Innovations in Incidence Geometry
Volume 14 (2015), Pages 27–42
ISSN 1781-6475

On the autotopism group of the commutative Dickson semifield $\mathcal{K}$ and the stabilizer of the Ganley unital embedded in the semifield plane $\Pi(\mathcal{K})$

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Abstract

Two methods, one algebraic and one geometric, are given to determine the autotopism group of the commutative Dickson semifield, completing a result of Sandler. The collineation stabilizer subgroup of the embedded Ganley unital is exhibited as a semidirect product whose factor groups are semidirect products of abelian groups.

Keywords: Dickson semifield, semifield plane, autotopism group, stabilizer of the embedded Ganley unital, automorphism group of unitary block design, inversive plane
MSC 2010: 12K10, 20B25, 51A35, 51E05, 51B10

1 Introduction

In the study of a unital of order n, that is, a 2-$(n^3+1, n+1, 1)$ design, for n an integer greater than 2, a basic problem is the determination of its automorphism group. If the unital is embedded in a projective plane of order n^2, one begins by studying the collineation subgroup of the plane stabilizing the unital. For example, if U is the classical unital of order q defined by a hermitian curve in the classical plane $\mathrm{PG}(2, q^2)$, then its stabilizer, $\mathrm{Col}(U)$, is given by $\mathrm{P\Gamma U}(3, q^2)$, a result from classical group theory (see [5]). The question is then whether

*This work was partially supported by a grant from the Research Grant Council of the HKSAR, China (Project number: HKU7046/13P)

the stabilizer, which is a subgroup of the automorphism group of the design, is actually the full group. For the classical unital, the question is answered affirmatively in 1972 in the fundamental paper of O'Nan [15], in which $\mathrm{Col}(U)$ is proven to be equal to the automorphism group $\mathrm{Aut}(U)$.

In the case of the Ganley unitals [9], which are polar unitals in the commutative Dickson semifield planes ([7], see also [14]), the same conclusion has been reached in [12, 13]. However, there remains the problem of determination of the collineation groups of the planes and the subgroups stabilizing the unitals. This is Problem 21 in the list of open problems presented in [3]. Partial results on the collineation groups have been obtained earlier in 1962 by Sandler [17]. In this paper we give a complete solution to the problem.

Let $q = p^e$ be a prime power where p is an odd prime and e an integer greater than 1. For any non-square δ in the finite field $\mathbb{F}_q$ and non-identity automorphism $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$, let $\mathcal{K} = \mathcal{K}(\delta, \sigma)$ be the (commutative) Dickson semifield as defined in [17]. The autotopism group A of $\mathcal{K}$ is computed in [17] only for the case when $\sigma^2 \neq \mathrm{id}$, and is shown to be solvable.

Now let $\Pi(\mathcal{K})$ be the projective plane coordinatized by $\mathcal{K}$. It is shown in [2] that the collineation group $\mathrm{Col}(\Pi(\mathcal{K}))$ is given by $\mathcal{A} \ltimes (\mathcal{S} \ltimes \mathcal{T})$, where $\mathcal{T}$ is the group of translations, $\mathcal{S}$ is the group of shears and $\mathcal{A}$ is the group of autotopisms of $\Pi(\mathcal{K})$ (see also [9]). Since $\mathcal{T} \cong \mathcal{K}(+) \times \mathcal{K}(+)$ and $\mathcal{S} \cong \mathcal{K}(+)$, and the two autotopism groups A and $\mathcal{A}$ are isomorphic, the determination of $\mathrm{Col}(\Pi(\mathcal{K}))$ is complete with the determination of either A or $\mathcal{A}$.

The aim of this paper is to determine the autotopism group A of $\mathcal{K}$ when $\sigma^2 = \mathrm{id}$, and to complete the computation of the collineation stabilizer subgroup $\mathrm{Col}(\mathcal{U})$ of the Ganley unital $\mathcal{U}$ in the Dickson semifield plane $\Pi(\mathcal{K})$ begun in Ganley [9].

The plan of the paper is as follows: In Section 2, we review standard facts on semifields, the (commutative) Dickson semifield $\mathcal{K}$, their autotopism groups, the projective planes they coordinatize, and the collineation groups of these planes. In Section 3, we determine the autotopism group A of $\mathcal{K}$ for the case $\sigma^2 = \mathrm{id}$ (Theorem 3.1) using algebraic computations similar to those employed in Sandler [17]. In Section 4, we recall the definition of the Ganley unital $\mathcal{U}$ embedded in the Dickson semifield plane $\Pi(\mathcal{K})$, and exhibit the structure of $\mathrm{Col}(\mathcal{U})$ as a semidirect product whose factor groups are semidirect products of abelian groups (Theorems 4.1, 4.2). In Section 5 we describe an alternative geometric approach to the study of A and $\mathrm{Col}(\mathcal{U})$. This requires the methodology and results from [12] to determine $\mathrm{Col}(\mathcal{U})$ via inversive plane geometry, and a key structure theorem from [13] to retrieve the autotopism group $\mathcal{A}$ of $\Pi(\mathcal{K})$ from $\mathrm{Col}(\mathcal{U})$.

2 The Dickson semifield and the Dickson semifield plane

2.1 Semifield and semifield plane

A *finite semifield* $K(+,\cdot)$ is a finite algebraic system with two binary operations, addition and multiplication, which satisfy the following axioms:
1. $K(+)$ is a group with identity 0.
2. For any $a, b \in K$, if $a \cdot b = 0$, then $a = 0$ or $b = 0$.
3. There is an element $1 \in K$ such that for any $a \in K$, $1 \cdot a = a \cdot 1 = a$.
4. For any $a, b, c \in K$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

In this paper we refer to a finite semifield simply as a semifield and we usually write ab for $a \cdot b$. In [14], it is shown that K is a vector space over the *middle nucleus* N of K, where N is the set $\{a \in K \mid (xa)y = x(ay) \text{ for all } x, y \in K\}$ and is a field. Thus the cardinality of a semifield is a prime power. Two semifields $K_1(+_1, \cdot_1)$ and $K_2(+_2, \cdot_2)$ are said to be *isotopic* if there exists an ordered triple (P, Q, R) of additive bijections from K_1 to K_2 such that $P(x) \cdot_2 Q(y) = R(x \cdot_1 y)$ for all $x, y \in K_1$. The triple (P, Q, R) is called an *isotopism* from K_1 to K_2. An isotopism (P, Q, R) from a semifield K to itself is called an *autotopism*. The set of autotopisms of a semifield forms a group A under the operation $(P_1, Q_1, R_1) \circ (P_2, Q_2, R_2) = (P_1 \circ P_2, Q_1 \circ Q_2, R_1 \circ R_2)$. It is a conjecture of Hughes [11] that the autotopism group of any semifield is solvable.

A semifield K can be used to coordinatize a projective plane $\Pi(K)$. The set of points of $\Pi(K)$ is the set $K \times K$ together with the set $\{(m) \mid m \in K\}$ and a point denoted by (∞) where ∞ is a symbol not in K. The set of lines of $\Pi(K)$ is given by $\{[m, k] \mid m, k \in K\} \cup \{[x] \mid x \in K\} \cup \{[\infty]\}$, where $[m, k] = \{(m)\} \cup \{(x, y) \in K \times K \mid mx + y = k\}$, $[x] = \{(\infty)\} \cup \{(x, y) \mid y \in K\}$, and $[\infty] = \{(\infty)\} \cup \{(m) \mid m \in K\}$. We note that $\Pi(K)$ is a projective plane of order the cardinality of K. Points not on $[\infty]$ are *affine points*. The geometric significance of isotopism is the following result of Albert ([2, Theorem 6]): Two semifields coordinatize isomorphic planes if and only if they are isotopic.

The collineation group $\mathrm{Col}(\Pi(K))$ of a semifield plane $\Pi(K)$ is well-known (see [2]) to consist of translations, shears and autotopisms defined as follows. Let $\mathcal{T} = \{\tau(a, b) \mid a, b \in K\}$ be the group of *translations* given by

$$\tau(a,b)\colon \begin{cases} (x,y) \longmapsto (x+a, y+b), & [m,k] \longmapsto [m, k+ma+b], \\ (m) \longmapsto (m), & [x] \longmapsto [x+a], \\ (\infty) \longmapsto (\infty), & [\infty] \longmapsto [\infty]. \end{cases} \tag{1}$$

Let $\mathcal{S} = \{\varsigma(c) \mid c \in K\}$ be the group of *shears* given by

$$\varsigma(c)\colon \begin{cases} (x,y) \longmapsto (x, -cx+y), & [m,k] \longmapsto [m+c,k], \\ (m) \longmapsto (m+c), & [x] \longmapsto [x], \\ (\infty) \longmapsto (\infty), & [\infty] \longmapsto [\infty]. \end{cases} \tag{2}$$

Note that $\mathcal{T} \cong K(+) \times K(+)$ and $\mathcal{S} \cong K(+)$. Let $\mathcal{A}$ be the group of collineations fixing (∞), (0) and $(0,0)$. By [14, Theorem 3.3.3],

$$\mathcal{A} = \{\gamma(P,Q,R) \mid (P,Q,R) \in A\},$$

where $\gamma(P,Q,R)$ is defined by

$$\gamma(P,Q,R)\colon \begin{cases} (x,y) \longmapsto (Q(x), R(y)), & [m,k] \longmapsto [P(m), R(k)], \\ (m) \longmapsto (P(m)), & [x] \longmapsto [Q(x)], \\ (\infty) \longmapsto (\infty), & [\infty] \longmapsto [\infty]. \end{cases} \tag{3}$$

The set $\mathcal{A}$ is a group under composition of maps whose elements are also called *autotopisms*. Indeed, the autotopism group A of K is isomorphic to $\mathcal{A}$ via the isomorphism sending $(P,Q,R) \in A$ to $\gamma(P,Q,R) \in \mathcal{A}$. We call $\mathcal{A}$ the *autotopisms group* of $\Pi(K)$. By [2, Theorems 5 and 7] (see also [9]), $\mathrm{Col}(\Pi(K)) = \mathcal{A} \ltimes (\mathcal{S} \ltimes \mathcal{T})$.

A note on notations: For the composition $f \circ g$ of two maps we will write fg when there is no danger of confusion.

2.2 The Dickson semifield and the Dickson semifield plane

We now focus our attention on the (commutative) Dickson semifields [6, 7]. Consider the finite field $\mathbb{F}_q$, where $q = p^e$ for an odd prime p, and $e > 1$. Then there is a non-square element δ in $\mathbb{F}_q$ and a non-identity automorphism σ of $\mathbb{F}_q$. Note that $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$ is given by

$$u^\sigma = u^{p^s}, \quad 0 < s < e. \tag{4}$$

Define multiplication in the two-dimensional vector space of column vectors over $\mathbb{F}_q$ by

$$\begin{pmatrix} x' \\ x'' \end{pmatrix} \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} x'y' + \delta x''^\sigma y''^\sigma \\ x'y'' + x''y' \end{pmatrix}. \tag{5}$$

The resulting algebraic system is a commutative semifield denoted by $\mathcal{K}(\sigma, \delta)$ and is called a *Dickson semifield* of order q^2. Note that if $\sigma = \mathrm{id}$ the system is isomorphic to $\mathbb{F}_{q^2}$. For any $\sigma_1, \sigma_2 \in \mathrm{Aut}(\mathbb{F}_q) \setminus \{\mathrm{id}\}$ and non-squares $\delta_1, \delta_2 \in \mathbb{F}_q$, $\mathcal{K}(\sigma_1, \delta_1)$ and $\mathcal{K}(\sigma_2, \delta_2)$ are isomorphic if and only if $\sigma_1 = \sigma_2$, and are isotopic if and only if $\sigma_1 = \sigma_2$ or $\sigma_1 = {\sigma_2}^{-1}$ ([17, Theorem 2]; see also [4]). Since $\mathcal{K}(\sigma, \delta_1)$ and $\mathcal{K}(\sigma, \delta_2)$ are isomorphic, without loss of generality we may take δ to be a generator of $\mathbb{F}_q^*$, and write $\mathcal{K}(\sigma)$ instead of $\mathcal{K}(\sigma, \delta)$.

In case $\sigma^2 \neq \mathrm{id}$, Sandler ([17, Theorem 3]) computed generators for the autotopism group $A(\sigma)$ of $\mathcal{K}(\sigma)$ and show that the group is solvable. We recall his results. Let (P, Q, R) be an autotopism of $\mathcal{K}(\sigma)$. By [17, Lemma on p. 190], P and Q take the forms:

$$\begin{aligned} P\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} &\longmapsto M(P) \begin{pmatrix} x'^{\varepsilon_P} \\ x''^{\varepsilon_P} \end{pmatrix}, \\ Q\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} &\longmapsto M(Q) \begin{pmatrix} x'^{\varepsilon_Q} \\ x''^{\varepsilon_Q} \end{pmatrix}, \end{aligned} \tag{6}$$

for some $\varepsilon_P, \varepsilon_Q \in \mathrm{Aut}(\mathbb{F}_q)$ and some 2-by-2 non-singular matrices $M(P)$ and $M(Q)$. As for R, since (P, Q, R) is an autotopism, $P(x)Q(y) = R(xy)$ for all $x, y \in \mathcal{K}(\sigma)$. In particular, if $x = \left(\begin{smallmatrix} x' \\ x'' \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, then

$$P \begin{pmatrix} x' \\ x'' \end{pmatrix} Q \begin{pmatrix} 1 \\ 0 \end{pmatrix} = R \begin{pmatrix} x' \\ x'' \end{pmatrix}. \tag{7}$$

Thus, if $M(P) = \left(\begin{smallmatrix} p_1 & p_2 \\ p_3 & p_4 \end{smallmatrix}\right)$ and $M(Q) = \left(\begin{smallmatrix} q_1 & q_2 \\ q_3 & q_4 \end{smallmatrix}\right)$, then by (5), the equation (7) implies

$$\begin{aligned} R\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto &\begin{pmatrix} p_1 q_1 & p_2 q_1 \\ p_1 q_3 + p_3 q_1 & p_2 q_3 + p_4 q_1 \end{pmatrix} \begin{pmatrix} x'^{\varepsilon_P} \\ x''^{\varepsilon_P} \end{pmatrix} \\ &+ \begin{pmatrix} \delta {p_3}^\sigma {q_3}^\sigma & \delta {p_4}^\sigma {q_3}^\sigma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x'^{\varepsilon_P \sigma} \\ x''^{\varepsilon_P \sigma} \end{pmatrix}. \end{aligned} \tag{8}$$

Sandler now considered the case when $\sigma^2 \neq \mathrm{id}$ and proceeded to show that $p_2 = p_3 = q_2 = q_3 = 0$, $p_1 \neq 0$, $p_4 \neq 0$, $q_1 \neq 0$, $q_4 = q_1 p_4 / p_1$, $\delta^\varepsilon p_1 {p_1}^\sigma q_1 = \delta ({p_4}^\sigma)^2 {q_1}^\sigma$ and $\varepsilon_P = \varepsilon_Q$. It follows that (P, Q, R) satisfies the following equations:

$$\begin{aligned} P\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} &\longmapsto M(P) \begin{pmatrix} x'^{\varepsilon} \\ x''^{\varepsilon} \end{pmatrix}, \\ Q\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} &\longmapsto M(Q) \begin{pmatrix} x'^{\varepsilon} \\ x''^{\varepsilon} \end{pmatrix}, \\ R\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} &\longmapsto M(R) \begin{pmatrix} x'^{\varepsilon} \\ x''^{\varepsilon} \end{pmatrix}, \end{aligned} \tag{9(a)}$$

where $\varepsilon \in \mathrm{Aut}(\mathbb{F}_q)$, and $M(P)$, $M(Q)$, $M(R)$ are non-singular matrices such that

$$M(P) = \begin{pmatrix} p_1 & 0 \\ 0 & p_4 \end{pmatrix}, \ M(Q) = \frac{q_1}{p_1} M(P), \text{ and } M(R) = q_1 M(P) \qquad (9)(b)$$

for some $p_1, p_4, q_1 \in \mathbb{F}_q^*$ satisfying

$$\delta^{\varepsilon} p_1 {p_1}^{\sigma} q_1 = \delta ({p_4}^{\sigma})^2 {q_1}^{\sigma}. \qquad (9)(c)$$

It is straightforward to check that an ordered triple (P, Q, R) of additive mappings between $\mathcal{K}(\sigma)$ satisfying (9) is an autotopism of $\mathcal{K}(\sigma)$ and that the set of such triples is a subgroup of $A(\sigma)$ regardless of whether σ^2 is the identity or not. When σ^2 is not the identity, the above arguments by Sandler show that the subgroup is in fact the full group $A(\sigma)$. Furthermore, generators of $A(\sigma)$ are given whereby it is shown that $A(\sigma)$ is solvable. From the discussion in [17], the structure of the group can be determined.

We shall complete the computations for $A(\sigma)$ in case $\sigma^2 = \mathrm{id}$ in Section 3. Furthermore, we show in Section 5 that when $\sigma^2 \neq \mathrm{id}$, $A(\sigma)$ can also be written as a semidirect product with factors a semidirect product of cyclic groups and a cyclic group via a design theoretic approach.

3 The autotopism group $A(\sigma)$ of $\mathcal{K}(\sigma)$

In this section we compute $A(\sigma)$ when $\sigma^2 = \mathrm{id}$. Let $A''(\sigma)$ be the set of autotopisms (P, Q, R) defined as follows:

$$\begin{aligned} P\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} &\longmapsto M(P) \begin{pmatrix} {x'}^{\varepsilon} \\ {x''}^{\varepsilon} \end{pmatrix}, \\ Q\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} &\longmapsto M(Q) \begin{pmatrix} {x'}^{\varepsilon} \\ {x''}^{\varepsilon} \end{pmatrix}, \\ R\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} &\longmapsto M(R) \begin{pmatrix} {x'}^{\varepsilon\sigma} \\ {x''}^{\varepsilon} \end{pmatrix}, \end{aligned} \qquad (10)(a)$$

where $\varepsilon \in \mathrm{Aut}(\mathbb{F}_q)$, and $M(P)$, $M(Q)$, $M(R)$ are non-singular matrices such that

$$M(P) = \begin{pmatrix} 0 & p_2 \\ p_3 & 0 \end{pmatrix}, \ M(Q) = \frac{q_3}{p_3} M(P), \text{ and } M(R) = \begin{pmatrix} \delta {p_3}^{\sigma} {q_3}^{\sigma} & 0 \\ 0 & p_2 q_3 \end{pmatrix} \qquad (10)(b)$$

for $p_3, p_2, q_3 \in \mathbb{F}_q^*$ satisfying

$$\delta \delta^{\varepsilon\sigma} p_3 {p_3}^{\sigma} {q_3}^{\sigma} = {p_2}^2 q_3. \qquad (10)(c)$$

Theorem 3.1. *When $\sigma^2 = \mathrm{id}$, the autotopism group $A(\sigma) = A'(\sigma) \cup A''(\sigma)$, where $A'(\sigma)$ is the group of autotopisms satisfying* (9) *and $A''(\sigma)$ is the set of autotopisms (P, Q, R) satisfying* (10).

Proof. Let (P, Q, R) be an autotopism. Recall that P, Q and R satisfy (6) and (8). Since $M(P)$ and $M(Q)$ are non-singular matrices,

$$p_1p_4 - p_2p_3 \neq 0 \text{ and } q_1q_4 - q_2q_3 \neq 0. \tag{11}$$

We deduce more necessary conditions. Write $\varepsilon = \varepsilon_P$. Let $\begin{pmatrix} a' \\ a'' \end{pmatrix} \in \mathcal{K}(\sigma)$. Since $\begin{pmatrix} \delta a''^{\sigma} \\ a' \end{pmatrix} = \begin{pmatrix} a' \\ a'' \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by (5) and (P, Q, R) is an autotopism,

$$R \begin{pmatrix} \delta a''^{\sigma} \\ a' \end{pmatrix} = P \begin{pmatrix} a' \\ a'' \end{pmatrix} Q \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{12}$$

By (6), (8) and (5), the equation (12) becomes

$$\begin{pmatrix} p_2q_1a'^{\varepsilon} + \delta p_4{}^{\sigma} q_3{}^{\sigma} a'^{\varepsilon\sigma} + \delta^{\varepsilon} p_1q_1a''^{\sigma\varepsilon} + \delta\delta^{\varepsilon\sigma} p_3{}^{\sigma} q_3{}^{\sigma} a''^{\sigma\varepsilon\sigma} \\ (p_2q_3 + p_4q_1)a'^{\varepsilon} + (\delta^{\varepsilon} p_1q_3 + \delta^{\varepsilon} p_3q_1)a''^{\sigma\varepsilon} \end{pmatrix}$$
$$= \begin{pmatrix} p_1q_2a'^{\varepsilon} + \delta p_3{}^{\sigma} q_4{}^{\sigma} a'^{\varepsilon\sigma} + p_2q_2a''^{\varepsilon} + \delta p_4{}^{\sigma} q_4{}^{\sigma} a''^{\varepsilon\sigma} \\ (p_1q_4 + p_3q_2)a'^{\varepsilon} + (p_2q_4 + p_4q_2)a''^{\varepsilon} \end{pmatrix}. \tag{13}$$

Note that (13) is true for all $a', a'' \in \mathbb{F}_q$. In particular, (13) is true when $a' = 0$ or $a'' = 0$. By comparing the first coordinates and the second coordinates of both sides of (13) when $a' = 0$ or $a'' = 0$, we obtain

$$p_2q_1a'^{\varepsilon} + \delta p_4{}^{\sigma} q_3{}^{\sigma} a'^{\varepsilon\sigma} = p_1q_2a'^{\varepsilon} + \delta p_3{}^{\sigma} q_4{}^{\sigma} a'^{\varepsilon\sigma}, \tag{14)(a}$$
$$(p_2q_3 + p_4q_1)a'^{\varepsilon} = (p_1q_4 + p_3q_2)a'^{\varepsilon}, \tag{14)(b}$$
$$\delta^{\varepsilon} p_1q_1a''^{\sigma\varepsilon} + \delta\delta^{\varepsilon\sigma} p_3{}^{\sigma} q_3{}^{\sigma} a''^{\sigma\varepsilon\sigma} = p_2q_2a''^{\varepsilon} + \delta p_4{}^{\sigma} q_4{}^{\sigma} a''^{\varepsilon\sigma}, \tag{14)(c}$$
$$(\delta^{\varepsilon} p_1q_3 + \delta^{\varepsilon} p_3q_1)a''^{\sigma\varepsilon} = (p_2q_4 + p_4q_2)a''^{\varepsilon}. \tag{14)(d}$$

Collecting like terms of (14)(c), we have, since $\sigma \neq \mathrm{id}$ and $\sigma^2 = \mathrm{id}$,

$$(\delta^{\varepsilon} p_1q_1 - \delta p_4{}^{\sigma} q_4{}^{\sigma})a''^{\sigma\varepsilon} + (\delta\delta^{\varepsilon\sigma} p_3{}^{\sigma} q_3{}^{\sigma} - p_2q_2)a''^{\varepsilon} = 0. \tag{15}$$

Note that since (15) is a polynomial in a'' of degree less than q which vanishes for all $a'' \in \mathbb{F}_q$, it is the zero polynomial. Hence, $\delta^{\varepsilon} p_1q_1 - \delta p_4{}^{\sigma} q_4{}^{\sigma} = \delta\delta^{\varepsilon\sigma} p_3{}^{\sigma} q_3{}^{\sigma} - p_2q_2 = 0$. Applying similar arguments to the other equations of (14), we obtain

$$\begin{gathered} p_2q_1 = p_1q_2, \quad p_4q_3 = p_3q_4, \\ p_2q_3 + p_4q_1 = p_1q_4 + p_3q_2, \\ \delta^{\varepsilon} p_1q_1 = \delta p_4{}^{\sigma} q_4{}^{\sigma}, \quad \delta\delta^{\varepsilon\sigma} p_3{}^{\sigma} q_3{}^{\sigma} = p_2q_2, \\ p_1q_3 + p_3q_1 = 0, \quad p_4q_2 + p_2q_4 = 0. \end{gathered} \tag{16}$$

We need a necessary condition on ε and ε_Q. Let $\begin{pmatrix} 0 \\ x'' \end{pmatrix} \in \mathcal{K}(\sigma)$. Since $\begin{pmatrix} 0 \\ x'' \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \delta x''^{\sigma} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ x'' \end{pmatrix}$ by (5) and (P, Q, R) is an autotopism,

$$P\begin{pmatrix} 0 \\ x'' \end{pmatrix} Q\begin{pmatrix} 0 \\ 1 \end{pmatrix} = P\begin{pmatrix} 0 \\ 1 \end{pmatrix} Q\begin{pmatrix} 0 \\ x'' \end{pmatrix}. \tag{17}$$

Expanding (17) using (6), (5) and then comparing the first coordinates of both sides, we obtain

$$p_2 q_2 (x''^{\varepsilon_Q} - x''^{\varepsilon}) + \delta {p_4}^{\sigma} {q_4}^{\sigma} (x''^{\varepsilon_Q} - x''^{\varepsilon})^{\sigma} = 0. \tag{18}$$

From (11) and (16), we obtain either

$$\begin{array}{lcl} p_3 = p_2 = q_3 = q_2 = 0, & & p_1 = p_4 = q_1 = q_4 = 0, \\ p_1 \neq 0,\ p_4 \neq 0,\ q_1 \neq 0, & \text{or} & p_3 \neq 0,\ p_2 \neq 0,\ q_3 \neq 0, \\ q_4 = q_1 p_4 / p_1, & & q_2 = q_3 p_2 / p_3, \\ \delta^{\varepsilon} p_1 {p_1}^{\sigma} q_1 = \delta ({p_4}^{\sigma})^2 {q_1}^{\sigma}, & & \delta \delta^{\varepsilon\sigma} p_3 {p_3}^{\sigma} {q_3}^{\sigma} = {p_2}^2 q_3. \end{array} \tag{19}$$

Using this information in (18) we conclude that $\varepsilon (= \varepsilon_P) = \varepsilon_Q$. There are two cases in (19). In the former case, (6), (8) imply (9), and in the latter case, (6)and (8) imply (10). It is routine to verify that any (P, Q, R) satisfying either (9) or (10) is an autotopism. This completes the proof of the theorem. □

We now show that $\mathcal{A}''(\sigma)$ is a coset of $\mathcal{A}'(\sigma)$ and deduce from Theorem 3.1 the following:

Theorem 3.2. *The following statements hold when* $\sigma^2 = \mathrm{id}$:

(a) $\mathcal{A}(\sigma) = \mathcal{A}'(\sigma) \cup \gamma(P^*, Q^*, R^*)\mathcal{A}'(\sigma)$, *where* $\mathcal{A}'(\sigma)$ *is the group of autotopisms given by* (9) *and* $\gamma(P^*, Q^*, R^*)$ *is given by*

$$\begin{aligned} P^* &\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -\delta \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x'^{\sigma} \\ x''^{\sigma} \end{pmatrix}, \\ Q^* &\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x'^{\sigma} \\ x''^{\sigma} \end{pmatrix}, \\ R^* &\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto \begin{pmatrix} -\delta & 0 \\ 0 & -\delta \end{pmatrix}\begin{pmatrix} x' \\ x''^{\sigma} \end{pmatrix}. \end{aligned} \tag{20}$$

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(b) $\mathcal{A}(\sigma)$ *is solvable.*

(c) *The order of* $\mathcal{A}(\sigma)$ *is* $4e(q-1)^2$.

Proof. (a) We prove $\mathcal{A}''(\sigma) = \gamma(P^*, Q^*, R^*)\mathcal{A}'(\sigma)$. Let $\gamma(P_1, Q_1, R_1) \in \mathcal{A}''(\sigma)$. Then $\gamma(P_1, Q_1, R_1) = \gamma(P^*, Q^*, R^*)\gamma(P_2, Q_2, R_2)$ where

$$P_2\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto \begin{pmatrix} -{p_3}^{\sigma} & 0 \\ 0 & (-\delta^{-1}p_2)^{\sigma} \end{pmatrix} \begin{pmatrix} x'^{\sigma\varepsilon} \\ x''^{\sigma\varepsilon} \end{pmatrix},$$

$$Q_2\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto \begin{pmatrix} {q_3}^{\sigma} & 0 \\ 0 & (\delta^{-1}p_2q_3/p_3)^{\sigma} \end{pmatrix} \begin{pmatrix} x'^{\sigma\varepsilon} \\ x''^{\sigma\varepsilon} \end{pmatrix},$$

$$R_2\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto \begin{pmatrix} (-p_3q_3)^{\sigma} & 0 \\ 0 & (-\delta^{-1}p_2q_3)^{\sigma} \end{pmatrix} \begin{pmatrix} x'^{\sigma\varepsilon} \\ x''^{\sigma\varepsilon} \end{pmatrix}.$$

In order to check that $\gamma(P_2, Q_2, R_2) \in \mathcal{A}'(\sigma)$, note that since $\sigma^2 = \mathrm{id}$ and $\delta\delta^{\varepsilon\sigma}p_3{p_3}^{\sigma}{q_3}^{\sigma} = {p_2}^2q_3$ by (10)(c), we have

$$\begin{aligned} \delta^{\varepsilon\sigma}(-{p_3}^{\sigma})(-{p_3}^{\sigma})^{\sigma}{q_3}^{\sigma} = \delta^{\varepsilon\sigma}{p_3}^{\sigma}p_3{q_3}^{\sigma} &= \delta^{-1}{p_2}^2q_3 \\ &= \delta(((-\delta^{-1}p_2)^{\sigma})^2)^{\sigma}({q_3}^{\sigma})^{\sigma}. \end{aligned}$$

Hence, $\gamma(P_2, Q_2, R_2) \in \mathcal{A}'(\sigma)$ and $\gamma(P_1, Q_1, R_1) \in \gamma(P^*, Q^*, R^*)\mathcal{A}'(\sigma)$.

Conversely, let $\gamma(P_3, Q_3, R_3) \in \mathcal{A}'(\sigma)$. Then $\gamma(P^*, Q^*, R^*)\gamma(P_3, Q_3, R_3) = \gamma(P^* \circ P_3, Q^* \circ Q_3, R^* \circ R_3)$ where

$$P^* \circ P_3\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -\delta{p_4}^{\sigma} \\ -{p_1}^{\sigma} & 0 \end{pmatrix} \begin{pmatrix} x'^{\varepsilon\sigma} \\ x''^{\varepsilon\sigma} \end{pmatrix},$$

$$Q^* \circ Q_3\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto \begin{pmatrix} 0 & \delta(q_1p_4/p_1)^{\sigma} \\ {q_1}^{\sigma} & 0 \end{pmatrix} \begin{pmatrix} x'^{\varepsilon\sigma} \\ x''^{\varepsilon\sigma} \end{pmatrix},$$

$$R^* \circ R_3\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto \begin{pmatrix} -\delta p_1q_1 & 0 \\ 0 & -\delta(p_4q_1)^{\sigma} \end{pmatrix} \begin{pmatrix} x'^{\varepsilon} \\ x''^{\varepsilon\sigma} \end{pmatrix}$$

for some $p_1, p_4, q_1 \in \mathbb{F}_q^*$ and $\varepsilon \in \mathrm{Aut}(\mathbb{F}_q)$ such that $\delta^{\varepsilon}p_1{p_1}^{\sigma}q_1 = \delta({p_4}^{\sigma})^2{q_1}^{\sigma}$. To check that $\gamma(P^*, Q^*, R^*)\gamma(P_3, Q_3, R_3) \in \mathcal{A}''(\sigma)$, first note that such an element satisfies (10)(a) and (10)(b). Furthermore, since $\delta^{\varepsilon}p_1{p_1}^{\sigma}q_1 = \delta({p_4}^{\sigma})^2{q_1}^{\sigma}$,

$$\begin{aligned} \delta(\delta^{\varepsilon\sigma})^{\sigma}(-{p_1}^{\sigma})(-{p_1}^{\sigma})^{\sigma}({q_1}^{\sigma})^{\sigma} = \delta(\delta^{\varepsilon}p_1{p_1}^{\sigma}q_1) &= \delta\delta({p_4}^{\sigma})^2{q_1}^{\sigma} \\ &= (-\delta{p_4}^{\sigma})^2{q_1}^{\sigma}, \end{aligned}$$

and so $\gamma(P^*, Q^*, R^*)\gamma(P_3, Q_3, R_3)$ satisfies (10)(c) and is in $\mathcal{A}''(\sigma)$.

(b) By (a), $A(\sigma)$ is an index two extension of $A'(\sigma)$. Since $\mathcal{A}'(\sigma)$ is identical to $\mathcal{A}(\sigma)$ when $\sigma^2 \neq \mathrm{id}$, which is solvable by Sandler, $\mathcal{A}(\sigma)$ is solvable.

(c) By (a), $|\mathcal{A}(\sigma)| = 2|\mathcal{A}'(\sigma)|$. By Sandler, $|\mathcal{A}'(\sigma)| = 2e(q-1)^2$. The result follows. □

Part (b) of the theorem gives positive evidence for the conjecture of Hughes mentioned in Section 2.1.

4 The Ganley unital $\mathcal{U}$ and the colineation stabilizer subgroup $\mathrm{Col}(\mathcal{U})$

In [9], the Dickson semifield $\mathcal{K} = \mathcal{K}(\sigma)$ is shown to admit an involutory automorphism α defined by

$$\alpha\colon \begin{pmatrix} x' \\ x'' \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} x' \\ -x'' \end{pmatrix}. \tag{21}$$

It follows (by [8, Theorem 5]) that $\Pi(\mathcal{K})$ admits a unitary polarity ρ given by

$$\rho\colon \begin{cases} (\infty) \longleftrightarrow [\infty], \\ \left(\begin{pmatrix} x' \\ x'' \end{pmatrix}\right) \longleftrightarrow \left[\begin{pmatrix} x' \\ -x'' \end{pmatrix}\right], \\ \left(\begin{pmatrix} x' \\ x'' \end{pmatrix}, \begin{pmatrix} y' \\ y'' \end{pmatrix}\right) \longleftrightarrow \left[\begin{pmatrix} x' \\ -x'' \end{pmatrix}, \begin{pmatrix} -y' \\ y'' \end{pmatrix}\right]. \end{cases} \tag{22}$$

Denote by $\mathcal{U} = \mathcal{U}(\sigma)$ the polar unital defined by ρ, whose points are the absolute points of ρ and whose blocks are the non-absolute lines of ρ. By [13], any unitary polarity in $\Pi(\mathcal{K})$ is conjugate to ρ. We call $\mathcal{U}$ the *Dickson–Ganley unital*, or simply the *Ganley unital*. In [12] it is shown that $\mathcal{U}$ is not isomorphic to the classical unital (see [*loc. cit.*, Corollary 3.3 and Corollary 4.5]; only a few special cases were known previously in [9]). Furthermore, it is shown that the automorphism group $\mathrm{Aut}(\mathcal{U})$ is isomorphic to the colineation stabilizer $\mathrm{Col}(\mathcal{U})$ (see [*loc. cit.*, Corollary 7.9]). The group $\mathrm{Col}(\mathcal{U})$ is studied in [9] and is given by (see [*loc. cit.*, Lemma 2]):

$$\mathrm{Col}(\mathcal{U}) = \left\{ \gamma(P,Q,R)\varsigma(\alpha(a))\tau(a,b) \in \mathrm{Col}(\Pi(\mathcal{K})) \;\middle|\; \begin{array}{l} a,b \in \mathcal{K}; \alpha(a)a = b + \alpha(b); \\ P\alpha = \alpha Q; R\alpha = \alpha R \end{array} \right\}. \tag{23}$$

Since $\mathrm{Col}(\Pi(\mathcal{K})) = \mathcal{A} \ltimes (\mathcal{S} \ltimes \mathcal{T})$, straightforward computations using (23) show that

$$\mathrm{Col}(\mathcal{U}) \cap \mathcal{A} = \{\gamma(P,Q,R) \in \mathcal{A} \mid P\alpha = \alpha Q; R\alpha = \alpha R\}, \tag{24}$$

and

$$\begin{aligned} &\mathrm{Col}(\mathcal{U}) \cap (\mathcal{S} \ltimes \mathcal{T}) \\ &= \left\{ \varsigma\left(\begin{pmatrix} a' \\ -a'' \end{pmatrix}\right) \tau\left(\begin{pmatrix} a' \\ a'' \end{pmatrix}, \begin{pmatrix} \frac{1}{2}(a'^2 - \delta(a''^2)^\sigma) \\ b'' \end{pmatrix}\right) \in \mathcal{S} \ltimes \mathcal{T} \;\middle|\; a', a'', b'' \in \mathbb{F}_q \right\}. \end{aligned} \tag{25}$$

When $\sigma^2 \neq \mathrm{id}$, using (9) and (24), we obtain

$$\mathrm{Col}(\mathcal{U}) \cap \mathcal{A} = \{\gamma(P,Q,R) \in \mathcal{A} \mid (P,Q,R) \text{ given in (9) with } p_1 = q_1\}. \tag{26}$$

Indeed, in this case, by Sandler's result, any autotopism satisfies (9), and then by (24), $p_1 = q_1$. Conversely, it is straightforward to show that any autotopism satisfying (9) with $p_1 = q_1$ must satisfy (24). Similarly, when $\sigma^2 = \mathrm{id}$, using Theorem 3.1 and (24), we obtain

$$\mathrm{Col}(\mathcal{U}) \cap \mathcal{A} = \left\{ \gamma(P,Q,R) \in \mathcal{A} \;\middle|\; \begin{array}{l} (P,Q,R) \text{ given by (9) with } p_1 = q_1; \\ \text{or by (10) with } p_3 = -q_3 \end{array} \right\}. \tag{27}$$

Note also that by [1, Theorem 3.9], $\mathrm{Col}(\mathcal{U}) \cap (\mathcal{S} \ltimes \mathcal{T})$ is regular on the affine points of $\mathcal{U}$ and is normal in $\mathrm{Col}(\mathcal{U})$.

We now exhibit $\mathrm{Col}(\mathcal{U})$ as a semidirect product whose factors are given by (24) and (25), and that these factors are both semidirect products of cyclic or abelian groups.

First we note the following formula for composition in $\mathrm{Col}(\Pi(\mathcal{K}))$:

$$\begin{aligned} &\gamma(P_1,Q_1,R_1)\varsigma(c_1)\tau(a_1,b_1)\gamma(P,Q,R)\varsigma(c)\tau(a,b) \\ &= \gamma(P_1P,Q_1Q,R_1R)\varsigma(P^{-1}(c_1)+c)\tau(Q^{-1}(a_1)+a, cQ^{-1}(a_1)+R^{-1}(b_1)+b). \end{aligned}$$

Next we consider the following two subgroups of $\mathcal{S} \ltimes \mathcal{T}$:

$$B' = \left\{ \varsigma\left(\begin{pmatrix} a' \\ 0 \end{pmatrix}\right) \tau\left(\begin{pmatrix} a' \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}(a'^2) \\ 0 \end{pmatrix}\right) \in \mathcal{S} \ltimes \mathcal{T} \;\middle|\; a' \in \mathbb{F}_q \right\} \cong \mathbb{F}_q(+), \tag{28}$$

$$\begin{aligned} B'' = &\left\{ \varsigma\left(\begin{pmatrix} 0 \\ -a'' \end{pmatrix}\right) \tau\left(\begin{pmatrix} 0 \\ a'' \end{pmatrix}, \begin{pmatrix} \frac{1}{2}(-\delta(a''^2)^\sigma) \\ b'' \end{pmatrix}\right) \in \mathcal{S} \ltimes \mathcal{T} \;\middle|\; a'', b'' \in \mathbb{F}_q \right\} \\ &\cong \mathbb{F}_q(+) \times \mathbb{F}_q(+). \end{aligned} \tag{29}$$

Then we have the following:

Theorem 4.1. *We have* $\mathrm{Col}(\mathcal{U}) = (\mathrm{Col}(\mathcal{U}) \cap \mathcal{A}) \ltimes (\mathrm{Col}(\mathcal{U}) \cap (\mathcal{S} \ltimes \mathcal{T}))$. *Moreover,* $\mathrm{Col}(\mathcal{U}) \cap (\mathcal{S} \ltimes \mathcal{T}) = B' \ltimes B''$*, where* B', B'' *are given respectively by* (28) *and* (29)*, and is solvable and of order* q^3.

Proof. To show that $\mathrm{Col}(\mathcal{U}) = (\mathrm{Col}(\mathcal{U}) \cap \mathcal{A}) \ltimes (\mathrm{Col}(\mathcal{U}) \cap (\mathcal{S} \ltimes \mathcal{T}))$, we exhibit a homomorphism f_1 from $\mathrm{Col}(\mathcal{U})$ to itself such that $f_1(\mathrm{Col}(\mathcal{U})) = \mathrm{Col}(\mathcal{U}) \cap \mathcal{A}$, $f_1|_{\mathrm{Col}(\mathcal{U})\cap\mathcal{A}} = 1_{\mathrm{Col}(\mathcal{U})\cap\mathcal{A}}$ and $\ker(f_1) = (\mathrm{Col}(\mathcal{U}) \cap (\mathcal{S} \ltimes \mathcal{T}))$ (apply e.g. [16, Lemma 7.20]). Take the homomorphism $f_1\colon \mathrm{Col}(\mathcal{U}) \to \mathrm{Col}(\mathcal{U})$ defined by $f_1(\gamma(P,Q,R)\varsigma(\alpha(a))\tau(a,b)) = \gamma(P,Q,R)$ for any $\gamma(P,Q,R)\varsigma(\alpha(a))\tau(a,b) \in \mathrm{Col}(\mathcal{U})$. It is straightforward to verify that f_1 satisfies our requirement.

Similarly, take the homomorphism $f_2\colon \operatorname{Col}(\mathcal{U})\cap(\mathcal{S}\ltimes\mathcal{T}) \to \operatorname{Col}(\mathcal{U})\cap(\mathcal{S}\ltimes\mathcal{T})$ given by

$$f_2\left(\varsigma\left(\begin{pmatrix} a' \\ -a'' \end{pmatrix}\right)\tau\left(\begin{pmatrix} a' \\ a'' \end{pmatrix}, \begin{pmatrix} \frac{1}{2}(a'^2-\delta(a''^2)^\sigma) \\ b'' \end{pmatrix}\right)\right) = \varsigma\left(\begin{pmatrix} a' \\ 0 \end{pmatrix}\right)\tau\left(\begin{pmatrix} a' \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}(a'^2) \\ 0 \end{pmatrix}\right).$$

Then $f_2(\operatorname{Col}(\mathcal{U})\cap(\mathcal{S}\ltimes\mathcal{T})) = B'$, $f_2|_{B'} = 1_{B'}$ and $\ker(f_2) = B''$. It follows that $\operatorname{Col}(\mathcal{U})\cap(\mathcal{S}\ltimes\mathcal{T})$ is $B'\ltimes B''$, and is therefore of order q^3. Since $\operatorname{Col}(\mathcal{U})\cap(\mathcal{S}\ltimes\mathcal{T})$ is a semidirect product of two abelian groups, it is solvable. □

Next we consider $\operatorname{Col}(\mathcal{U})\cap\mathcal{A}$. We define the following subgroups of $\mathcal{A}$:

$$E = \{\gamma(P,Q,R)\in\mathcal{A} \mid (P,Q,R) \text{ given by (9) with } p_1 = q_1 = 1\}, \tag{30}$$

$$D' = \left\{\gamma(P,Q,R)\in\mathcal{A} \;\middle|\; (P,Q,R) \text{ given by (9) with } p_1 = q_1 = {p_4}^\sigma, \varepsilon = \mathrm{id}\right\}, \tag{31}$$

$$D'' = \left\{\gamma(P,Q,R)\in\mathcal{A} \;\middle|\; \begin{array}{l}(P,Q,R) \text{ given either by (9) with } p_1 = q_1 = {p_4}^\sigma, \\ \varepsilon = \mathrm{id}; \text{ or by (10) with } p_2 = \delta {p_3}^\sigma = -\delta {q_3}^\sigma, \varepsilon = \sigma\end{array}\right\}. \tag{32}$$

For geometric interpretations of these groups, see Section 5.

We have the following:

Theorem 4.2. (a) $E \cong \mathbb{Z}/2e\mathbb{Z}$.

(b) $D' \cong \mathbb{Z}/(q-1)\mathbb{Z}$.

(c) $D'' \cong \mathbb{Z}/2(q-1)\mathbb{Z}$.

(d) *When* $\sigma^2 \neq \mathrm{id}$, $\operatorname{Col}(\mathcal{U})\cap\mathcal{A}$ *is* $E\ltimes D'$, *and hence solvable and of order* $2e(q-1)$.

(e) *When* $\sigma^2 = \mathrm{id}$, $\operatorname{Col}(\mathcal{U})\cap\mathcal{A}$ *is* $E\ltimes D''$, *and hence solvable and of order* $4e(q-1)$.

Proof. (a) Consider $\gamma(P,Q,R)\in E$ with $p_4 = \delta^{p^{e-s}(p-1)/2}$ and $x^\varepsilon = x^p$. By a straightforward computation $\gamma(P,Q,R)^i \in E$ with $p_4 = \delta^{p^{e-s}(p^i-1)/2}$ and $x^\varepsilon = x^{p^i}$. The order of $\gamma(P,Q,R)$ is the smallest positive integer j such that $p^j \equiv 0 \pmod q$ and $(p^j-1)/2 \equiv 0 \pmod{q-1}$, and thus $j = 2e$. By solving (9)(c) and $p_1 = q_1 = 1$, we check that any element in E is of the form $\gamma(P,Q,R)^i$. Hence, E is isomorphic to $\mathbb{Z}/2e\mathbb{Z}$.

(b) D' is generated by $\gamma(P,Q,R)\in D'$ with $p_1 = \delta$. Since δ is a generator of $\mathbb{F}_q^*$, D' is isomorphic to $\mathbb{Z}/(q-1)\mathbb{Z}$.

(c) D'' is generated by $\gamma(P,Q,R) \in D'' \setminus D''$ with $p_3 = -1, p_2 = -\delta$. Since $\gamma(P,Q,R)^2$ is the generator of D' stated in (b), D'' is indeed isomorphic to $\mathbb{Z}/2(q-1)\mathbb{Z}$.

(d) Similar to the proof of Theorem 4.1, consider $f_3\colon \operatorname{Col}(\mathcal{U}) \cap \mathcal{A} \to \operatorname{Col}(\mathcal{U}) \cap \mathcal{A}$ given by $f_3(\gamma(P,Q,R)) = \gamma(P',Q',R')$ where $p_1' = 1$ and $p_4' = p_4/(p_1{}^{\sigma^{-1}})$. Then $f_3(\operatorname{Col}(\mathcal{U}) \cap \mathcal{A}) = E$, $f_3|_E = 1_E$ and $\ker(f_3) = D'$.

(e) Consider $f_4\colon \operatorname{Col}(\mathcal{U}) \cap \mathcal{A} \to \operatorname{Col}(\mathcal{U}) \cap \mathcal{A}$ given by $f_4|_{E \ltimes D'} = f_3$, and if $\gamma(P,Q,R) \in \operatorname{Col}(\mathcal{U}) \cap \mathcal{A} \setminus (E \ltimes D')$, then $f_4(\gamma(P,Q,R)) = \gamma(P',Q',R')$ where (P',Q',R') is given by (9) with $p_1' = q_1' = 1$, $p_4' = p_2{}^{\sigma}/(\delta^{\sigma} p_3)$, and $\varepsilon' = \varepsilon\sigma$. Then $f_4(\operatorname{Col}(\mathcal{U}) \cap \mathcal{A}) = E$, $f_4|_E = 1_E$ and $\ker(f_4) = D''$. $\square$

The structure of $\operatorname{Col}(\mathcal{U})$ is completely determined by Theorem 4.1 and Theorem 4.2: it is a semidirect product of solvable groups, hence solvable, with order equals $2eq^3(q-1)$ when $\sigma^2 \neq \mathrm{id}$, and $4eq^3(q-1)$ when $\sigma^2 = \mathrm{id}$.

5 An alternative geometric approach

In this section we describe an alternative geometric approach to obtain the results in the previous sections. Furthermore, we give an alternative description of the structure of $\mathcal{A}$ when $\sigma^2 \neq \mathrm{id}$.

We construct from the Dickson–Ganley unital $\mathcal{U}$ a design S as in Wilbrink [18] which is isomorphic to the residual $\mathcal{I}^{[1,0]^t}$ of the classical inversive plane $\mathcal{I}$. We then construct a group homomorphism from $\operatorname{Col}(\mathcal{U})$ to $\operatorname{Aut}(S)$. Since $\operatorname{Aut}(S)$ is isomorphic to $\operatorname{Aut}(\mathcal{I}^{[1,0]^t})$, we can gain information on the structure of subgroups of $\operatorname{Col}(\mathcal{U})$ by studying their images in $\operatorname{Aut}(\mathcal{I}^{[1,0]^t})$. This leads to another proof of Theorem 4.2.

In more details, the design S is defined as follows (see [12] for full details). The points of S are the q^2 non-absolute lines on (∞). The blocks of S are the equivalence classes $\langle [m,k] \rangle$ of the non-absolute lines missing (∞). Recall that $[m_1,k_1] \sim [m_2,k_2]$ if and only if $m_1 = m_2$ and the first coordinates of k_1 and k_2 equal. Thus $\langle [m,k] \rangle$ consists of the q parallel lines described by [12, Lemma 4.1]. Incidence of S is defined as follows: $[x]$ is incident with $\langle [m,k] \rangle$ in S if and only if $[x]$ meets $[m,k]$ at an absolute point. By [12, Section 6], S is a 2-$(q^2, q+1, q)$ design.

Consider the classical inversive plane $\mathcal{I} = (\mathcal{X}, \mathcal{C})$ of order q, whose points are points of the projective line $\mathrm{PG}(1,q^2)$ and whose circles are all sublines $\mathrm{PG}(1,q)$ in $\mathrm{PG}(1,q^2)$ [10]. Recall (see also [12, Lemma 6.1]) that the circle set $\mathcal{C}$ is given by $\mathcal{C} = \{c_{a_0,a_1,b} \mid a_0, a_1 \in \mathbb{F}_q, b \in \mathbb{F}_{q^2}; a_0, a_1, b \text{ not all zeros}; a_0 a_1 - b^{q+1} \neq 0\}$, where $c_{a_0,a_1,b} = \{[z,w]^t \in \mathrm{PG}(1,q^2) \mid a_0 z^{q+1} + b z^q w + b^q z w^q + a_1 w^{q+1} = 0\}$.

Let $\kappa \in \mathbb{F}_{q^2}$ be a zero of $X^2 - \delta \in \mathbb{F}_q[X]$. Then any element of $\mathbb{F}_{q^2}$ can be written as $a_1 + \kappa a_2$ where $a_1, a_2 \in \mathbb{F}_q$. By [12, Theorem 6.2], the map

$$H\colon S \longrightarrow \mathcal{I}^{[1,0]^t} \tag{33}$$

defined by

$$H\colon \begin{cases} \left[\begin{pmatrix} x' \\ x'' \end{pmatrix}\right] \longmapsto \begin{bmatrix} x' + \kappa x''^{\sigma} \\ 1 \end{bmatrix}, \\ \left\langle \left[\begin{pmatrix} m' \\ m'' \end{pmatrix}, \begin{pmatrix} k' \\ k'' \end{pmatrix}\right] \right\rangle \longmapsto c_{-1,-2k',m'-\kappa m''^{\sigma}} \end{cases} \tag{34}$$

is an isomorphism. This induces the group isomorphism ([12, (6.4)])

$$h\colon\ \mathrm{Aut}(S) \longrightarrow \mathrm{Aut}(\mathcal{I}^{[1,0]^t}), \tag{35}$$

where for any $\Psi \in \mathrm{Aut}(S)$, $h(\Psi)$ is defined by

$$h(\Psi)\colon [z,1]^t \longmapsto H(\Psi(H^{-1}([z,1]^t))) \tag{36}$$

for $[z,1]^t \in \mathcal{I}^{[1,0]^t}$. Next we define a group homomorphism,

$$\Lambda\colon\ \mathrm{Col}(\mathcal{U}) \longrightarrow \mathrm{Aut}(S), \tag{37}$$

as follows. For any $\Phi \in \mathrm{Col}(\mathcal{U})$, consider the automorphism of S given by

$$\Lambda(\Phi)\colon [x] \longmapsto \Phi([x]) \tag{38}$$

for $[x] \in S$. Note that by definition Λ is a group homomorphism. We study $\mathrm{Col}(\mathcal{U})$ via the homomorphism

$$h \circ \Lambda\colon\ \mathrm{Col}(\mathcal{U}) \longrightarrow \mathrm{Aut}(\mathcal{I}^{[1,0]^t}). \tag{39}$$

Since $\mathcal{A}$ fixes the line $[0]$, it can be shown that $(h \circ \Lambda)|_{\mathrm{Col}(\mathcal{U}) \cap \mathcal{A}}$ is injective. Since any autotopism $\gamma(P,Q,R) \in \mathrm{Col}(\mathcal{U}) \cap \mathcal{A}$ has the from given in (6) and (8), we have

$$\begin{aligned} &(h \circ \Lambda)(\gamma(P,Q,R))\colon \\ &\qquad [u + \kappa v, 1]^t \longmapsto [q_1 u^{\varepsilon_Q} + q_2 (v^{\sigma^{-1}})^{\varepsilon_Q} + \kappa(q_3 u^{\varepsilon_Q} + q_4 v^{\sigma^{-1}\varepsilon_Q})^{\sigma}, 1]^t. \end{aligned} \tag{40}$$

Since $(h \circ \Lambda)(\gamma(P,Q,R)) \in \mathrm{Aut}(\mathcal{I}^{[1,0]^t})$ and $P = \alpha Q \alpha^{-1}$, a tedious computation shows $(h \circ \Lambda|_{\mathrm{Col}(\mathcal{U}) \cap \mathcal{A}})(\mathrm{Col}(\mathcal{U}) \cap \mathcal{A})$ is $\{\varphi \in \mathrm{Aut}(\mathcal{I}^{[1,0]^t}) \mid \varphi([z,1]^t) = [a_1 z^{\epsilon}, 1]^t$ for some $a_1 \in \mathbb{F}_q^*, \epsilon \in \mathrm{Aut}(\mathbb{F}_{q^2})\}$ or $\{\varphi \in \mathrm{Aut}(\mathcal{I}^{[1,0]^t}) \mid \varphi([z,1]^t) = [(a_1 + \kappa a_2) z^{\epsilon}, 1]^t$

for some $a_1, a_2 \in \mathbb{F}_q, a_1a_2 = 0, \epsilon \in \mathrm{Aut}(\mathbb{F}_{q^2})\}$, depending on whether $\sigma^2 \neq \mathrm{id}$ or not. We retrieve $\mathrm{Col}(\mathcal{U}) \cap \mathcal{A}$ ((26) and (27)). Theorem 4.2 can be obtained again by studying $h \circ \Lambda|_{\mathrm{Col}(\mathcal{U})\cap\mathcal{A}}$.

To retrieve $\mathcal{A}$, we need the result [13, the seventh paragraph of the introduction and Theorem 1.2] that any polar unital in $\Pi(\mathcal{K})$ is conjugate to $\mathcal{U}$. Recall from [13] that any unitary polarity is conjugate to one that maps $(0,0)$ to $[0,0]$, and that ([13, Theorem 1.2]) any unitary polarity mapping $(0,0)$ to $[0,0]$ is conjugate to ρ by an element of the group $\mathcal{G}_\rho$ given by

$$G_\rho = \{\gamma(P,Q,R) \in \mathcal{A} \mid P,Q,R, \text{ given by (9) with } \varepsilon = \mathrm{id}, p_4 = 1\}. \qquad (41)$$

Note that $G_\rho \cong \mathbb{F}_q^*$. Then, by [13, Corollary 1.3], any autotopism of $\Pi(\mathcal{K})$ is a unique composition of an element of $\mathrm{Col}(\mathcal{U}) \cap \mathcal{A}$ and an element of G_ρ. Thus we are able to determine $\mathcal{A}$ and hence recover the result of Sandler and Theorem 3.1. Furthermore, it is straighforward to verify the following alternative description of the structure of $\mathcal{A}$; we omit the computational details here.

Theorem 5.1. *When $\sigma^2 \neq \mathrm{id}$, $\mathcal{A}(\sigma) = (E \ltimes D') \ltimes G_\rho$ where E and D' are defined by* (30) *and* (31).

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