\varepsilon^{\mathrm{gr}}_{n,n}(l_X) := \{\varepsilon^{\mathrm{gr}}_{n,n}(Y)\}_{Y \in l_X}$ is a non-singular conic of $\mathrm{PG}(W_{2n+1,n})$. Moreover, if Y is an n-element of $\mathcal{B}_n$ not containing X then $\langle\varepsilon^{\mathrm{gr}}_{n,n}(l_X)\rangle \cap \varepsilon^{\mathrm{gr}}_{n,n}(Y) = 0$. Thus, $\varepsilon^{\mathrm{gr}}_{n,n}$ is a Veronesean embedding of $\mathcal{B}_{n,n}$ in the subspace $W^{\mathrm{gr}}_{2n+1,n} := \langle\varepsilon^{\mathrm{gr}}_{n,n}(\mathcal{B}_{n,n})\rangle$ of $W_{2n+1,n}$ spanned by $\varepsilon^{\mathrm{gr}}_{n}(\Delta_n)$. We call $\varepsilon^{\mathrm{gr}}_{n,n}$ the *Grassmann embedding* of $\mathcal{B}_{n,n}$.

The spin embedding. The geometry $\mathcal{B}_{n,n}$ also admits a projective embedding, namely the *spin embedding* $\varepsilon^{\mathrm{spin}}_n : \mathcal{B}_{n,n} \to V_{2^n} := V(2^n, \mathbb{F})$. We refer the reader to Buekenhout and Cameron [7] for a concise description of this embedding. It is worth mentioning that when $\mathrm{char}(\mathbb{F}) \neq 2$ the embedding $\varepsilon^{\mathrm{spin}}_n$ is relatively universal (Blok and Brouwer [2]; also Cooperstein and Shult [15]). Hence it is absolutely universal, since $\mathcal{B}_{n,n}$ admits the absolutely universal embedding (Kasikova and Shult [20]).

The Veronese-spin embedding. Let ν_{2^n} be the usual quadratic Veronesean map from $V_{2^n} = V(2^n, \mathbb{F})$ to $V(\binom{2^n+1}{2}, \mathbb{F})$, which maps a vector $(x_1, \ldots, x_{2^n})$ of V_{2^n} onto the vector

$$(x_1^2, \ldots, x_{2^n}^2, x_1x_2, \ldots, x_1x_{2^n}, x_2x_3, \ldots, x_2x_{2^n}, \ldots, x_{2^n-1}x_{2^n}).$$

The mapping ν_{2^n} defines a Veronesean embedding of $\mathrm{PG}(V_{2^n})$ in $V(\binom{2^n+1}{2}, \mathbb{F})$, which we also denote by the symbol ν_{2^n}. The composition $\varepsilon^{\mathrm{ver}}_n := \nu_{2^n} \cdot \varepsilon^{\mathrm{spin}}_n$ is a Veronesean embedding of $\mathcal{B}_{n,n}$ in a subspace W^{ver}_n of $V(\binom{2^n+1}{2}, \mathbb{F})$. We call it the *Veronese-spin embedding* of $\mathcal{B}_{n,n}$.

Homogeneity. Let $G := \mathrm{Spin}(2n+1, \mathbb{F})$, namely G is the universal Chevalley group of type B_n defined over $\mathbb{F}$. We recall that the adjoint group of type B_n is $\overline{G} := \mathrm{SO}(2n+1, \mathbb{F})$ $(= \mathrm{PSO}(2n+1, \mathbb{F}))$. If $\mathrm{char}(\mathbb{F}) \neq 2$ then G is a non-split central extension of $\overline{G}$ by a group of order 2 while if $\mathrm{char}(\mathbb{F}) = 2$ then $G = \overline{G}$.

Each of the embeddings $\varepsilon^{\mathrm{gr}}_{n,k}$, $\varepsilon^{\mathrm{spin}}_n$ and $\varepsilon^{\mathrm{ver}}_n$ is G-homogeneous. The vector space V_{2^n}, regarded as a G-module via $\varepsilon^{\mathrm{spin}}_n$, is called the *spin module*. We shall denote this module by the symbol W^{spin}_n. We call $W^{\mathrm{ver}}_n = \langle\varepsilon^{\mathrm{ver}}_n(\mathcal{B}_{n,n})\rangle$ the *Veronese-spin module* for G. We call $W^{\mathrm{gr}}_{2n+1,k}$ a *Grassmann module* for G. When $\mathrm{char}(\mathbb{F}) \neq 2$ the group G acts as $\overline{G}$ in W^{ver}_n as well as in $W^{\mathrm{gr}}_{2n+1,k}$ for every k, but it acts faithfully in W^{spin}_n.

Weyl embeddings. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the fundamental dominant weights for the root system of type B_n, numbered in the usual way (see the picture at the beginning of this subsection). For $\lambda = \lambda_1, \ldots, \lambda_n$ or $\lambda = 2\cdot\lambda_n$, let $V_n(\lambda)$ be the Weyl module with λ as the highest weight. An embedding $\varepsilon_{n,\lambda}$ of $\mathcal{B}_{n,k}$ (for $\lambda = \lambda_k$ or $k = n$ and $\lambda = 2\lambda_n$) can be created in $V_n(\lambda)$ as follows. Let v_0 be a highest weight vector of $V_n(\lambda)$. Then the G-orbit of $\langle v_0\rangle$ corresponds to the set of points of $\mathcal{B}_{n,k}$ and, if P_k is the minimal fundamental parabolic subgroup of G of type k and L_0 is the P_k-orbit of $\langle v_0\rangle$, then the G-orbit of L_0 corresponds to the set of lines of $\mathcal{B}_{n,k}$. If X_0 is the k-element of $\mathcal{B}_n$ corresponding to $\langle v_0\rangle$ then $\varepsilon_{n,\lambda}$ maps $g(X_0)$ to $g(\langle v_0\rangle)$, for every $g \in G$. If $\lambda = \lambda_k$ then $\varepsilon_{n,\lambda}$ is projective. In particular, it is well known that $\varepsilon_{n,\lambda_n} \cong \varepsilon_n^{\mathrm{spin}}$, namely $V_n(\lambda_n) \cong W_n^{\mathrm{spin}}$. On the other hand, $\varepsilon_{n,2\lambda_n}$ is Veronesean, as one can see by computing L_0 explicitly. We denote the embedding $\varepsilon_{n,2\lambda_n}$ by the symbol $\varepsilon_{n,n}^{\mathrm{W}}$ and we call it the *Weyl Veronesean embedding* of $\mathcal{B}_{n,n}$. We extend this notation to the case $k < n$: when $k < n$ we put $\varepsilon_{n,k}^{\mathrm{W}} = \varepsilon_{n,\lambda_k}$ and we call $\varepsilon_{n,k}^{\mathrm{W}}$ the *Weyl embedding* of $\mathcal{B}_{n,k}$.

We have $\dim(V_n(2\lambda_n)) = \binom{2n+1}{n}$ and $\dim(V_n(\lambda_k)) = \binom{2n+1}{k}$ when $k < n$, as one can check by using the Weyl dimension formula (see e.g. Humphreys [19, 24.3]). Hence

Proposition 3.1. $\dim(\varepsilon_{n,k}^{\mathrm{W}}) = \binom{2n+1}{k}$ *for* $k = 1, 2, \ldots, n$.

Moreover, the G-module $W_{2n+1,k}^{\mathrm{gr}}$ is a homomorphic image of $V_n(\lambda)$, where $\lambda = \lambda_k$ when $k < n$ and $\lambda = 2\lambda_n$ when $k = n$. (See Blok [1, section 9]; also Carter [12]). In other words:

Proposition 3.2. $\varepsilon_{n,k}^{\mathrm{W}} \geq \varepsilon_{n,k}^{\mathrm{gr}}$ *for* $k = 1, 2, \ldots, n$.

Symplectic embeddings. When $\mathbb{F}$ is a perfect field of characteristic 2 the building $\mathcal{B}_n$ is isomorphic to the building of type C_n associated to a non-degenerate alternating form α on $V_{2n} := V(2n, \mathbb{F})$, the elements of $\mathcal{B}_n$ of type k being now regarded as k-subspaces of V_{2n} totally isotropic for α. Thus, we can also define a projective embedding $\varepsilon_{n,k}^{\mathrm{Sp}}$ of $\mathcal{B}_{n,k}$ in a subspace of $W_{2n,k} := \bigwedge^k V_{2n}$, which maps every totally isotropic k-subspace $\langle v_1, \ldots, v_k\rangle$ of V_{2n} onto the point $\langle v_1 \wedge \cdots \wedge v_k\rangle$ of $\mathrm{PG}(W_{2n,k})$. This embedding is G-homogeneous (recall that $\mathrm{Spin}(2n+1, \mathbb{F}) \cong \mathrm{Sp}(2n, \mathbb{F})$ when $\mathbb{F}$ is a perfect field of characteristic 2). We call $\varepsilon_{n,k}^{\mathrm{Sp}}$ a *symplectic* embedding of $\mathcal{B}_{n,k}$. We warn that the embedding $\varepsilon_{n,k}^{\mathrm{Sp}}$ is projective for $k = n$ too.

Obviously, $\varepsilon_{n,1}^{\mathrm{Sp}}$ is the natural embedding of $\mathcal{B}_{n,1}$ as a polar space of symplectic type in V_{2n}.

If $k > 1$ then $W_{2n,k}^{\mathrm{Sp}} := \langle \varepsilon_{n,k}^{\mathrm{Sp}}(\Delta_k)\rangle$ is a proper subspace of $W_{2n,k}$. In fact $\dim(W_{2n,k}^{\mathrm{Sp}}) = \binom{2n}{k} - \binom{2n}{k-2}$ while $\dim(W_{2n,k}) = \binom{2n}{k}$.

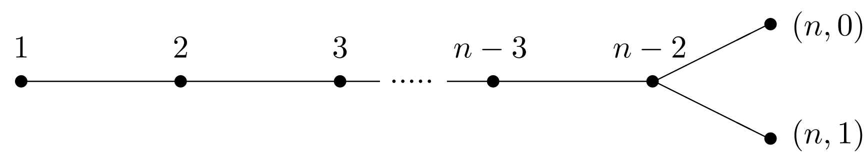
Let $k = n > 2$. Then the spin embedding $\varepsilon_n^{\mathrm{spin}}$ is a proper quotient of $\varepsilon_{n,n}^{\mathrm{Sp}}$ (Blok, Cardinali and De Bruyn [3]; also Cardinali and Lunardon [8]). Actually

$\varepsilon_n^{\mathrm{spin}} < \varepsilon_{n,n}^{\mathrm{Sp}}$, since $\dim(\varepsilon_{n,n}^{\mathrm{Sp}}) = \binom{2n}{n} - \binom{2n}{n-2} > 2^n = \dim(\varepsilon_n^{\mathrm{spin}})$. When $2 < |\mathbb{F}|$ the embedding $\varepsilon_{n,n}^{\mathrm{Sp}}$ is the hull of $\varepsilon_n^{\mathrm{spin}}$ (Cooperstein [13] for the finite case and De Bruyn and Pasini [18] for the general case), hence it is universal. The case $\mathbb{F} = \mathbb{F}_2$ is exceptional. Indeed when $\mathbb{F} = \mathbb{F}_2$ the universal projective embedding of $\mathcal{B}_{n,n}$ has dimension equal to $(2^n+1)(2^{n-1}+1)/3 > \binom{2n}{n} - \binom{2n}{n-2} = \dim(\varepsilon_{n,n}^{\mathrm{Sp}})$ (Li [21]; also Blokhuis and Brouwer [6]).

Finally, let $n = 2$. We have $\varepsilon_{2,1}^{\mathrm{Sp}} \cong \varepsilon_2^{\mathrm{spin}} < \varepsilon_{2,2}^{\mathrm{Sp}} \cong \varepsilon_{2,1}^{\mathrm{gr}} \cong \varepsilon_{2,1}^{\mathrm{W}}$.

3.2. Grassmannians of the building of type D_n and their embeddings

We denote by $\mathcal{D}_n$ the building of type D_n defined over $\mathbb{F}$, where $n \geq 3$. Note that, given a non-singular quadratic form q^+ of Witt index n in $V_{2n} = V(2n, \mathbb{F})$, the non-trivial subspaces of V_{2n} totally singular for q^+, with their dimensions taken as types, form a non-thick building $\overline{\mathcal{D}}_n$ of Coxeter type C_n. The building $\mathcal{D}_n$ is obtained from $\overline{\mathcal{D}}_n$ by dropping the elements of type $n-1$ and partitioning the set of n-elements in two families, marked by two distinct types, say $(n,0)$ and $(n,1)$, where two elements X and Y of type $(n,0)$ and $(n,1)$ are declared to be incident precisely when $\dim(X \cap Y) = n-1$.

(D_n) 

We allow $n = 3$. Recall that the Coxeter diagram D_3 is the same as A_3, but with the usual types $1, 2, 3$ replaced with $(3,0), 1$ and $(3,1)$ respectively. In other words, $\mathcal{D}_3 = \mathrm{PG}(3, \mathbb{F})$, the elements of $\mathcal{D}_3$ of type 1, $(3,0)$ and $(3,1)$ being respectively the lines, the points and the planes of $\mathrm{PG}(3, \mathbb{F})$ (or lines, planes and points, if we prefer so).

For $k < n$, the k-Grassmannian $\overline{\mathcal{D}}_{n,k}$ of $\overline{\mathcal{D}}_n$ is defined just in the same way as the k-Grassmannian $\mathcal{B}_{n,k}$ of $\mathcal{B}_n$. We put $\mathcal{D}_{n,k} := \overline{\mathcal{D}}_{n,k}$ and we call $\mathcal{D}_{n,k}$ the *k-Grassmannian* of $\mathcal{D}_n$.

The 1-Grassmannian $\mathcal{D}_{n,1}$ of $\mathcal{D}_n$ is the polar space defined by q^+ on V_{2n}. Regarded V_{2n} as a hyperplane of $V_{2n+1} = V(2n+1, \mathbb{F})$, we can accordingly regard $\mathcal{D}_{n,1}$ as a hyperplane of the polar space $\mathcal{B}_{n,1}$ (which is defined in V_{2n+1}). Similarly, $\mathcal{D}_{n,k}$ is the subgeometry induced by $\mathcal{B}_{n,k}$ on the set of k-subspaces of V_{2n}.

Note that the points of $\mathcal{D}_{n,n-1}$ are the $\{(n,0),(n,1)\}$-flags of $\mathcal{D}_n$ while the lines of $\mathcal{D}_{n,n-1}$ correspond to flags of $\mathcal{D}_n$ of type $\{n-2,(n,0)\}$ or $\{n-2,(n,1)\}$.

In particular, $\mathcal{D}_{3,2}$ is the so-called root-subgroup geometry of $\mathrm{SL}(4, \mathbb{F})$, with the point-plane flags of $\mathrm{PG}(3, \mathbb{F})$ as points and the line-plane and point-line flags of $\mathrm{PG}(3, \mathbb{F})$ as lines.

Remark 3.3. When $k = n - 1$ the conventions adopted above are not so consistent with the terminology commonly used in the literature. If we followed the custom, we should rather call $\mathcal{D}_{n,n-1}$ the $\{(n, 0), (n, 1)\}$-Grassmannian of $\mathcal{D}_n$. Moreover, the types $(n, 0)$ and $(n, 1)$ are usually replaced by $n - 1$ and n. However, in the context of this paper our slightly unusual conventions make life easier.

Remark 3.4. Two more Grassmannians of $\mathcal{D}_n$ should be mentioned, which cannot be regarded as Grassmannians of $\overline{\mathcal{D}}_n$, namely the $(n, 0)$- and $(n, 1)$-Grassmannians, called *half-spin geometries* in the literature. We are not going to discuss them in this paper, but we feel compelled to say at least a few words on them. They are mutually isomorphic and can be constructed as follows. Consider the $(n, 0)$-Grassmannian $\mathcal{D}_{n,(n,0)}$, to fix ideas. The points of $\mathcal{D}_{n,(n,0)}$ are the elements of $\mathcal{D}_n$ of type $(n, 0)$ while the elements of type $n - 2$ are taken as lines. The geometry $\mathcal{D}_{n,(n,0)}$ admits a 2^{n-1}-dimensional projective embedding η, called the *half-spin embedding* (see e.g. Buekenhout and Cameron [7]). Moreover, $\mathrm{grk}(\mathcal{D}_{n,(n,0)}) = 2^{n-1}$ (Cooperstein and Shult [15]). Hence η is relatively universal. Moreover, $\mathcal{D}_{n,(n,0)}$ admits the universal embedding (Kasikova and Shult [20]). Hence η is absolutely universal.

Grassmann and Weyl embeddings.

We have remarked that $\mathcal{D}_{n,k}$ is a subgeometry of $\mathcal{B}_{n,k}$ for $k = 1, 2, \ldots, n - 1$. Accordingly, the Grassmann embedding $\varepsilon^{\mathrm{gr}}_{n,k}$ of $\mathcal{B}_{n,k}$ induces on $\mathcal{D}_{n,k}$ a projective embedding $\eta^{\mathrm{gr}}_{n,k}$ in a subspace $W^{\mathrm{Sp}}_{n,k} := \langle \eta^{\mathrm{gr}}_{n,k}(\mathcal{D}_{n,k}) \rangle$ of $W_{2n,k} = \bigwedge^k V_{2n}$. We call $\eta^{\mathrm{gr}}_{n,k}$ the *Grassmann embedding* of $\mathcal{D}_{n,k}$.

Let $\mu_1, \ldots, \mu_{n-2}, \mu_{n,0}$ and $\mu_{n,1}$ be the fundamental dominant weights of the root system of type D_n, corresponding to the nodes $1, 2, \ldots, n - 2, (n, 0)$ and $(n, 1)$ of the D_n-diagram in the obvious way. Put $\mu_{n-1} := \mu_{n,0} + \mu_{n,1}$. Then for $k = 1, 2, \ldots, n - 1$ the Weyl module $V_n(\mu_k)$ hosts a projective embedding $\eta^{\mathrm{W}}_{n,k}$ of $\mathcal{D}_{n,k}$. We call $\eta^{\mathrm{W}}_{n,k}$ the *Weyl embedding* of $\mathcal{D}_{n,k}$. With the help of the Weyl dimension formula we can check that $\dim(V_n(\mu_k)) = \binom{2n}{k}$. Therefore:

Proposition 3.5. $\dim(\eta^{\mathrm{W}}_{n,k}) = \binom{2n}{k}$ *for* $k = 1, 2, \ldots, n - 1$.

An analogue of Proposition 3.2 also holds:

Proposition 3.6. $\eta^{\mathrm{W}}_{n,k} \geq \eta^{\mathrm{gr}}_{n,k}$ *for* $k = 1, 2, \ldots, n - 1$.

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3.3. A list of the embeddings defined in this section

1. (*Grassmann*) $\varepsilon^{\mathrm{gr}}_{n,k} : \mathcal{B}_{k,n} \to W^{\mathrm{gr}}_{2n+1,k} \subseteq W_{2n+1,k}$. Projective when $k < n$ and Veronesean when $k = n$. It is a quotient of $\varepsilon^{\mathrm{W}}_{n,k}$ (item 4 of this list).
2. (*Spin*) $\varepsilon^{\mathrm{spin}}_{n} : \mathcal{B}_{n,n} \to W^{\mathrm{spin}}_{n} = V_n(\lambda_n)$. Projective. Dimension equal to 2^n.
3. (*Veronese-spin*) $\varepsilon^{\mathrm{ver}}_{n} : \mathcal{B}_{n,n} \to W^{\mathrm{ver}}_{n} \subseteq V(\binom{2^n+1}{2}, \mathbb{F})$. Veronesean.
4. (*Weyl*) $\varepsilon^{\mathrm{W}}_{n,k} : \mathcal{B}_{k,n} \to V_n(\lambda)$, with $\lambda = \lambda_k$ for $k < n$ and $\lambda = 2 \cdot \lambda_n$ when $k = n$. Projective when $k < n$ and Veronesean when $k = n$. In any case it has dimension equal to $\binom{2n+1}{k}$.
5. (*Symplectic*) $\varepsilon^{\mathrm{Sp}}_{n,k} : \mathcal{B}_{n,k} \to W^{\mathrm{Sp}}_{2n,k}$. It only exists when $\mathbb{F}$ is a perfect field of characteristic 2. It is projective, with dimension equal to $\binom{2n}{k} - \binom{2n}{k-2}$.
6. (*Grassmann*) $\eta^{\mathrm{gr}}_{n,k} : \mathcal{D}_{k,n} \to W^{\mathrm{gr}}_{2n,k} \subseteq W_{2n,k}$, $k < n$. Projective. It is a quotient of $\eta^{\mathrm{W}}_{n,k}$ (see below).
7. (*Weyl*) $\eta^{\mathrm{W}}_{n,k} : \mathcal{D}_{k,n} \to V_n(\mu_k)$, where $k < n$ and $\mu_{n-1} = \mu_{n,0} + \mu_{n,1}$. Projective. Dimension equal to $\binom{2n}{k}$.

4. More on the previous embeddings

4.1. Grassmann and Weyl embeddings

It is well known that $\varepsilon^{\mathrm{gr}}_{n,1} \cong \varepsilon^{\mathrm{W}}_{n,1}$ and $\eta^{\mathrm{gr}}_{n,1} \cong \eta^{\mathrm{W}}_{n,1}$ for any choice of the field $\mathbb{F}$. So, throughout this subsection we assume $k > 1$.

Theorem 4.1. *Let* $\mathrm{char}(\mathbb{F}) \neq 2$. *Then:*

(1) $\varepsilon^{\mathrm{gr}}_{n,k} \cong \varepsilon^{\mathrm{W}}_{n,k}$ *for every* $k = 2, 3, \ldots, n$.

(2) $\eta^{\mathrm{gr}}_{n,k} \cong \eta^{\mathrm{W}}_{n,k}$ *for every* $k = 2, 3, \ldots, n-1$.

Sketch of the proof. Both (1) and (2) can be proved by exploiting the fact that the Weyl modules $V_n(\lambda_2), \ldots, V_n(\lambda_{n-1}), V_n(2 \cdot \lambda_n), V_n(\mu_1), \ldots, V_n(\mu_{n-2})$ and $V_n(\mu_{n,0} + \mu_{n,1})$ are irreducible when $\mathrm{char}(\mathbb{F}) \neq 2$, but in [9] we have used a different, more elementary argument to prove (1). The bulk of the proof given in [9] is to show that if $\mathrm{char}(\mathbb{F}) \neq 2$ then the set of points of $\mathcal{B}_{n,k}$ is a generating set for the k-Grassmannian $\mathcal{G}_k$ of $\mathrm{PG}(V_{2n+1}) \cong \mathrm{PG}(2n, \mathbb{F})$ in the sense of Subsection 2.3. Having proved this, let γ_k be the natural embedding of $\mathcal{G}_k$ in $W_{2n+1,k}$, mapping every k-subspace $\langle v_1, \ldots, v_k \rangle$ of V_{2n+1} onto $\langle v_1 \wedge \cdots \wedge v_k \rangle$. Then $\gamma_k(\mathcal{G}_k)$ spans $W_{2n+1,k}$. Moreover γ_k induces $\varepsilon^{\mathrm{gr}}_{n,k}$ on $\mathcal{B}_{n,k}$. Therefore, since $\mathcal{B}_{n,k}$ generates $\mathcal{G}_k$, the image of $\mathcal{B}_{n,k}$ by $\varepsilon^{\mathrm{gr}}_{n,k}$ spans $W_{2n+1,k}$. However $\dim(W_{2n+1,k}) = \binom{2n+1}{k} = \dim(V_n(\lambda))$ (where $\lambda = \lambda_k$ if $k < n$ and $\lambda = 2\lambda_n$ when $k = n$). Claim (1) follows. Claim (2) can be proved by the same argument used in [9] for (1), modulo a few little modifications. □

For the rest of this subsection we assume that $\mathrm{char}(\mathbb{F}) = 2$. We firstly consider the building $\mathcal{B}_n$. We recall that its elements are the subspaces of V_{2n+1} totally singular for a given quadratic form q of Witt index n. As $\mathrm{char}(\mathbb{F}) = 2$, the radical N_0 of the sesquilinearization of q is 1-dimensional. We call N_0 the *nucleus* of q.

We need to fix some notation. As in the sketch of the proof of Theorem 4.1, we denote by $\mathcal{G}_k$ the k-Grassmannian of $\mathrm{PG}(V_{2n+1})$ and by γ_k the natural embedding of $\mathcal{G}_k$ in $W_{2n+1,k}$. Recall that $\mathcal{B}_{n,k}$ is contained in $\mathcal{G}_k$ and γ_k induces $\varepsilon^{\mathrm{gr}}_{n,k}$ on $\mathcal{B}_{n,k}$. Given an element X of $\mathcal{B}_n$ of type $k-1$ (recall that we have assumed that $k > 1$) let $\mathrm{Res}^+(X)$ be its upper residue, formed by the elements of $\mathcal{B}_n$ of type $k, k+1, \ldots, n$ incident to X. Then $\mathrm{Res}^+(X)$ is the building of an orthogonal polar space of rank $n-k+1$ defined in $X^\perp/X$. We denote this polar space by $\mathrm{Res}^+_k(X)$. Let $\mathcal{G}_k(X)$ be the subspace of $\mathcal{G}_k$ formed by the k-subspaces of V_{2n+1} that contain X and are contained in $X^\perp$ and let $W_{2n+1,k}(X)$ be the subspace of $W_{2n+1,k}$ spanned by $\mathcal{G}_k(X)$. Then $\dim(W_{2n+1,k}(X)) = 2(n-k+1)+1$, $\gamma_k(\mathcal{G}_k(X)) = \mathrm{PG}(W_{2n+1,k}(X))$ and $\varepsilon^{\mathrm{gr}}_{n,k}$ embeds $\mathrm{Res}^+_k(X)$ in $W_{2n+1,k}(X)$ as the polar space associated to a non-singular quadratic form q_X of $W_{2n+1,k}(X)$ of Witt index $n-k+1$. So, $\varepsilon^{\mathrm{gr}}_{n,k}(\mathrm{Res}^+_k(X))$ spans $W_{2n+1,k}(X)$. Let N_X be the nucleus of q_X and let $\mathcal{N}^{\mathrm{gr}}_{n,k}$ be the subspace of $W_{2n+1,k}$ spanned by the 1-dimensional subspaces N_X for X a $(k-1)$-element of $\mathcal{B}_{n,k}$. Clearly, $\mathcal{N}^{\mathrm{gr}}_{n,k}$ is contained in $W^{\mathrm{gr}}_{2n+1,k} = \langle \varepsilon^{\mathrm{gr}}_{n,k}(\mathcal{B}_{n,k})\rangle$ and it is stabilized by the group G ($= \mathrm{Spin}(2n+1,\mathbb{F})$). Finally, we denote by $\iota^{\mathrm{gr}}_{n,k-1}$ the mapping sending every point X of $\mathcal{B}_{n,k-1}$ to N_X. The following is proved in [9]:

Theorem 4.2. *Let $k > 1$ and $\mathrm{char}(\mathbb{F}) = 2$. Then the following hold:*

(1) *The subspace $\mathcal{N}^{\mathrm{gr}}_{n,k}$ of $W^{\mathrm{gr}}_{2n+1,k}$ defines a quotient of the embedding $\varepsilon^{\mathrm{gr}}_{n,k}$. We have $\dim(W^{\mathrm{gr}}_{2n+1,k}/\mathcal{N}^{\mathrm{gr}}_{n,k}) = \binom{2n}{k} - \binom{2n}{k-2}$. If $\mathbb{F}$ is perfect then $\varepsilon^{\mathrm{gr}}_{n,k}/\mathcal{N}^{\mathrm{gr}}_{n,k} \cong \varepsilon^{\mathrm{Sp}}_{n,k}$. If $\mathbb{F}$ is non-perfect then $\varepsilon^{\mathrm{gr}}_{n,k}/\mathcal{N}^{\mathrm{gr}}_{n,k}$ is projective when $k < n$ and laxly projective when $k = n$.*

(2) *The mapping $\iota^{\mathrm{gr}}_{n,k-1}$ is a projective embedding of $\mathcal{B}_{n,k-1}$ in $\mathcal{N}^{\mathrm{gr}}_{n,k}$. We have $\dim(\mathcal{N}^{\mathrm{gr}}_{n,k}) = \binom{2n}{k-1} - \binom{2n}{k-3}$ (with the usual convention that $\binom{2n}{-1} = 0$, when $k = 2$). If $\mathbb{F}$ is perfect then $\iota^{\mathrm{gr}}_{n,k-1} \cong \varepsilon^{\mathrm{Sp}}_{n,k-1}$.*

(3) $\dim(\varepsilon^{\mathrm{gr}}_{n,k}) = \binom{2n+1}{k} - \binom{2n+1}{k-2}$.

Sketch of the proof. Claim (3) immediately follows from (1) and (2) (recall that $\dim(\varepsilon^{\mathrm{gr}}_{n,k}) = \dim(W^{\mathrm{gr}}_{2n+1,k})$ by definition). As for (1) and (2), assume firstly that

$\mathbb{F}$ is perfect. Under this assumption, claim (2) is obtained in [9] with the help of straightforward calculations. As for (1), in [9] we firstly prove that $\mathcal{N}^{\mathrm{gr}}_{n,k}$ satisfies $(Q1)$ of Section 2. When $k < n$ this is enough to conclude that $\varepsilon^{\mathrm{gr}}_{n,k}/\mathcal{N}_{n,k}$ is a projective embedding. When $k = n$ we also need (Q2), but this property is fairly easy to prove. On the other hand, (Q3) fails to hold for any line of $\mathcal{B}_{n,n}$. Hence $\varepsilon^{\mathrm{gr}}_{n,n}/\mathcal{N}^{\mathrm{gr}}_{n,n}$ is a projective embedding, as explained in Subsection 2.2. Finally, the isomorphism $\varepsilon^{\mathrm{gr}}_{n,k}/\mathcal{N}^{\mathrm{gr}}_{n,k} \cong \varepsilon^{\mathrm{Sp}}_{n,k}$ is proved by a direct algebraic argument.

Suppose now that $\mathbb{F}$ is non-perfect. Then we can replace $\mathcal{B}_n$ with a suitable sub-building defined over $\mathbb{F}_2$. As $\mathbb{F}_2$ is perfect, claims (1) and (2) hold for that sub-building. Turning back to $\mathcal{B}_n$, we obtain the statements of (1) and (2) in the non-perfect case. $\square$

By (3) of Theorem 4.2 we immediately obtain the following:

Corollary 4.3. *Let* $k > 1$ *and* $\mathrm{char}(\mathbb{F}) = 2$. *Then* $\varepsilon^{\mathrm{gr}}_{n,k} < \varepsilon^{\mathrm{W}}_{n,k}$.

Let $k > 1$ and $\mathrm{char}(\mathbb{F}) = 2$. Let $\pi_{n,k}$ be the projection of $\varepsilon^{\mathrm{W}}_{n,k}$ onto $\varepsilon^{\mathrm{gr}}_{n,k}$. By Theorem 4.2, if $k = 2$ then $\dim(\ker(\pi_{n,k})) = 1$ while $\dim(\ker(\pi_{n,k})) = \dim(V_n(\lambda_{k-2}))$ when $k > 2$. Moreover $\dim(\pi^{-1}_{n,k}(\mathcal{N}^{\mathrm{gr}}_{n,k})) = \dim(V_n(\lambda_{k-1}))$. A more clear picture is offered in [11], where the following is proved.

Theorem 4.4. *Let* $k > 1$ *and* $\mathrm{char}(\mathbb{F}) = 2$. *Then* $\pi^{-1}_{n,k}(\mathcal{N}^{\mathrm{gr}}_{n,k}) \cong V_n(\lambda_{k-1})$ *(isomorphism of G-modules). Moreover, if* $k > 2$ *then* $\ker(\pi_{n,k}) \cong V_n(\lambda_{k-2})$.

Sketch of the proof. Put $V := V_n(\lambda_k)$ when $k < n$ and $V = V_n(2\lambda_n)$ when $k = n$. Let v_0 be a highest weight vector of V, let $\mathfrak{L}$ be the Lie algebra of G and $\mathfrak{A}$ the enveloping associative algebra of $\mathfrak{L}$. It is proved in [11] that an element $\mathfrak{a}_k \in \mathfrak{A}$ exists such that $\mathfrak{A}(\mathfrak{a}_k(v_0)) \cong V_n(\lambda_{k-1})$, with $\mathfrak{a}_k(v_0)$ in the role of highest weight vector of $V_n(\lambda_{k-1})$. By the very same argument, with V replaced by $\mathfrak{A}(\mathfrak{a}_k(v_0))$, an element $\mathfrak{a}_{k-1} \in \mathfrak{A}$ exists such that $\mathfrak{A}(\mathfrak{a}_{k-1}\mathfrak{a}_k(v_0)) \cong V_n(\lambda_{k-2})$, with the convention that $V_n(\lambda_0)$ ($= V_n(\lambda_{k-2})$ when $k = 2$) is the trivial 1-dimensional module. It turns out that $\pi_{n,k}(\mathfrak{a}_{k-1}\mathfrak{a}_k(v_0)) = 0$ and $\pi_{n,k}(\mathfrak{a}_k(v_0)) \in \mathcal{N}^{\mathrm{gr}}_{n,k}$. Hence $\mathfrak{A}(\mathfrak{a}_{k-1}\mathfrak{a}_k(v_0)) \subseteq \ker(\pi_{n,k})$ and $\mathfrak{A}(\mathfrak{a}_k(v_0)) \subseteq \pi^{-1}_{n,k}(\mathcal{N}^{\mathrm{gr}}_{n,k})$. We know by Theorem 4.2 that $\dim(\pi^{-1}_{n,k}(\mathcal{N}^{\mathrm{gr}})) = \binom{2n+1}{k-1} = \dim(V_n(\lambda_{k-1}))$ and $\dim(\ker(\pi_{n,k})) = (\pi_{n,k}) = \binom{2n+1}{k-2} = \dim(V_n(\lambda_{k-2}))$. Moreover $\mathfrak{A}(\mathfrak{a}_{k-1}\mathfrak{a}_k(v_0)) \subseteq \ker(\pi_{n,k})$ and $\mathfrak{A}(\mathfrak{a}_k(v_0)) \subseteq \pi^{-1}_{n,k}(\mathcal{N}^{\mathrm{gr}}_{n,k})$. By these inclusions and the isomorphisms $\mathfrak{A}(\mathfrak{a}_{k-1}\mathfrak{a}_k(v_0)) \cong V_n(\lambda_{k-2})$ and $\mathfrak{A}(\mathfrak{a}_k(v_0)) \cong V_n(\lambda_{k-1})$ we obtain that $\mathfrak{A}(\mathfrak{a}_{k-1}\mathfrak{a}_k(v_0)) = \ker(\pi_{n,k})$ and $\mathfrak{A}(\mathfrak{a}_k(v_0)) = \pi^{-1}_{n,k}(\mathcal{N}^{\mathrm{gr}}_{n,k})$, by comparing dimensions. Hence $\pi^{-1}_{n,k}(\mathcal{N}^{\mathrm{gr}}_{n,k}) \cong V_n(\lambda_{k-1})$ and $\ker(\pi_{n,k}) \cong V_n(\lambda_{k-2})$. $\square$

Turning back to Theorem 4.2, that theorem also has the following interesting consequence:

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Corollary 4.5. *Let $k < n$ and let $\mathbb{F}$ be a perfect field of characteristic 2. Then $\varepsilon^{\mathrm{Sp}}_{n,k}$ is not universal. More explicitly, $\varepsilon^{\mathrm{Sp}}_{n,k} < \varepsilon^{\mathrm{gr}}_{n,k}$.*

Remark 4.6. The hypothesis that $k < n$ is essential for the conclusion of Corollary 4.5. Indeed, although $\varepsilon^{\mathrm{Sp}}_{n,n}$ is a proper quotient of $\varepsilon^{\mathrm{gr}}_{n,n}$, the latter is Veronesean rather than projective. So, the fact that $\varepsilon^{\mathrm{Sp}}_{n,n}$ is a quotient of $\varepsilon^{\mathrm{gr}}_{n,n}$ in the sense of Subsection 2.2 does not imply that $\varepsilon^{\mathrm{Sp}}_{n,n}$ is not universal. In fact, $\varepsilon^{\mathrm{Sp}}_{n,n}$ is universal when $2 < |\mathbb{F}| < \infty$, as remarked at the end of Subsection 3.1.

Still assuming that $\mathrm{char}(\mathbb{F}) = 2$, we now turn to $\mathcal{D}_n$. With q^+ as in Subsection 3.2, the sesquilinearization α of the quadratic form q^+ is a non-degenerate alternating form of V_{2n}. Let $\mathcal{C}_n$ be the building of type C_n associated to α. Then, for every $k = 1, 2, \ldots, n-1$, the k-Grassmannian $\mathcal{D}_{n,k}$ of $\mathcal{D}_n$ is a subspace (Subsection 2.3) of the k-Grassmannian $\mathcal{C}_{n,k}$ of $\mathcal{C}_n$. Let $\varepsilon^{\mathrm{Sp}}_{n,k}$ be the projective embedding of $\mathcal{C}_{n,k}$ mapping every k-subspace $\langle v_1, \ldots, v_k\rangle$ of V_{2n} totally isotropic for α onto the 1-dimensional subspace $\langle v_1 \wedge \cdots \wedge v_k\rangle$ of $W_{2n,k} = \wedge^k V_{2n}$ (compare the definition of symplectic embeddings in Subsection 3.1, but note that now $\mathcal{C}_n$ need not be isomorphic to $\mathcal{B}_n$, since $\mathbb{F}$ might be non-perfect). It is well known that $\varepsilon^{\mathrm{Sp}}_{n,k}$ is a projective embedding of $\mathcal{C}_{n,k}$, $\dim(\varepsilon^{\mathrm{Sp}}_{n,k}) = \binom{2n}{k} - \binom{2n}{k-2}$ and $\varepsilon^{\mathrm{Sp}}_{n,k}$ induces $\eta^{\mathrm{gr}}_{n,k}$ on $\mathcal{D}_{n,k}$. Therefore $\dim(\eta^{\mathrm{gr}}_{n,k}) \leq \binom{2n}{k} - \binom{2n}{k-2}$. Consequently, $\eta^{\mathrm{gr}}_{n,k} < \eta^{\mathrm{W}}_{n,k}$. A sharper statement is proved in [9], namely the following.

Theorem 4.7. *Let $1 < k < n$ and $\mathrm{char}(\mathbb{F}) = 2$. Then $\dim(\eta^{\mathrm{gr}}_{n,k}) = \binom{2n}{k} - \binom{2n}{k-2}$.*

Sketch of the proof. Suppose firstly that $\mathbb{F}$ is perfect. Let $\mathcal{G}_k$ be the k-Grassmannian of $\mathrm{PG}(V_{2n})$. Then both $\mathcal{D}_{n,k}$ and $\mathcal{C}_{n,k}$ are subgeometries of $\mathcal{G}_k$. As $\mathcal{D}_{n,k} \subseteq \mathcal{C}_{n,k}$, we have $\langle \mathcal{D}_{n,k}\rangle_{\mathcal{G}_k} \subseteq \langle \mathcal{C}_{n,k}\rangle_{\mathcal{G}_k}$. The crucial step of the proof is to prove the reverse inclusion $\langle \mathcal{C}_{n,k}\rangle_{\mathcal{G}_k} \subseteq \langle \mathcal{D}_{n,k}\rangle_{\mathcal{G}_k}$. Having poved this, we obtain that $\langle \mathcal{D}_{n,k}\rangle_{\mathcal{G}_k} = \langle \mathcal{C}_{n,k}\rangle_{\mathcal{G}_k}$ and the statement of the theorem follows, recalling that the natural embedding of $\mathcal{G}_k$ in $W_{2n,k}$ induces $\varepsilon^{\mathrm{Sp}}_{n,k}$ on $\mathcal{C}_{n,k}$ and $\eta^{\mathrm{gr}}_{n,k}$ on $\mathcal{D}_{n,k}$.

The inclusion $\langle \mathcal{C}_{n,k}\rangle_{\mathcal{G}_k} \subseteq \langle \mathcal{D}_{n,k}\rangle_{\mathcal{G}_k}$ is proved in [9] by exploiting the fact that, since $\mathcal{C}_n \cong \mathcal{B}_n$ (because $\mathbb{F}$ is assumed to be perfect) and $\mathcal{D}_{n,1}$ is a hyperplane of $\mathcal{B}_{n,1}$, the subgeometry $\mathcal{D}_{n,1}$ is in fact a hyperplane of $\mathcal{C}_{n,1}$. We refer the reader to [9] for the details of this proof.

When $\mathbb{F}$ is non-perfect the conclusion of the theorem can be obtained as in the proof of Theorem 4.2, by descent to the prime subfield $\mathbb{F}_2$ of $\mathbb{F}$. □

Remark 4.8. We have defined $\eta^{\mathrm{gr}}_{n,k}$ with the help of the Grassmann embedding $\varepsilon^{\mathrm{Sp}}_{n,k}$ of $\mathcal{C}_{n,k}$. It is worth remarking that $\varepsilon^{\mathrm{Sp}}_{n,k}$ is in fact a Weyl embedding (see e.g. Blok [1]).

4.2. The Veronese-spin embedding

The next theorem is Theorem 1 of [10].

Theorem 4.9. *We have $\varepsilon_n^{\mathrm{ver}} \cong \varepsilon_{n,n}^{\mathrm{W}}$ for every choice of the field $\mathbb{F}$.*

Sketch of the proof. The group $\mathrm{SL}(2^n,\mathbb{F})$ can be lifted from $V_{2n} = V(2^n,\mathbb{F})$ to $V\big(\binom{2^n+1}{2},\mathbb{F}\big)$ via the Veronesean quadratic map. Thus, $V\big(\binom{2^n+1}{2},\mathbb{F}\big)$ can be regarded as an $\mathrm{SL}(2^n,\mathbb{F})$-module. One can prove that this module is isomorphic to the Weyl module $V(2\cdot\omega_1)$ for $\mathrm{SL}(2^n,\mathbb{F})$, where $\omega_1,\ldots,\omega_{2^n-1}$ are the fundamental dominant weights of the root system of type A_{2^n-1}. Next, regarding $V(2\cdot\omega_1)$ as a $\mathrm{Spin}(2n+1,\mathbb{F})$-module, as we can in view of the inclusion $\mathrm{Spin}(2n+1,\mathbb{F}) \leq \mathrm{SL}(2^n,\mathbb{F})$, we can recognize $V(2\cdot\lambda_n)$ inside $V(2\cdot\omega_1)$ as a $\mathrm{Spin}(2n+1,\mathbb{F})$-submodule. Finally, it is proved that the isomorphism $V(2\cdot\omega_1)\cong V\big(\binom{2^n+1}{2},F\big)$ induces an isomorphism from $V(2\cdot\lambda_n)$ to W_n^{ver}. $\square$

Assumption. For the rest of this subsection we assume that $\mathrm{char}(\mathbb{F}) = 2$.

We firstly recall a number of known facts about the quadratic Veronesean map $\nu_{2^n}\colon V_{2^n} \to V\big(\binom{2^n+1}{2},\mathbb{F}\big)$. This map induces a Veronesean embedding of $\mathrm{PG}(2^n-1,\mathbb{F})$ in $V\big(\binom{2^n+1}{2},\mathbb{F}\big)$, which we still denote by ν_{2^n}. The image $\nu_{2^n}(\mathrm{PG}(2^n-1,\mathbb{F}))$ of $\mathrm{PG}(2^n-1,\mathbb{F})$ by ν_{2^n} is called a *Veronesean variety*. As noticed in the sketch of the proof of Theorem 4.9, the group $\mathrm{SL}(2^n,\mathbb{F})$ lifts to $V\big(\binom{2^n+1}{2},\mathbb{F}\big)$. Clearly, it stabilizes the Veronesean variety $\nu_{2^n}(\mathrm{PG}(2^n-1,\mathbb{F}))$. For every line l of $\mathrm{PG}(2^n-1,\mathbb{F})$ let n_l be the nucleus of the conic $\nu_{2^n}(l)$. The *nucleus subspace* of $V\big(\binom{2^n+1}{2},\mathbb{F}\big)$ relative to ν_{2^n} is the subspace $\mathcal{N}$ of $V\big(\binom{2^n+1}{2},\mathbb{F}\big)$ spanned by the nuclei n_l, for l a line of $\mathrm{PG}(2^n-1,\mathbb{F})$ (Thas and Van Maldeghem [26]). The subspace $\mathcal{N}$ is stabilized by $\mathrm{SL}(2^n,\mathbb{F})$ in its action on $V\big(\binom{2^n+1}{2},\mathbb{F}\big)$. Moreover, $\mathcal{N}\cap\langle\nu_{2^n}(l)\rangle = n_l$ for every line l of $\mathrm{PG}(2^n-1,\mathbb{F})$. Hence $\mathcal{N}\cap\nu_{2n}(p) = 0$ for every point p of $\mathrm{PG}(2^{n-1},\mathbb{F})$.

Put $\mathcal{N}_{n,0}^{\mathrm{ver}} := \mathcal{N}_{2^n}\cap W_n^{\mathrm{ver}}$. For every $(n-1)$-element $X\in\mathcal{B}_{n,n-1}$ of $\mathcal{B}_n$, let l_X be the line of $\mathcal{B}_{n,n}$ correspondig to X and let n_X be the nucleus of the conic $\varepsilon_n^{\mathrm{ver}}(l_X) = \nu_{2^n}(\varepsilon_n^{\mathrm{spin}}(l_X))$. We put $\mathcal{N}_{n,1}^{\mathrm{ver}} := \langle n_X\rangle_{X\in\mathcal{B}_{n,n-1}}$.

Clearly, $\mathcal{N}_{n,0}^{\mathrm{ver}} \supseteq \mathcal{N}_{n,1}^{\mathrm{ver}}$ and both these subspaces are stabilized by the group $G = \mathrm{Spin}(2n+1,\mathbb{F})$ in its action on W_n^{ver} ($\cong V(2\cdot\lambda_n)$ by Theorem 4.9). We can also define two mappings $\varepsilon_n^{\mathrm{ver}}/\mathcal{N}_{n,0}^{\mathrm{ver}}$ and $\varepsilon_n^{\mathrm{ver}}/\mathcal{N}_{n,1}^{\mathrm{ver}}$ from $\mathcal{B}_{n,n}$ to the set of 1-dimensional linear subspaces of $W_n^{\mathrm{ver}}/\mathcal{N}_{n,0}^{\mathrm{ver}}$ and $W_n^{\mathrm{ver}}/\mathcal{N}_{n,1}^{\mathrm{ver}}$ respectively and a mapping $\iota^{\mathrm{ver}}_{n,n-1} : \mathcal{B}_{n,n-1} \to \mathcal{N}^{\mathrm{ver}}_{n,1}$, as follows:

$$(\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,0})(X) := \langle \varepsilon^{\mathrm{ver}}_n(X), \mathcal{N}^{\mathrm{ver}}_{n,0}\rangle/\mathcal{N}^{\mathrm{ver}}_{n,0} \text{ for every point } X \text{ of } \mathcal{B}_{n,n};$$
$$(\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,1})(X) := \langle \varepsilon^{\mathrm{ver}}_n(X), \mathcal{N}^{\mathrm{ver}}_{n,1}\rangle/\mathcal{N}^{\mathrm{ver}}_{n,1} \text{ for every point } X \text{ of } \mathcal{B}_{n,n};$$
$$\iota^{\mathrm{ver}}_{n,n-1}(X) := n_X \text{ for every point } X \text{ of } \mathcal{B}_{n,n-1}.$$

Lemma 4.10. *Let $n = 2$.*

(1) *Suppose that $\mathbb{F}$ is perfect. Then $\mathcal{N}^{\mathrm{ver}}_{2,0}$ defines a quotient of $\varepsilon^{\mathrm{ver}}_2$ and we have $\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{2,0} \cong \varepsilon^{\mathrm{spin}}_2$.*

(2) *In any case, $\mathcal{N}^{\mathrm{ver}}_{2,1}$ defines a quotient of $\varepsilon^{\mathrm{ver}}_2$ and $\dim(\varepsilon^{\mathrm{ver}}_2/\mathcal{N}^{\mathrm{ver}}_{2,1}) = 5$. If $\mathbb{F}$ is perfect then $\varepsilon^{\mathrm{ver}}/\mathcal{N}^{\mathrm{ver}}_{2,1} \cong \varepsilon^{\mathrm{sp}}_{2,2}$. If $\mathbb{F}$ is non-perfect then $\varepsilon^{\mathrm{ver}}/\mathcal{N}^{\mathrm{ver}}_{2,1}$ is laxly projective.*

(3) *$\iota^{\mathrm{ver}}_{2,1} \cong \varepsilon^{\mathrm{W}}_{2,1}$ $(\cong \varepsilon^{\mathrm{gr}}_{2,1})$.*

Sketch of the proof. Claim (3) is Lemma 2 of [10]. It can be rephrased as follows: $\iota^{\mathrm{ver}}_{2,1}(\mathcal{B}_{2,1})$ is a copy of the quadric $\mathcal{B}_{2,1} \cong Q(4,\mathbb{F})$ in $\mathrm{PG}(\mathcal{N}^{\mathrm{ver}}_{2,1}) \cong \mathrm{PG}(4,\mathbb{F})$ (notation as in Payne and Thas [24]). One of the points of $\mathrm{PG}(\mathcal{N}^{\mathrm{ver}}_{2,1})$ is the nucleus of the quadric $\iota^{\mathrm{ver}}_{2,1}(\mathcal{B}_{2,1})$. Both these claims admit straightforward proofs. (We warn that in [10] it is assumed that $\mathbb{F}$ is perfect, but this hypothesis plays no role in the proof of (3).) Moreover, denoted by $\mathcal{N}^{\mathrm{ver}}_{2,2}$ the nucleus of the quadric $\iota^{\mathrm{ver}}_{2,1}(\mathcal{B}_{2,1})$, it is not difficult to see that $\mathcal{N}^{\mathrm{ver}}_{2,2} = \ker(\pi_{2,2})$, where $\pi_{2,2}$ is the projection of $W^{\mathrm{ver}}_2 = V_2(2\lambda_2)$ onto $W^{\mathrm{gr}}_{2,2}$, as in Theorem 4.4. Accordingly, $\mathcal{N}^{\mathrm{ver}}_{2,1}/\mathcal{N}^{\mathrm{ver}}_{2,2} = \mathcal{N}^{\mathrm{gr}}_{2,2}$ (notation as in Theorem 4.2). Claim (2) now follows from Theorem 4.2.

Turning to (1), the crucial step in the proof of this claim is to prove that, for every line l of $\mathcal{B}_{2,2}$, the intersection of $\mathcal{N}^{\mathrm{ver}}_{2,0}$ with the plane $\langle C_l\rangle$ spanned by the conic $C_l := \varepsilon^{\mathrm{ver}}_2(l)$ is just the nucleus n_l of C_l. This is proved in [10, Lemma 3.2] under the hypothesis that $\mathbb{F}$ is perfect, exploiting the fact that, when the underlying field is perfect, all lines through the nucleus of a conic are tangent to the conic. Having proved that $\mathcal{N}^{\mathrm{ver}}_{2,0} \cap \langle C_l\rangle = n_l$, it readily follows that $\mathcal{N}^{\mathrm{ver}}_{2,0}$ defines a quotient of $\varepsilon^{\mathrm{ver}}_2$ and $\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{2,0} \cong \varepsilon^{\mathrm{spin}}_2$. □

So far we have defined $\mathcal{N}^{\mathrm{ver}}_{n,0}$, $\mathcal{N}^{\mathrm{ver}}_{n,1}$ and $\iota^{\mathrm{ver}}_{n,n-1}$. In the proof of Lemma 4.10 we have also defined $\mathcal{N}^{\mathrm{ver}}_{2,2}$ as the nucleus of the quadric $\iota^{\mathrm{ver}}_{2,1}(\mathcal{B}_{2,1})$. This definition can be generalized as follows.

Let $n > 2$. Given a k-element X of $\mathcal{B}_n$ with $k \leq n-2$, the upper residue $\mathrm{Res}^+(X)$ of X in $\mathcal{B}_n$ is a building of type B_{n-k} with $\{k+1,\ldots,n\}$ as the set of types. We can define the *n-Grassmannian* $\mathrm{Res}^+_n(X)$ of $\mathrm{Res}^+(X)$ by taking the n-elements of $\mathrm{Res}^+(X)$ as points and the lines of $\mathcal{B}_{n,n}$ contained in $\mathrm{Res}^+(X)$ as lines.

Let $k = n-2$. Then $\mathrm{Res}^+_n(X)$ is isomorphic to the symplectic generalized quadrangle $W(3,\mathbb{F})$. We call it a *quad* of $\mathcal{B}_{n,n}$. We have $\dim(\langle\varepsilon^{\mathrm{spin}}_n(\mathrm{Res}^+_n(X))\rangle) = 4$, namely $\varepsilon^{\mathrm{spin}}_n$ embeds $\mathrm{Res}^+_n(X)$ in the 4-space $\langle\varepsilon^{\mathrm{spin}}_n(\mathrm{Res}^+_n(X))\rangle$ as a copy of $W(3,\mathbb{F})$. By claim (3) of Lemma 4.10, $\iota^{\mathrm{ver}}_{n,n-1}(\mathrm{Res}^+_n(X))$ is a copy of $Q(4,\mathbb{F})$ in the 5-dimensional subspace $\langle\iota^{\mathrm{ver}}_{n,n-1}(\mathrm{Res}^+_n(X))\rangle$ of W^{ver}_n. We denote by n_X the nucleus of the quadric $\iota^{\mathrm{ver}}_{n,n-1}(\mathrm{Res}^+_n(X))$ in $\langle\iota^{\mathrm{ver}}_{n,n-1}(\mathrm{Res}^+_n(X))\rangle$ and we put $\mathcal{N}^{\mathrm{ver}}_{n,2} = \langle n_X\rangle_{X\in\mathcal{B}_{n,n-2}}$, where we write $X \in \mathcal{B}_{n,n-2}$ to say that X is an $(n-2)$-element of $\mathcal{B}_n$. Clearly, $\mathcal{N}^{\mathrm{ver}}_{n,2} \subseteq \mathcal{N}^{\mathrm{ver}}_{n,1}$ and $\mathcal{N}^{\mathrm{ver}}_{n,2}$ is stabilized by G.

We can also introduce one more mapping, which could not be defined when $n=2$: we denote by $\iota^{\mathrm{ver}}_{n,n-2} : \mathcal{B}_{n,n-2} \to \mathcal{N}^{\mathrm{ver}}_{n,2}$ the mapping sending every point X of $\mathcal{B}_{n,n-2}$ to n_X.

Theorem 4.11. *Let $n \geq 2$. Then the following hold.*

(1) *If $\mathbb{F}$ is perfect then $\mathcal{N}^{\mathrm{ver}}_{n,0}$ defines a quotient of $\varepsilon^{\mathrm{ver}}_n$. In any case, $\mathcal{N}^{\mathrm{ver}}_{n,1}$ and $\mathcal{N}^{\mathrm{ver}}_{n,2}$ define quotients of $\varepsilon^{\mathrm{ver}}_n$.*

(2) *Let $\mathbb{F}$ be perfect. Then $\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,0} \cong \varepsilon^{\mathrm{spin}}_n$.*

(3) *If $\mathbb{F}$ is perfect then $\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,1} \cong \varepsilon^{\mathrm{sp}}_{n,n}$. When $\mathbb{F}$ is non-perfect then $\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,1}$ is laxly projective of dimension $\binom{2n}{n} - \binom{2n}{n-2}$.*

(4) *We have $\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,2} \cong \varepsilon^{\mathrm{gr}}_{n,n}$.*

(5) *The mapping $\iota^{\mathrm{ver}}_{n,n-1}$ is a projective embedding of $\mathcal{B}_{n,n-1}$ in $\mathcal{N}^{\mathrm{ver}}_{n,1}$. Moreover $\iota^{\mathrm{ver}}_{n,n-1} \cong \varepsilon^{\mathrm{W}}_{n,n-1}$.*

(6) *Let $n > 2$. Then $\iota^{\mathrm{ver}}_{n,n-2}$ is a projective embedding of $\mathcal{B}_{n,n-2}$ in $\mathcal{N}^{\mathrm{ver}}_{n,2}$. Moreover $\iota^{\mathrm{ver}}_{n,n-2} \cong \varepsilon^{\mathrm{W}}_{n,n-2}$.*

Sketch of the proof. All the above claimed are proved in [10], but under the assumption that $\mathbb{F}$ is perfect. However, this assumption is only needed for claim (2) and the first claims of (1) and (3). We firstly sketch the proof of (2), then we shall turn to the rest.

Suppose that $\mathbb{F}$ is perfect. We know by Lemma 4.10 that for every quad Q of $\mathcal{B}_{n,n}$ the image $\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,0}(Q)$ of Q by $\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,0}$ spans a 4-dimensional vector space of $W^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,0}$. This fact, combined with a result of De Bruyn [16, Theorem 1.6], implies that $\dim(\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,0}) = 2^n$. It follows that $\varepsilon^{\mathrm{ver}}_n/\mathcal{N}^{\mathrm{ver}}_{n,0} \cong \varepsilon^{\mathrm{spin}}_n$.

As for the remaining claims of the theorem, a proof simpler than in [10] can be given with the help of [11]. It follows from the main result of [11] that $\mathcal{N}^{\mathrm{ver}}_{n,2} = \ker(\pi_{n,n}) \cong V_n(\lambda_{n-2})$. Claims (6) and (4) readily follow from this fact and Theorem 4.2. It also follows that $\mathcal{N}^{\mathrm{ver}}_{n,1}/\mathcal{N}^{\mathrm{ver}}_{n,2} = \mathcal{N}^{\mathrm{gr}}_{n,n}$. Hence (3) holds by Theorem 4.2. Finally, $\mathcal{N}^{\mathrm{ver}}_{n,1} \cong V_n(\lambda_{n-1})$ still by the main result of [11]. Claim (5) follows. $\square$

The previous construction of $\mathcal{N}^{\mathrm{ver}}_{n,2}$ and $\iota^{\mathrm{ver}}_{n,n-2}$ from $\mathcal{N}^{\mathrm{ver}}_{n,1}$ and $\iota^{\mathrm{ver}}_{n,n-1}$ can be generalized as follows. Suppose that for a given $k < n-1$ an embedding $\iota^{\mathrm{ver}}_{n,k+1}$ of $\mathcal{B}_{n,k+1}$ in a suitable submodule $\mathcal{N}^{\mathrm{ver}}_{n,n-k-1}$ of W^{ver}_n has been defined in such a way that for every k-element $X \in \mathcal{B}_{n,k}$ of $\mathcal{B}_n$, the subspace $W_X := \langle \iota^{\mathrm{ver}}_{n,k+1}(\mathrm{Res}^+_n(X))\rangle$ is $(2(n-k)+1)$-dimensional, and the restriction of $\iota^{\mathrm{ver}}_{n,k+1}$ to $\mathrm{Res}^+_n(X) \cong Q(2(n-k),\mathbb{F})$ is isomorphic to the natural embedding of $Q(2(n-k),\mathbb{F})$ in $V(2(n-k)+1,\mathbb{F})$. Thus $\iota^{\mathrm{ver}}_{n,k+1}(\mathrm{Res}^+_n(X))$ is a quadric in $\mathrm{PG}(W_X)$ ($\cong \mathrm{PG}(2(n-k),\mathbb{F})$). Let n_X be its nucleus. Put $\mathcal{N}^{\mathrm{ver}}_{n,n-k} = \langle n_X \rangle_{X\in\mathcal{B}_{n,k}}$ and let $\iota_{n,k}$ be the mapping which maps every $X \in \mathcal{B}_{n,k}$ to n_X.

We have $\mathcal{N}^{\mathrm{ver}}_{n,n-k} \cong V_n(\lambda_k)$ (isomorphism of G-modules) by the main result of [11]. Therefore:

Theorem 4.12. *For $0 < k < n-2$, the mapping $\iota_{n,k}$ is a projective embedding of $\mathcal{B}_{n,k}$ in $\mathcal{N}^{\mathrm{ver}}_{n,n-k}$, isomorphic to the Weyl embedding $\varepsilon^{\mathrm{W}}_{n,k}$.*

5. Universality

By Kasikova and Shult [20], the geometry $\mathcal{B}_{n,k}$ admits the universal projective embedding for any $k = 1,2,\ldots,n$ and $\mathcal{D}_{n,k}$ admits the universal projective embedding for $k = 1,2,\ldots,n-2$. The theory developed in [20] cannot be applied to $\mathcal{D}_{n,n-1}$, however this geometry admits the universal projective embedding by Blok and Pasini [5]. It is well known that $\varepsilon^{\mathrm{W}}_{n,1}$ ($\cong \varepsilon^{\mathrm{gr}}_{n,1}$) and $\eta^{\mathrm{W}}_{n,1}$ ($\cong \eta^{\mathrm{gr}}_{n,1}$) are universal (Tits [28, chapter 8]). So, we may assume $k > 1$. The following conjecture is quite natural:

Conjecture 5.1. For $k = 2,3,\ldots,n-1$ both $\varepsilon^{\mathrm{W}}_{n,k}$ and $\eta^{\mathrm{W}}_{n,k}$ are universal.

As for the case $k = n$, we have already remarked that $\varepsilon^{\mathrm{spin}}_n$ is universal when $\mathrm{char}(\mathbb{F}) \neq 2$, but $\varepsilon^{\mathrm{spin}}_n$ is projective whereas in this paper we are more interested in Veronesean embeddings of $\mathcal{B}_{n,n}$. We know nothing on the existence of the absolutely universal Veronesean embedding of $\mathcal{B}_{n,n}$. However, we may ask whether $\varepsilon^{\mathrm{W}}_{n,n}$ is relatively universal or not.

We begin our exposition with an elementary result on quasi-Veronesean embeddings of projective spaces, to be exploited more than one time in this section. Next we will address two special cases of Conjecture 5.1. Eventually, we will turn to $\varepsilon^{\mathrm{W}}_{n,n}$.

5.1. A lemma on quasi-Veronesean embeddings of $\mathrm{PG}(d,\mathbb{F})$

We need some preliminaries on 3-subspaces and 3-generating sets of point-line geometries. We say that a subset S of the point-set $\mathcal{P}$ of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is a 3-*subspace* of Γ, if S contains every line l of Γ such that $|l \cap S| \geq 3$. Intersections of 3-subspaces are 3-subspaces. Hence we can consider the 3-*span* $\langle X \rangle_\Gamma^{(3)}$ of a subset $X \subseteq \mathcal{P}$, defined as the smallest 3-subspace of Γ containing X. We say that X 3-*generates* Γ if $\langle X \rangle_\Gamma^{(3)} = \mathcal{P}$. The 3-*generating rank* $\mathrm{grk}_3(\Gamma)$ of Γ is the size of a smallest 3-generating set of Γ. Clearly, if Γ admits a quasi-Veronesean embedding ε then $\dim(\varepsilon) \leq \mathrm{grk}_3(\Gamma)$. If moreover $\dim(\varepsilon) = \mathrm{grk}_3(\Gamma) < \infty$ then ε is relatively universal.

Lemma 5.2. *Let $\mathbb{F} \neq \mathbb{F}_2$. Then $\mathrm{grk}_3(\mathrm{PG}(d,\mathbb{F})) = \binom{d+2}{2}$ for every integer $d > 0$.*

The proof is very elementary. We refer to [9, Section 4.4] for it. Lemma 5.2 immediately implies the following.

Corollary 5.3. *Let $\mathbb{F} \neq \mathbb{F}_2$ and let d be a positive integer.*

(1) *Every quasi-Veronesean embedding of $\mathrm{PG}(d,\mathbb{F})$ is at most $\binom{d+2}{2}$-dimensional.*

(2) *The Veronesean embedding of $\mathrm{PG}(d,\mathbb{F})$ in $V\left(\binom{d+2}{2}, \mathbb{F}\right)$ induced by the usual quadratic map is relatively universal.*

Remark 5.4. Remark 2.2 makes it clear that the restriction $\mathbb{F} \neq \mathbb{F}_2$ cannot be dropped from Lemma 5.2 and Corollary 5.3. Note also that, if $\Gamma = (\mathcal{P}, \mathcal{L})$ is a point-line geometry with all lines of size 3 then every subset of $\mathcal{P}$ is a 3-subspace. Hence $\mathrm{grk}_3(\Gamma) = |\mathcal{P}|$.

5.2. Two special cases of Conjecture 5.1

The next theorem is proved in [9, Section 4].

Theorem 5.5. *Let $\mathbb{F}$ be a perfect field of positive characteristic or a number field.*

(1) *If $n > 2$ then both $\varepsilon_{n,2}^{\mathrm{W}}$ and $\eta_{n,2}^{\mathrm{W}}$ are universal.*

(2) *Let $n > 3$ and $\mathbb{F} \neq \mathbb{F}_2$. Then both $\varepsilon_{n,3}^{\mathrm{W}}$ and $\eta_{n,3}^{\mathrm{W}}$ are universal.*

Sketch of the proof. For $k = 2$ or 3, with $k < n$, let $\rho_{n,k} : \mathcal{B}_{n,k} \to W_{\rho_{n,k}}$ and $\sigma_{n,k} : \mathcal{D}_{n,k} \to W_{\sigma_{n,k}}$ be given projective embeddings of $\mathcal{B}_{n,k}$ and $\mathcal{D}_{n,k}$ respectively, for some $\mathbb{F}$-vector spaces $W_{\rho_{n,k}}$ and $W_{\sigma_{n,k}}$. In order to avoid repetitions, we introduce four auxiliary symbols $\mathcal{X}$, ξ, $\mathcal{Y}$ and υ, to be read either as $\mathcal{B}$, ρ, $\mathcal{D}$ and σ respectively or as $\mathcal{D}$, σ, $\mathcal{B}$ and ρ, both interpretations being allowed, except that $\mathcal{X} = \mathcal{D}$ only if $n > k+1$. With this convention, let H be a hyperplane of the polar space $\mathcal{X}_{n,1}$ such that the polar space $\mathcal{X}_{n,1,H}$ induced by $\mathcal{X}_{n,1}$ on H is isomorphic to $\mathcal{Y}_{n,1}$ if $\mathcal{X} = \mathcal{B}$ and to $\mathcal{Y}_{n-1,1}$ if $\mathcal{X} = \mathcal{D}$. Let $\mathcal{X}_{n,k,H}$ be the subgeometry of $\mathcal{X}_{n,k}$ induced on the set of k-elements of $\mathcal{X}_n$ contained in H. Then $\mathcal{X}_{n,k,H}$ is isomorphic to either $\mathcal{Y}_{n,k}$ or $\mathcal{Y}_{n-1,k}$, according to whether $\mathcal{X}$ stands for $\mathcal{B}$

or $\mathcal{D}$. The embedding $\xi_{n,k}$ induces on $\mathcal{X}_{n,k,H}$ a projective embedding $\xi_{n,k,H}$: $\mathcal{X}_{n,k,H} \to W_{\xi_{n,k},H}$ where $W_{\xi_{n,k},H} := \langle \xi_{n,k}(\mathcal{X}_{n,k,H}) \rangle$. So, if we know an upper bound for the dimension of $\upsilon_{n,k}$ (when $\mathcal{X} = \mathcal{B}$) or $\upsilon_{n-1,k}$ (when $\mathcal{X} = \mathcal{D}$), then we also know an upper bound for $\dim(W_{\xi_{n,k},H})$.

Let a be a point of $\mathcal{X}_{n,1}$ exterior to H and $\mathcal{X}_{n,k,a}$ the subgeometry of $\mathcal{X}_{n,k}$ induced on the set of k-elements of $\mathcal{X}_n$ incident to a. Then $\mathcal{X}_{n,k,a}$ is isomorphic to the $(k-1)$-Grassmannian $\mathcal{X}_{n-1,k-1}$ of $\mathcal{X}_{n-1}$. Let $\xi_{n,k,a} : \mathcal{X}_{n,k,a} \to W_{\xi_{n,k},a}$ be the embedding induced by $\xi_{n,k}$ on $\mathcal{X}_{n,k,a}$, where $W_{\xi_{n,k},a} := \langle \xi_{n,k}(\mathcal{X}_{n,k,a}) \rangle$. This embedding can be regarded as a projective embedding of $\mathcal{X}_{n-1,k-1}$. So, if we know an upper bound for the dimension of a projective embedding of $\mathcal{X}_{n-1,k-1}$, then we also know an upper bound for $\dim(W_{\xi_{n,k},a})$.

When $k = 2$ let l_0 be a line of $\mathcal{X}_{n,1}$ not contained in $H \cup a^\perp$ and such that $a^\perp \cap l_0 \neq H \cap l_0$. Put $S_2 := \{l_0\} \cup \mathcal{X}_{n,2,a} \cup \mathcal{X}_{n,2,H}$.

When $k = 3$ the subgeometry $\mathcal{X}_{n,1,a,H}$ of $\mathcal{X}_{n,1}$ induced on $a^\perp \cap H$ is isomorphic to the polar space $\mathcal{X}_{n-1,1}$. This polar space admits a generating set of $f(n)$ points, where $f(n) = 2n-1$ or $f(n) = 2n-2$ according to whether $\mathcal{X}$ stands for $\mathcal{B}$ or $\mathcal{D}$. Hence the same holds for $\mathcal{X}_{n,1,a,H}$. Let $\{p_1, \dots, p_{f(n)}\}$ be a spanning set of $f(n)$ points of $\mathcal{X}_{n,1,a,H}$. For every $i = 1, \dots, f(n)$ let α_i be a plane of $\mathcal{X}_{n,1}$ through p_i such that $\alpha_i \cap H \cap a^\perp = \{p_i\}$. Put $S_3 := \{\alpha_i\}_{i=1}^{f(n)} \cup \mathcal{X}_{n,3,a} \cup \mathcal{X}_{n,3,H}$.

It is proved in [9] that S_k spans $\mathcal{X}_{n,k}$, both for $k = 2$ and $k = 3$. So, if we know an upper bound d_1 for the dimension of a projective embedding of $\mathcal{X}_{n-1,k-1}$ and an upper bound d_2 for the dimension of $\upsilon_{n,k}$ (when $\mathcal{X} = \mathcal{B}$) or $\upsilon_{n-1,k}$ (when $\mathcal{X} = \mathcal{D}$), then we obtain that $\dim(\xi_{n,k}) \le d_1 + d_2 + 1$ when $k = 2$ and $\dim(\xi_n) \le d_1 + d_2 + f(n)$ when $k = 3$. In this way, by an inductive argument and going back and forth from $\mathcal{D}_n$ to $\mathcal{B}_n$ we can compute upper bounds for $\dim(\rho_{n,k})$ and $\dim(\sigma_{n,k})$ for $k = 2$ and $k = 3$ and any $n > k$, provided that we know upper bounds for $\dim(\sigma_{3,2})$ and $\dim(\sigma_{4,3})$. Explicitly, assume the following:

(1*) $\dim(\sigma_{3,2}) \le 15$ $(= \binom{6}{2})$;

(2*) $\dim(\sigma_{4,3}) \le 56$ $(= \binom{8}{3})$.

Then $\dim(\sigma_{n,k}) \le \binom{2n}{k}$ and $\dim(\rho_{n,k}) \le \binom{2n+1}{k}$ for $k = 2$ as well as $k = 3$. As $\dim(\eta^{\mathrm{W}}_{n,k}) = \binom{2n}{k}$ and $\dim(\varepsilon^{\mathrm{W}}_{n,k}) = \binom{2n+1}{k}$, the universality of $\eta^{\mathrm{W}}_{n,2}$, $\varepsilon^{\mathrm{W}}_{n,2}$, $\eta^{\mathrm{W}}_{n,3}$ and $\varepsilon^{\mathrm{W}}_{n,3}$ follows.

So, (1*) and (2*) remain to be proved. It follows from Völklein [30] that (1*) holds true when $\mathbb{F}$ is either a perfect field of positive characteristic or a number field. So, claim (1) of the theorem is proved.

Let us turn to (2*). We still assume that $\mathbb{F}$ is either perfect of positive characteristic or a number field, but now we also suppose that $\mathbb{F} \neq \mathbb{F}_2$. Let a and

b be two non-collinear points of $\mathcal{D}_{4,1}$. Let $\mathcal{D}_{4,3,a}$ and $\mathcal{D}_{4,3,b}$ be the subgeometries induced by $\mathcal{D}_{4,3}$ on set of the planes of the polar space $\mathcal{D}_{4,1}$ containing a or b respectively. (Note that the planes of $\mathcal{D}_{4,1}$ are just the points of $\mathcal{D}_{4,3}$.) Let $\sigma_{4,3,a}$ and $\sigma_{4,3,b}$ be the restrictions of $\sigma_{4,3}$ to $\mathcal{D}_{4,3,a}$ and $\mathcal{D}_{4,3,b}$ respectively. We have $\mathcal{D}_{4,3,a} \cong \mathcal{D}_{4,3,b} \cong \mathcal{D}_{3,2}$. Therefore, by (1^*), each of $\sigma_{4,3,a}$ and $\sigma_{4,3,b}$ is at most 15-dimensional. Let $S_{a,b}$ be the set of planes of $\mathcal{D}_{4,1}$ contained in $a^\perp \cap b^\perp$. It is shown in [9] that $S_{a,b}$ can be regarded as the direct sum of two copies of $\mathrm{PG}(3,\mathbb{F})$ and $\sigma_{4,3}$ induces a quasi-Veronesean embedding on each of them. The embedding induced by $\sigma_{4,3}$ on $S_{a,b}$, being the direct sum of two quasi-Veronesean embeddings of $\mathrm{PG}(3,\mathbb{F})$, is at most 20-dimensional by Corollary 5.3 (which can be applied, since we have assumed that $\mathbb{F} \neq \mathbb{F}_2$). So far we have constructed a subset $S := \mathcal{D}_{4,3,a} \cup \mathcal{D}_{4,3,b} \cup S_{a,b}$ of the point-set of $\mathcal{D}_{4,1}$ such that $\dim(\langle\sigma_{4,3}(S)\rangle) \leq 50$. However, S is not yet a generating set of $\mathcal{D}_{4,3}$. As shown in [9], in order to generate $\mathcal{D}_{4,3}$ we only must add to S a suitable set of six points of $\mathcal{D}_{4,3}$. So, $\dim(\sigma_{4,3}) \leq 50 + 6 = 56$, as claimed in (2^*). □

Remark 5.6. There are two main obstacles to overcome when we try to adapt the strategy described above to the case $k > 3$. Firstly, it is not clear how to define an analogous of the set $\{\alpha_i\}_{i=1}^{f(n)}$, used in the above sketch to construct S_3. Starting from a generating set of a polar space as $\mathcal{X}_{n,1,a,H}$ does not seem to work. We should rather consider a generating set of the geometry $\mathcal{X}_{n,k-2,a,H} \cong \mathcal{X}_{n-1,k-2}$ induced by $\mathcal{X}_{n,k-2}$ on the set of $(k-2)$-elements of $\mathcal{X}_n$ contained in $a^\perp \cap H$, but if we choose this way then we need to know the generating rank of $\mathcal{X}_{n-1,k-2}$. We may inductively assume that we already know the absolutely universal embedding of $\mathcal{X}_{n-1,k-2}$, but the generating rank of $\mathcal{X}_{n-1,k-2}$ might be larger than the dimension of that embedding. In particular, it might also depend on the underlying field $\mathbb{F}$ (see Blok and Pasini [4]; see also below, Remark 5.7).

Secondly, in order to start the induction we should prove that $\dim(\sigma_{k+1,k}) \leq \binom{2k+2}{k}$ for every projective embedding $\sigma_{k+1,k}$ of $\mathcal{D}_{k+1,k}$, at least for suitable choices of $\mathbb{F}$. This is perhaps the hardest point.

Remark 5.7. Cooperstein [14] has proved that if $\mathbb{F}$ is a prime field then $\mathcal{B}_{n,2}$ and $\mathcal{D}_{n,2}$ have generating ranks equal to $\binom{2n+1}{2}$ and $\binom{2n}{2}$ respectively. Claim (1) of Theorem 5.5 follows from this fact too, but provided that $\mathbb{F}$ is a prime field. The arguments exploited by Cooperstein in [14] do not seem to work for larger fields.

Remark 5.8. When $\mathbb{F} = \mathbb{C}$ (the field of complex numbers), if $\varepsilon^{\mathrm{W}}_{n,k}$ is not universal then its hull is infinite dimensional. The same holds for $\eta^{\mathrm{W}}_{n,k}$.

Indeed, let $\mathbb{F} = \mathbb{C}$ and let $\tilde{\varepsilon} : \mathcal{B}_{n,k} \to \widetilde{W}$ be the hull of $\varepsilon^{\mathrm{W}}_{n,k}$. Put $\lambda = \lambda_k$ if $k < n$ and $\lambda = 2 \cdot \lambda_n$ if $k = n$. Let $f : \widetilde{W} \to V_n(\lambda)$ be the projection of $\tilde{\varepsilon}$ onto $\varepsilon^{\mathrm{W}}_{n,k}$ and $\tilde{v}_0 \in f^{-1}(v_0)$, where v_0 is a highest weight vector for $V_n(\lambda)$. Then $\tilde{v}_0$ has just the same properties as v_0, namely $h(\tilde{v}_0) = \lambda(h) \cdot \tilde{v}_0$ for every element h of the Cartan subalgebra of the Lie algebra $\mathfrak{L}_G$ of G and $X_\alpha(\tilde{v}_0) = 0$ for every positive root α, where X_α is the 1-dimensional subalgebra of $\mathfrak{L}_G$ corresponding to α. Hence $\widetilde{W}$ is a quotient of the cyclic $\mathfrak{L}_G$-module $Z(\lambda)$ (notation as in Humphreys [19]). If π is the projection of $Z(\lambda)$ onto $\widetilde{W}$, then $\ker(\pi)$ is a submodule of the maximal proper submodule $J(\lambda)$ of $Z(\lambda)$. Clearly, $\dim(\widetilde{W})$ is equal to the index $|Z(\lambda) : \ker(\pi)|$ of $\ker(\pi)$ in $Z(\lambda)$.

Suppose that $\dim(\widetilde{W})$ is finite, namely $|Z(\lambda) : \ker(\pi)| < \infty$. Since $\mathbb{F} = \mathbb{C}$, every finite dimensional reducible $\mathfrak{L}_G$-module is completely reducible. Hence $Z(\lambda)/\ker(\pi)$ splits as a direct sum $Z(\lambda)/\ker(\pi) = X \oplus J(\lambda)/\ker(\pi)$, for a suitable submodule X of $Z(\lambda)/\ker(\pi)$. Clearly, $X = \overline{X}/\ker(\pi)$ for a submodule $\overline{X}$ of $Z(\lambda)$. If $J(\lambda)/\ker(\pi) \neq 0$ then $\overline{X}$ is a proper submodule of $Z(\lambda)$ not contained in $J(\lambda)$, contrary to the maximality of $J(\lambda)$. Therefore $\ker(\pi) = J(\lambda_k)$, namely f is an isomorphism.

So, if $\tilde{\varepsilon} \not\cong \varepsilon^{\mathrm{W}}_{n,k}$ then $\dim(\tilde{\varepsilon})$ is infinite. We cannot claim that this is impossible, but it is hard to believe, at least when $k < n$.

5.3. Positive and negative results on $\varepsilon^{\mathrm{W}}_{n,n}$

We shall now turn to $\varepsilon^{\mathrm{W}}_{n,n}$ but only focusing on the following two special cases: $n = 2$ with $\mathbb{F}$ a finite field of odd order; any n, but with $\mathrm{char}(\mathbb{F}) = 2$. The first case is settled by the following theorem (Theorem 5 of [10]).

Theorem 5.9. *Let $n = 2$. Let $\mathbb{F}$ be a finite field of odd order $q > 3$. Then $\varepsilon^{\mathrm{W}}_{2,2}$ is relatively universal.*

Sketch of the proof. Recall that $\varepsilon^{\mathrm{W}}_{2,2} \cong \varepsilon^{\mathrm{ver}}_2$ $(= \nu_4 \cdot \varepsilon^{\mathrm{spin}}_2)$ by Theorem 4.9. The spin embedding $\varepsilon^{\mathrm{spin}}_2$ embeds $\mathcal{B}_{2,2}$ in $V_4 = V(4, \mathbb{F})$ as a copy of the generalized quadrangle $W(3, \mathbb{F})$ of symplectic type. The quadratic Veronesean map ν_4 induces a Veronesean embedding $\bar{\nu}_4$ of $W(3, \mathbb{F})$. Since $W(3, \mathbb{F})$ and $\mathrm{PG}(3, \mathbb{F})$ have the same set of points, the embeddings ν_4 and $\bar{\nu}_4$ coincide if regarded just as functions from that set of points to $\mathrm{PG}(9, \mathbb{F})$. In particular, $\dim(\bar{\nu}_4) = \dim(\nu_4) = 10$. Nevertheless, ν_4 and $\bar{\nu}_4$ are different embeddings, since they have different domains, namely $\mathrm{PG}(3, \mathbb{F})$ and $W(3, \mathbb{F})$ respectively. This difference becomes fully clear if we consider their hulls: the presheaf associated to $\bar{\nu}_4$, to be used to construct the hull of $\bar{\nu}_4$, is a proper sub-presheaf of the one associated to ν_4. Thus, although ν_4 is relatively universal when $\mathbb{F} \neq \mathbb{F}_2$ (Corollary 5.3), it might happen that the hull of $\bar{\nu}_4$ is larger than $\bar{\nu}_4$ even if $\mathbb{F} \neq \mathbb{F}_2$.

The embedding $\bar{\nu}_4$ is G-homogeneous (note that $G = \mathrm{Spin}(5, \mathbb{F}) \cong \mathrm{Sp}(4, \mathbb{F})$). Hence the hull of $\bar{\nu}_4$ is G-homogeneous. Thus, in order to prove the theorem, we only must prove that if $\nu : W(3,\mathbb{F}) \to W$ is a G-homogeneous Veronesean embedding of $W(3,\mathbb{F})$ such that $\nu \geq \bar{\nu}_4$, then $\dim(\nu) = 10 = \dim(\bar{\nu}_4)$. Let ν be such an embedding. We recall that the hyperbolic lines of $W(3,\mathbb{F})$ are lines of $\mathrm{PG}(3,\mathbb{F})$ (see Payne and Thas [24] for the definition of hyperbolic lines). Moreover, every line of $\mathrm{PG}(3,\mathbb{F})$ either is totally isotropic (namely it is a line of the generalized quadrangle $W(3,\mathbb{F})$) or it is a hyperbolic line of $W(3,\mathbb{F})$. All lines of $\mathrm{PG}(3,\mathbb{F})$ are mapped by ν_4 onto conics. Therefore, since $\nu \geq \bar{\nu}_4$ and $\nu_4(p) = \bar{\nu}_4(p)$ for every point p of $\mathrm{PG}(3,\mathbb{F})$, if l is a hyperbolic line of $W(3,\mathbb{F})$ either $\nu(l)$ is a conic or $\dim(\langle\nu(l)\rangle) > 3$. As G is transitive on the set of hyperbolic lines of $W(3,\mathbb{F})$ and ν is assumed to be G-homogeneous, the same situation occurs for all hyperbolic lines. So, if we can prove that ν maps a hyperbolic line of $W(3,\mathbb{F})$ onto a conic of $\mathrm{PG}(W)$ then ν is also a Veronesean embedding of $\mathrm{PG}(3,\mathbb{F})$. Hence it is at most 10-dimensional by Corollary 5.3, and we are done.

In [10], by combinatorial arguments and some elementary group theory, it is proved that when $\mathbb{F}$ is a finite field of odd order $q > 3$ the assumption that $\dim(\langle\nu(l)\rangle) > 3$ for a hyperbolic line l of $W(3,\mathbb{F})$ leads to a contradiction, which is what we need to finish the proof. We are not going to expose the arguments used in [10] to prove the above. We only say that, if K is the kernel of the projection of ν onto $\bar{\nu}_4$, the hypothesis that $q > 3$ is exploited to prove that $K \subseteq \langle\nu(l)\rangle$ for every hyperbolic line l of $W(3,\mathbb{F})$. A contradiction is eventually obtained from this fact. $\square$

When $\mathrm{char}(\mathbb{F}) = 2$ we have the following, where we assemble Theorems 6 and 7 of [10].

Theorem 5.10. *Let* $\mathrm{char}(\mathbb{F}) = 2$.

(1) *Let* $n = 2$. *Then* $\varepsilon^{\mathrm{W}}_{2,2}$ *is not relatively universal.*

(2) *Let* $\mathbb{F}$ *be perfect. Then* $\varepsilon^{\mathrm{W}}_{n,n}$ *is not relatively universal, for any* $n \geq 2$.

Sketch of the proof. When $\mathrm{char}(\mathbb{F}) = 2$ and either $n = 2$ or $\mathbb{F}$ is perfect, the embedding $\varepsilon^{\mathrm{spin}}_n$ is not universal. Indeed when $\mathbb{F}$ is perfect and $n > 2$ then $\varepsilon^{\mathrm{spin}}_n$ is a proper quotient of $\varepsilon^{\mathrm{Sp}}_{n,n}$, as remarked at the end of Subsection 3.1. When $n = 2$, $\mathcal{B}_{2,2} \cong \mathcal{C}_{2,1}$, whence $\varepsilon^{\mathrm{spin}}_2 \cong \varepsilon^{\mathrm{Sp}}_{2,1}$. The latter embedding is never universal, no matter if $\mathbb{F}$ is perfect or not (see e.g. De Bruyn and Pasini [17] for the non-perfect case). Let $\hat{\varepsilon} : \mathcal{B}_{n,n} \to \widehat{W}$ be the hull of $\varepsilon^{\mathrm{spin}}_n$. By the above, $d := \dim(\widehat{W}) > 2^n$. Let ν_d be the quadratic Veronesean map from $\widehat{W}$ to $V(\binom{d+1}{2}, \mathbb{F})$ and put $\hat{\varepsilon}^{\mathrm{ver}} = \nu_d \cdot \hat{\varepsilon}$. Then $\varepsilon^{\mathrm{ver}}_n$ ($\cong \varepsilon^{\mathrm{W}}_{n,n}$ by Theorem 4.9) is a proper quotient of $\hat{\varepsilon}^{\mathrm{ver}}$. $\square$

Conjecture 5.11. We guess that the hypothesis that $\mathbb{F}$ is perfect is superfluous in claim (2) of Theorem 5.10.

Problems 5.12. (1) Compute the hull of $\varepsilon^{\mathrm{W}}_{2,2}$ for $\mathbb{F} = \mathbb{F}_3$.

(2) Can we remove the hypothesis that $\mathbb{F}$ is finite from Theorem 5.9?

(3) Let $\mathbb{F}$ be such that $\varepsilon^{\mathrm{W}}_{2,2}$ is relatively universal (whence $\mathrm{char}(\mathbb{F}) \neq 2$). Does the universality of $\varepsilon^{\mathrm{W}}_{2,2}$ imply that $\varepsilon^{\mathrm{W}}_{n,n}$ is relatively universal for any $n > 2$?

(4) Does $\mathcal{B}_{n,n}$ admit the absolutely universal Veronesean embedding?

(5) Corollary 5.3, when $\mathbb{F} \neq \mathbb{F}_2$ the quadric Veronesean embedding of $\mathrm{PG}(d, \mathbb{F})$ in $V(\binom{d+2}{2}, \mathbb{F})$ is relatively universal. Is it absolutely universal?

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