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A characterization of point shadows of residues in building geometries

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Abstract

We characterize shadows of residues in building shadow spaces by local properties. The results are applied to buildings with diagram $\mathsf{Y}_{l,m,n}$ (including E_k, $k \in \{6,7,8\}$, and D_k) to show that every subspace isomorphic to the shadow of a residue is the shadow of a residue.

Keywords: building, incidence geometry, point-line geometry, shadow space, building Grassmannian
MSC 2010: 51E24

1 Introduction and definitions

There are many characterizations of shadows of residues and apartments of residues in building shadow spaces seemingly dependent on the specific geometry (e.g. [3, 2, 7]). In the present paper we give a general characterization of shadows of residues in a building shadow space (Theorems 2.1 and 3.1) and then apply our results to buildings with diagram $\mathsf{Y}_{l,m,n}$ (Theorem 6.1). As an immediate corollary we obtain the following (for notation see Subsection 1.2).

Corollary 1.1. *Suppose $\mathcal{B}$ is a building with Coxeter diagram M over a set I and $\Sigma = (\mathcal{P}, \mathcal{L})$ is its shadow space. Let $\phi : \Sigma|\mathcal{P}_O \to \Sigma|\mathcal{P}_1$ be an isomorphism, where O is a residue of $\mathcal{B}$ and $\mathcal{P}_1$ is a subspace of Σ. Below, all subscripts are positive integers.*

(1) *Suppose Σ is of type $\mathsf{A}_{n,k}$ with $n \geq 3$ and $1 \leq k \leq n$, of type $\mathsf{D}_{n,k}$ with $n \geq 4$ and $k \in \{1, n, n-1\}$, or of type $\mathsf{E}_{n,k}$ with $n \in \{6,7,8\}$ and $k \in \{1,2,n\}$. Then $\mathcal{P}_1 = \mathcal{P}_{O'}$ for a residue O' of $\mathcal{B}$ with $M_O \cong M_{O'}$, where the isomorphism stabilizes the point node.*

(2) *Suppose* $\mathrm{typ}(O) = I - \{\alpha\}$. *Suppose further that* Σ *is of type* $\mathsf{D}_{n,\{n-1,n\}}$ *and* $\alpha \in \{1, \dots, n-3\}$*,* Σ *is of type* $\mathsf{D}_{n,\{1,n\}}$ *and* $\alpha = n-1$*, or* Σ *is of type* $\mathsf{E}_{6,\{1,6\}}$ *and* $\alpha = 2$*. Then* $\mathcal{P}_1 = \mathcal{P}_{O'}$ *for a residue* O' *of* $\mathcal{B}$ *with* $\mathrm{typ}(O) = \mathrm{typ}(O')$.

1.1 Incidence geometries and point-line geometries

A *point-line geometry* $\Sigma = (\mathcal{P}, \mathcal{L})$ is a bipartite graph with parts $\mathcal{P}$ and $\mathcal{L}$ labelled "points" and "lines"; we write $\mathrm{Pts}(\Sigma) = \mathcal{P}$ and $\mathrm{Lin}(\Sigma) = \mathcal{L}$. Let $\Sigma = (\mathcal{P}, \mathcal{L})$ and $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ be point-line geometries. Suppose that (1) $\mathcal{P}_1 \subseteq \mathcal{P}$, (2) $\mathcal{L}_1 \subseteq \mathcal{L}$, and (3) if a point $p \in \mathcal{P}_1$ and a line $L \in \mathcal{L}_1$ are incident in Σ_1, then p and L are incident in Σ. Then Σ_1 is a *subgeometry* of Σ. We say that a subgeometry Σ_1 of Σ is an *induced subgeometry* of Σ if a point $p \in \mathcal{P}_1$ and a line $L \in \mathcal{L}_1$ are incident in Σ_1 if and only if they are incident in Σ. A subgeometry $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ of Σ is a *full subgeometry* of Σ if (1) Σ_1 is an induced subgeometry of Σ and (2) for every $L \in \mathcal{L}_1$, we have $\mathcal{P}_L \subseteq \mathcal{P}_1$, where $\mathcal{P}_L$ is the set of points of Σ incident with L. A subgeometry $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ of $\Sigma = (\mathcal{P}, \mathcal{L})$ is full if and only if, for every line $L \in \mathcal{L}_1$, $(\mathcal{P}_1)_L = \mathcal{P}_L$. A morphism of point-line geometries is a morphism of the underlying graphs that maps points to points and lines to lines; monomorphisms, epimorphisms, and isomorphisms are defined similarly. If $\phi : \Sigma_1 \to \Sigma$ is a morphism of point-line geometries, then the image of Σ_1 under ϕ is a subgeometry of Σ in which a point p and a line L are incident if and only if $p = \phi(p')$ and $L = \phi(L')$ for an incident point-line pair $\{p', L'\}$ of Σ_1.

Lemma 1.2. *Suppose* $\phi : \Sigma = (\mathcal{P}, \mathcal{L}) \to \Sigma' = (\mathcal{P}', \mathcal{L}')$ *is a morphism of point-line geometries. If* $\phi(\mathcal{P}_L) = (\mathcal{P}')_{\phi(L)}$ *for all* $L \in \mathcal{L}'$*, then the image of* Σ *under* ϕ *is a full subgeometry of* Σ'.

Proof. Let $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ be the image of Σ under ϕ, and suppose $L' \in \mathcal{L}_1$ and $p' \in (\mathcal{P}')_{L'}$. Let $L \in \mathcal{L}$ be such that $L' = \phi(L)$. Since $\phi(\mathcal{P}_L) = (\mathcal{P}')_{\phi(L)}$, $p' \in (\mathcal{P}')_{\phi(L)}$. Therefore $p' = \phi(p)$ for some $p \in \mathcal{P}_L$ and p' is incident with L' in Σ'. □

1.2 Builidings and building Grassmannians

In this paper buildings are chamber systems. For a chamber system $\mathcal{C}$, we denote $\mathrm{typ}(\mathcal{C})$ the type set of $\mathcal{C}$ and, for a set of chambers X of $\mathcal{C}$, we denote $\mathrm{typ}(X)$ the type set of the chamber subsystem of $\mathcal{C}$ induced on X.

Let M be a Coxeter matrix over a set I and let W be a Coxeter group of type M with generators $\{s_i \mid i \in I\}$. A *building* of type M is a chamber system $\mathcal{B}$ of type M (that is, for every $\{i, j\} \subseteq I$, every residue of $\mathcal{B}$ of type $\{i, j\}$ is a generalized m_{ij}-gon) satisfying the following condition.

(P) If two galleries w and w' of reduced types $t_1 \dots t_n$ and $t'_1 \dots t'_{n'}$ have the same initial and terminal chambers, then $s_{t_1} \dots s_{t_n} = s_{t'_1} \dots s_{t'_{n'}}$ in W.

All buildings have the following properties.

(Bld1) If $\{Q_\alpha \mid \alpha \in S\}$ is a family of residues of $\mathcal{B}$ and $\cap_{\alpha \in S} Q_\alpha \neq \emptyset$, then $\cap_{\alpha \in S} Q_\alpha$ is a residue of $\mathcal{B}$ of type $\cap_{\alpha \in S} \mathrm{typ}(Q_\alpha)$.

(Bld2) Suppose R, P, and Q are residues of $\mathcal{B}$ such that $R \cap P \neq \emptyset$, $R \cap Q \neq \emptyset$, and $P \cap Q \neq \emptyset$. Then $R \cap P \cap Q \neq \emptyset$.

For these and other properties of buildings see [13, 14, 9, 11, 12, 15]. We now define shadow spaces and Grassmann geometries of buildings (cf. [8, Chapter 5], [4], and [1, Sections 2.5 and 11.4]). Suppose $\mathcal{B}$ is a building of type M over a set I, $|I| \geq 1$. Let $I' \subseteq I$, let $J = I - I'$, and assume that $I' \neq I$. For a set of chambers X of $\mathcal{B}$, let $\mathrm{Sh}_{I'}(X)$ be the set of residues of $\mathcal{B}$ of type I' meeting X. Let $\mathcal{P} = \{\mathrm{Sh}_{I'}(X) \mid X \text{ is a residue of } \mathcal{B}, \mathrm{typ}(X) = I'\}$ and let $\mathcal{L} = \{\mathrm{Sh}_{I'}(X) \mid X \text{ is a panel of } \mathcal{B}, \mathrm{typ}(X) \subseteq J\}$. We remark that here we distinguish between an I'-residue X of $\mathcal{B}$ and the corresponding point $p = \{X\}$ of $\mathcal{P}$, which is a set whose only element is X. The J*-shadow space* of $\mathcal{B}$ is the point-line geometry $\mathrm{ShSp}(\mathcal{B}, J) = (\mathcal{P}, \mathcal{L})$ with the incidence being the symmetrization of the strict containment of sets. We denote $\mathcal{G}(\mathcal{B}, J)$ the point-collinearity graph of $\mathrm{ShSp}(\mathcal{B}, J)$.

The geometry $\mathrm{ShSp}(\mathcal{B}, J)$ is the truncation to types 1 and 2 of the J-Grassmann geometry $\mathrm{ShGm}(\mathcal{B}, J)$ defined as follows. For $T \subseteq I$, let $\mathrm{K}_{I,I'}(T)$ denote the union of the vertex sets of the connected components of $M|T$ meeting J. The J*-Grassmann geometry* $\mathrm{ShGm}(\mathcal{B}, J)$ of $\mathcal{B}$ is an incidence geometry (that is a multipartite graph with labeled parts) over the type set $\{k \in \mathbb{N} \mid 1 \leq k \leq |\mathrm{K}_{I,I'}(I)|\}$. The objects of $\mathrm{ShGm}(\mathcal{B}, J)$ are the sets $\mathrm{Sh}_{I'}(R)$ with R a residue of $\mathcal{B}$ and $\mathrm{K}_{I,I'}(\mathrm{typ}(R))$ finite; the object $O = \mathrm{Sh}_{I'}(R)$ is of type k in $\mathrm{ShGm}(\mathcal{B}, J)$ if and only if $|\mathrm{K}_{I,I'}(\mathrm{typ}(R))| = k - 1$; the incidence is the symmetrization of the strict containment of sets. The set of objects of $\mathrm{ShGm}(\mathcal{B}, J)$ will be denoted $\mathrm{ShSet}(\mathcal{B}, J)$. For $O \in \mathrm{ShSet}(\mathcal{B}, J)$, the set of all residues R of $\mathcal{B}$ with $\mathrm{Sh}_{I'}(R) = O$ ordered by inclusion has at least one minimal element $O_\#$ and a unique maximal element $O^\#$. We have $\mathrm{typ}(O_\#) = \mathrm{K}_{I,I'}(\mathrm{typ}(R))$ and $\mathrm{typ}(O^\#) = \mathrm{typ}(O_\#) \cup L$, where $L \subset I' - \mathrm{typ}(O_\#)$ consists of the types in I' not adjacent in M to any vertices in $\mathrm{typ}(O_\#)$ (see [13, Chapter 12], [4], or [6]). For $O \in \mathrm{ShSet}(\mathcal{B}, J)$ and for every residue R of $\mathcal{B}$ with $\mathrm{Sh}_{I'}(R) = O$, we let $M_O = M_R := M|\mathrm{typ}(O_\#)$.

A *plane* of $\mathrm{ShGm}(\mathcal{B}, J)$ (or of $\mathrm{ShSp}(\mathcal{B}, J)$) is any object of type 3. For $\Gamma = \mathrm{ShGm}(\mathcal{B}, J)$, the sets of points, lines, and planes will be denoted $\mathrm{Pts}(\Gamma)$, $\mathrm{Lin}(\Gamma)$, and $\mathrm{Pln}(\Gamma)$. If $\pi \in \mathrm{Pln}(\Gamma)$, then $|\mathrm{typ}(\pi_\#)| = 2$ and either $\mathrm{typ}(\pi_\#) \subseteq J$ or, else, $\mathrm{typ}(\pi_\#) \cap J \neq \emptyset$ and $M|\mathrm{typ}(\pi_\#)$ is connected. A point-line geometry $\Sigma' \cong \mathrm{ShSp}(\mathcal{B}, J)$ will be called a J-shadow space of $\mathcal{B}$; a geometry $\Gamma' \cong \mathrm{ShGm}(\mathcal{B}, J)$

is a J-Grassmann geometry of $\mathcal{B}$.

For future reference we introduce the following notation that will be referred to as **Hypothesis (A)**. We let $\Gamma = \mathrm{ShGm}(\mathcal{B}, J)$, let $\Sigma = \mathrm{ShSp}(\mathcal{B}, J)$, let $\mathcal{O} = \mathrm{ShSet}(\mathcal{B}, J)$, let $\sim_\Gamma$ denote the incidence relation in Γ, and let τ denote the labeling map of Γ assigning to each object $O \in \mathrm{ShSet}(\mathcal{B}, J)$ its type k. For $O \in \mathcal{O}$ we denote $\mathrm{typ}_\#(O) = \mathrm{typ}(O_\#)$, $\mathrm{typ}^\#(O) = \mathrm{typ}(O^\#)$, and we let $\mathcal{T} = \{\mathrm{typ}_\#(O) \mid O \in \mathcal{O}\}$. We let $\mathcal{P} = \mathrm{Pts}(\Gamma)$, $\mathcal{L} = \mathrm{Lin}(\Gamma)$, and $\Pi = \mathrm{Pln}(\Gamma)$. For $X \in \mathcal{O}$ and for $\mathcal{Y} \in \{\mathcal{P}, \mathcal{L}, \Pi\}$, we let $\mathcal{Y}_X = \{O \in \mathcal{Y} \mid O \sim_\Gamma X\}$. For a residue R of $\mathcal{B}$, we let $\mathcal{Y}_R = \mathcal{Y}_O$, where $O = \mathrm{Sh}_{I'}(R)$. For R a residue of $\mathcal{B}$ or $R \in \mathcal{O}$, we let $\Sigma_R = (\mathcal{P}_R, \mathcal{L}_R)$ be the induced subgeometry of Σ.

Remark 1.3. Suppose hypothesis (A) holds.

1. Suppose X, Y, and Z are objects of Γ of types k_1, k_2, and k_3, where $k_1 < k_2 < k_3$. If X is incident with Y, and Y is incident with Z, then X is incident with Z.
2. The shadow space Σ is a partial linear space ([4, Corollary 5.4] and [1, Lemma 11.4.6]). For collinear points p and q of Γ, the unique line on p and q will be denoted $\langle p, q\rangle$.
3. All singular subspaces of Σ are projective spaces ([5, Corollary 3.14]; [1, Theorem 11.5.13 and Proposition 11.5.15]). If X is a singular subspace of finite rank, then $X = \mathcal{P}_R$ for a residue of $\mathcal{B}$ such that $M|\mathrm{typ}(R)$ is of type A_k and $\mathrm{typ}(R) \cap J$ is one of its end nodes ([5, Corollary 3.15]).
4. For every $O \in \mathcal{O}$, $\mathcal{P}_O$ is a convex subspace of Σ ([4, Corollary 6.2]; [1, Proposition 11.4.9]).
5. For $X, Y \in \mathcal{O}$, $X \sim_\Gamma Y$ if and only if $X^\# \cap Y^\# \neq \emptyset$ and either $\mathrm{typ}_\#(X) \subsetneq \mathrm{typ}_\#(Y)$ or $\mathrm{typ}_\#(Y) \subsetneq \mathrm{typ}_\#(X)$ (see [13, Theorem 12.15] or [4, Propositon 4.5]).

Suppose hypothesis (A) holds. For $\mathcal{P}_1 \subseteq \mathcal{P}$, we denote $\mathcal{L}|\mathcal{P}_1$ the set of lines of Σ meeting $\mathcal{P}_1$ in at least two distinct points, and we denote $\Sigma|\mathcal{P}_1 = (\mathcal{P}_1, \mathcal{L}|\mathcal{P}_1)$ the subgeometry induced in Σ on $\mathcal{P}_1 \cup (L|\mathcal{P}_1)$. For a residue R of $\mathcal{B}$, we let $\Sigma(R) = \mathrm{ShSp}(R, J \cap \mathrm{typ}(R))$.

Remark 1.4. Let R be a residue of $\mathcal{B}$ with $|\mathrm{typ}_\#(R)| \geq 2$. By Remark 1.3(4) and [5, Proposition 3.6] (see also [1, Proposition 11.4.9]) $\Sigma|\mathcal{P}_R = \Sigma_R$ and $\Sigma_R \cong \Sigma(R)$, where the isomorphism takes a point or line X of Σ_R to the point or line Y of $\Sigma(R)$ with $Y^\# = X^\# \cap R$.

2 Characterization of shadows of residues: Theorem 2.1

Theorem 2.1. *Suppose hypothesis (A) holds and let $T \in \mathcal{T}$ be such that $|T| \geq 2$. Suppose $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ is a connected full subgeometry of Σ satisfying conditions* (Shd-loc) *and* (Shd-pln)*:*

(Shd-loc) *For every $p \in \mathcal{P}_1$ there is an object O_p of Γ on p with $\mathrm{typ}_{\#}(O_p) = T$ such that $\mathcal{L}_p \cap \mathcal{L}_1 = \mathcal{L}_p \cap \mathcal{L}_{O_p}$.*

(Shd-pln) *For every pair of distinct collinear points p and q of Σ_1, and for every plane $\pi \in \Pi$ incident with p and q, we have $\mathcal{L}_1 \cap \mathcal{L}_p \cap \mathcal{L}_\pi - \{\langle p, q\rangle\} \neq \emptyset$ if and only if $\mathcal{L}_1 \cap \mathcal{L}_q \cap \mathcal{L}_\pi - \{\langle p, q\rangle\} \neq \emptyset$.*

Then $\Sigma_1 = \Sigma|\mathcal{P}_O$, where $O \in \mathcal{O}$ and $\mathrm{typ}_{\#}(O) = T$.

We need two lemmas.

Lemma 2.2. *Suppose the hypothesis of Theorem 2.1 holds and let $p, q \in \mathcal{P}_1$ be distinct collinear points of Σ_1.*

(i) *If $|T| = 2$, then $O_p = O_q$.*

(ii) *If $|T| \geq 3$, then $\Pi_{\langle p,q\rangle} \cap \Pi_{O_p} = \Pi_{\langle p,q\rangle} \cap \Pi_{O_q}$.*

Proof. By condition (Shd-loc) O_p and O_q are both incident with $\langle p, q\rangle$. If $|T| = 2$, let $\pi = O_p$; if $|T| \geq 3$, let $\pi \in \Pi_{\langle p,q\rangle} \cap \Pi_{O_p}$.

By [4, Lemma 5.6(2)] applied to π, there is a line L of π on p distinct from $\langle p, q\rangle$. By Remark 1.3(1) $L \in \mathcal{L}_{O_p}$, therefore by condition (Shd-loc) $L \in \mathcal{L}_1$. By condition (Shd-pln) there is a line $N \in \mathcal{L}_1 \cap L_q \cap \mathcal{L}_\pi - \{\langle p, q\rangle\}$, and by condition (Shd-loc) $N \in \mathcal{L}_{O_q}$. Since $\langle p, q\rangle$ and N are distinct lines of π on q incident with O_q, by [4, Lemma 5.6(1)] $\pi \subseteq O_q$.

If $|T| = 2$ we have shown that $O_p = O_q$. If $|T| \geq 3$, then the above shows that $\Pi_{\langle p,q\rangle} \cap \Pi_{O_p} \subseteq \Pi_{\langle p,q\rangle} \cap \Pi_{O_q}$; the reverse inclusion follows by symmetry. □

Lemma 2.3. *Assume that the hypothesis of Theorem 2.1 holds and $|T| \geq 2$. Then, for all $p, q \in \mathcal{P}_1$, we have $O_p = O_q$.*

Proof. Since Σ_1 is connected it suffices to show that $O_p = O_q$ for any two distinct collinear points p and q of Σ_1. For $|T| = 2$ this is Lemma 2.2(i). Suppose $|T| \geq 3$. By condition (Shd-loc) $\langle p, q\rangle$ is incident with both O_p and O_q. For $r \in \{p, q\}$, let $Z_r = r^{\#} \cap \langle p, q\rangle^{\#} \cap O_r^{\#}$; by Remark 1.3(5) and by property (Bld2) of buildings $Z_r \neq \emptyset$. Let $\mathcal{S}_r$ denote the set of all planes π of Γ such that $\pi^{\#} \cap Z_r \neq \emptyset$ and

$$\mathrm{typ}_{\#}(\langle p, q\rangle) \subseteq \mathrm{typ}_{\#}(\pi) \subseteq T. \tag{2.1}$$

We claim that $\mathcal{S}_r = \Pi_{\langle p,q\rangle} \cap \Pi_{O_r}$. By the definition of $\mathcal{S}_r$ we have $\mathcal{S}_r \subseteq \Pi_{\langle p,q\rangle} \cap \Pi_{O_r}$. Suppose $\pi \in \Pi_{\langle p,q\rangle} \cap \Pi_{O_r}$. By Remark 1.3(1) $\pi \in \Pi_r$, therefore $\pi \in \Pi_r \cap \Pi_{\langle p,q\rangle} \cap \Pi_{O_r}$ and by Remark 1.3(5) and property (Bld2) of buildings $\pi^{\#} \cap Z_r \neq \emptyset$ and (2.1) holds. By Lemma 2.2 $\Pi_{\langle p,q\rangle} \cap \Pi_{O_p} = \Pi_{\langle p,q\rangle} \cap \Pi_{O_q}$, therefore $\mathcal{S}_p = \mathcal{S}_q$. By [4, Corollary 7.4] applied to Z_p and Z_q this implies that $O_p^{\#} = O_q^{\#}$. □

Proof of Theorem 2.1. By Lemma 2.3 there is $O \in \mathcal{O}$ with $\mathrm{typ}_{\#}(O) = T$ such that, for all $p \in \mathcal{P}_1$, $O_p = O$. We show that O is as in the conclusion. Since Σ_1 and $\Sigma|\mathcal{P}_O$ are induced subgeometries of Σ, it suffices to prove that $\mathcal{P}_1 \cup \mathcal{L}_1 = \mathcal{P}_O \cup (\mathcal{L}|\mathcal{P}_O)$. By condition (Shd-loc) and the definition of O,

(1) *for every point $p \in \mathcal{P}_1$ and for every $L \in \mathcal{L}_p$, $L \in \mathcal{L}_1$ if and only if $L \in \mathcal{L}|\mathcal{P}_O$.*

By the definition of O we have $\mathcal{P}_1 \subseteq \mathcal{P}_O$. Suppose $p \in \mathcal{P}_O - \mathcal{P}_1$. Since by Remark 1.4 $\Sigma|\mathcal{P}_O \cong \Sigma(O)$ is connected, we can assume that p is collinear in $\Sigma|\mathcal{P}_O$ with $q \in \mathcal{P}_1$. Therefore by (1) $\langle p, q\rangle \in \mathcal{L}_1$. Since Σ_1 is a full subgeometry of Σ, $p \in \mathcal{P}_1$, a contradiction. Therefore

(2) $\mathcal{P}_1 = \mathcal{P}_O$.

Let $L \in \mathcal{L}_1 \cup (\mathcal{L}|\mathcal{P}_O)$. By (2) L is incident with some $p \in \mathcal{P}_1$, therefore by (1) $L \in \mathcal{L}_1$ if and only if $L \in \mathcal{L}|\mathcal{P}_O$. □

3 Characterization of shadows of residues: Theorem 3.1

Suppose hypothesis (A) holds. Let O be an object of Γ and let $T = \mathrm{typ}_{\#}(O)$. If O is a point, or if the graph $M|T$ is a string and $T \cap J$ is one of its end nodes, then we say that O is an object of Γ of the *first kind*. All other objects of Γ are objects of the *second kind*.

Suppose O is an object of Γ and let $k = \tau(O)$. We denote Γ_O^- the subgeometry of Γ induced on the set $\{O' \in \mathcal{O} \mid O' \sim_\Gamma O \text{ and } \tau(O') < k\}$ and we denote Γ_O^+ the subgeometry of Γ induced on the set $\{O' \in \mathcal{O} \mid O' \sim_\Gamma O \text{ and } k < \tau(O')\}$ (cf. [8, Chapter 5]); for $p \in \mathcal{P}$, we let $\Gamma_p = \Gamma_p^+$. When they exist, the points, lines, and planes of Γ_O^- are the points, lines, and planes of Γ incident with O; the points, lines, and planes of Γ_O^+ are the objects of Γ of types $k+1$, $k+2$, and $k+3$ incident with O.

For a building $\mathcal{B}$ with the diagram M over a type set I we consider the following conditions.

(Bld-iso) *For every $X \subseteq I$, all residues of $\mathcal{B}$ of type X are isomorphic to each other.*

(Bld-str) *For every residue R of $\mathcal{B}$ with $M|\mathrm{typ}(R)$ a string, and for every $\{i\}$-shadow space Δ of R with i an end node of $M|\mathrm{typ}(R)$, every monomorphism of Δ to itself which is surjective on points is an isomorphism.*

Theorem 3.1. *Suppose hypothesis* (A) *holds, all m_{ij} are integers, and $\mathcal{B}$ satisfies* (Bld-iso) *and* (Bld-str). *Let $\phi : \Sigma|\mathcal{P}_O \to \Sigma_1$ be an isomorphism, where O is an object of Γ and Σ_1 is a subgeometry of Σ. Suppose*

(Phi-str) *for every object U of Γ of the first kind with $\mathcal{P}_U \subseteq \mathcal{P}_O$, we have $\phi(\mathcal{P}_U) = \mathcal{P}_{U'}$, where $U' \in \mathcal{O}$ and $\mathrm{typ}_\#(U') = \mathrm{typ}_\#(U)$.*

Then $\Sigma_1 = \Sigma|\mathcal{P}_{O'}$, where $O' \in \mathcal{O}$ and $\mathrm{typ}_\#(O') = \mathrm{typ}_\#(O)$.

Remark 3.2. Every $L \in \mathcal{L}_O$ is an object of Γ of the first kind, therefore by (Phi-str) $\phi(\mathcal{P}_L) = \mathcal{P}_{L'}$ for some $L' \in \mathcal{L}$. Since L, and hence $\phi(L)$, are incident with at least two points of Γ and by Remark 1.3(2) Σ is a partial linear space, $L' = \phi(L)$. Therefore by Lemma 1.2 Σ_1 is a full subgeometry of Σ.

Our proof of Theorem 3.1 consists of Propositions 3.3, 3.4, and the conclusion. The conclusion is proved below; Proposition 3.3 and 3.4 will be proved in Sections 4 and 5 respectively. We let $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $T = \mathrm{typ}_\#(O)$.

Proposition 3.3. *Suppose hypothesis (A) holds. Let π be a plane of Γ of the second kind with $\mathrm{typ}_\#(\pi) = \{i, j\}$ and $m_{ij} \in \mathbb{Z}$. Let $\{i', j'\} \subseteq J$ be such that $m_{ij} = m_{i'j'}$, and let $\xi : \{i, j\} \to \{i', j'\}$ be a bijection. Suppose $\phi : \Sigma|\mathcal{P}_\pi \to \Sigma_1$ is an isomorphism, where Σ_1 is a full subgeometry of Σ, and*

(Phi-lin) *for every $L \in \mathcal{L}|\mathcal{P}_\pi$ and $k \in \{i, j\}$, if $\mathrm{typ}_\#(L) = \{k\}$ then $\mathrm{typ}_\#(\phi(L)) = \{\xi(k)\}$.*

Then $\Sigma_1 = \Sigma|\mathcal{P}_{\pi'}$, where $\pi' \in \Pi$ and $\mathrm{typ}_\#(\pi') = \{i', j'\}$.

Proposition 3.4. *Suppose the hypothesis of Theorem 3.1 holds and let $n = |T|$. Suppose O is of the second kind, $n \geq 3$, and the statement of Theorem 3.1 is true for every object U of the second kind with $|\mathrm{typ}_\#(U)| \leq n - 1$. Then*

(1) Σ_1 *satisfies condition* (Shd-loc) *of Theorem 2.1.*

(2) Σ_1 *satisfies condition* (Shd-pln) *of Theorem 2.1.*

Proof of Theorem 3.1. Suppose first that O is an object of the first kind. Then by (Phi-str) there exists an object O' with $\mathrm{typ}_\#(O) = \mathrm{typ}_\#(O')$ such that $\phi(\mathcal{P}_O) = \phi(\mathcal{P}_{O'})$. Therefore ϕ induces a monomorphism $\alpha : \Sigma|\mathcal{P}_O \to \Sigma|\mathcal{P}_{O'}$. Since $\mathrm{typ}_\#(O) = \mathrm{typ}_\#(O')$, by (Bld-iso) and Remark 1.4 there exists an isomorphism $\beta : \Sigma|\mathcal{P}_{O'} \to \Sigma|\mathcal{P}_O$. By (Bld-str) $\alpha \circ \beta : \Sigma|\mathcal{P}_{O'} \to \Sigma|\mathcal{P}_{O'}$ is an isomorphism, therefore so is α.

Suppose now that O is an object of the second kind an let $n = |T|$; we have $n \geq 2$. We use induction on n. If $n = 2$, then the conclusion holds by

Proposition 3.3. Suppose the statement is true for all $n \leq k$, where $k \in \mathbb{Z}$ and $k \geq 2$, and suppose $n = k + 1$. We claim that Σ_1 satisfies the hypothesis of Theorem 2.1, therefore the conclusion holds. By Remark 1.4 Σ_1 is connected and by Remark 3.2 Σ_1 is a full subgeometry of Σ. By Proposition 3.4 Σ_1 satisfies conditions (Shd-loc) and (Shd-pln). $\square$

4 Proof of Proposition 3.3

Suppose $\mathcal{C}$ is a chamber system of type $M = (m_{ij})$ over I. Let $W = (c_0, \ldots, c_n)$ be a gallery in $\mathcal{C}$ and suppose that, for $i \in \{0, \ldots, n-1\}$, the label of the edge $\{c_i, c_{i+1}\}$ is $\{t_i\}$. The type of W is $\mathrm{t}(W) = t_0 \ldots t_{n-1}$, a word in the free monoid of words on I. For $\{i, j\} \subseteq I$, we denote $p(i, j)$ the word $iji \ldots$ of length m_{ij}.

Let $G = (V, E)$ be a graph. Suppose that $n \geq 0$ is an integer. A *walk of length* n in G is a sequence of vertices $w = (p_0, \ldots, p_n)$ such that, for all $i \in \{0, \ldots, n-1\}$, $\{p_i, p_{i+1}\} \in E$; we denote $\mathrm{l}(w)$ the length of w. A *segment* of w is a concatenation $v = w' \circ u \circ w''$, where w', u, and w'' are walks; we also say that the walk u is a segment of w. The walk w is *circular* if $p_0 = p_n$. A *circuit* is a circular walk in which no two vertices are equal. Suppose $F = (V', E')$ is a subgraph of G and let $x \in V$. The subgraph F is *strongly gated in G with respect to x* if there is a vertex $g \in V'$ such that, for every vertex $y \in V'$, $\mathrm{d}_G(x, y) = \mathrm{d}_G(x, g) + \mathrm{d}_F(g, y)$; we say that g is the *gate* of x in F and we write $g = \mathrm{gate}_F(x)$. If F is a subgraph of G and $X \subseteq V$, then we let $\mathrm{Gate}_F(X) = \{\mathrm{gate}_F(x) \mid x \in X\}$. A subgraph is strongly gated in G if it is strongly gated with respect to every vertex of G. Every residue of a building $\mathcal{B}$ is strongly gated in $\mathcal{B}$ (see [12, 10, 9]).

Suppose hypothesis (A) holds and let $\mathcal{G} := \mathcal{G}(\mathcal{B}, J)$. Every gallery $W = (c_0 \ldots, c_n)$ of $\mathcal{B}$ determines a walk $\mathrm{w}_{\mathcal{G}}(W)$ in $\mathcal{G}$, consisting of $p \in \mathcal{P}$ with $p^{\#}$ the residues of $\mathcal{B}$ of type I' traversed by W; we assume that no two consecutive vertices of $\mathrm{w}_{\mathcal{G}}(W)$ are equal.

Remark 4.1. Suppose hypothesis (A) holds and $w = (p_0, \ldots, p_n)$ is a walk in $\mathcal{G}$; let $c_0 \in p_0^{\#}$, and let $c_n \in p_n^{\#}$. Every $p^{\#}$, $p \in \mathcal{P}$, is a connected chamber system and by [5, Lemma 3.10], if $\langle p, q \rangle \in \mathcal{L}$, then there is a chamber in $p^{\#}$ connected by an edge to a chamber in $q^{\#}$. Therefore, there is a gallery $W = (c_0, c_1') \circ W_1 \circ (c_1, c_2') \circ W_2 \circ \cdots \circ (c_{n-1}, c_n') \circ W_n$ from c_0 to c_n in $\mathcal{B}$ such that, for every $i \in \{1, \ldots, n\}$, $\{c_i', c_i\} \subseteq p_i^{\#}$ and W_i is a walk from c_i' to c_i in $p_i^{\#}$. We have $\mathrm{w}_{\mathcal{G}}(W) = w$; in general, the gallery W is not unique.

For $k \in J$, let $\mathcal{L}_k = \{L \in \mathcal{L} \mid \mathrm{typ}_{\#}(L) = \{k\}\}$. Suppose $w = (p_0, \ldots, p_n)$ is a walk in $\mathcal{G}$, and every edge of w spans a line in $\cup_{k \in H} \mathcal{L}_k$, $H \subseteq J$. Then we say that w is an $\{\mathcal{L}_k \mid k \in H\}$-*walk*. If, for every $k \in H$, there is at least one edge of w spanning a line in $\mathcal{L}_k$, then we say that w is a *strictly* $\{\mathcal{L}_k \mid k \in H\}$*-walk* in $\mathcal{G}$. The $\{\mathcal{L}_k \mid k \in H\}$- and strictly $\{\mathcal{L}_k \mid k \in H\}$-circular walks, circuits, and segments are defined similarly.

Remark 4.2. Suppose hypothesis (A) holds. Let w be a circular strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-walk such that $w = w' \circ w''$, where $w' = (p_0, \dots, p_k)$, $w'' = (p_k, \dots, p_n)$, $\langle p_{k-1}, p_k \rangle \in \mathcal{L}_i$, $\langle p_k, p_{k+1} \rangle \in \mathcal{L}_j$ and, for all $\{i, j\} \subseteq \{0, \dots, k\}$ and all $\{i, j\} \subseteq \{k, \dots, n\}$, $p_i \neq p_j$. Then there exists a segment of w containing p_k which is a strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-circuit: it is the shortest circular segment of w containing (p_{k-1}, p_k, p_{k+1}).

Proposition 4.3. *Suppose hypothesis (A) holds. Let $\{i, j\} \subseteq J$ and suppose $m_{ij} \in \mathbb{Z}$. Let w be a strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-circuit in $\mathcal{G}$ with the smallest possible length among all strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-circuits of $\mathcal{G}$. Then $w = \mathrm{w}_\mathcal{G}(C)$ for a circuit C in $\mathcal{B}$ with $\mathrm{t}(C) = p(i, j) \circ p(j, i)^{-1}$.*

The proof of Proposition 4.3 consists of Lemmas 4.4 and 4.5 below. First, we introduce some notation. Suppose the hypothesis of Proposition 4.3 holds and $w = (p_0, \dots, p_n)$. Since w is a strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-circuit, there is a vertex p_k of w, such that the edges $\{p_{k-1}, p_k\}$ and $\{p_k, p_{k+1}\}$ span lines in $\mathcal{L}_j$ and $\mathcal{L}_i$ respectively, where the indices are added modulo n. Renumbering the vertices of w if neccessary, we can assume that $\langle p_{n-1}, p_0 \rangle \in \mathcal{L}_j$ and $\langle p_0, p_1 \rangle \in \mathcal{L}_i$; we let $p = p_0$. Let W be defined as in Remark 4.1 with $c_0 = c_n$. We let $x = c_0$, $x' = c'_1$, $y = c'_0$, and $y' = c_{n-1}$. Since $\langle p_0, p_1 \rangle \in \mathcal{L}_i$ and $\langle p_{n-1}, p_0 \rangle \in \mathcal{L}_j$, the edges $\{x, x'\}$ and $\{y', y\}$ are labelled $\{i\}$ and $\{j\}$ respectively.

Lemma 4.4. *Suppose the hypothesis of Proposition 4.3 holds. Then $\mathrm{l}(w)$ cannot be odd.*

Proof. Suppose $\mathrm{l}(w)$ is odd, let $l = (\mathrm{l}(w) - 1)/2$, let $q' = p_l$, and let $q'' = p_{l+1}$. Let $a = c_l$ and let $b = c'_{l+1}$; then $\{a, b\}$ is an edge of $\mathcal{B}$ labelled $\{k\}$ for some $k \in \{i, j\}$. Let Q be the panel of $\mathcal{B}$ of type $\{k\}$ containing $\{a, b\}$, let $z \in \mathrm{Gate}_{p^\#}(Q)$, and let $c = \mathrm{gate}_Q(z)$; then $z = \mathrm{gate}_{p^\#}(c)$ (see, for example, [4, Lemma 2.2]). We are going to construct a strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-circuit of length less than $\mathrm{l}(w)$.

Let $U = (c_0, c'_1) \circ W_1 \circ \dots \circ (c_{l-1}, c'_l) \circ W_l$ and $V = W_{l+1} \circ (c_{l+1}, c'_{l+2}) \circ \dots \circ (c_{n-1}, c'_0)$. Then U is a walk from x to a, V is a walk from b to y, and $W = U \circ (a, b) \circ V \circ W_0$. We define $u = \mathrm{w}_\mathcal{G}(U)$ and $v = \mathrm{w}_\mathcal{G}(V)$; then $\mathrm{l}(u) = l = \mathrm{l}(v)$, the first edge of u spans a line in $\mathcal{L}_i$, the last edge of v spans a line in $\mathcal{L}_j$, and $w = u \circ (q', q'') \circ v$.

Let W'_0 be a geodesic from x to z in $p^\#$ and let $U' = (W'_0)^{-1} \circ U \circ (a, c)$. Let U'' be a geodesic from z to c, such that $\mathrm{t}(U'')$ is obtained from $\mathrm{t}(U')$ exclusively by deleting terms, without using any other operations; U'' exists by

[6, Corollary 2.2]. Similarly, let W_0'' be a geodesic from y to z in $p^{\#}$, let $V' = (c,b) \circ V \circ W_0''$, and let V'' be a geodesic from c to z, such that $\mathrm{t}(V'')$ is obtained from $\mathrm{t}(V')$ by deleting terms.

Define $u' = \mathrm{w}_{\mathcal{G}}(U')$, $u'' = \mathrm{w}_{\mathcal{G}}(U'')$, $v' = \mathrm{w}_{\mathcal{G}}(V')$, and $v'' = \mathrm{w}_{\mathcal{G}}(V'')$. Let $W' = U'' \circ V''$ and let $w' = \mathrm{w}_{\mathcal{G}}(W')$. We show that w' is a strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-circular walk and $\mathrm{l}(w') < \mathrm{l}(w)$. This will be done in steps.

(1) $\mathrm{l}(u'') \geq 1$ *and* $\mathrm{l}(v'') \geq 1$.

Suppose $\mathrm{l}(u'') = 0$ or $\mathrm{l}(v'') = 0$. Then $z = c$. Since w is a strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-circuit, $\mathrm{typ}(U)$ or $\mathrm{typ}(V)$ or both contain $l \in \{i,j\}$ such that $l \neq k$; without loss of generality assume that l is in $\mathrm{typ}(U)$. Then $(p_0, \ldots, p_l, p_0)$ is a strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-circuit of length less than $\mathrm{l}(w)$, since it does not contain p_{l+1}; a contradiction.

(2) *Either* $c = a$ *and* $\mathrm{l}(u') = \mathrm{l}(u)$ *or, else,* $c \neq a$*,* $\mathrm{l}(u') = \mathrm{l}(u) + 1$*, and the last letter* k *is deleted from* $\mathrm{t}(U')$ *when moving to* $\mathrm{t}(U'')$*. Similarly, either* $c = b$ *and* $\mathrm{l}(v') = \mathrm{l}(v)$ *or, else,* $c \neq b$*,* $\mathrm{l}(v') = \mathrm{l}(v)+1$*, and the first letter* k *is deleted from* $\mathrm{t}(V')$ *when moving to* $\mathrm{t}(V'')$*.*

We prove the statement regarding u. Suppose $c \neq a$; then $\mathrm{l}(u') = \mathrm{l}(u)+1$. Since U'' is a geodesic from z to $\mathrm{gate}_Q(z)$, the last edge of U'' cannot be labelled $\{k\}$.

(3) $\mathrm{l}(u'') = \mathrm{l}(u)$ *and* $\mathrm{l}(v'') = \mathrm{l}(v)$*.*

We prove the statement regarding $\mathrm{l}(u'')$. By (1) and (2) $1 \leq \mathrm{l}(u'') \leq \mathrm{l}(u)$. Suppose that $\mathrm{l}(u'') < \mathrm{l}(u)$. If the first edge of u'' spans a line in $\mathcal{L}_i$, then let $W'' = U'' \circ (c,b) \circ V \circ W_0''$. If the first edge of u'' spans a line in $\mathcal{L}_j$, then let $W'' = U'' \circ (c,a) \circ U^{-1} \circ W_0'$. Let $w'' = \mathrm{w}_{\mathcal{G}}(W'')$. Then w'' is a circular strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-walk, and either its first edge spans a line in $\mathcal{L}_i$ and its last edge spans a line in $\mathcal{L}_j$, or the other way around. Using that $\mathrm{l}(u) = \mathrm{l}(v)$, we obtain $\mathrm{l}(w'') = \mathrm{l}(u'') + \mathrm{l}(v) + 1 < \mathrm{l}(u) + \mathrm{l}(v) + 1 = \mathrm{l}(w)$. By Remark 4.2 this contradicts the minimality of w.

(4) *The first edge of* u'' *spans a line in* $\mathcal{L}_i$ *and the last edge of* v'' *spans a line in* $\mathcal{L}_j$*.*

We prove the statement regarding u''. By (2) there are two possibilities. If $c = a$, then $\mathrm{l}(u') = \mathrm{l}(u)$. By (3) $\mathrm{l}(u'') = \mathrm{l}(u)$, therefore no letters i or j are deleted when switching from $\mathrm{t}(U')$ to $\mathrm{t}(U'')$. Suppose $c \neq a$. Then by (2) $\mathrm{l}(u') = \mathrm{l}(u) + 1$ and the last letter k is deleted when switching from $\mathrm{t}(U')$ to $\mathrm{t}(U'')$. By (3) $\mathrm{l}(u'') = \mathrm{l}(u)$, therefore no other letters i of j were deleted. Since by (1) $\mathrm{l}(u'') \geq 1$, the first letter i in $\mathrm{typ}(U')$ remains in $\mathrm{typ}(U'')$.

We can now finish the proof of the lemma. By (4) w' is a circular strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-walk and by (3) $\mathrm{l}(w') = \mathrm{l}(u) + \mathrm{l}(v) \leq \mathrm{l}(w) - 1$. By Remark 4.2 this contradicts the minimality of $\mathrm{l}(w)$. $\square$

Lemma 4.5. *If the hypothesis of Proposition 4.3 holds and* $\mathrm{l}(w)$ *is even, then the conclusion of Proposition 4.3 holds.*

Proof. Let $l = (1/2)\mathrm{l}(w)$ and let $q = p_l$. Let $z \in \mathrm{Gate}_{p^\#}(q^\#)$ and let $c = \mathrm{gate}_{q^\#}(z)$; then $z = \mathrm{gate}_{p^\#}(c)$. Let $a = c'_l$ and let $b = c_l$; then W_l is a gallery from a to b. Let $U = (c_0, c'_1) \circ W_1 \circ \cdots \circ (c_{l-1}, c'_l)$ and let $V = (c_l, c'_{l+1}) \circ W_{l+1} \circ \ldots (c_{n-1}, c'_0)$; then U is a walk from x to a, V is a walk from b to y, and $W = U \circ W_l \circ V \circ W_0$. Define $u = \mathrm{w}_{\mathcal{G}}(U)$ and $v = \mathrm{w}_{\mathcal{G}}(V)$; then u is walk from p to q, v is a walk from q to p, $\mathrm{l}(u) = l = \mathrm{l}(v)$, the first edge of u spans a line in $\mathcal{L}_i$, the last edge of v spans a line in $\mathcal{L}_j$, and $w = u \circ v$.

Let W'_0 and W''_0 be geodesics from x to z and from y to z, and let W'_l and W''_l be geodesics from a to c and from b to c. Define $U' = (W'_0)^{-1} \circ U \circ W'_l$. There exists a geodesic U'' from z to c, such that $\mathrm{t}(U'')$ is obtained from $\mathrm{t}(U')$ by deleting terms. Similarly, let $V' = (W''_l)^{-1} \circ V \circ W''_0$ and let V'' be a geodesic from c to z, such that $\mathrm{t}(V'')$ is obtained from $\mathrm{t}(V')$ by deleting terms.

Define $u' = \mathrm{w}_{\mathcal{G}}(U')$, $u'' = \mathrm{w}_{\mathcal{G}}(U'')$, $v' = \mathrm{w}_{\mathcal{G}}(V')$, and $v'' = \mathrm{w}_{\mathcal{G}}(V'')$.

(1) $\mathrm{l}(u'') = \mathrm{l}(u)$*, and the first edge of* u'' *spans a line in* $\mathcal{L}_i$*. Similarly,* $\mathrm{l}(v'') = \mathrm{l}(v)$*, and the last edge of* v'' *spans a line in* $\mathcal{L}_j$*.*

We prove the statement regarding u''. Suppose $\mathrm{l}(u'') < \mathrm{l}(u)$. If the first edge of u'' spans a line in $\mathcal{L}_i$, let $W' = U'' \circ (W''_l)^{-1} \circ V \circ W''_0$. If the first edge of u'' spans a line in $\mathcal{L}_j$, let $W' = U'' \circ (W'_l)^{-1} \circ U^{-1} \circ W'_0$. Define $w' = \mathrm{w}_{\mathcal{G}}(W')$. Then w' is a circular $\{\mathcal{L}_i, \mathcal{L}_j\}$-walk, and either its first edge spans a line in $\mathcal{L}_i$ and its last edge spans a line in $\mathcal{L}_j$, or the other way around. Using that $\mathrm{l}(u) = \mathrm{l}(v)$, we have $\mathrm{l}(w') = \mathrm{l}(u'') + \mathrm{l}(v) < \mathrm{l}(u) + \mathrm{l}(v) = \mathrm{l}(w)$. By Remark 4.2 this contradicts the minimality of w. Therefore $\mathrm{l}(u'') = \mathrm{l}(u)$ and $\mathrm{t}(U'')$ is obtained from $\mathrm{t}(U')$ by deleting letters not contained in $\{i, j\}$. This implies that the first edge of u'' spans a line in $\mathcal{L}_i$.

(2) *Let* x'' *be the chamber that follows* z *in the walk* U''*, and let* y'' *be the chamber that preceeds* z *in the walk* V''*. Then the edge* $\{z, x''\}$ *is labelled* $\{i\}$ *and the edge* $\{z, y''\}$ *is labelled* $\{j\}$*.*

Since $(U'')^{-1}$ is a geodesic from c to $z = \mathrm{gate}_{p^\#}(c)$, we have $x'' \notin p^\#$. Therefore the conclusion follows from (1). The proof for V'' is similar.

(3) *Let* a' *be the chamber that preceeds* c *in the walk* U''*, and let* b' *be the chamber that follows* c *in the walk* V''*. Then* $a' \neq b'$ *and the labels of the edges* $\{a', c\}$ *and* $\{c, b'\}$ *are distinct elements of* $\{\{i\}, \{j\}\}$*.*

Since the walks U'' and $(V'')^{-1}$ are geodescis from z to $c = \mathrm{gate}_{q^\#}(z)$, the edges $\{a', c\}$ and $\{c, b'\}$ do not lie in $q^\#$. Suppose that the label of both $\{a', c\}$ and $\{c, b'\}$ is $\{k\}$, $k \in \{i, j\}$, and let Q be the panel of $\mathcal{B}$ of type $\{k\}$ on c. Since U'' and $(V'')^{-1}$ are geodesics from z to c, we have $\mathrm{d}(z, a') = \mathrm{d}(z, c) - 1 = \mathrm{d}(z, b')$, therefore $a' = \mathrm{gate}_Q(z) = b'$. Let U''' be the initial segment of U'' begining with z and ending with a', let V''' be the terminal segment of V'' begining with b' and ending with z, let $W' = U''' \circ V'''$, and let $w' = \mathrm{w}_{\mathcal{G}}(W')$. By (1) $\mathrm{l}(w') = \mathrm{l}(u) - 1 + \mathrm{l}(v) - 1 < \mathrm{l}(w)$, and the first and last edges of w' span lines in $\mathcal{L}_i$ and $\mathcal{L}_j$ respectively. By Remark 4.2 this contradicts the minimality of w.

Let R be the residue of $\mathcal{B}$ of type $\{i, j\}$ containing z and let $g = \mathrm{gate}_R(c)$.

(4) *z and g are opposite in R, and $c = \mathrm{gate}_{q^\#}(g)$.*

Since $g = \mathrm{gate}_R(c)$, g lies on a geodesic from c to z in $\mathcal{B}$. Therefore, g lies on a geodesic from z to c in $\mathcal{B}$. Since $\mathrm{gate}_{q^\#}(z) = c$, $\mathrm{gate}_{q^\#}(g) = c$. By (2) $x'', y'' \in R$. Since U'' is a geodesic from z to c, $\mathrm{d}_{\mathcal{B}}(c, x'') = \mathrm{d}_{\mathcal{B}}(c, z) - 1$. Since $g = \mathrm{gate}_R(c)$, this implies $\mathrm{d}_R(g, x'') = \mathrm{d}_R(g, z) - 1$. Similarly, $\mathrm{d}_R(g, y'') = \mathrm{d}_R(g, z) - 1$. Therefore z and g are opposite in R.

(5) *Suppose $g \neq c$. Let Y be a geodesic from g to c in $\mathcal{B}$, and let c' be the vertex of Y that preceeds c. Then $c' = a'$ or $c' = b'$.*

All letters appearing in $\mathrm{t}(U'')$ belong to $I' \cup \{i, j\}$, therefore z and c are contained in one residue R' of $\mathcal{B}$ of type $I' \cup \{i, j\}$, and $R \subseteq R'$. By convexity of R' all vertices of Y lie in R', therefore the label of the edge $\{c', c\}$ is contained in $I' \cup \{i, j\}$. Since by (4) $c = \mathrm{gate}_{q^\#}(g)$, we have $c' \notin q^\#$, therefore the label of $\{c', c\}$ is $\{i\}$ or $\{j\}$. Let $\{k\}$ be the label of $\{a', c\}$, let $\{l\}$ be the label of $\{b', c\}$, and let $\{m\}$ be the label of $\{c', c\}$. By (3) $m \in \{k, l\}$. Suppose $m = k$ and let Q be the panel of $\mathcal{B}$ of type $\{k\}$ on c. Since a' and c' lie on geodesics from z to c, we have $\mathrm{d}_{\mathcal{B}}(z, a') = \mathrm{d}_{\mathcal{B}}(z, c) - 1 = \mathrm{d}_{\mathcal{B}}(z, c')$, therefore $a' = \mathrm{gate}_Q(z) = c'$. Similarly, if $m = l$ then $b' = c'$.

(6) *$c = g$.*

Suppose $g \neq c$; then by (5) $c' = a'$ or $c' = b'$. Suppose $c' = a'$. Let S be the initial segment of V'' starting with c and ending with y'', and let Z be a geodesic from y'' to g; then $Z \circ Y$ and S^{-1} are geodesics from y'' to c in $\mathcal{B}$. Let A be an apartment of $\mathcal{B}$ containing c and z; since A is a convex induced chamber subsystem of $\mathcal{B}$, the galleries $Z \circ Y$ and S^{-1} lie in A. Let r be the reflection in the Coxeter group of A stabilizing the edge $\{a', c\}$. Then r must stabilize exactly one edge of S. Therefore, $S = S_1 \circ (d, d') \circ S_2$, where $\{d, d'\}$ is stabilized by r.

Let S_1' be the image of S_1 under r; then S_1' is a walk from a' to d'. Let U_1'' be the initial segment of U'' that begins with z and ends with a'. Define $s_1' = \mathrm{w}_{\mathcal{G}}(S_1')$, $s_2 = \mathrm{w}_{\mathcal{G}}(S_2)$, $u_1'' = \mathrm{w}_{\mathcal{G}}(U_1'')$ and let $w' = u_1'' \circ s_1' \circ s_2 \circ (y'', z)$. Then w' is a circular $\{\mathcal{L}_i, \mathcal{L}_j\}$-walk, and by (1) the first and last edges of w'' span lines in $\mathcal{L}_i$ and $\mathcal{L}_j$ respectively and $\mathrm{l}(u_1'') = \mathrm{l}(u) - 1$. Therefore $\mathrm{l}(w') = \mathrm{l}(u_1'') + \mathrm{l}(s_1') + \mathrm{l}(s_2) + 1 \leq \mathrm{l}(u) - 1 + \mathrm{l}(v) < \mathrm{l}(w)$, by Remark 4.2 a contradiction with the minimality of w. The case $c' = b'$ is similar.

(7) *The walks u and v^{-1} are geodesics from p to q in $\mathcal{G}$.*

We prove the statement regarding u. By (1) $\mathrm{l}(u) = \mathrm{l}(u'')$. We show that u'' is a geodesic from p to q in $\mathcal{G}$. By (6) the gallery U'' lies in R. Let $\Delta = \mathrm{ShSp}(R, \{i, j\})$; the points and lines of Δ are the chambers and panels of R respectively, and the point-collinearity graph of Δ is the graph of the chamber system R with the lables removed. Let $\psi : \Sigma|\mathcal{P}_R \to \Delta$ be the isomorphism of Remark 1.4; the image under ψ of the walk u'' is U''. Since U'' is a geodesic in R, and by Remark 1.3(4) $\mathcal{P}_R$ is a convex subspace of Σ, u'' is a geodesic in $\mathcal{G}$.

(8) *All vertices of w are in $\mathcal{P}_R$.*

By (7) u and v^{-1} are geodesics from p to q in $\mathcal{G}$, and by by Remark 1.3(4) $\mathcal{P}_R$ is a convex subspace of Σ, therefore (8) holds.

We can now finish the proof of the lemma. Let Δ and ψ be as in the proof of (7) and let G be the point-collinearity graph of Δ; let $\mathcal{L}'_i$ and $\mathcal{L}'_j$ be the sets of the panels of R of types $\{i\}$ and $\{j\}$ respectively. By (8) $\psi(w)$ is a shortest strictly $\{\mathcal{L}'_i, \mathcal{L}'_j\}$-circuit of G. Since R is a generalized m_{ij}-gon, $\psi(w) = C$ for a circuit C of R of type $p(i, j) \circ p(j, i)^{-1}$. Then $w = \mathrm{w}_{\mathcal{G}}(C)$. □

Corollary 4.6. *Suppose hypothesis* (A) *holds, let $\{i, j\} \subseteq J$, and let w be a strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-circuit of $\mathcal{G}$ of length $2m_{ij}$. Then there exists a plane π of Γ with $\mathrm{typ}_{\#}(\pi) = \{i, j\}$ such that all vertices of w lie in $\mathcal{P}_\pi$.*

Proof. By Proposition 4.3 there is a circuit C of $\mathcal{B}$ of type $p(i, j) \circ p(j, i)^{-1}$ such that $w = \mathrm{w}_{\mathcal{G}}(C)$. Let $\pi \in \Pi$ be such that $\mathrm{typ}_{\#}(\pi) = \{i, j\}$ and $\pi^{\#}$ contains C. □

The following is immediate from Remark 1.4 and [5, Proposition 3.8(2)].

Lemma 4.7. *Suppose hypothesis* (A) *holds, let $\{i, j\} \subseteq J$, and let π be a plane of Γ with $\mathrm{typ}_{\#}(\pi) = \{i, j\}$. Then $\Sigma|\pi$ is a generalized $2m_{ij}$-gon with exactly two lines on each point; its point-collinearity graph is isomorphic to the graph of the residue $\pi_{\#}$ with the labels removed.* □

Proof of Proposition 3.3. By Lemma 4.7 $\mathcal{G}|\mathcal{P}_\pi$ contains a strictly $\{\mathcal{L}_i, \mathcal{L}_j\}$-circuit w_0 of length $2m_{ij}$. Let $w = (p_0, \ldots, p_n)$ be the image of w_0 under ϕ. By (Phi-lin) w is a strictly $\{\mathcal{L}_{i'}, \mathcal{L}_{j'}\}$-circuit of $\mathcal{G}$ of length $2m_{ij}$, therefore by Corollary 4.6 there is a plane π' of Γ, such that $\mathrm{typ}_{\#}(\pi') = \{i', j'\}$ and $\{p_0, \ldots, p_n\} \subseteq \mathcal{P}_{\pi'}$. We claim that

(1) $\mathcal{P}_1 \subseteq \mathcal{P}_{\pi'}$.

To prove (1) we consider two cases. Suppose first that $m_{ij} = 2$; then $\mathrm{l}(w) = 4$. By Lemma 4.7 Σ_1 is a grid, therefore it is spanned by the vertices of w. By Remark 1.3(4) $\mathcal{P}_{\pi'}$ is a subspace of Σ, therefore (1) holds.

Suppose now that $m_{ij} \geq 3$ and let $\mathcal{G}_1$ denote the point-collinearity graph of Σ_1. Let $q \in \mathcal{P}_1$ and let $w' = (q_0, \ldots, q_l)$ be a geodesic from $q_0 = p_0$ to $q_l = q$ in $\mathcal{G}_1$. By Lemma 4.7 and condition (Phi-lin), $l \leq m_{ij}$, the edges of w' span lines belonging alternately to $\mathcal{L}_{i'}$ and $\mathcal{L}_{j'}$, and the walk w' can be completed to a strictly $\{\mathcal{L}_{i'}, \mathcal{L}_{j'}\}$-circuit w'' of $\mathcal{G}_1$ of length $2m_{ij}$. By Corollary 4.6 there is a plane π'' of Γ, such that $\mathrm{typ}_{\#}(\pi'') = \{i', j'\}$ and $\mathcal{P}_{\pi''}$ contains the vertices of w''. We are going to show that $\pi' = \pi''$.

Suppose first that w and w'' share two edges on p_0. Since $\mathrm{typ}_{\#}(\langle p_0, p_1\rangle) \neq \mathrm{typ}_{\#}(\langle p_0, p_n\rangle)$, $\langle p_0, p_1\rangle$ and $\langle p_0, p_n\rangle$ are two distinct lines incident with both π' and π''. Therefore by [4, Lemma 5.6(1)] $\pi' = \pi''$.

Suppose now that $\{p_1, p_n\} \neq \{q_1, q_n\}$. First, replacing w'' with $(w'')^{-1}$ if necessary, we can assume that $q_n \notin \{p_1, p_n\}$. Then replacing w with w^{-1} if necessary, we can assume that $\langle p_0, p_1\rangle \in \mathcal{L}_\alpha$ and $\langle p_0, q_n\rangle \in \mathcal{L}_\beta$, where $\{\alpha, \beta\} = \{i', j'\}$. By hypothesis $m_{ij} \geq 3$, therefore every gallery of $\mathcal{B}$ of type iji or jij is a geodesic and can be extended to a circuit of length $2m_{ij}$ of type $p(i,j) \circ p(j,i)^{-1}$. Therefore by condition (Phi-lin) every walk in $\mathcal{G}_1$ of length 3, whose edges span lines belonging alternately to $\mathcal{L}_{i'}$ and $\mathcal{L}_{j'}$, can be extended to a strictly $\{\mathcal{L}_{i'}, \mathcal{L}_{j'}\}$-circuit of length $2m_{ij}$. Let w_1 and w_2 be strictly $\{\mathcal{L}_{i'}, \mathcal{L}_{j'}\}$-circuits of $\mathcal{G}_1$ of length $2m_{ij}$ containing segments (p_2, p_1, p_0, q_n) and (p_1, p_0, q_n, q_{n-1}) respectively. By Corollary 4.6, for each $\gamma \in \{1, 2\}$, there is a plane π_γ of Γ, such that $\mathrm{typ}_{\#}(\pi_\gamma) = \{i', j'\}$ and the shadow $\mathcal{P}_{\pi_\gamma}$ contains the vertices of w_γ. The planes π_1 and π_2 share the lines $\langle p_0, p_1\rangle$ and $\langle p_0, q_n\rangle$. Since $\langle p_0, p_1\rangle \in \mathcal{L}_\alpha$ and $\langle p_0, q_n\rangle \in \mathcal{L}_\beta$, $\langle p_0, p_1\rangle \neq \langle p_0, q_n\rangle$. Therefore by [4, Lemma 5.6(1)] $\pi_1 = \pi_2$. Similarly, $\pi_1 = \pi'$ and $\pi_2 = \pi''$. Therefore, $\pi' = \pi''$ and $q \sim_\Gamma \pi'$.

(2) $\mathcal{P}_1 = \mathcal{P}_{\pi'}$.

Suppose $q \in \mathcal{P}_{\pi'} - \mathcal{P}_1$. Since $\Sigma|\mathcal{P}_{\pi'}$ is connected, we can assume that q is collinear with a point $r \in \mathcal{P}_1$ via a line $L_1 \in \mathcal{L}|\mathcal{P}_{\pi'}$. By Lemma 4.7 r lies on exactly two lines L_1, L_2 of $\Sigma|\mathcal{P}_{\pi'}$ and on exactly two lines N_1, N_2 of Σ_1. Since, for every $i \in \{1, 2\}$, N_i is incident with at least two points in $\mathcal{P}_1$, and by Remark 1.3(4) $\mathcal{P}_{\pi'}$ is a subspace of Σ, we have $\{L_1, L_2\} = \{N_1, N_2\}$. By Remark 3.2 Σ_1 is a full subgeometry of Σ, therefore $q \in \mathcal{P}_1$, a contradiction with the choice of q. □

5 Proof of Proposition 3.4

Here and in Subsections 5.1–5.2 we prove preliminary results. We complete the proof of Proposition 3.4 in Subsection 5.3.

Lemma 5.1. *Suppose the hypothesis of Proposition 3.4 holds. Then, for every object U of $\Gamma_{\mathcal{O}}^-$, ϕ induces an isomorphism $\Sigma|\mathcal{P}_U \to \Sigma|\mathcal{P}_{U'}$, where $U' \in \mathcal{O}$ and*

$\mathrm{typ}_{\#}(U') = \mathrm{typ}_{\#}(U)$.

Proof. If U is an object of the second kind, then $|\mathrm{typ}_{\#}(U)| < |T|$ and the restriction $\phi|\mathcal{P}_U$ satisfies the hypothesis of Theorem 3.1, therefore the conclusion holds by hypothesis. Suppose U is an object of the first kind. Then by condition (Phi-str) of Theorem 3.1 there is $U' \in \mathcal{O}$ such that $\phi(\mathcal{P}_U) = \mathcal{P}_{U'}$ and $\mathrm{typ}_{\#}(U) = \mathrm{typ}_{\#}(U')$. By Remark 1.4, condition (Bld-iso), and condition (Bld-str), U' is as in the conclusion. □

Corollary 5.2. *Suppose the hypothesis of Proposition 3.4 holds. Then ϕ induces a monomorphism $\overline{\phi} : \Gamma_O^- \to \Gamma$ defined by, for all objects U of Γ_O^-, $\overline{\phi}(U) = U'$, where U' is as in Lemma 5.1.* □

5.1 An auxiliary proposition

Suppose hypothesis (A) holds. Let $p \in \mathcal{P}$ and let $H \subseteq J$. We let $\Gamma(p, H)$ be the subgeometry induced in Γ on the set of all objects U of Γ incident with p, that have the property that $\mathrm{typ}_{\#}(U) \cap J = H$. If U is an object of $\Gamma(p, H)$, then its type in $\Gamma(p, H)$ is $\tau(U) - |H|$. The objects of $\Gamma(p, H)$ of types 1 and 2 will be called points and lines of $\Gamma(p, H)$ respectively. We denote $\Sigma(p, H)$ and $\mathcal{O}_{p,H}$ the point-line truncation and the set of objects of $\Gamma(p, H)$. For $U \in \mathcal{O}$ incident with p, we denote $\Gamma(p, H)_U^-$ and $\Sigma(p, H)_U$ the subgeometries of $\Gamma(p, H)$ and $\Sigma(p, H)$ induced on the sets of objects belonging to Γ_U^-.

For $H \subseteq I$, define $\overline{H} := \mathrm{typ}^{\#}(U)$, where $U \in \mathcal{O}$ and $\mathrm{typ}_{\#}(U) \subseteq H \subseteq \mathrm{typ}^{\#}(U)$. Recall that $p^{\#}$ is an I'-residue of $\mathcal{B}$. Suppose now that $H \subseteq J$. Let $H' = I' \cap \overline{H}$, $H'' = I' - H'$, $\Phi(p, H) = \mathrm{ShGm}(p^{\#}, H'')$, and $\Delta(p, H) = \mathrm{ShSp}(p^{\#}, H'')$. For $U \in \mathcal{O}$, we define $U_{p,H} = \mathrm{Sh}_{H'}(U^{\#} \cap p^{\#})$. For $T \subseteq I$ we let $T_H = \mathrm{typ}_{\#}(U_{p,H})$ and $T^H = \mathrm{typ}^{\#}(U_{p,H})$, where $U \in \mathcal{O}$ is such that $\mathrm{typ}_{\#}(U) \subseteq T \subseteq \mathrm{typ}^{\#}(U)$; both T_H and T^H are subsets of I'.

The purpose of this subsection is to prove the following.

Proposition 5.3. *Suppose the hypothesis of Proposition 3.4 holds. Let $p \in \mathcal{P}_O$, let $q = \phi(p)$, and let $H \subseteq J \cap T$. Suppose that* (C1) *or* (C2) *holds.*

(C1) $|H| = 1$ *and $O_{p,H}$ is an object of $\Phi(p, H)$ of the first kind.*

(C2) $|H| \in \{1, 2\}$ *and $O_{p,H}$ is an object of $\Phi(p, H)$ of the second kind.*

Then ϕ induces isomorphisms

(1) $\Sigma(p, H)_O \to \Sigma(q, H)_{O'}$, *where $O' \in \mathcal{O}$ and $\mathrm{typ}^{\#}(O') = H \cup T^H$.*

(2) $\Delta(p, H)_{O_{p,H}} \to \Delta(q, H)_{O_{q,H}}$, *where $O_{q,H}$ is an object of $\Phi(q, H)$ with $\mathrm{typ}^{\#}(O_{q,H}) = T^H$.*

The proof of Proposition 5.3 consists of Lemmas 5.4–5.9 below. First, we will show that Parts (1) and (2) of the conclusion are equivalent. Suppose hypothesis (A) holds. We say that $S \subseteq I$ is I'*-closed* if $S = \mathrm{typ}^{\#}(U)$ for some $U \in \mathcal{O}$. For $S \subseteq I$ with $\mathrm{typ}_{\#}(U) \subseteq S \subseteq \mathrm{typ}^{\#}(U)$ we let $\mathrm{Cl}_{I,I'}(S) := \overline{S}$, where $\overline{S}$ is as defined before Proposition 5.3, and we call this the I'*-closure* of S in I. If D is the diagram graph of $\mathcal{B}$ (that is, the diagram M with the edge labels removed), then $\mathrm{typ}^{\#}(U) = \mathrm{typ}_{\#}(U) \cup (I' - X)$, where $X = \cup\{D_{0,1}(k) \mid k \in \mathrm{typ}_{\#}(U)\}$ and $D_{0,1}(k)$ denotes the set of vertices equal or adjacent to k in D.

Lemma 5.4. *Suppose hypothesis* (A) *holds, D is the diagram graph of $\mathcal{B}$, and $H \subseteq J$. We use the notation of the beginning of this subsection.*

(1) *Suppose T is an I'-closed subset of I and $T \cap J = H$. Then $T \cap I'$ is an H'-closed subset of I'.*

(2) *Suppose S is an H'-closed subset of I' and $T = H \cup S$. Then T is an I'-closed subset of I.*

(3) *Let $T \subseteq I$ and $S \subseteq I'$. Let $T' = \mathrm{Cl}_{I',I'-S}(T \cap I')$, $K = \mathrm{K}_{I,I'}(T)$, and $K' = \mathrm{K}_{I',I'-S}(T')$. Suppose $T \cap S \subseteq K$. Then the graph $D|K'$ is the union of the connected components of the graph $D|(K \cap I')$ meeting S nontrivially.*

(4) *Under the hypothesis of Proposition 5.3 the graph $D|T_H$ is the union of the connected components of $D|(T \cap I')$ meeting H'' nontrivially.* □

Lemma 5.5. *Suppose hypothesis* (A) *holds, let $p \in \mathcal{P}$, and let $H \subseteq J$.*

(1) *Let $\psi_{p,H} : \Gamma(p,H) \to \Phi(p,H)$ be defined by, for every $U \in \mathcal{O}_{p,H}$, $\psi_{p,H}(U) = \mathrm{Sh}_{H'}(U^{\#} \cap p^{\#})$. Then $\psi_{p,H}$ is an isomorphism.*

(2) *Let $V \in \mathcal{O}$, $p \in \mathcal{P}_V$, and $H \subseteq \mathrm{typ}_{\#}(V) \cap J$. Then $\psi_{p,H}$ induces an isomorphism $\Gamma(p,H)^-_V \to \Phi(p,H)^-_{V_{p,H}}$.*

Proof. Part (1) is immediate from Remark 1.3(5), Lemma 5.4 (1) and (2), and properties (Bld1) and (Bld2) of buildings. To prove (2) we need to show that U is an object of $\Gamma(p,H)^-_V$ if and only if $U' := \mathrm{Sh}_{H'}(U^{\#} \cap p^{\#})$ is an object of $\Phi(p,H)^-_{V_{p,H}}$. Let $S = \mathrm{typ}_{\#}(V)$. Suppose U is an object of $\Gamma(p,H)$, let $N = \mathrm{typ}_{\#}(U)$ and $N' = \mathrm{typ}_{\#}(U')$; we have $N' = N - H$.

Suppose first that U is an object of $\Gamma(p,H)^-_O$. By property (Bld2) of buildings $U^{\#} \cap V^{\#} \cap p^{\#} \neq \emptyset$, therefore $(U')^{\#} \cap V^{\#}_{p,H} \neq \emptyset$. Since $N \subseteq S$, by Lemma 5.4(3) $N' \subseteq S_H$. Therefore by Remark 1.3(5) U' is incident with $V_{p,H}$ in $\Phi(p,H)$.

Suppose now that U' is an object of $\Phi(p,H)^-_{V_{p,H}}$. Then $(U')^{\#} \cap V^{\#}_{p,H}$ contains a residue of $V^{\#}_{p,H}$ of type $S^H - S_H$, therefore $(U')^{\#}$ meets every residue of $V^{\#}_{p,H}$ of type S_H nontrivially. Since by Lemma 5.4(4) $S_H \subseteq S$, we obtain that $(U')^{\#} \cap V^{\#} \neq \emptyset$. Therefore, using Lemma 5.4(1), $U^{\#} \cap V^{\#} \neq \emptyset$. Since $N' \subsetneq S_H$ and $H \subseteq S$, $N \subsetneq S$. This shows that U is an object of $\Gamma(p,H)^-_V$. □

From Lemma 5.5(2) applied to O we immediately obtain the following.

Corollary 5.6. *Conclusions* (1) *and* (2) *of Proposition 5.3 are equivalent.* □

Suppose the hypothesis of Proposition 5.3 holds, except possibly for (C1) and (C2). By Corollary 5.2 and Lemma 5.5(2) $\psi_{q,H} \circ \overline{\phi} \circ \psi_{p,H}^{-1}$ induces a monomorphism $\phi^* : \Phi(p,H)^-_{O_{p,H}} \to \Phi(q,H)$.

Lemma 5.7. *If the hypothesis of Proposition 5.3 holds and condition* (C1) *is satisfied, then Part* (2) *of the conclusion of Proposition 5.3 holds.*

Proof. Suppose $H = \{i\}$. First we prove that

$\phi^*(\mathrm{Pts}(\Phi(p,H)_{O_{p,H}}) = \mathrm{Pts}(\Phi(q,H))_{O_{q,H}}$ *for an object* $O_{q,H}$ *of* $\Phi(q,H)$ *with* $\mathrm{typ}^\#(O_{q,H}) = T^H$.

Let $S = H \cup T^H$ and let $U \in \mathcal{O}$ be such that $U^\#$ is the residue of $\mathcal{B}$ of type S containing $O^\#_{p,H}$. By condition (C1) U is of the first kind and by hypothesis O is of the second kind, therefore $U \neq O$. Using Lemma 5.4(4) and Remark 1.3(5) we see that U is an object of Γ^-_O, therefore by Lemma 5.1 ϕ induces a bijection $(\mathcal{L}|\mathcal{P}_U) \to (\mathcal{L}|\mathcal{P}_{U'})$ taking lines with $\mathrm{typ}_\#$ equal to H to lines with $\mathrm{typ}_\#$ equal to H, where $U' \in \mathcal{O}$ and $\mathrm{typ}^\#(U') = S$. By Lemma 5.5(2) the object $O_{q,H} := \mathrm{Sh}_{H'}((U')^\# \cap q^\#)$ is as claimed.

By Remark 1.4 and (Bldg-iso) $\Phi(p,H)_{O_{p,H}} \cong \Phi(q,H)_{O_{q,H}}$. Therefore by Corollary 5.2, condition (Bld-str), and the claim above ϕ induces an isomorphism $\Delta(p,H)_{O_{p,H}} \to \Delta(q,H)_{O_{q,H}}$. □

Suppose hypothesis (A) holds and let $\phi : \Sigma|\mathcal{P}_O \to \Sigma$ be a monomorphism, where $O \in \mathcal{O}$. We consider the following extension of (Phi-str).

(Phi-all) *for every object* U *of* Γ^-_O*, we have* $\phi(\mathcal{P}_U) = \mathcal{P}_{U'}$*, where* $U' \in \mathcal{O}$ *and* $\mathrm{typ}_\#(U') = \mathrm{typ}_\#(U)$.

Suppose the hypothesis of Proposition 5.3 holds, except possibly for (C1) and (C2). By (Bld-iso) there exists an isomorphism $\lambda : \Phi(p,H) \to \Phi(q,H)$. Let $O_{p,H,\lambda} = \lambda(O_{p,H})$ and let $\theta : \Delta(q,H)_{O_{p,H,\lambda}} \to \Delta(q,H)$ be the monomorphism induced by $\phi^* \circ \lambda^{-1}$.

Lemma 5.8. *Suppose the hypothesis of Proposition 5.3 holds, except possibly for conditions* (C1) *and* (C2)*, and suppose* $|H| \in \{1,2\}$. *Then* θ *satisfies* (Phi-all).

Proof. Let U be an object of $\Phi(q,H)^-_{O_{p,H,\lambda}}$ and let $S_0 = \mathrm{typ}^\#(U)$. Let $S = H \cup S_0$, and let $V \in \mathcal{O}$ be such that $\mathrm{typ}^\#(V) = S$ and $(\lambda^{-1}(U))^\# \subseteq V^\#$; then by Lemma 5.5 V is an object of $\Gamma(p,H)^-_O$. Let $V' = \overline{\phi}(V)$; then V' is an object of $\Gamma(q,H)$ and $\mathrm{typ}(V') = S$. Define $U' := \psi_{q,H}(V')$. By Lemma 5.5(1) to show that U' is as in the conclusion it suffices to show that $\overline{\phi}(\mathcal{A}) = \mathcal{A}'$, where

$\mathcal{A} = \mathrm{Pts}(\Gamma(p, H))_V$ and $\mathcal{A}' = \mathrm{Pts}(\Gamma(q, H))_{V'}$. Since by Lemma 5.1 ϕ satisfies condition (Phi-all), and the incidence is inclusion, $\overline{\phi}(\mathcal{A}) \subseteq \mathcal{A}'$. It remains to prove the reverse inclusion.

Case 1. Suppose $|H| = 1$. Then the elements of $\mathrm{Pts}(\Gamma(p, H))$ and $\mathrm{Pts}(\Gamma(q, H))$ are lines of Γ with $\mathrm{typ}_{\#}$ equal to H, and all lines of Γ incident with V and V' have $\mathrm{typ}_{\#}$ equal to H, therefore $\mathcal{A} = \mathcal{L}_V \cap \mathcal{L}_p$ and $\mathcal{A}' = \mathcal{L}_{V'} \cap \mathcal{L}_q$. By Lemma 5.1 $\phi(\mathcal{L}_V) = \mathcal{L}_{V'}$, therefore $\phi(\mathcal{A}) = \mathcal{A}'$.

Case 2. Suppose $|H| = 2$. Then the elements of $\mathrm{Pts}(\Gamma(p, H))$ and $\mathrm{Pts}(\Gamma(q, H))$ are planes of Γ of the second kind with $\mathrm{typ}_{\#}$ equal to H. Let $\pi' \in \mathcal{A}'$ and let $\phi_{\pi'}$ be the restriction of ϕ^{-1} to $\Sigma|\mathcal{P}_{\pi'}$. Then $\phi_{\pi'}$ satisfies the hypothesis of Proposition 3.3. Indeed, let $\Sigma' = (\mathcal{P}', \mathcal{L}')$ be the image of $\Sigma|\mathcal{P}_{\pi'}$ under $\phi_{\pi'}$. Since ϕ satisfies condition (Phi-str), Σ' is a full subgeometry of Σ and condition (Phi-lin) holds. By Proposition 3.3 there is a plane $\pi \in \mathcal{A}$, such that $\overline{\phi}(\pi) = \pi'$. Therefore $\mathcal{A}' \subseteq \overline{\phi}(\mathcal{A})$. □

Lemma 5.9. *Suppose that the hypothesis of Proposition 5.3 holds and condition* (C2) *is satisfied, then Part* (2) *of the conclusion of Proposition 5.3 holds.*

Proof. The morphism θ satisfies the hypothesis of Theorem 3.1. Indeed, by Lemma 5.8 θ satisfies (Phi-str) and, since $\mathcal{B}$ satisfies (Bld-iso) and (Bld-str), so does the building q. By Lemma 5.4(4) $(H \cup T_H) \subseteq T$ and, since $H \neq \emptyset$, $|T_H| < T$. Therefore the object $O_{q,H}$ exists by the inductive hypothesis. □

5.2 Intersections of pairs of residues $Q_{q,\{i\}}$

The object of this section is to prove the following.

Proposition 5.10. *Suppose the hypothesis of Proposition 3.4 holds. Let $p \in \mathcal{P}_O$ and let $q = \phi(p)$. For every $i \in T \cap J$, let $O_{q,\{i\}}$ be the object of $\Phi(q, \{i\})$ whose existence is asserted in Proposition 5.3. Then $O^{\#}_{q,\{i\}} \cap O^{\#}_{q,\{j\}} \neq \emptyset$ for all $\{i, j\} \subseteq J \cap T$.*

Lemma 5.11. *Suppose that the hypothesis of Theorem 3.1 holds. Let $p \in \mathcal{P}_O$, let $q = \phi(p)$, let $\{i, j\} \subseteq T \cap J$, and suppose $L_i \in \mathrm{Pts}(\Gamma(p, \{i\}))$ and $L_j \in \mathrm{Pts}(\Gamma(p, \{j\}))$ are such that $L_i^{\#} \cap L_j^{\#} \neq \emptyset$. Then $\phi(L_i)^{\#} \cap \phi(L_j)^{\#} \cap q^{\#} \neq \emptyset$.*

Proof. For $\alpha \in \{i, j\}$, let $L'_\alpha = \phi(L_\alpha)$; by condition (Phi-str), for $\alpha \in \{i, j\}$, $\mathrm{typ}_{\#}(L'_\alpha) = \{\alpha\}$. Let $\pi \in \mathrm{Pts}(\Gamma(p, \{i, j\}))$ be such that $\pi^{\#} \cap p^{\#} = L_i^{\#} \cap L_j^{\#} \cap p^{\#}$. By Proposition 3.3 ϕ induces an isomorphism $\Sigma|\mathcal{P}_\pi \to \Sigma|\mathcal{P}_{\pi'}$, where $\pi' \in \mathrm{Pts}(\Gamma(q, \{i, j\}))$. Since the incidence is inclusion, π' is incident with L'_i, L'_j, and q. Therefore $(L'_i)^{\#} \cap (L'_j)^{\#} \cap q^{\#} = (\pi')^{\#} \cap q^{\#} \neq \emptyset$. □

Using notation of Proposition 5.3, for $H \subseteq J \cap T$, let $O'_{q,H} = \psi^{-1}_{q,H}(O_{q,H})$; then $(O'_{q,H})^{\#}$ is the residue of $\mathcal{B}$ of type $H \cup T^H$ containing $O^{\#}_{q,H}$.

Recall that for $S \subseteq I$ and $U \in \mathcal{O}$ with $\mathrm{typ}_{\#}(U) \subseteq S \subseteq \mathrm{typ}^{\#}(U)$ we denote $\mathrm{Cl}_{I,I'}(S) = \mathrm{typ}^{\#}(U)$, the I'-closure of S in I (see Subsection 5.1). For a residue R of $\mathcal{B}$ we define the *I'-closure* $\mathrm{Cl}_{I,I'}(R)$ of R in $\mathcal{B}$ to be the residue of $\mathcal{B}$ of type $\mathrm{Cl}_{I,I'}(\mathrm{typ}(R))$ containing R.

Proof of Proposition 5.10. Let $\{i,j\} \in T \cap J$. Suppose $O_{p,\{i,j\}}$ is an object of $\Phi(p,\{i,j\})$ of the first kind. Then, since $\mathrm{typ}_{\#}(O_{p,\{i,j\}})$ is the union of the vertex sets of the connected components of $D|(T \cap I')$ meeting $D(i) \cup D(j)$ nontrivially, $|(D(i) \cup D(j)) \cap T \cap I'| \leq 1$. Therefore Cases 1 and 2 below exhaust all the possibilities.

Case 1. $|(D(i) \cup D(j)) \cap T \cap I'| \leq 1$.

First, we prove two claims.

(1) *For every line $N_i \in \mathrm{Pts}(\Gamma(q,\{i\}))_{O'_{q,\{i\}}}$, there is a line $N_j \in \mathrm{Pts}(\Gamma(q,\{j\}))_{O'_{q,\{j\}}}$ such that $N_i^{\#} \cap N_j^{\#} \neq \emptyset$.*

Let $N_i \in \mathrm{Pts}(\Gamma(q,\{i\}))_{O'_{q,\{i\}}}$ and let $L_i \in \mathrm{Pts}(\Gamma(p,\{i\}))_O$ be such that $N_i = \phi(L_i)$. By (Bld2) and Remark 1.3(5) the intersection $L_i^{\#} \cap p^{\#} \cap O^{\#}$ is nonempty, therefore there is a line $L_j \in \mathrm{Pts}(\Gamma(p,\{j\}))_O$ such that $L_i^{\#} \cap L_j^{\#} \cap p^{\#} \cap O^{\#} \neq \emptyset$. Let $N_j = \phi(L_j)$; then $N_j \in \mathrm{Pts}(\Gamma(q,\{j\}))_{O'_{q,\{j\}}}$. Since $L_i^{\#} \cap L_j^{\#} \neq \emptyset$, by Lemma 5.11 ${N_i}^{\#} \cap {N_j}^{\#} \neq \emptyset$.

Let $S = \{i,j\} \cup (I' - D(i) \cap D(j))$. Then

(2) $\mathrm{Cl}_{I',I' \cap S}(O^{\#}_{q,\{i\}}) = \mathrm{Cl}_{I',I' \cap S}(O^{\#}_{q,\{j\}})$.

First, we show that $\mathrm{Sh}_S(O^{\#}_{q,\{i\}}) = \mathrm{Sh}_S(O^{\#}_{q,\{j\}})$. Indeed, let $R \in \mathrm{Sh}_S(O^{\#}_{q,\{i\}})$, and let $N_i \in \mathrm{Pts}(\Gamma(q,\{i\})_{O'_{q,\{i\}}}$ be such that $N_i \cap R \cap O^{\#}_{q,\{i\}} \neq \emptyset$. By (1) there is a line $N_j \in \mathrm{Pts}(\Gamma(q,\{j\})_{O'_{q,\{j\}}}$, such that ${N_i}^{\#} \cap {N_j}^{\#} \neq \emptyset$. Since ${N_j}^{\#} \cap {N_i}^{\#} \neq \emptyset$ and ${N_i}^{\#} \subseteq R$, we have ${N_j}^{\#} \subseteq R$. By property (Bld2) of buildings ${N_j}^{\#} \cap O^{\#}_{q,\{j\}} \neq \emptyset$, therefore $R \cap O^{\#}_{q,\{j\}} \neq \emptyset$. Since $\mathrm{Sh}_S(O^{\#}_{q,\{i\}}) = \mathrm{Sh}_S(O^{\#}_{q,\{j\}})$, and $O^{\#}_{q,\{i\}} \cup O^{\#}_{q,\{j\}} \subseteq q$, by properties (Bld1) and (Bld2) of buildings $\mathrm{Sh}_{I' \cap S}(O^{\#}_{q,\{i\}}) = \mathrm{Sh}_{I' \cap S}(O^{\#}_{q,\{j\}})$. Therefore, by [13, Theorem 12.15] or by [4, Proposition 4.5] (2) holds.

We let $X = \mathrm{Cl}_{I',I' \cap S}(O^{\#}_{q,\{i\}})$ and let $T' = \mathrm{typ}(X)$. Up to interchanging i and j, there are two subcases of Case 1.

Case 1.1. $D(i) \cap T \cap I' = D(j) \cap T \cap I'$.

Recall that, for $k \in J$, $\overline{\{k\}} := \mathrm{Cl}_{I,I'}(\{k\})$. We claim that

(3) *for every* $k \in \{i,j\}$, $T' \cap \overline{\{k\}} = T^{\{k\}} \cap \overline{\{k\}}$.

Let $K' = \mathrm{K}_{I',I'\cap S}(T')$. By the hypothesis of Case 1.1, for every $k \in \{i,j\}$, $K' = T_{\{k\}}$, therefore

$$\begin{aligned} T' \cap \overline{\{k\}} &= (K' \cup [(I' \cap S) - \cup_{\alpha \in K'} D_{0,1}(\alpha)]) \cap \overline{\{k\}} \\ &= (T_{\{k\}} \cup [(I' \cap \overline{\{k\}}) - \cup_{\alpha \in T_{\{k\}}} D_{0,1}(\alpha)]) \cap \overline{\{k\}} \\ &= T^{\{k\}} \cap \overline{\{k\}}. \end{aligned}$$

Let $N_i \in \mathrm{Pts}(\Gamma(q,\{i\}))_{O'_{q,\{i\}}}$. By (1) there is a line $N_j \in \mathrm{Pts}(\Gamma(q,\{j\}))_{O'_{q,\{j\}}}$, such that ${N_i}^{\#} \cap {N_j}^{\#} \neq \emptyset$. Since $O^{\#}_{q,\{i\}} \cup O^{\#}_{q,\{j\}} \subseteq X$, by property (Bld2) of buildings ${N_i}^{\#} \cap {N_j}^{\#} \cap X \neq \emptyset$. By (3), for every $k \in \{i,j\}$, ${N_k}^{\#} \cap X \subseteq O^{\#}_{q,\{k\}}$. Therefore $O^{\#}_{q,\{i\}} \cap O^{\#}_{q,\{j\}} \neq \emptyset$.

Case 1.2. $D(j) \cap T \cap I' = \emptyset$.

In this case $\mathrm{typ}^{\#}(O_{q,\{j\}}) = T^{\{j\}} = I' \cap \overline{\{j\}}$, therefore the set $\mathrm{Pts}(\Gamma(q,\{j\}))_{O'_{q,\{j\}}}$ consists of exactly one line, which we denote N. We have $q^{\#} \cap N^{\#} = O^{\#}_{q,\{j\}}$. By (1), for every line $N_{\{i\}} \in \mathrm{Pts}(\Gamma(q,\{i\}))_{O'_{q,\{i\}}}$, we have ${N_{\{i\}}}^{\#} \cap O^{\#}_{q,\{j\}} \neq \emptyset$. Since the residue $O^{\#}_{q,\{i\}}$ is $(I' \cap \overline{\{i\}})$-closed in I', by [13, Theorem 12.15] or by [4, Proposition 4.5] $O^{\#}_{q,\{i\}} \cap O^{\#}_{q,\{j\}} \neq \emptyset$.

Case 2. $O_{p,\{i,j\}}$ *is an object of* $\Phi(p,\{i,j\})$ *of the second kind.*

Let $H = \{i,j\}$. By Proposition 5.3 ϕ induces an isomorphism $\Sigma(p,H)_O \to \Sigma(q,H)_{O'_{q,H}}$, where $O'_{q,H} = \psi^{-1}_{q,H}(O_{q,H})$, $O_{q,H} \in \mathrm{ShSet}(q^{\#}, H'')$, $\mathrm{typ}^{\#}(O_{q,H}) = T^H$. We claim that

(4) *for every* $k \in H$, $\mathrm{Pts}(\Gamma(q,\{k\}))_{O'_{q,H}} = \mathrm{Pts}(\Gamma(q,\{k\}))_{O'_{q,\{k\}}}$.

Suppose $L' \in \mathrm{Pts}(\Gamma(q,\{k\}))_{O'_{q,\{k\}}}$. Let $L \in \mathrm{Pts}(\Gamma(p,\{k\}))_O$ be such that $L' = \phi(L)$, let $\pi \in \mathrm{Pts}(\Gamma(p,H))_O$ be a plane incident with L, and let $\pi' = \overline{\phi}(\pi)$. Then $(\pi')^{\#} \cap O^{\#}_{q,H} \neq \emptyset$ and $(\pi')^{\#} \cap q^{\#} \subseteq (L')^{\#} \cap q^{\#}$. Therefore $(L')^{\#} \cap O^{\#}_{q,H} \neq \emptyset$. Suppose $L' \in \mathrm{Pts}(\Gamma(q,\{k\}))_{O'_{q,H}}$ and let $\pi' \in \mathrm{Pts}(\Gamma(q,H))_{O'_{q,H}}$ be a plane incident with L'. Then $\pi' = \overline{\phi}(\pi)$ for some plane $\pi \in \mathrm{Pts}(\Gamma(p,H))_O$. By Lemma 5.1 $L' = \phi(L)$ for a line $L \in \mathrm{Pts}(\Gamma(p,\{k\}))_\pi$, therefore $L' \in \mathrm{Pts}(\Gamma(q,\{k\})_{O'_{q,\{k\}}}$.

Since (4) holds and the residue $O^{\#}_{q,\{k\}}$ is $(I' \cap \overline{\{k\}})$-closed in I', by [13, Theorem 12.15] or by [4, Proposition 4.5] $\mathrm{Cl}_{I',I'\cap\overline{\{k\}}}(O^{\#}_{q,H}) = O^{\#}_{q,\{k\}}$. Therefore $O^{\#}_{q,H} \subseteq O^{\#}_{q,\{k\}}$ for all $k \in H$, which implies $O^{\#}_{q,\{i\}} \cap O^{\#}_{q,\{j\}} \neq \emptyset$. □

5.3 Final part of the proof of Proposition 3.4

Recall that $\overline{T} = \mathrm{Cl}_{I,I'}(T)$ (see Subsection 5.1).

Proof of Proposition 3.4(1). Let $S = T \cap J$, let $p \in \mathcal{P}_O$, and let $q = \phi(p)$. By Proposition 5.3, for every $i \in S$, there is an object $O_{q,\{i\}}$ of $\Phi(q, \{i\})$ such that ϕ induces an isomorphism $\Delta(p, \{i\})_{O_{p,\{i\}}} \to \Delta(q, \{i\})_{O_{q,\{i\}}}$. Let $R_q = \cap_{i \in S} O^{\#}_{q,\{i\}}$. By Proposition 5.10, for all $i, j \in S$, we have $O^{\#}_{q,\{i\}} \cap O^{\#}_{q,\{j\}} \neq \emptyset$ and by the definition of $\mathrm{ShGm}(\mathcal{B}, J)$ the set S is finite. Therefore, by properties (Bld1) and (Bld2) of buildings $R_q \neq \emptyset$ and $\mathrm{typ}(R_q) = \cap_{i \in S} T^{\{i\}}$. We claim that

(1) $\mathrm{typ}(R_q) = \overline{T} \cap I'$.

By the definition of $T^{\{i\}}$, for every $i \in S$, $\overline{T} \cap I' \subseteq T^{\{i\}}$, therefore $\overline{T} \cap I' \subseteq \cap_{i \in S} T^{\{i\}}$. Suppose $i \in I' - \overline{T}$. Since $\overline{T} = T \cup [I' - \cup_{\alpha \in T} D_{0,1}(\alpha)]$, we have $i \in D_{0,1}(j)$ for some $j \in T$. Suppose first that $j \in T \cap J$. We have $T^{\{j\}} = T_{\{j\}} \cup [I' - D_{0,1}(j) - \cup_{\alpha \in T_{\{j\}}} D_{0,1}(\alpha)]$. Since $i \notin \overline{T}$, we have $i \notin T_{\{j\}}$ and, since $i \in D_{0,1}(j)$, we have $i \notin T^{\{j\}} - T_{\{j\}}$. Therefore, $i \notin T^{\{j\}}$.

Suppose now that $j \in T \cap I'$. Let K be the connected component of $D|(T \cap I')$ containing j, and let $k \in T \cap J$ be such that $D_{0,1}(k) \cap K \neq \emptyset$. We claim that $i \notin T^{\{k\}}$. We have $T^{\{k\}} = T_{\{k\}} \cup [I' - D_{0,1}(k) - \cup_{\alpha \in T_{\{k\}}} D_{0,1}(\alpha)]$. Since $i \notin \overline{T}$, we have $i \notin T_{\{k\}}$. By Lemma 5.4(3) the set $T_{\{k\}}$ is the union of the vertex sets of the connected components of $D|(T \cap I')$ that meet $D_{0,1}(k)$ at a nonempty set. Therefore, $j \in K \subseteq T_{\{k\}}$ and $i \notin T^{\{k\}} - T_{\{k\}}$. This shows that $i \notin T^{\{k\}}$.

We can now finish the proof of Proposition 3.4(1). Let R'_q be the residue of $\mathcal{B}$ of type $\overline{T}$ containing R_q, and let $O_q = \mathrm{Sh}_{I'}(R'_q)$. By (1) $R'_q \cap q^{\#} = R_q$ therefore, for every $i \in S$, $\mathrm{Cl}_{I', I' \cap \overline{\{i\}}}(R_q) = (O_{q,\{i\}})^{\#}$ and $\mathrm{Pts}(\Phi(q, \{i\}))_{R_q} = \mathrm{Pts}(\Phi(q, \{i\}))_{O_{q,\{i\}}}$. This shows that O_q is as required in (Shd-loc). □

To prove Proposition 3.4(2) we need the following two results. Recall that $\overline{\phi}$ was defined in Corollary 5.2 and ϕ^* was defined before Lemma 5.7.

Lemma 5.12. *Suppose the hypothesis of Proposition 3.4 holds. Let $p \in \mathcal{P}_O$, let $q = \phi(p)$, let O_q be the object of Γ as in* (Shd-loc) *(existing by Proposition 3.4(1)) and let $R_q = q^{\#} \cap (O_q)^{\#}$. Then, for every $H \subseteq T \cap J$ with $|H| = 2$, ϕ^* induces a bijection* $\mathrm{Pts}(\Phi(p, H))_{O_{p,H}} \to \mathrm{Pts}(\Phi(q, H))_{R_q}$.

Proof. Suppose $H = \{i, j\}$ and let $\mathcal{P}_0 = \phi^*(\mathrm{Pts}(\Phi(p, H))_{O_{p,H}})$.

(1) $\mathcal{P}_0 \subseteq \mathrm{Pts}(\Phi(q, H))_{R_q}$.

Let $\pi \in \mathrm{Pts}(\Gamma(p, H))_O$ and let $\pi' = \overline{\phi}(\pi)$. Let $L \in \mathrm{Pts}(\Gamma(p, \{i\}))_O$ and $N \in \mathrm{Pts}(\Gamma(p, \{j\}))_O$ be two distinct lines of π on p, let $L' = \phi(L)$ and $N' = \phi(N)$. Then $(L')^{\#} \cap R_q \neq \emptyset$, $(N')^{\#} \cap R_q \neq \emptyset$, and by Lemma 5.11 $(L')^{\#} \cap (N')^{\#} \neq \emptyset$, therefore by property (Bld2) of buildings $(L')^\# \cap (N')^\# \cap R_q \neq \emptyset$. By Lemma 5.1 $(\pi')^\# \cap q^\# = (L')^\# \cap (N')^\# \cap q$, therefore $(\pi')^\# \cap R_q \neq \emptyset$.

(2) *If $O_{p,H}$ is an object of $\Phi(p,H)$ of the second kind, then $\mathcal{P}_0 = \mathrm{Pts}(\Phi(q,H))_{R_q}$.* Let $O_{q,H}$ be the object of $\Phi(q,H)$ existing by Proposition 5.3, such that $\mathcal{P}_0 = \mathrm{Pts}(\Phi(q,H))_{O_{q,H}}$. Then, comparing $\mathrm{typ}(R_q)$ and $\mathrm{typ}^\#(O_{q,H})$, by (1) and [13, Theorem 12.15] or by [4, Proposition 4.5] $(O_{q,H})^\# = \mathrm{Cl}_{I',H'}(R_q)$, therefore (2) holds.

(3) *If $O_{p,H}$ is an object of $\Phi(p,H)$ of the first kind, then $\mathrm{Pts}(\Phi(q,H))_{R_q} \subseteq \mathcal{P}_0$.* If $O_{p,H}$ is an object of $\Phi(p,H)$ of the first kind, then $|(D(i) \cup D(j)) \cap (T \cap I')| \leq 1$. Therefore, up to interchanging i and j, there are two possibilities: either $D(i) \cap T \cap I' = D(j) \cap T \cap I'$ or $|D(i) \cap T \cap I'| = 1$ and $D(j) \cap T \cap I' = \emptyset$. In both cases

$$D(i) \cap (T \cap I') = (D(i) \cup D(j)) \cap (T \cap I') \tag{5.1}$$

Let $\pi' \in \mathrm{Pts}(\Gamma(q,H))$ be such that $(\pi')^\# \cap R_q \neq \emptyset$. Let $L' \in \mathrm{Pts}(\Gamma(q,\{i\}))_{\pi'}$. Since $(\pi')^\# \cap q^\# \subseteq (L')^\# \cap q^\#$ and $(\pi')^\# \cap R_q \neq \emptyset$, we have $(L')^\# \cap R_q \neq \emptyset$. Therefore $L' = \phi(L)$ for some line $L \in \mathrm{Pts}(\Gamma(p,\{i\}))_O$. Let $\pi_L \in \mathrm{Pts}(\Gamma(p,H))_O$ be a plane incident with L and let $\pi'_L = \overline{\phi}(\pi_L)$. By Lemma 5.1 $\mathrm{typ}_\#(\pi') = \mathrm{typ}_\#(\pi'_L)$. We have $(\pi')^\# \cap q^\# \subseteq (L')^\# \cap q^\#$ and $(\pi'_L)^\# \cap q^\# \subseteq (L')^\# \cap q^\#$. By (1) $(\pi'_L)^\# \cap R_q \neq \emptyset$, therefore by equation (5.1) $(\pi')^\# \cap R_q = (L')^\# \cap R_q = (\pi'_L)^\# \cap R_q$. This shows $(\pi')^\# \cap (\pi'_L)^\# \neq \emptyset$, therefore $\pi' = \pi'_L$. □

Lemma 5.13. *Suppose the hypothesis of Proposition 3.4 holds. Let π be a plane of Γ incident with two distinct intersecting lines in $\mathcal{L}_1$. Then $\pi = \overline{\phi}(\pi')$ for some plane π' of Γ_O^-.*

Proof. Suppose $\{L, N\} \subseteq \mathcal{L}_1$ are distinct intersecting lines of Γ incident with π, let q be the common point of L and N, let $p = \phi^{-1}(q)$, and let $H = \mathrm{typ}_\#(\pi) \cap J$. Either π is of the first kind and $\mathrm{typ}_\#(L) = \mathrm{typ}_\#(N) = H$ or π is of the second kind and $H = \mathrm{typ}_\#(L) \cup \mathrm{typ}_\#(N)$. Since ϕ satisfies (Phi-str), $\mathrm{typ}_\#(L) \cup \mathrm{typ}_\#(N) \subseteq T \cap J$. Therefore in both cases $H \subseteq T \cap J$.

Case 1. π is a plane of the first kind. By Proposition 5.3 there is an object $O_{q,H}$ of $\Phi(q,H)$ such that ϕ induces an isomorphism $\Delta(p,H)_{O_{p,H}} \to \Delta(q,H)_{O_{q,H}}$. Let $L_1 = \psi_{q,H}(L)$, $N_1 = \psi_{q,H}(N)$, and $\pi_1 = \psi_{q,H}(\pi)$. Then $\{L_1, N_1\} \subseteq \mathrm{Pts}(\Phi(q,H))_{O_{q,H}}$ and $\pi_1 \in \mathrm{Lin}(\Phi(q,H))$. Since the line π_1 of $\Phi(q,H)$ is incident with two distinct points of $\mathrm{Pts}(\Phi(q,H))_{O_{q,H}}$, by Remark 1.3(4) π_1 is incident with $O_{q,H}$, therefore the conclusion holds.

Case 2. π is a plane of the second kind. Let O_q be as in (Shd-loc) (O_q exists by Proposition 3.4(1)) and let $R_q = q^\# \cap (O_q)^\#$. Then $L^\# \cap R_q \neq \emptyset$ and $N^\# \cap R_q \neq \emptyset$. Since $\pi^\# \cap q^\# = L^\# \cap N^\# \cap q^\# \neq \emptyset$, by (Bld2)

$L^\# \cap N^\# \cap R_q \neq \emptyset$ and $\pi^\# \cap R_q \neq \emptyset$. Therefore the conclusion follows by Lemma 5.12. □

Proof of Proposition 3.4(2). Let p and q be distinct collinear points of Σ_1 and let $\pi \in \Pi$ be incident with p and q. We show that $\mathcal{L}_1 \cap \mathcal{L}_p \cap \mathcal{L}_\pi - \{\langle p, q\rangle\} \neq \emptyset$ implies $\mathcal{L}_1 \cap \mathcal{L}_q \cap \mathcal{L}_\pi - \{\langle p, q\rangle\} \neq \emptyset$; the converse follows by symmetry.

Let $N = \langle p, q\rangle$; then $N \in \mathcal{L}_1$ and, since π is incident with two distinct points of N, by Remark 1.3(4) π is incident with N. Suppose $L_p \in \mathcal{L}_1 \cap \mathcal{L}_p \cap \mathcal{L}_\pi - \{N\}$. Then π is incident with two distinct lines $\{L_p, N\} \subseteq \mathcal{L}_1$, therefore by Lemma 5.13 $\pi = \overline{\phi}(\pi')$ for some plane π' of Γ_O^-. Let $q' = \phi^{-1}(q)$ and let $N' = \phi^{-1}(N)$. By [4, Lemma 5.6(2)] every point of π' lies on at least two distinct lines incident in Γ with π', therefore there is a line $L_{q'}$ of π' on q' distinct from N'. By Remark 1.3(1) $L_{q'}$ is incident with O. Let $L_q = \phi(L_{q'})$. Then $L_q \in \mathcal{L}_1 - \{N\}$ and L_q is incident with π and q. □

6 Application of Theorem 3.1 to simply laced diagrams

We consider the diagram $\mathsf{Y}_{l,m,n}$. If $m \geq 0$ we assume that the three arms of the diagram $\mathsf{Y}_{l,m,n}$ are labelled $(-l, \dots, -1, 0)$, $(0, 1', \dots, m')$, and $(0, 1, \dots, n)$, where 0 is the label of the branching node; we let $I_1 = \{-l, \dots, -1, 0\}$, $I_2 = \{0, 1, \dots, n\}$, and $I_3 = \{1', \dots, m'\}$. We denote $\mathsf{Y}_{l,-1,n}$ the diagram $\mathsf{A}_l \times \mathsf{A}_n$ with the two connected components labelled $(-l, \dots, -1)$ and $(1, \dots, n)$, and we let $I_1 = \{-l, \dots, -1\}$, $I_2 = \{1, \dots, n\}$, and $I_3 = \emptyset$. We define $I = I_1 \cup I_2 \cup I_3$. The diagrams A_k, D_k, and E_k are examples of the diagram $\mathsf{Y}_{l,m,n}$. For a building $\mathcal{B}$ with diagram M we consider the following generalization of (Bld-iso).

(Bld-iso') *For all residues R and Q of $\mathcal{B}$, if there is an isomorphisms of diagrams $\xi : M|\mathrm{typ}(R) \to M|\mathrm{typ}(Q)$, then there is an isomorphism of chamber systems $\xi' : R \to Q$ mapping, for every $i \in \mathrm{typ}(R)$, the edges labelled $\{i\}$ to the edges labelled $\{\xi(i)\}$.*

Theorem 6.1. *Suppose hypothesis* (A) *holds, $\mathcal{B}$ has property* (Bld-iso')*, and* $\mathsf{M} = \mathsf{Y}_{l,m,n}$*, where l, m, and n are integers, $l > 0$, $n > 0$, $m \geq -1$. Let $\phi : \Sigma|\mathcal{P}_O \to \Sigma|\mathcal{P}_1$ be an isomorphism, where O is a residue of $\mathcal{B}$ and $\mathcal{P}_1$ is a subspace of Σ.*

(1) *Suppose $m \geq 0$ and $J = \{m'\}$ or, else, $m = -1$ and $J = \{-1, 1\}$. Then $\mathcal{P}_1 = \mathcal{P}_{O'}$ for a residue O' of $\mathcal{B}$ with $M_O \cong M_{O'}$ and the isomorphism of diagrams takes $\mathrm{typ}_\#(O) \cap J$ to $\mathrm{typ}_\#(O') \cap J$.*

(2) *Suppose that $J = \{-l, n\}$ and $\mathrm{typ}(O) = I - \{\alpha'\}$ with $1 \leq \alpha \leq m$. Suppose further that either M is D_k and $k \geq 4$ or, else, M is E_6 and $m = 1$. Then*

$\mathcal{P}_1 = \mathcal{P}_{O'}$ for a residue O' of $\mathcal{B}$ with $M_O = M_{O'}$.

Remark 6.2. If in Part (1) of Theorem 6.1 we allow I_1 or I_2 to be infinite, then the conclusion fails. Indeed, suppose I_1 is infinite and let U be an object of Γ with $\mathrm{typ}^{\#}(U) = I - \{1\}$; then $\mathcal{P}_U$ is a maximal singular subspace of Σ. The geometry $\Sigma|\mathcal{P}_U$ is a projective space of infinite rank, therefore it has a proper subspace X such that $\Sigma|X \cong \Sigma|\mathcal{P}_U$. The subspace X is not the shadow of any object of Γ.

Under the hypothesis of Theorem 6.1 we let $\Sigma_1 = \Sigma|\mathcal{P}_1$.

Lemma 6.3. *Suppose the hypothesis of Theorem 6.1 holds, except possibly for the condition* (Bld-iso'). *Suppose U is an object of Γ of the first kind, and $U = O$ or U is in Γ_O^-. Then ϕ induces an isomorphism $\Sigma|\mathcal{P}_U \to \Sigma|\mathcal{P}_{U'}$, where U' is a residue of $\mathcal{B}$ with $M_{U'} \cong \mathsf{A}_k$, $k = |\mathrm{typ}_{\#}(U)|$.*

Proof. Let $\Sigma' = (\mathcal{P}', \mathcal{L}')$ be the image of Σ_U under ϕ. Since the diagram M_U is of type A_k, Σ' is a projective space ([13]). By Remark 1.3(4) $\mathcal{P}_U$ is a subspace of $\Sigma|\mathcal{P}_O$, and by hypothesis $\mathcal{P}_1$ is a subspace of Σ, therefore $\mathcal{P}'$ is a subspace of Σ. The conclusion follows from Remark 1.3(3). □

Suppose the hypothesis of Theorem 6.1 holds, except possibly for the condition (Bld-iso'). First, suppose that part (1) of the hypothesis holds and O is an object of Γ of the second kind. Then $\mathrm{typ}^{\#}(O) = I - \{-1-a, 1+b\}$, where $1 \le a \le l$ and $1 \le b \le n$, and the maximal singular subspaces of Σ_O are point shadows of residues of $\mathcal{B}$ of types $A = I - \{-1-a, 1\}$ and $B = I - \{-1, 1+b\}$. Here, a and b are integers; for instance if $a = l$ and $b = n-1$, then $\mathrm{typ}^{\#}(O) = I - \{-1-l, 1+n-1\} = I - \{n\}$. Let $A' = I - \{-1, 1+a\}$, let $B' = I - \{-1-b, 1\}$, and let $\mathcal{M}_-$, $\mathcal{M}_+$, $\mathcal{M}'_-$, $\mathcal{M}'_+$ be the sets of objects U of Γ with $\mathrm{typ}^{\#}(U)$ equal to A, B, B', A' respectively. Suppose now that part (2) of the hypothesis holds and let $\mathcal{M}_-$, $\mathcal{M}_+$, $\mathcal{M}'_-$, and $\mathcal{M}'_+$ be the sets of objects U of Γ with $\mathrm{typ}^{\#}(U)$ equal to $I - \{\alpha', -l\}$, $I - \{\alpha', n\}$, $I - \{-l, -1\}$, and $I - \{n, 1\}$. The maximal singular subspaces of Σ_O are point shadows of objects in $\mathcal{M}_- \cup \mathcal{M}_+$. In both cases, by Lemma 6.3, if $U \in \mathcal{M}_- \cup \mathcal{M}_+$ then $U' \in \mathcal{M}_- \cup \mathcal{M}_+ \cup \mathcal{M}'_- \cup \mathcal{M}'_+$. Suppose that, for all $\{U, V\} \subseteq \mathcal{M}_- \cup \mathcal{M}_+$ incident with O, $\mathrm{typ}^{\#}(U') = \mathrm{typ}^{\#}(V')$ if and only if $\mathrm{typ}^{\#}(U) = \mathrm{typ}^{\#}(V)$. If, in addition, $(U, V) \in \mathcal{M}_- \times \mathcal{M}_+$, implies $(U', V') \in (\mathcal{M}_- \times \mathcal{M}_+) \cup (\mathcal{M}'_+ \times \mathcal{M}'_-)$ under part (1) of the hypothesis and $(U', V') \in (\mathcal{M}_- \times \mathcal{M}_+) \cup (\mathcal{M}_+ \times \mathcal{M}_-)$ under part (2) of the hypothesis, then we say that ϕ *preserves classes*. If ϕ preserves classes for all U and V in a set $S \subseteq \mathcal{M}_- \cup \mathcal{M}_+$, then we say that ϕ *preserves classes for* S. For a point $p \in \mathcal{P}_O$, we denote $\mathcal{M}_{O,p}$ the set of spaces in $\mathcal{M}_- \cup \mathcal{M}_+$ incident with O and p.

Lemma 6.4. *Suppose either the hypothesis of Theorem 6.1(1) holds with $m = -1$ and O is an object of the second kind or the hypothesis of Theorem 6.1(2) holds. Then ϕ preserves classes.*

Proof. Since $\Sigma|\mathcal{P}_O$ is connected and, for every pair of collinear points $\{p, q\} \subseteq \mathcal{P}_O$, $\mathcal{M}_{O,p} \cap \mathcal{M}_{O,q} \neq \emptyset$, it suffices to show that for every $p \in \mathcal{P}_O$, ϕ preserves classes for $\mathcal{M}_{O,p}$.

Let $k = |I|$. Let $p \in \mathcal{P}_O$, let $q = \phi(p)$, and let $\Delta_{p,O}$ and Δ_q be the geometries of lines and projective planes of Γ on p incident with O, and of lines and projective planes of Γ on q respectively. Then $\Delta_{p,O}$ is the disjoint union of two projective spaces $\Delta^-_{p,O}$ and $\Delta^+_{p,O}$, whose point sets $\mathcal{L}^-$ and $\mathcal{L}^+$ are intesections of shadows of some $U \in \mathcal{M}_-$ and $V \in \mathcal{M}_+$ with the shadow of O. The projective dimensions of $\Delta^-_{p,O}$ and $\Delta^+_{p,O}$ are both $k-3$ if the hypothesis of Theorem 6.1(2) holds and Σ is not $\mathsf{D}_{k,\{k-1,k\}}$, are both α if Σ is $\mathsf{D}_{k,\{k-1,k\}}$, and are $a-1$ and $b-1$ if $\mathsf{M} = \mathsf{A}_m \times \mathsf{A}_n$.

The geometry Δ_q is the disjoint union of geometries Δ^-_q and Δ^+_q, whose points are the lines of Γ on q corresponding to residues of types $I - \{-l, n-1\}$ and $I - \{n, -(l-1)\}$ if the hypothesis of Theorem 6.1(2) holds, and of types $I - \{-2, 1\}$ and $I - \{-1, 2\}$ if $\mathsf{M} = \mathsf{A}_l \times \mathsf{A}_n$. We have the following cases.

Case 1. $\mathsf{M} = \mathsf{A}_l \times \mathsf{A}_n$ or Σ is $\mathsf{D}_{k,\{k-1,k\}}$; then Δ^-_q and Δ^+_q are projective spaces.

Case 2. Σ is $\mathsf{D}_{k,\{1,k\}}$ and $l \geq 2$; then Δ^+_q is a projective space and Δ^-_q is a line Grassmannian of a projective space;

Case 3. Σ is $\mathsf{E}_{6,\{1,6\}}$; then Δ^-_q and Δ^+_q are polar spaces of type D_4.

Let $\{\mathcal{N}^-, \mathcal{N}^+\} = \{\phi(\mathcal{L}^-), \phi(\mathcal{L}^+)\}$. Suppose $\mathcal{N}^-$ and $\mathcal{N}^+$ are subspaces of the same Δ^δ_q, $\delta \in \{+, -\}$. Then, checking Cases 1–3, there exist $x \in \mathcal{N}^-$ and $y \in \mathcal{N}^+$ collinear in Δ^δ_q, a contradiction since no two points of $\mathcal{L}^-$ and $\mathcal{L}^+$ are collinear in $\Delta_{p,O}$. Therefore we can assume that $\mathcal{N}^- \subseteq \mathrm{Pts}(\Delta^-_q)$ and $\mathcal{N}^+ \subseteq \mathrm{Pts}(\Delta^+_q)$.

Let $\mathcal{S}_-$, $\mathcal{S}'_-$ and $\mathcal{S}_+$, $\mathcal{S}'_+$ be the sets of nontrivial shadows of the elements of $\mathcal{M}_-$, $\mathcal{M}'_-$ in $\mathrm{Pts}(\Delta^-_q)$ and of the elements of $\mathcal{M}_+$, $\mathcal{M}'_+$ in $\mathrm{Pts}(\Delta^+_q)$. In Case 1, if $M = \mathsf{A}_l \times \mathsf{A}_n$ then the set of singular subspaces of Δ^-_q and Δ^+_q of projective dimensions $a-1$ and $b-1$ is $\mathcal{S}_- \cup \mathcal{S}_+ \cup \mathcal{S}'_- \cup \mathcal{S}'_+$; if Σ is of type $\mathsf{D}_{k,\{k-1,k\}}$, then the set of singular subspaces of dimension α is $\mathcal{S}_- \cup \mathcal{S}_+$. In Case 2 with $l \geq 3$ the set of singular subspaces of Δ^-_q and Δ^+_q of projective dimension $k-3$ is $\mathcal{S}_- \cup \mathcal{S}_+$. Therefore the proof is complete in these cases.

Consider Case 2 with $l = 2$ and Case 3. If Σ is $\mathsf{D}_{k,\{1,k-1\}}$ the set of singular subspaces of Δ^+_q of projective dimension $k-3$ is $\mathcal{S}_+$, and if Σ is $\mathsf{E}_{6,\{1,6\}}$ then it is $\mathcal{S}_+ \cup \mathcal{S}'_+$; in both cases the set of singular subspaces of Δ^-_q of projective dimension $k-3$ is $\mathcal{S}_- \cup \mathcal{S}'_-$. For every $L \in \mathcal{L}^+$, let S_L be the set of lines

$M \in \mathrm{Pts}(\Delta_p^-)$ lying in a grid of $\Sigma|\mathcal{P}_O$ with L; then $S_L \in \mathcal{S}'_-$ and $\mathrm{codim}(S_L \cap \mathcal{L}^-, \mathcal{L}^-) = 1$. Suppose by way of contradiction $\mathcal{N}^- \in \mathcal{S}'_-$. For $N \in \mathrm{Pts}(\Delta_q^+)$, let S_N be the set of lines $M \in \mathrm{Pts}(\Delta_q^-)$ lying in a grid of Σ with N; then $S_N \in \mathcal{S}'_-$. Therefore $\mathrm{codim}(S_N \cap \mathcal{N}^-, \mathcal{N}^-) = 0$ for exactly one $N \in \mathrm{Pts}(\Delta_q^+)$ and $\mathrm{codim}(S_N \cap \mathcal{N}^-, \mathcal{N}^-) \in \{2, 4\}$ for the rest, a contradiction. This shows $\mathcal{N}^- \in \mathcal{S}_-$. In case (3) $\mathcal{N}^+ \in \mathcal{S}_+$ by symmetry. □

Lemma 6.5. *Suppose the hypothesis of Theorem 6.1(1) holds and O is an object of Γ of the second kind. Then ϕ preserves classes.*

Proof. We use induction on m. For $m = -1$ the statement is true by Lemma 6.4. Suppose the statement is true for $m = k$, where $k \geq -1$, and let $m = k + 1$. Since the geometry $\Sigma|\mathcal{P}_O$ is connected and, for any two collinear points p and q of $\Sigma|\mathcal{P}_O$, $\mathcal{M}_{O,p} \cap \mathcal{M}_{O,q} \neq \emptyset$, it suffices to show that, for every $p \in \mathcal{P}_O$, ϕ preserves classes for $\mathcal{M}_{O,p}$.

Since O is an object of Γ of the second kind and $m \geq 0$, $|\mathrm{typ}_{\#}(O)| \geq 3$. Let $p \in \mathcal{P}_O$, let $q = \phi(p)$, and let $H = \{m'\}$. For $r \in \{p, q\}$, let $\Sigma(r, H)$, $\Delta(r, H)$, $\Phi(r, H)$ be defined as in Subsection 5.1; let $O_{p,H}$ be an object of $\Phi(p, H)$ such that $O^{\#}_{p,H} = p^{\#} \cap O^{\#}$. Since by Lemma 6.3 planes are mapped to planes, ϕ induces a monomorphism $\Sigma(p, H)_O \to \Sigma(q, H)$. Therefore by Lemma 5.5(2) ϕ induces a monomorphism $\Delta(p, H)_{O_{p,H}} \to \Delta(q, H)$. By condition (Bld-iso') there is an isomorphism of chamber systems $\lambda : q \to p$ and, composed with ϕ, it induces a monomorphism $\theta : \Delta(q, H)_{O_{p,H,\lambda}} \to \Delta(q, H)$, where $O_{p,H,\lambda}$ is an object of $\Phi(q, H)$ such that $O^{\#}_{p,H,\lambda} = \lambda^{-1}(O^{\#}_{p,H})$. We show that θ satisfies the induction hypothesis, therefore preserves classes. Since (Bld-iso') holds for $\mathcal{B}$, it holds for the building q. Since $m \geq 0$, the geometry $\Phi(q, H)$ is of the required type with $m = k$, and $O_{p,H}$ is its object of the second kind. Let $\Delta'(q, H) = (\mathcal{L}', \mathcal{P}')$ be the image of $\Delta(q, H)_{O_{p,H,\lambda}}$ under θ. Since all planes of Γ are projective planes, $\mathcal{P}_1$ is a subspace of Σ, and $\Sigma_1 = \Sigma|\mathcal{P}_1$, we obtain that $\mathcal{L}'$ is a subspace of $\Delta(q, H)$ and $\Delta'(q, H) = \Delta(q, H)|\mathcal{L}'$. □

Proof of Theorem 6.1. For objects of the first kind the conclusion holds by Lemma 6.3. Suppose O is an object of the second kind. By hypothesis the diagram of $\mathcal{B}$ is finite, all m_{ij} are integers, and condition (Bld-iso') holds. If R is a residue of $\mathcal{B}$ such that $M|\mathrm{typ}(R)$ is a string and j is one of its end nodes, then the $\{j\}$-shadow space of R is a projective space. Therefore condition (Bld-str) holds. If ϕ satisfies condition (Phi-str) of Theorem 3.1, then ϕ satisfies the hypothesis of Theorem 3.1 and the conclusion holds.

Suppose ϕ does not satisfy (Phi-str). By Lemmas 6.4 and 6.5 ϕ preserves classes. Let ξ be a map defined by $\xi(i) = -i$ for all $i \in I_1 \cup I_2$ and $\xi(i) = i$ for all $i \in I_3$. If Part (1) of the hypothesis holds and $\mathrm{typ}^{\#}(O) = I - \{-1-a, 1+b\}$, then let X be an object of Γ with $\mathrm{typ}^{\#}(X) = I - \{-1-b, 1+a\}$. If Part (2) of the hypothesis holds, then let $X = O$. In both cases, applying (Bld-iso') to $X_{\#}$ and $O_{\#}$, there is an isomorphism of geometries $\mu : \Gamma_X^- \to \Gamma_O^-$ such that, for every object U in Γ_X^- of the first kind, $\mu(U)$ is an object of Γ_O with $\mathrm{typ}_{\#}(\mu(U)) = \xi(\mathrm{typ}_{\#}(U))$. Then $\phi_\mu = \phi \circ \mu$ satisfies (Phi-str), therefore it satisfies the hypothesis of Theorem 3.1. $\square$

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