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Neighborhood distinguishing coloring in graphs

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Abstract

In the case of a finite dimensional vector space V, any ordered basis can be used to give distinct codes for elements of V. Chartrand et al [1] introduced coding for vertices of a finite connected graph using distance. A binary coding of vertices of a graph (connected or disconnected) was suggested in [2]. Motivated by these papers, a new type of coding, called neighborhood distinguishing coloring code, is introduced in this paper. A study of this code is initiated.

Keywords: Neighborhood distinguishing coloring code, neighborhood distinguishing coloring number of a graph

MSC 2000: 05C15

1 Introduction

Definition 1.1. Let $G = (V, E)$ be a simple connected graph. Two adjacent vertices are referred to as neighbors of each other in G. The set of neighbors of a vertex v is called the open neighborhood of v (or simply the neighborhood of v) and is denoted by $N(v)$.

Definition 1.2. Let $G = (V, E)$ be a simple connected graph. A proper vertex coloring of a graph G is an assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are colored differently. A proper coloring can be considered as a function $C : V(G) \to N$ (where N is the set of positive integers) such that $C(u) \neq C(v)$ if u and v are adjacent in G. If each color used is one of k given colors, then we refer to the coloring as *k-coloring*. Suppose that c is a k-coloring of a graph G, where each color is one of the integers $1, 2, 3, \ldots, k$ that are being used. If V_i ($1 \leq i \leq k$) is the set of vertices in G colored i (where one or more of these set may be empty), then each nonempty set V_i is called a color class and the nonempty elements of $\{V_1, V_2, \dots, V_k\}$ produce a partition of $V(G)$. This is called a proper color partition.

Let $G = (V, E)$ be a simple connected graph. Chartrand et al [1] introduced a coding for vertices of G as follows: Let S be a subset of $V(G)$. For a fixed ordering of S, say, $S = \{v_1, v_2, \dots, v_r\}$, the code of a vertex $u \in V(G)$ with respect to S is defined as

$$\mathrm{code}(u) = (d(u, v_1), d(u, v_2), \dots, d(u, v_r)).$$

The set S is called a resolving set if the codes of any two vertices of G with respect to S are distinct. The minimum cardinality of a resolving set of G is called the metric dimension of G. This concept and its manifestations were studied by many. Instead of involving distances, Suganthi [2] introduced a binary coding using neighborhoods. A subset S of V is called a neighborhood resolving set if for some order of $S = \{u_1, u_2, \dots, u_r\}$, the codes of the vertices defined by $\mathrm{code}(u) = (a_1, a_2, \dots, a_r)$ where $a_i = 1$ if $u \in N(u_i)$ and 0 otherwise, are distinct. A detailed study of neighborhood resolving sets has been made in [2]. A question was raised whether a partition of the vertex set of a graph may be used for defining codes for vertices. An attempt to answer this question led to the study of neighborhood distinguishing coloring codes. If $\pi = \{V_1, V_2, \dots, V_r\}$ is a partition of the vertex set $V(G)$, then a code for a vertex u in $V(G)$ may be defined as

$$\mathrm{code}(u) = \big(|N(u) \cap (V_1)|, |N(u) \cap (V_2)|, \dots, |N(u) \cap (V_r)|\big).$$

The partition is said to be distinguishing if the codes of distinct vertices of G are distinct. For neighborhood distinguishing code, instead of taking any arbitrary partition, proper color partitions of $V(G)$ are considered. A study of this new type of coding is made in this paper.

2 Main Results

Definition 2.1. Let $\pi = \{V_1, V_2, \dots, V_k\}$ be a proper color partition of G. Fixing this order of π, for each $u \in V(G)$, we assign a code denoted by $C(u)$ as $C(u) = \{|N(u) \cap V_i|, i = 1, 2, \dots, k\}$. Then π is called a *neighborhood distinguishing coloring* (abbreviated as NDC) if $C(u) \neq C(v)$ for all distinct $u, v \in V$.

Definition 2.2. The minimum cardinality of a neighborhood distinguishing coloring of a graph G is called the *neighborhood distinguishing coloring number of G*, and it is denoted by $\chi_{\mathrm{NDC}}(G)$. Also, a neighborhood distinguishing color partition of G with $\chi_{\mathrm{NDC}}(G)$ elements is called a χ_{NDC}-partition of G.

Definition 2.3. Let $G = (V, E)$ be a simple connected graph. A walk whose initial and terminal vertices are distinct is an open walk; otherwise, it is a closed walk. A walk in a graph G in which no vertex is repeated is called a path. A path with n vertices denoted by P_n.

Definition 2.4. Let $G = (V, E)$ be a simple connected graph. A graph G is $complete$ if every two distinct vertices in the graph are adjacent. The complete graph of order n is denoted by K_n.

Definition 2.5. Let $G = (V, E)$ be a simple connected graph. A graph G is a bipartite graph if it is possible to partition $V(G)$ into two subsets U and W, called partite sets, such that every edge of G joints a vertex U and a vertex of W. A bipartite graph having partite sets U and W is a complete bipartite graph if every vertex of U is adjacent to every vertex of W. If the partite sets U and W of a complete bipartite graph contain s and t vertices, then this graph is denoted by $K_{s,t}$ or $K_{t,s}$. The graph $K_{1,t}$ is called a star.

Theorem 2.6. *A graph G has NDC if and only if any two non-adjacent vertices of G do not have the same neighborhood.*

Proof. Let G admit NDC. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a NDC. Let $x, y \in V(G)$ be distinct vertices. Then $C(x) \neq C(y)$. Therefore, there exists i, $1 \leq i \leq k$ such that $|N(x) \cap V_i| \neq |N(y) \cap V_i|$. Hence $N(x) \neq N(y)$.

Conversely, suppose for any x and y which are non-adjacent, $N(x) \neq N(y)$. Let $\pi = \{\{u_1\}, \{u_2\}, \ldots, \{u_k\}\}$ where $V(G) = \{u_1, u_2, \ldots, u_k\}$. If u_i and u_j, $1 \leq i, j \leq k$, $i \neq j$ are adjacent then $C(u_i)$ has 0 in the i^{th} place and $C(u_j)$ has 1 in the i^{th} place. Therefore $C(u_i) \neq C(u_j)$. If u_i and u_j are non-adjacent then there exists u_k such that $u_k \in N(u_i)$ and $u_k \notin N(u_j)$ or vice versa. Hence $C(u_i) \neq C(u_j)$. Therefore, π is a NDC. □

Corollary 2.7. *Let G be a graph which admits neighborhood distinguishing coloring. Then $\pi = \{\{u_1\}, \{u_2\}, \ldots, \{u_n\}\}$ is an NDC with $V(G) = \{u_1, u_2, \ldots, u_n\}$.*

Remark 2.8. (a) For any graph G, $\chi(G) \leq \chi_{\mathrm{NDC}}(G)$.

(b) $\chi_{\mathrm{NDC}}(G) > 1$. For if $\chi_{\mathrm{NDC}}(G) = 1$, then $G = \overline{K}_n$ and $\overline{K}_n$ has no NDC.

(c) If $G = K_n$, the complete graph of order n, then $\chi_{\mathrm{NDC}}(G) = n$.

(d) $2 \leq \chi_{\mathrm{NDC}}(G) \leq n$, with $|V(G)| = n$, and the bounds are sharp, since $\chi_{\mathrm{NDC}}(P_4) = 2$ and $\chi_{\mathrm{NDC}}(K_n) = n$.

(e) A graph admitting NDC has at most one isolated vertex.

Definition 2.9. Let G be a graph. The Mycielskian $\mu(G)$ of G is the graph obtained as follows: Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. Attach vertices $u'_1, u'_2, \ldots, u'_n$

and v. Make u'_i adjacent with all the neighbors of u_i in G $(1 \leq i \leq n)$ and make v adjacent with $u'_1, u'_2, \ldots, u'_n$. The resulting graph is called the Mycielskian of G and is denoted by $\mu(G)$.

Theorem 2.10. *Let G be a graph which admits NDC. Let $\mu(G)$ be the Mycielskian of G. Then $\chi_{\text{NDC}}(\mu(G)) = \chi_{\text{NDC}}(G) + 1$.*

Proof. Let G be a graph which admits NDC. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a χ_{NDC}-partition of G. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and

$$V(\mu(G)) = \{u_1, u_2, \ldots, u_n, u'_1, u'_2, \ldots, u'_n, v\}.$$

Let $\pi' = \big\{V_1 \cup \{v\}, V_2, \ldots, V_k, \{u'_1, u'_2, \ldots, u'_n\}\big\}$. Then it can be verified that π' is a χ_{NDC}-partition of $\mu(G)$. Therefore, $\chi_{\text{NDC}}(\mu(G)) = \chi_{\text{NDC}}(G) + 1$. □

Corollary 2.11. *Given any positive integer k, there exists a triangle free connected graph G such that $\chi_{\text{NDC}}(G) = k$. More precisely, if $k = 1$ or 2, take $G = K_1$ or K_2; if $k \geq 3$, take $G = \mu^{(k-3)}(C_5)$; then G is triangle free, connected and $\chi_{\text{NDC}}(G) = k$.*

Definition 2.12. Let $G = (V, E)$ be a simple connected graph. The degree of a vertex v in a graph G is the number of vertices in G that are adjacent to v. Thus the degree of a vertex v is the number of the vertices in its neighborhood $N(v)$. The largest degree among the vertices of G is called the maximum degree of G and is denoted by $\Delta(G)$.

Theorem 2.13. *Suppose G admits NDC, and $|V(G)| = n$. Then*

$$n \leq (\Delta + 1)^{\chi_{\text{NDC}}(G)}.$$

Proof. Suppose $n > (\Delta+1)^{\chi_{\text{NDC}}(G)}$. Let $\chi_{\text{NDC}}(G) = k$. Then any χ_{NDC}-partition π of G can yield at most $(\Delta+1)^k$ distinct codes. Since $n > (\Delta+1)^k, \pi$ cannot be a distinguishing coloring, a contradiction. Therefore, $n \leq (\Delta + 1)^{\chi_{\text{NDC}}(G)}$. □

Definition 2.14. Let $G = (V, E)$ be a simple connected graph. The graph G^+ is defined as the graph obtained from G by adjoining exactly one pendant vertex at each of the vertices of G.

Theorem 2.15. *Any graph can be embedded in a graph which admits NDC.*

Proof. Let G be a graph. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. If G admits NDC, then we are through. Suppose G does not admit NDC. Consider G^+. Let $V(G^+) = \{u_1, u_2, \ldots, u_n, u'_1, u'_2, \ldots, u'_n\}$ where u'_i is a pendant vertex adjacent with $u_i, 1 \leq i \leq n$. Let $x, y \in V(G^+)$ and let x and y be distinct and non-adjacent. If $x = u'_i$ and $y = u'_j$, then x and y are non-adjacent and $N(x) \neq N(y)$.

If $x = u_i$ and $y = u'_j$ $(i \neq j)$ then x and y are non-adjacent, $u'_j \in N(y)$ and $u'_j \notin N(x)$. Therefore $N(x) \neq N(y)$. Let $x = u_i$ and $y = u_j$. Then $u'_i \in N(u_i)$ and $u'_i \notin N(u_j)$. Therefore $N(x) \neq N(y)$. Therefore G^+ admits NDC. Hence the theorem. □

Definition 2.16 (Embedding index of a graph which does not admit NDC)**.** Let G be a graph which does not admit NDC. Let H be a graph which admits NDC such that G is an induced subgraph of H, and H is a graph with the smallest order satisfying these properties. Then $|V(H)| - |V(G)|$ is called the NDC embedding index of G.

Remark 2.17. Let G be a graph in which t pairs of non-adjacent vertices have the same neighborhood. Let S be the set of vertices formed by these t pairs. Attach one pendant vertex at each of the vertices of S. Let H be the resulting graph. Then H admits NDC and the embedding index of G is at most $|S|$.

Remark 2.18. (a) There are graphs in which $\chi(G) = \chi_{\mathrm{NDC}}(G)$, e.g. C_5.

(b) Given a positive integer k, there exists a connected graph G such that $\chi_{\mathrm{NDC}}(G) - \chi(G) = k$.

Proof. **Case (i):** k is even.

Let $G = K_{1,(k+2)/2}$. Let $V(G) = \{u, v_1, v_2, \ldots, v_{(k+2)/2}\}$. Subdivide each edge of G exactly once. Let H be the resulting graph. Let the new vertices in H be $\{w_1, w_2, \ldots, w_{(k+2)/2}\}$. Then

$$\big\{\{u, v_1\}, \{v_2\}, \ldots, \{v_{(k+2)/2}\}, \{w_1\}, \{w_2\}, \ldots, \{w_{(k+2)/2}\}\big\}$$

is a χ_{NDC}-partition of H and hence $\chi_{\mathrm{NDC}}(H) = k + 2$. Since H is a tree $\chi(H) = 2$. Therefore, $\chi_{\mathrm{NDC}}(H) - \chi(H) = k + 2 - 2 = k$.

Case (ii): k is odd.

Let $G = K_{1,k}$. Let $V(G) = \{u, v_1, v_2, \ldots, v_{(k+3)/2}\}$. Subdivide each edge of G except the last one exactly once. Let H be the resulting graph and the new vertices in H be $\{w_1, w_2, \ldots, w_{(k+1)/2}\}$. Then

$$\big\{\{u, v_1\}, \{v_2\}, \ldots, \{v_{(k+3)/2}\}, \{w_1\}, \{w_2\}, \ldots, \{w_{(k+1)/2}\}\big\}$$

is a χ_{NDC}-partition of H and hence $\chi_{\mathrm{NDC}}(H) = k + 2$. Since H is a tree $\chi(H) = 2$. Therefore, $\chi_{\mathrm{NDC}}(H) - \chi(H) = k + 2 - 2 = k$. □

(c) Even in graphs in which no two non-adjacent vertices have the same neighbor, $\chi(G)$ need not be equal to $\chi_{\mathrm{NDC}}(G)$. For example, in P_5, $\chi(G) = 2$, and $\chi_{\mathrm{NDC}}(G) = 3$.

3 Areas for further study

(a) Properties of NDC-partitions.

(b) A vertex u is said to be

  (i) NDC-fixed if $\{u\}$ appears in every $\chi_{\rm NDC}$-partition.

  (ii) NDC-free if $\{u\}$ appears in some $\chi_{\rm NDC}$-partition and does not appear in some other $\chi_{\rm NDC}$-partition.

  (iii) NDC-totally free if $\{u\}$ does not appear in any $\chi_{\rm NDC}$-partition.

(c) Characterize fixed, free and totally free vertices in a graph which admits NDC.

(d) Characterize graphs which admit a unique $\chi_{\rm NDC}$-partition.

(e) Find a necessary and sufficient condition for a partition to be NDC.

(f) Characterize graphs G for which $\chi_{\rm NDC}(G) = \chi(G)$.

References

[1] **G. Chartrand**, **L. Eroh**, **M. Johnson** and **O.R. Oellermann**, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.* **105** (2000), 99–113.

[2] **S. Suganthi**, A Study on Resolving sets in Graphs, *Thesis submitted to Madurai Kamaraj University*.

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