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Finite elation Laguerre planes admitting a two-transitive group on their set of generators

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Abstract

We investigate finite elation Laguerre planes admitting a group of automorphisms that is two-transitive on the set of generators. We exclude the sporadic cases of socles in two-transitive groups, as well as the alternating and Suzuki groups and the cases with abelian socle (except for the smallest ones, where the Laguerre planes are Miquelian of order at most four). The remaining cases are dealt with in a separate paper. We prove that a finite elation Laguerre plane is Miquelian if its automorphism group is two-transitive on the set of generators. Equivalently, each translation generalized quadrangle of order q with a group of automorphisms acting two-transitively on the set of lines through the base point is classical.

Keywords: Laguerre plane, elation group, translation generalized quadrangle, oval, generalized oval, pseudo-oval, two-transitive group, socle
MSC 2000: 51E25, 51B15, 20B20, 51E12

1 Introduction

A finite *Laguerre plane* $\mathcal{L}$ *of order* n is an orthogonal array of strength 3 on n symbols (levels), $n+1$ constraints and index 1, cf. [1], or equivalently, a transversal design $\mathrm{TD}_1(3, n+1, n)$. Since we have a more geometric point of view we rather use the term Laguerre plane instead of orthogonal array or transversal design, see Section 2 for an explicit definition.

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Models of finite Laguerre planes can be obtained as follows. Let $\mathcal{O}$ be an oval in the Desarguesian projective plane $\mathcal{P}_2 = \mathrm{PG}(2, q)$, for a prime power q. Embed $\mathcal{P}_2$ into 3-dimensional projective space $\mathcal{P}_3 = \mathrm{PG}(3, q)$ and let v be a point of $\mathcal{P}_3$ not belonging to $\mathcal{P}_2$. Then P consists of all points of the cone with base $\mathcal{O}$ and vertex v except the point v. Generators are the traces of lines of $\mathcal{P}_3$ through v that are contained in the cone. Circles are obtained by intersecting P with planes of $\mathcal{P}_3$ not passing through v. In this way one obtains an *ovoidal Laguerre plane of order* q. If the oval $\mathcal{O}$ one starts off with is a conic, one obtains the *Miquelian Laguerre plane of order* q. All known finite Laguerre planes of odd order are Miquelian and all known finite Laguerre planes of even order are ovoidal. In fact, it is a long standing problem whether or not these are the only finite Laguerre planes. (There are many non-ovoidal infinite Laguerre planes though.)

Some partial results in this direction were obtained by combining the classifications for finite projective planes of small orders and their ovals, see Section 2 for a description of the relation of Laguerre planes to projective planes and ovals. In this way it was shown that a Laguerre plane of order at most nine must be ovoidal, see Theorem 2.2 below. In [7] and [30] it was shown by a computer search that translation Laguerre planes and elation Laguerre planes of order 16 must be ovoidal.

Finite elation Laguerre planes were introduced in [35] and [26], see Section 2 for a description of the structure of finite elation Laguerre planes. They are characterized by the existence of a group of automorphisms that acts trivially on the set of generators and regularly on the set of circles. This group of automorphisms, which we call the *elation group* of the Laguerre plane, is unique and potentially plays a role analogous to the translation group of finite translation planes. In fact, elation Laguerre planes are linked to dual translation planes since such Laguerre planes can be described as dual translation planes with collections of certain ovals, see Section 2.

Every ovoidal Laguerre plane is an elation Laguerre plane, but there are infinite non-ovoidal elation Laguerre planes; see, for example, [25]. Hence elation Laguerre planes form a proper generalization of the notion of ovoidal Laguerre planes. In [14], elation Laguerre planes were further characterized as weakly Miquelian Laguerre planes, that is, those Laguerre planes in which a certain variation M2 of Miquel's configuration, which characterizes the Miquelian Laguerre planes, is satisfied. From this perspective, elation Laguerre planes are 'closest' to the Miquelian planes. Finally, a finite elation Laguerre plane of order q is also equivalent to a generalized oval (or pseudo-oval) with $q + 1$ points and thus to a translation generalized quadrangle of order q, i.e., with parameters (q, q). See [4], [12], [36], and Remark 2.7 below for a brief discussion of this relationship. All of this indicates that elation Laguerre planes form a nice subclass of Laguerre planes and that, if there are finite non-ovoidal Laguerre planes, elation Laguerre planes certainly are a natural class to look for them.

Doubly transitive groups of automorphisms have been investigated for various geometries, see for example [6], [23], [11], [9], [13], [8], [36, 8.5]. In this note we follow the program for finite translation planes to construct such planes from information about a suitable group in the translation complement of the collineation group and to classify all arising planes. The most homogeneous assumption one can make is that the automorphism group of the elation Laguerre plane is doubly transitive on the set of generators. Formulated in the language of translation generalized quadrangles [36, Theorem 8.5.1] shows that such an elation Laguerre plane of even order must be Miquelian, see also Theorem 3.2. Our main result is that if the automorphism group of the elation Laguerre plane is doubly transitive on the set of generators, then the Laguerre plane is Miquelian. Furthermore, the socles of the stabilizer of a circle are determined, see Main Theorem 3.10 and its Corollary 3.11 in terms of translation generalized quadrangles.

It should be noted that, on the one hand, there are infinite Laguerre planes that are not elation Laguerre planes but whose automorphism groups are doubly transitive on the set of generators, see [17]. On the other hand, there are infinite elation Laguerre planes whose automorphism groups are not doubly transitive on the set of generators, see for example [16]. However, the classification of topological, locally compact, 2- or 4-dimensional Laguerre planes admitting large groups of automorphisms (see [28, Section 5] and the references given there) shows that such an elation Laguerre plane is Miquelian if its automorphism group is doubly transitive on the set of generators.

2 Elation Laguerre planes

Explicitly, a finite Laguerre plane $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ of order n, $n \geq 2$, consists of a set P of $n(n+1)$ points, a set $\mathcal{C}$ of n^3 circles and a set $\mathcal{G}$ of $n+1$ generators (or parallel classes), where circles and generators are both subsets of P, such that the following three axioms are satisfied:

(G) $\mathcal{G}$ partitions P and each generator contains n points.

(C) Each circle intersects each generator in precisely one point.

(J) Three points no two of which are on the same generator can be uniquely joined by a circle.

The *internal incidence structure* $\mathbb{A}_x$ at any point x of a Laguerre plane has the collection of all points not on the generator through x as point set and, as lines, all circles passing through x (without the point x) and all generators not passing through x. From the definition of a Laguerre plane it readily follows that each internal incidence structure is an affine plane of order n, the *derived affine plane at* x. The projective completion $\mathbb{P}_x$ of $\mathbb{A}_x$ will be called the *derived projective plane at* x.

A circle C, not incident with the distinguished point x, induces an oval in $\mathbb{P}_x$: we delete the unique point incident with C and the generator $[x]$ through x and add the point ω at infinity that corresponds to the set of generators. Note that each oval arising in this way from circles of $\mathcal{L}$ passes through the common point ω and has the line at infinity of $\mathbb{A}_x$ as a tangent. Thus a Laguerre plane corresponds to a projective plane with sufficiently many of these ovals, pairwise intersecting in at most two affine points. This planar description of a Laguerre plane must then be extended by the points of one generator where one has to adjoin a new point to each line and to each oval of the affine plane, as above.

Using Segre's result [24] that every oval in a finite Desarguesian projective plane of odd order is a conic, the following characterization of finite Miquelian Laguerre planes was obtained in [5] or [21, VII.2].

Theorem 2.1. *A finite Laguerre plane of odd order with one Desarguesian derivation is Miquelian.* □

For small orders this and the results of [27] and [29] imply the following.

Theorem 2.2. *A Laguerre plane of order at most ten is ovoidal and, in fact, Miquelian except in case of order eight.* □

An automorphism of a Laguerre plane $\mathcal{L}$ is a permutation of the point set that maps circles onto circles and generators to generators. All automorphisms of $\mathcal{L}$ form a group with respect to composition, the automorphism group $\mathrm{Aut}(\mathcal{L})$ of $\mathcal{L}$. This group acts on the set $\mathcal{G}$ of generators; the kernel of that action is denoted by Δ. The collection of all automorphisms that fix each generator globally but fix no circle, together with the identity forms a normal subgroup E in $\mathrm{Aut}(\mathcal{L})$, see [26]. If $\mathcal{L}$ is an elation Laguerre plane then E has maximal order and is the elation group of $\mathcal{L}$, acting regularly on the set of circles. An element of E induces an elation with center ω in the derived projective plane at any of its fixed points. Indeed, one has the following, compare [25].

Theorem 2.3. *Let $\mathcal{L}$ be an elation Laguerre plane of finite order q.*

1. *Each derived projective plane of $\mathcal{L}$ is a dual translation plane; the translation center is the point ω at infinity of vertical lines.*

2. *The order q is a prime power.*

3. *If q is a prime then $\mathcal{L}$ is Miquelian.*

4. *Each oval induced by a circle of $\mathcal{L}$ in a derived projective plane passes through the translation center ω and has the line at infinity as a tangent.* □

For elation Laguerre planes of order 16 a computer search was conducted in [30] and the following result was obtained.

Theorem 2.4. *An elation Laguerre plane of order 16 is ovoidal.* □

Extending the usual representation of dual translation planes, a description of elation Laguerre planes in terms of a matrix-valued map was developed in [26, Theorem 3], see also [35]. Let $M(3m, m; \mathbb{F})$ denote the set of all $3m \times m$ matrices over $\mathbb{F}$, and let ∞ be any symbol not in $\mathbb{F}^m$.

Theorem 2.5. *Let $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ be a elation Laguerre plane of order $q = r^e$. There are a divisor m of e and a matrix-valued map $D : \mathbb{F}^m \cup \{\infty\} \to M(3m, m; \mathbb{F})$ where $\mathbb{F} := \mathbb{F}_{r^{e/m}}$, such that $\mathcal{L}$ can be represented in the following form.*

1. *The point set is $P = (\mathbb{F}^m \cup \{\infty\}) \times \mathbb{F}^m$,*

2. *the generators are the verticals $\{a\} \times \mathbb{F}^m$ of P for $a \in \mathbb{F}^m \cup \{\infty\}$,*

3. *the set of circles is $\mathcal{C} = \{K_c \mid c \in \mathbb{F}^{3m}\}$, where a circle K_c is described as $K_c = \{ (x, c \cdot D(x)) \in P \mid x \in \mathbb{F}^m \cup \{\infty\} \}$.*

4. *The elation group E consists of all maps $(x, y) \mapsto (x, y + c \cdot D(x))$, for $c \in \mathbb{F}^{3m}$.*

5. *For each $t \in \mathbb{F} \setminus \{0\}$ the map $\delta_t : (x, y) \mapsto (x, t \cdot y)$ belongs to Δ (i.e., is an automorphism that fixes each generator globally) and fixes the circle K_0.*

The special value $m = 1$ yields an ovoidal Laguerre plane. Also note (cf. [26, 3.5 a)]) that every elation Laguerre plane of order q has a representation as in Theorem 2.5 over the prime field $\mathbb{F}_r$, that is, for $m = e$.

Remark 2.6. For any point x on the circle K, the stabilizer Δ_K induces a group of homologies in the derived projective plane $\mathbb{P}_x$. By [18, Theorem 1.12] these homologies form a subgroup of the multiplicative group of the kernel of the translation plane obtained as dual of $\mathbb{P}_x$. This subgroup is the multiplicative group of a field $\mathbb{F}$ (embedded in $\mathbb{F}_q$), and we can represent the elation Laguerre plane over $\mathbb{F}$ as in the theorem above. In analogy to the situation in translation planes we call the largest field over which $\mathcal{L}$ can be represented as in Theorem 2.5 the *kernel of $\mathcal{L}$*. The group Δ_K is the multiplicative group of the kernel of $\mathcal{L}$ and thus cyclic; its order divides $q - 1$.

Remark 2.7. In the representation of an elation Laguerre plane over $\mathbb{F} = \mathbb{F}_r$ where $r = p^{e/m}$ (so that $q = r^m$) as in the theorem above the elation group E is a $3m$-dimensional vector space over $\mathbb{F}$, and the stabilizer E_x of each point x on K_0 is a $2m$-dimensional vector subspace of E. The geometric axioms of a Laguerre plane imply that the collection of $q+1$ vector subspaces of E is a dual pseudo-oval in the notation of [12] and [15].

Under a duality the E_x yield a family of $q+1$ vector subspaces of dimension m in $\mathbb{F}^{3m}$. Passing over to projective notation one sees that, geometrically, a finite elation Laguerre plane of order q as described above is equivalent to a $(q+1)$-set of $(m-1)$-dimensional subspaces in the $(3m-1)$-dimensional projective space $\mathrm{PG}(3m-1, r)$ over $\mathbb{F}$, compare [4], [35] and [26, Theorem 4]. In [36] such a set is called a generalized oval. More precisely, a *generalized oval* in $\mathrm{PG}(3m-1, r)$ is a collection of $r^m + 1$ projective subspaces π_i of dimension $m-1$ such that any three of the π_i generate the entire $\mathrm{PG}(3m-1, r)$. It follows that for each $i = 0, \dots, r^m$ there is a $(2m-1)$-dimensional projective subspace τ_i, the tangent space of the generalized oval at π_i, that contains π_i and is disjoint from any π_j where $j \neq i$.

One obtains a translation generalized quadrangle of order q from a generalized oval, and on the other hand, every translation generalized quadrangle of order q arises from a generalized oval in this way, see [36, Section 3.5] or [22, Section 8.7]. In fact, a generalized oval is just a Kantor system (or fourgonal family) in an abelian group, cf. [37, 4.9.2, 4.9.5].

Note that the Lie geometry of a Laguerre plane of *odd* order q yields a generalized quadrangle of order q, see [22, Theorem 2.4.2]. However, this construction does not work when q is even. In the case of an elation Laguerre plane, the generalized quadrangle obtained in this way even is a translation generalized quadrangle.

3 Doubly transitive groups

We consider a finite elation Laguerre plane $\mathcal{L}$. We assume that $\mathrm{Aut}(\mathcal{L})$ is doubly transitive on the set $\mathcal{G}$ of generators of $\mathcal{L}$. Since Δ acts trivially on $\mathcal{G}$, our assumptions imply that the stabilizer $\mathrm{Aut}(\mathcal{L})_K$ is doubly transitive on $\mathcal{G}$ or, equivalently, on the points of the fixed circle K.

In order to make our results more readily applicable, we will study a two-transitive subgroup Γ of $\mathrm{Aut}(\mathcal{L})_K$. The pointwise stabilizer (i.e., the intersection of Γ with Δ) will be denoted by $\Gamma_{[K]}$. Let $Q := \Gamma/\Gamma_{[K]}$ and $\pi\colon \Gamma \to Q$ be the natural homomorphism.

As the group $\mathrm{Aut}(\mathcal{L})$ is two-transitive on $\mathcal{G}$, it is also point-transitive and in fact transitive on the incident point-circle pairs. Hence all derived planes of the elation Laguerre plane $\mathcal{L}$ are isomorphic to each other and $\mathcal{L}$ can be reconstructed as a coset geometry from $\mathrm{Aut}(\mathcal{L})$ ([10], see also [33]).

We begin with the ovoidal case.

Theorem 3.1. *A finite ovoidal Laguerre plane whose automorphism group is doubly transitive on the set of generators is Miquelian.*

Proof. An ovoidal Laguerre plane $\mathcal{L}$ of order q is embedded in 3-dimensional projective space $\mathrm{PG}(3, q)$. Thus each derived affine plane is Desarguesian. If $\mathcal{L}$ has odd order then $\mathcal{L}$ is Miquelian by Theorem 2.1. If q is even then the transitivity assumption on $\mathcal{L}$ implies that the oval $\mathcal{O}$ in $\mathrm{PG}(2, 2^h)$, which forms the base of the cone that is the point set of $\mathcal{L}$, has a collineation group which is doubly transitive on $\mathcal{O}$. The two-transitive ovals are known to be conics, see [19, Theorem 1.3 and the remark following it]. But then $\mathcal{L}$ is again Miquelian. □

In the even order case the above Theorem has been generalized in [36, Theorem 8.5.1] to generalized ovals (and thus to finite elation Laguerre planes) without the use of the classification of finite simple groups. We reformulate [36, Theorem 8.5.1] in the language used here:

Theorem 3.2. *A finite elation Laguerre plane of even order whose automorphism group is doubly transitive on the set of generators is Miquelian.*

Proof. As outlined in Remark 2.7 a finite elation Laguerre plane $\mathcal{L}$ of order q is equivalent to a generalized oval $\mathcal{O}$ in $\mathrm{PG}(3m-1, r)$ for some subfield $\mathbb{F}_r$ of $\mathbb{F}_q$ where $q = r^m$. If the automorphism group of $\mathcal{L}$ is doubly transitive on the set of generators, then $\mathcal{O}$ is a two-transitive generalized oval in the notation of [36]. By [36, Theorem 8.5.1] the translation generalized quadrangle $\mathrm{T}(\mathcal{O})$ which arises from a two-transitive generalized oval $\mathcal{O}$ is isomorphic to the classical generalized quadrangle $\mathrm{Q}(4, q)$ whose points and lines are the points and lines of a nonsingular quadric of projective index 1 in $\mathrm{PG}(4, q)$. Hence $\mathcal{O}$ is classical. But then the elation Laguerre plane is Miquelian. □

The authors of [36] have informed us that the following argument should be added in order to complete the proof of their Theorem 8.5.1. The set-up in that proof (we are using the notation of [36, Theorem 8.5.1] here; their q corresponds to our r and n is our m) is that α is an involution which leaves $\mathcal{O}$ invariant and fixes precisely one element $\pi \in \mathcal{O}$; such an α is considered as an element of $\mathrm{P\Gamma L}(3n, q)$. The following lines are to show that α is induced by a linear bijection of $\mathbb{F}_q^{3n}$.

Assume otherwise, then α induces a Baer involution on the $\mathrm{PG}(3n-1,q)$ containing the generalized oval $\mathcal{O}$. Let η be the nucleus of $\mathcal{O}$. The fixed points of α in $\mathrm{PG}(3n-1,q)$ are not all contained in the subspace $\langle \pi, \eta \rangle$, so there is a second space $\langle \pi', \eta \rangle$ fixed by α with $\pi' \in \mathcal{O}$. Hence π and π' are fixed by α. This contradiction leaves only the possibility that α is induced by a linear bijection.

Abelian socles

A first step towards the understanding of the double transitive groups is already due to Burnside [2, p. 202]; compare also [3, Prop. 5.2]:

Theorem 3.3. *If Ψ is a finite doubly transitive and effective group on v points, then Ψ contains a transitive normal subgroup Σ (the* socle *of G) and either Σ is elementary abelian or Σ is a non-abelian simple group.* □

For the case of an elementary abelian socle the following straightforward result (see [20, Lemma 19.3]) will be helpful.

Lemma 3.4. *Let r, s be primes and e, f be positive integers such that $r^e + 1 = s^f$. Then one of the following holds:*

1. $s = 2$, $e = 1$ *(r is a Mersenne prime);*
2. $r = 2$, $f = 1$ *(s is a Fermat prime);*
3. $r = 2$, $e = 3$, $s = 3$, $f = 2$.

Proposition 3.5. *Let $\mathcal{L}$ be a finite elation Laguerre plane admitting a group Γ of automorphisms fixing a circle K and two-transitive on K. If the socle of $Q := \Gamma/\Gamma_{[K]}$ is abelian then $\mathcal{L}$ is a Miquelian Laguerre plane of order $q \in \{2, 3, 4\}$.*

Proof. Let r^e be the order of $\mathcal{L}$. The socle has order s^f for some prime s and acts regular on $\mathcal{G}$, see Theorem 3.3. Thus we have $r^e + 1 = |\mathcal{G}| = s^f$, and Lemma 3.4 leaves three cases to consider.

Case 1: $s = 2$, $e = 1$. Then $\mathcal{L}$ has prime order $q = r$ and is Miquelian by Theorem 2.3. Any Sylow 2-subgroup S of $\mathrm{P\Gamma L}(2, r) = \mathrm{PGL}(2, r)$ has order $2(r + 1) = 2^{f+1}$, and contains a cyclic subgroup of order 2^f (from the multiplicative group of a field of order r^2 contained in the matrix ring $M(2, 2; \mathbb{F}_r)$). The elementary abelian socle Σ of Q then meets that cyclic group in a cyclic subgroup of order at least 2^{f-1}. This yields $f \le 2$, and $q = r = 3$ follows.

We note that $\mathrm{P\Gamma L}(2, 3) = \mathrm{PGL}(2, 3) \cong \mathrm{S}_4$ contains a unique candidate for Q, namely the alternating group $\mathrm{A}_4 \cong \mathrm{AGL}(1, 4)$.

Case 2: $r = 2$, $f = 1$. In this case $\mathcal{L}$ has even order and is Miquelian by Theorem 3.2.

The socle has prime order $s = 2^e + 1$ and thus is cyclic. Hence Q is contained in the affine group $\mathrm{A\Gamma L}(1, q) = \mathrm{AGL}(1, q)$ and then actually coincides with $\mathrm{AGL}(1, q)$ because it has the same order (being two-transitive). In particular, the stabilizer of a point in K is cyclic, and is a Sylow 2-subgroup of Q.

Let G be a Sylow 2-subgroup of Γ. The order of Δ_K divides $2^e - 1$, see Remark 2.6. Thus it is odd, and G has trivial intersection with Δ_K. Thus $G \cong \pi(G)$ is cyclic of order $q - 1 = 2^e$. Furthermore, G is faithfully and linearly represented on $E_\infty \cong \mathbb{F}_2^{2e}$, that is, we have an embedding $\rho\colon G \to \mathrm{GL}(2e, 2)$. Since a generator g of G has order 2^e and because $x \to x^2$ is injective in any field of characteristic 2, we see that 1 is the only eigenvalue of $\rho(g)$ and that $\rho(g)$ is unipotent in $\mathrm{GL}(2e, 2)$. Hence $(\rho(g) + I)^{2e} = 0$. If $e \geq 4$ one sees by induction that $2e \leq 2^{e-1}$ so that $\rho(g)^{2^{e-1}} = I$, contradicting the fact that the order of g is 2^e. Hence $e \leq 3$ and $q \in \{2, 4, 8\}$. As $8 + 1$ is not a prime, only the cases $q \in \{2, 4\}$ remain. (Note that Theorem 2.2 then again implies that $\mathcal{L}$ is Miquelian.)

We identify the group Q inside the group $\mathrm{P\Gamma L}(2, q)$ induced by the stabilizer of the circle in the Miquelian plane, as follows.

- For $q = 2$ we have $\mathrm{P\Gamma L}(2, 2) = \mathrm{PGL}(2, 2) \cong \mathrm{S}_3$, and $Q = \mathrm{PGL}(2, 2)$ follows.
- For $q = 4$ we have $\mathrm{P\Gamma L}(2, 4) \cong \mathrm{S}_5$, and Q is the normalizer of a Sylow 5-subgroup.

  Note that Q is substantially different from the other minimally two-transitive subgroup of S_5, namely the simple group A_5. One may interpret the group Q as the smallest (and non-simple) example $\mathrm{Sz}(2)$ of a Suzuki group.

Case 3: $r^3 = 8$, $s^f = 9$. Then $\mathcal{L}$ has order 8, is ovoidal by Theorem 2.2, and Miquelian by Theorem 3.1 or Theorem 3.2. In this case, the Sylow 3-subgroup Σ of Q is elementary abelian of order 9, and the Sylow 2-subgroups of Q induce subgroups of order 8 in $\mathrm{Aut}(\Sigma) \cong \mathrm{GL}(2, 3)$. One of the candidates is the multiplicative group of a field of order 9, this is cyclic and intersects each other group of order 8 in $\mathrm{GL}(2, 3)$ in a cyclic subgroup of order 4 at least because the Sylow 2-subgroups of $\mathrm{GL}(2, 3)$ have order 2^4.

The group $\mathrm{P\Gamma L}(2, 8)$ induced by the stabilizer of a circle in the Miquelian plane of order 8 has elementary abelian Sylow 2-subgroups, and contains no elements of order 4. Thus the present case is indeed impossible. □

The simple non-abelian case

The two-transitive groups with non-abelian socle are also known explicitly (thanks to the classification of finite simple groups). The list given in Table 1 can be found[1], for instance, in [3] or [13].

Σ	v	remarks/restrictions
A_n	n	$n \geq 6$, (two representations if $n = 6$)
$\mathrm{PSL}(d, f)$	$(f^d - 1)/(f - 1)$	$d \geq 2$, $(d, f) \notin \{(2, 2), (2, 3)\}$ (two representations if $d > 2$)
$\mathrm{PSU}(3, f^2)$	$f^3 + 1$	$f > 2$
$\mathrm{Sz}(2^{2a+1})$	$2^{4a+2} + 1$	$a > 0$, Suzuki groups: ${}^2\mathrm{B}_2(2^{2a+1})$
$\mathrm{R}(3^{2a+1})$	$3^{6a+3} + 1$	$a > 0$, Ree groups: ${}^2\mathrm{G}_2(3^{2a+1})$
$\mathrm{PSp}(2d, 2)$	$2^{2d-1} \pm 2^{d-1}$	$d > 2$
$\mathrm{PSL}(2, 11)$	11	(two representations)
A_7	15	(two representations)
$\mathrm{PSL}(2, 8)$	28	socle of $\mathrm{R}(3)$
M_n	n	Mathieu groups, $n \in \{11, 12, 22, 23, 24\}$ (two representations if $n = 12$)
M_{11}	12	Mathieu group
Co_3	276	Conway group
HS	176	Higman-Sims group (two representations)

Table 1: Non-abelian socles: all possibilities

Theorem 3.6. *Let Q be a finite group that acts two-transitively and faithfully on a set with v points. If the socle Σ is not abelian then Σ is one of the groups listed in Table 1.* □

Note that the value of f in the different cases in Table 1 will always be a prime power. The group Q will be contained in the automorphism group of its socle.

There are some isomorphisms between these groups. For example, one has $\mathrm{PSL}(2,4) \cong \mathrm{A}_5 \cong \mathrm{PSL}(2,5)$, $\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$, $\mathrm{PSL}(2,9) \cong \mathrm{A}_6$, and $\mathrm{PSL}(4,2) \cong \mathrm{A}_8$, which shows that a group may have two non-equivalent two-transitive permutation representations. The second permutation representation of A_6 arises from $\mathrm{PSp}(4,2) \cong \mathrm{S}_6$, cf. [34] or [32]. We have left out the natural action of A_5 in the first row because that action is equivalent to the action of $\mathrm{PSL}(2,4)$ occurring in the second (and we will treat the action of $\mathrm{PSL}(2,2^e)$ systematically in [31]).

[1] We have modified the names for the parameters to avoid confusion with our fixed meaning for q. Also, we use the order f^2 of the quadratic extension field for the unitary groups.

Proposition 3.7. *Let $\mathcal{L}$ be a finite elation Laguerre plane of order q admitting a group Γ of automorphisms fixing a circle K and acting two-transitively on K. Assume that $Q := \Gamma/\Gamma_{[K]}$ has non-abelian socle Σ. Then Σ is one of the groups listed in Table 2 (and q is a prime power).*

Σ	q	restrictions	reference
A_{q+1}	q	prime power $q \geq 4$	Proposition 3.8
$\mathrm{PSL}(2,q)$	q	$q \neq 2,3,\ d = \gcd(2,q-1)$	[31]
$\mathrm{PSU}(3,f^2)$	f^3	$f > 2,\ d = \gcd(3,f+1)$	[31]
$\mathrm{R}(3^{2a+1})$	3^{6a+3}	$a > 0$	[31]
$\mathrm{PSL}(2,8)$	3^3	socle of $\mathrm{R}(3)$	[31]
$\mathrm{PSp}(6,2)$	3^3		Proposition 3.9

Table 2: Non-abelian socles: the restricted list

Proof. If Q arises from a circle stabilizer in the automorphism group of an elation Laguerre plane $\mathcal{L}$ then $q = v-1$ must be a prime power r^e by Theorem 2.3. If $v-1$ is actually a prime r then $\mathcal{L}$ is the Miquelian plane of order r, and the socle Σ occurs as a subgroup of $\mathrm{PSL}(2,r)$. These two observations exclude all the Mathieu groups (M_n for $n \in \{11,12,22,23,24\}$), the Conway group Co_3 and the Higman-Sims group HS.

Of the other groups not occurring in families in Table 1 only the socle $\mathrm{PSL}(2,8)$ of $\mathrm{R}(3)$ acts on a set of the right size, that is, of the form 1 plus a proper prime power.

Finally, we discuss the groups occurring in families.

- $\mathrm{PSL}(d,f)$: If $(f^d-1)/(f-1) = f^{d-1}+\cdots+f+1 = f(f^{d-2}+\cdots+1)+1$ is of the form 1 plus a prime power, then the prime involved must be the same as in f and the sum in the parentheses $f^{d-2}+\cdots+1$ must be 1. Hence $d=2$.
- $\mathrm{PSp}(2d,2)$: Let $\varepsilon = \pm 1$. Then $r^e = 2^{2d-1}+\varepsilon 2^{d-1}-1 = (2^d-\varepsilon)(2^{d-1}+\varepsilon)$ and r is odd. Furthermore, because $d > 2$, we find that $2^d - \varepsilon \geq 2^d - 1 > 1$ and $2^{d-1}+\varepsilon \geq 2^{d-1}-1 > 1$. Hence $2^d - \varepsilon = r^l$ and $2^{d-1}+\varepsilon = r^k$ for some $1 \leq k \leq l < m$. But then r^k divides $p^l + p^k = 2^{d-1}\cdot 3$ so that $r=3$ and $k=1$. Thus $2^{d-1}+\varepsilon = 3$ which implies $\varepsilon = -1$ and $d=3$. Therefore only $\mathrm{PSp}(6,2)$ acting on a set of size $2^5-2^2 = 28 = 3^3+1$ can possibly occur.
- $\mathrm{Sz}(2^{2a+1})$: In this case $\mathcal{L}$ has even order and thus is Miquelian by Theorem 3.2. Since Sylow 2-subgroups of $\mathrm{Sz}(2^{2a+1})$ are non-abelian when $a > 1$ (see, for example, [18, Theorem 24.2]) whereas those of $\mathrm{PSL}(2,q)$,

the socle of a circle stabilizer in the automorphism group of the Miquelian Laguerre plane or order q, are, one sees that this case cannot occur. □

In the present paper, we are going to eliminate the groups A_n (for $n > 5$, see Proposition 3.8) and $\mathrm{PSp}(6, 2)$ (see Proposition 3.9) from Table 2. The remaining cases ($\mathrm{PSL}(2, q)$, unitary groups, Ree groups, and the socle $\mathrm{PSL}(2, 8)$ of the smallest Ree group $\mathrm{R}(3)$) are treated in a separate paper [31]. There we show that only the groups $\mathrm{PSL}(2, q)$ actually occur, and the corresponding Laguerre planes are Miquelian.

Proposition 3.8. *Let $\mathcal{L}$ be a finite elation Laguerre plane of order $q > 4$ admitting a group Γ of automorphisms fixing some circle K and two-transitive on K. Then the socle of $Q := \Gamma/\Gamma_{[K]}$ is not isomorphic to A_{q+1}.*

Proof. Aiming at a contradiction, we assume that the socle Σ of Q is isomorphic to A_{q+1}. Let $\pi\colon \Gamma \to Q$ be the restriction map, and let $G := \pi^{-1}(\Sigma) \le \Gamma$ denote the full pre-image of the socle. The order of that pre-image is then $|G| = \frac{(q+1)!}{2}\, d$ where $d = |\Gamma_{[K]}|$ divides $q - 1$, see Remark 2.6. In particular, $1 \le d \le q - 1$.

For $u \in K$ we consider the stabilizer G_u in G. Then $\pi(G_u) \le \Sigma$ is the stabilizer of u in Σ. Thus $\pi(G_u) \cong A_q$ is simple, and $|G_u| = \frac{q!}{2}\, d$. The group G_u acts on the set $[u] \smallsetminus \{u\}$ of $q - 1$ points. The kernel N_u of that action is not trivial because $(q - 1)! < |G_u|$.

As Δ_K acts semi-regularly outside K, the intersection $\Delta_K \cap N_u$ is trivial, and $\pi(N_u) \cong N_u$ is a normal subgroup of the simple group $\Sigma_u \cong \mathrm{A}_q$. We have just noted that N_u is not trivial; so $\pi(N_u) = \Sigma_u$ has order $\frac{q!}{2}$.

Now pick three different points $x, y, z \in K$. Then

$$|N_x \cap N_y| = \frac{|N_x| \cdot |N_y|}{|N_x N_y|} \ge \frac{|N_x|^2}{|G|} = \frac{q!}{2d(q+1)} \qquad \text{and}$$

$$|N_x \cap N_y \cap N_z| = \frac{|N_x \cap N_y| \cdot |N_z|}{|(N_x \cap N_y) N_z|} \ge \frac{q!}{2(d(q+1))^2} \ge \frac{q!}{2(q+1)^2(q-1)^2}\,;$$

we have used $d \le q - 1$ for the last inequality. The sequence whose n-th term is $\frac{n!}{2(n+1)^2(n-1)^2}$ is strictly increasing for $n \ge 5$. Therefore, $|N_x \cap N_y \cap N_z| \ge \frac{7!}{2 \cdot 8^2 \cdot 6^2} = \frac{35}{32} > 1$ for $q \ge 7$. This shows that $N_x \cap N_y \cap N_z$ is non-trivial. However, any automorphism in $N_x \cap N_y \cap N_z$ fixes every circle of $\mathcal{L}$ and thus must be the identity—a contradiction.

The case $q = 6$ cannot occur because it is not a prime power.

It remains to discuss $q = 5$; then $\mathcal{L}$ is Miquelian, see Theorem 2.2. In the automorphism group of the Miquelian Laguerre plane of order 5, the stabilizer of a circle induces a group isomorphic to $\mathrm{P\Gamma L}(2, 5) = \mathrm{PGL}(2, 5)$ on K, acting two-transitively with a socle isomorphic to $\mathrm{PSL}(2,5) \cong \mathrm{A}_5$. However, we are considering a two-transitive group Q with socle $\mathrm{A}_{q+1} = \mathrm{A}_6$, and arrive at a contradiction because $|\mathrm{A}_6| = 720 > 120 = |\mathrm{P\Gamma L}(2,5)|$. □

Proposition 3.9. *Let $\mathcal{L}$ be a finite elation Laguerre plane of order 27, and let Γ be a group of automorphisms fixing some circle K. Then $Q := \Gamma/\Gamma_{[K]}$ does not contain a normal subgroup isomorphic to $\mathrm{PSp}(6,2)$ that is two-transitive on K.*

Proof. Assume, to the contrary, that there is such a subgroup $\Sigma \cong \mathrm{PSp}(6,2)$ in Q. The order $2^9 \cdot 3^4 \cdot 5 \cdot 7 = 1451520$ of $\mathrm{PSp}(6,2)$ is divisible by 5 but $|\Delta_K|$ is not because it divides $27-1$. However, by Theorem 2.5.5 we have an involution $\delta \in \Delta_K$; clearly this involution is centralized by Γ.

We consider a Sylow 5-subgroup S of the pre-image $G := \pi^{-1}(\Sigma) \le \Gamma$. This group has order 5, and has orbits of lengths 1 or 5. Therefore S fixes at least 3 points on K, and for each fixed point $u \in K$ there is at least one more fixed point $v \in [u] \setminus \{u\}$, where $[u]$ denotes the generator containing u. The central involution δ acts semi-regularly on $[u] \setminus \{u\}$, so $\delta(v)$ is another fixed point of S. But then S fixes at least 7 points in $[u]$.

Let x, y, z be three distinct fixed points of S in K. If $x' \in [x]$ and $y' \in [y]$ are fixed by S then the circle $L(x', y', z)$ through x', y' and z is fixed by S. This gives 7^2 fixed circles through z.

Now let $w \in K \setminus \{x, y, z\}$ be arbitrary. Since $[w]$ has 27 points, there exists at least one point $w' \in [w]$ lying on two of the fixed circles. These two circles have the two points w' and z in common. Thus S fixes w', the generator $[w'] = [w]$, and then also the point w in $[w] \cap K$. We obtain that $S \le \Delta_K$, contradicting the fact that $|\Delta_K|$ is not divisible by 5. □

Main Theorem 3.10. *If the automorphism group of an elation Laguerre plane of order q contains a subgroup Γ fixing a circle and acting two-transitively on that circle, then the Laguerre plane is Miquelian.*

The socle of the group induced on the fixed circle is either isomorphic to $\mathrm{PSL}(2,q)$, or we have $q = 4$ and the socle is isomorphic to $\mathrm{AGL}(1,5)$.

Proof. The socle is either abelian, or a simple group contained in the list shown in Table 1. The abelian case has been discussed in Proposition 3.5. By the arguments given in Proposition 3.7 the list in Table 1 has been reduced to the list in Table 2. Propositions 3.8 and 3.9 have excluded the groups A_{q+1} (for $q > 4$) and $\mathrm{PSp}(6,2)$.

There remain the series $\mathrm{PSL}(2,q)$, $\mathrm{PSU}(3,f^2)$, and $\mathrm{R}(3^{2a+1})$ for prime powers q, f and positive integers a, respectively (i.e., the groups of Lie type $\mathrm{A}_1(q)$,

${}^2\mathrm{A}_2(f^2)$, and ${}^2\mathrm{G}_2(3^{2a+1})$), and the commutator group $\mathrm{R}(3)' \cong \mathrm{PSL}(2,8)$ with its transitive action on 28 points; these are all treated in [31]. □

Each translation generalized quadrangle of order q arises from a generalized oval $\mathcal{O}$ in $\mathrm{PG}(3n-1,r)$, where $q = r^n$ (see [21, 8.7.1]). Under the correspondence between translation generalized quadrangles of order q and elation Laguerre planes of order q, the action on the set of lines through the base point corresponds to the action on the set of generators of the Laguerre plane. Using that correspondence immediately gives us the following

Corollary 3.11. *If the automorphism group of a translation generalized quadrangle of order q is two-transitive on the set of lines through the base point then the quadrangle is isomorphic to the classical generalized quadrangle $\mathrm{Q}(4,q)$.*

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