A family of 2-arc transitive pentagraphs with unbounded valency

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Abstract

We construct polygonal graphs on the points of a generalized polygon in general position with respect to a polarity.

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1 Polygonal graphs

Let \((X, L, I)\) be a generalized \(n\)-gon with polarity \(\sigma\). Let \(Z\) be the set of points in general position with respect to \(\sigma\), i.e., \(Z = \{x \in X \mid d(x, x^\sigma) \geq n - 1\}\), with distances measured in the point-line incidence graph \(\Sigma\) of \((X, L, I)\). (Thus, if \(n\) is even then \(d(x, x^\sigma) = n - 1\) and if \(n\) is odd then \(d(x, x^\sigma) = n\) for \(x \in Z\).) Define a graph \(\Gamma\) with vertex set \(Z\) by letting distinct vertices \(x, y \in Z\) be adjacent (notation \(x \sim y\)) when \(x I y^\sigma\).

Theorem 1.1. If \(n\) is odd, then \(\Gamma\) has girth \(g \geq n\) and each edge is contained in a unique \(n\)-gon. If \(n\) is even, then \(\Gamma\) has girth \(g \geq n + 1\) and each 2-path is contained in a unique \((n + 1)\)-gon.

Proof. Let us first collect information about the vertex set \(Z\).

Step 1. If \(x_0 I x_1^\sigma I x_2 I \ldots I x_{l-1} I x_0 I x_1 I \ldots I x_{l-1} I x_0\) is a self-polar \(2l\)-circuit in \(\Sigma\), and \(l \leq n + 1\), then \(x_i \in Z\) \((0 \leq i \leq l - 1)\).

(Indeed, if \(d_{\Sigma}(x_i, x_{i}^\sigma) = m\), then we find an \((m + l)\)-circuit in \(\Sigma\), so that \(m + l \geq 2n\).)
Step 2. If \( n \) is even, and \( x \in \mathbb{Z} \), and \( x I x_{1} I \ldots I x_{n-1} I x^n \) is the unique path of length \( n - 1 \) joining \( x \) to \( x^n \) in \( \Sigma \), then \( x_i \not\in \mathbb{Z} \) \((1 \leq i \leq n - 2)\).

(Indeed, applying \( \sigma \) to this path, we find another path that must coincide with this path, so that \( x^\sigma_i = x_{n-i-1} \) \((1 \leq i \leq n - 2)\).)

Now look at the graph \( \Gamma \). Note that if \( x \sim y \sim z \) in \( \Gamma \), then \( x I y I z \) in \( \Sigma \).

Step 3. \( \Gamma \) does not have even circuits of length less than \( 2n \) and no odd circuits of length less than \( n \). In particular, if two vertices have distance less than \( n \) in \( \Gamma \), then there is a unique shortest path in \( \Gamma \) joining them.

(Indeed, if \( x_0 \sim x_1 \sim \ldots \sim x_{n-1} \sim x_0 \) is an \( l \)-circuit in \( \Gamma \), and \( l \) is even, then \( x_0 I x_{1} I x_{2} I \ldots I x_{n-1} I x_0 \) is an \( l \)-circuit in \( \Sigma \), and it follows that \( l \geq 2n \). If \( l \) is odd, then \( x_0 I x_{1} I x_{2} I \ldots I x_{n-1} I x_0 \) is an \( l \)-path in \( \Sigma \), and by Step 2 we have \( l \geq n \).)

Step 4. If \( n \) is odd, then each edge is contained in a unique \( n \)-gon.

(Indeed, if \( n \) is odd, and \( xy \) is an edge in \( \Gamma \), then \( d_{\Sigma}(x,y) = n - 1 \) and in \( \Sigma \) there is a unique geodesic \( x = x_0 I x_{1} I x_{2} I \ldots I x_{n-1} = y \) joining \( x \) and \( y \). This geodesic is part of the self-polar \( 2n \)-circuit

\[
    x_0 I x_{1} I x_{2} I \ldots I x_{n-1} I x_0 I x_1 I x_{2} I \ldots I x_{n-1} I x_0
\]

in \( \Sigma \). Thus, by Step 1, \( x_0 \sim x_1 \sim \ldots \sim x_{n-1} \sim x_0 \) is the unique \( n \)-gon on the edge \( xy \) in \( \Gamma \).)

Step 5. If \( n \) is even, then each \( 2 \)-path is contained in a unique \((n + 1)\)-gon.

(Indeed, if \( x \sim y \sim z \) in \( \Gamma \), then \( d_{\Sigma}(x,y) = d_{\Sigma}(y,z) = n \) (since by Step 2 the unique point on \( y^n \) that has distance \( n - 2 \) to \( y \) is not in \( \mathbb{Z} \)). Let \( x = x_0 I x_{1} I x_{2} I \ldots I x_{n-1} = z^n \) be the unique path of length \( n - 1 \) in \( \Sigma \) joining \( x \) to \( z^n \). Then

\[
    x_0 I x_{1} I x_{2} I \ldots I x_{n-1} I y I x_0 I x_1 I \ldots I x_{n-1} I y^n I x_0
\]

is a self-polar \((2n + 2)\)-circuit in \( \Sigma \). Thus, by Step 1, \( x_0 \sim x_1 \sim \ldots \sim x_{n-1} = z \sim y \sim x \) is the unique \((n + 1)\)-gon on the path \( x \sim y \sim z \) in \( \Gamma \).)

This completes the proof. \( \Box \)

If \((X, L, I)\) is a \(2m\)-gon, then \( \Gamma \) is a single edge. If \((X, L, I)\) is a \((2m + 1)\)-gon, then there are two possible polarities \(\sigma\); for one choice of \(\sigma\) the graph \(\Gamma\) consists of a single vertex; for the other choice it is a \((2m + 1)\)-gon itself.
2 Pentagraphs

Now let us specialize to the finite case \( n = 4 \), i.e., let \((X, L, I)\) be a generalized quadrangle of order \( q \) with a polarity \( \sigma \). Then \( 2q \) is a square, cf. Payne [4]. Examples exist when \( q \) is an odd power of 2, cf. Tits [9]. We define the graph \( \Gamma \) as before. As we shall see, \( \Gamma \) is a pentagraph, that is, any 2-path in \( \Gamma \) is contained in a unique pentagon. (For this concept, and other examples, and some theory, see Perkel [5, 6, 7, 8] and Ivanov [3].)

**Theorem 2.1.** \( \Gamma \) is a pentagraph of valency \( q \) on \( q^3 + q \) vertices, and has distance distribution diagram

\[
\begin{array}{cccccccc}
1 & q & 1 & q & 1 & q(q-1) & 1 & q(q-1) \\
q & 1 & q-1 & q & 1 & q-2 & 1 & q-1 \\
1 & q-1 & 1 & q(q-1)(q-2) & 1 & q-1 & 1 & q-1 \\
\end{array}
\]

*Proof.* Recall that a point or line is called absolute (for \( \sigma \)) if it is incident with its image (under \( \sigma \)). We shall use \( \sim \) for adjacency in \( \Gamma \), and \( \perp \) for collinearity in \((X, L)\).

**Step 1.** Each line contains a unique absolute point, and, dually, each point is on a unique absolute line.

(Indeed, if \( x \) is absolute, then \( x^\sigma \) is the only absolute line on \( x \), and if \( x \) is not absolute then the unique line on \( x \) meeting \( x^\sigma \) is the only absolute line on \( x \).)

**Step 2.** The set \( A \) of absolute points under \( \sigma \) is an ovoid in \((X, L)\). The graph \( \Gamma \) has \( v = q(q^2 + 1) \) vertices.

(Indeed, each \( l \in L \) meets \( A \) in a unique point. It follows that \( |A| = q^2 + 1 \). But \( |X| = (q + 1)(q^2 + 1) \).)

**Step 3.** \( \Gamma \) is regular of valency \( q \), and does not contain triangles. Adjacent vertices are non-collinear.

(Indeed, the neighbours of \( x \) are the \( q \) nonabsolute points of \( x^\sigma \).)

**Step 4.** \( \Gamma \) does not have quadrangles, and any two vertices at distance 2 determine a unique pentagon. Two vertices have distance 2 if and only if they are collinear and the line joining them is non_absolute.
Let us describe the distribution of vertices in $\Gamma$ around a vertex $x$. Let $m$ be the absolute line on $x$, and let $x' = m' = x'' \cap A$ be its absolute point. The vertex set of $\Gamma$ is partitioned into the following seven parts: $X_0 = \{x\}$, $X_1 = x'' \setminus A$, $X_2 = x' \setminus (A \cup m)$, $X_5 = m \setminus (A \cup \{x\})$, $X_{4a} = \{x'\} \setminus (A \cup m \cup x'')$, $X_{4b} = \{y \in X \setminus A \mid y \sim z \in X_1 \text{ and } yz \text{ is absolute}\}$, and $X_3$, consisting of the remaining points. Our aim is to show that $X_i$ consists of the vertices at distance $i$ from $x$ in $\Gamma$, where $X_{4a}$ and $X_{4b}$ are distinguished by the fact that points in $X_{4a}$ have neighbours in $X_5$. (Note however that for $q = 2$ we have $X_3 = \emptyset$, and the graph $\Gamma$ is the disjoint union of two pentagons. If $p$ is in the relation $4a$ to $x$, then $x$ is in relation $4b$ to $p$, i.e., relations $4a$ and $4b$ are paired, while the remaining relations are self-paired.)

**Step 5.** We have $|X_0| = 1$, $|X_1| = q$, $|X_2| = q(q - 1)$, $|X_3| = q(q - 1)(q - 2)$, $|X_{4a}| = |X_{4b}| = q(q - 1)$, $|X_5| = q - 1$.

(Indeed, the claims are clear for $X_1$ with $i \leq 2$. The only vertices that do not have distance 2 to some vertex of $X_1$, are the vertices that either are collinear to the point $x' = x'' \cap A$ (i.e., are in $X_{4a} \cup X_5$), or are joined to a vertex on $x''$ by an absolute line (i.e., are in $X_{4b}$). The absolute line $m$ on $x$ contains $q$ vertices, $q - 1$ other than $x$, and none of them is collinear to a point in $X_0 \cup X_1 \cup X_2$, so these vertices have distance at least 5 to $x$. The vertices adjacent to some vertex in $X_3$ are the $q(q - 1)$ vertices of $X_{4a}$. The vertices of $X_3$ are collinear to a unique vertex of $x''$, so this determines $|X_3|$.)

**Step 6.** Each vertex in $X_3 \cup X_{4b}$ has a unique neighbour in $X_{4a}$.

(Indeed, let $p \in X_3 \cup X_{4a}$. Then $p''$ does not pass through $x'$ (since $p \notin m$, i.e., $p \notin X_0 \cup X_3$), so $x'$ is collinear with a unique point $z \in p''$. The line $x'z$ is not absolute (since $z \notin m$ because $p \notin X_{4a} \cup X_1$) and the point $z$ is not absolute (since the line $x'z$ contains only one absolute point), so $z$ is the unique neighbour of $p$ in $X_1 \cup X_{4a}$. Clearly $z \in X_1$ iff $p \in X_2$.)

This proves everything claimed in the diagram. □

Now let us look at the special case where $q = 2^{2e} + 1$ and $(X, L)$ is the $Sp(4, q)$ generalized quadrangle. The centralizer in $Sp(4, q)$ of the polarity $\sigma$ is the Suzuki group $Sz(q)$ of order $(q^2 + 1)q^2(q - 1)$. This group is 2-transitive on
the set $A$ of absolute points, and 2-arc transitive on the graph $\Gamma$ (cf. Tits [9, Th. 6.1]). In this case we can be more precise about the stars in the diagram above.

**Theorem 2.2.** The graph $\Gamma$ is a 2-arc transitive pentagraph with distance distribution diagram

\[
\begin{align*}
1 & \quad q & \quad 1 & \quad q & \quad 1 & \quad q(q-1) & \quad 1 \\
1 & \quad q-1 & \quad q & \quad 1 & \quad q-1 & \quad 1 & \quad q(q-1)
\end{align*}
\]

**Proof.** If $p$ and $x$ are two non-collinear points, then $\{p, x\}^\perp$ is a hyperbolic line that meets $A$ in either 0 or 2 points (since $A$ is an ovoid, and all tangents to $A$ are totally isotropic lines). We shall talk about exterior and secant (hyperbolic) lines, respectively.

**Step 1.** Each vertex in $X_{4a} \cup X_{4b}$ has a unique neighbour in $X_{4b}$. 

(Indeed, if $p \in X_{4a}$ or $p \in X_{4b}$, then $\{p, x\}^\perp \cap A$ contains the point $x'$ (or $p'$, respectively), so this hyperbolic line is a secant, and there are precisely $q - 2$ points in $\Gamma$ at distance 2 from both $p$ and $x$.)

If $p \in X_3$, then $\{p, x\}^\perp \cap A$ contains either 0 or 2 points, so that $p$ has either 0 or 2 neighbours in $X_{4b}$. Let us call the set of vertices of the former (latter) kind $X_{3a}$ ($X_{3b}$, respectively).

**Step 2.** $X_{3a}$ is the set of vertices $p$ such that the line $xp$ is exterior. We have $|X_{3a}| = |X_{3b}| = \frac{1}{2}q(q - 1)(q - 2)$.

(Indeed, the lines joining $x$ to a point of $X_2 \cup X_5$ are the tangents (totally isotropic lines) on $x$, the lines joining $x$ to a point of $X_1 \cup X_3 \cup X_4$ are the exterior lines on $x$, and the lines joining $x$ to a point of $X_{3a}$ are the secants on $x$. But $A$ has $\frac{1}{2}q^2q^2 + 1$ secants, and the same number of exterior lines. (In fact, $l$ is secant iff $l^\perp$ is exterior.))

The planes meet the set $A$ either in one point: tangent planes, or in an oval (having $q + 1$ points): secant planes.
Step 3. If \( p \in X_2 \cup X_3 \cup X_{4a} \) then \( p \) has \( \frac{1}{2}q - 1 \) neighbours in \( X_{3a} \).

(Indeed, if \( p \in X_2 \cup X_3 \cup X_{4a} \), then \( x \notin p^\sigma \), and the plane \( \langle x, p^\sigma \rangle \) is a secant plane. In this plane, the point \( x \) is on one tangent, and on \( \frac{1}{2}q \) secants. One of these secants contains \( p' \); the remaining \( \frac{1}{2}q - 1 \) contain each one neighbour of \( p \).)

This determines the entire diagram. \( \square \)

Remark 2.3. The graph \( \Gamma \), and the fact that it is 2-arc transitive for \( Sz(q) \), was found independently by Fang Xin Gui, a student of Cheryl Praeger.

Remark 2.4. \( \text{Aut} \ \Gamma \) is not primitive: the spread \( \{a^\sigma \mid a \in A\} \) is a system of blocks of imprimitivity. However, \( \text{Aut} \ \Gamma \) acts 2-transitively on the set of blocks, so that we do not find a nontrivial graph structure on the quotient.

Remark 2.5. Of course we also get finite heptagraphs (of valency \( q = 3^{2e} + 1 \)) starting from a generalized hexagon (of type \( G_2(q) \)) with a polarity.

3 Addendum

The above was written in April 1992. In the meantime, Xin Gui Fang & C. E. Praeger [1, 2] appeared where the above graphs are found in the classification of certain 2-arc transitive graphs (and they refer to this work). As far as we know, the relation to generalized polygons with polarity still does not appear in the literature.

References


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