Disjoint unions of dimensional dual hyperovals

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Abstract

The notion of sub-objects and their disjoint union is introduced for a dimensional dual arc (Section 1.2). This naturally motivates a problem to decompose a dimensional dual hyperoval (DHO for short) into the disjoint union of some subdual arcs, including a subDHO (Section 1.3), because such an expression is useful to calculate its universal cover, as suggested by an elementary observation about its ambient space (Proposition 2.1). Under mild restrictions, a criterion is obtained for a DHO $\mathcal{B}_1$ of rank n over two element field to be extended to a DHO $\mathcal{A}$ of rank $n+1$ so that $\mathcal{A}$ is a disjoint union of $\mathcal{B}_1$ and some subDHO $\mathcal{B}_2$ of rank n (Theorem 1.3(i)). Under the choice of a complement to $\mathcal{B}_1$, such $\mathcal{A}$ as well as $\mathcal{B}_2$ are uniquely determined by $\mathcal{B}_1$, if they exist (Theorem 1.3(ii)). Several known families of DHOs are examined whether they can be extended to DHOs in the above form, but no example is found unless they are bilinear. If a subDHO $\mathcal{B}_1$ is bilinear over a specified complement, the criterion is satisfied, and thus there exists a unique pair $(\mathcal{A}, \mathcal{B}_2)$ of DHOs satisfying the above conditions (Corollary 1.4). This makes clear the meaning of the construction, called "extension" in [1, Section 5] for bilinear DHOs.

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1. Introduction

1.1. Dimensional dual arcs (DA) and hyperovals (DHO)

For a natural number n with $n \geq 2$, a *dimensional dual arc* (*DA*, for short) *of rank* n *over* a finite field $\mathbb{F}_q$ is a nonempty collection $\mathcal{A}$ of subspaces of a vector

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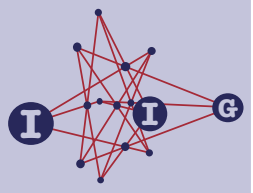


space U over $\mathbb{F}_q$ which satisfies the following conditions (A0)–(A2) (we allow the case where $\mathcal{A}$ consists of one or two members):

(A0) Each member X of $\mathcal{A}$ has vector dimension n;

(A1) $\dim(X \cap Y) = 1$ for any two distinct members X and Y of $\mathcal{A}$;

(A2) $X \cap Y \cap Z = \{0\}$ for any mutually distinct three members of $\mathcal{A}$.

The subspace of U spanned by all members of $\mathcal{A}$ is called the *ambient space* of $\mathcal{A}$ and denoted $\mathbf{U}(\mathcal{A})$ (instead of the usual notation $\mathbf{A}(\mathcal{A})$ as in [9]).

Every DA of rank n over $\mathbb{F}_q$ has at most $((q^n-1)/(q-1))+1 = \sum_{k=1}^{n-1} q^k + 2$ members. A DA of rank n over $\mathbb{F}_q$ is called a *dimensional dual hyperoval* (*DHO*, for short) if it consists of $((q^n-1)/(q-1))+1$ members.

For two DAs $\mathcal{A}_i$ of rank n over $\mathbb{F}_q$ ($i = 1, 2$) with $|\mathcal{A}_2| \geq 2$, we say that $\mathcal{A}_1$ *covers* $\mathcal{A}_2$ (or $\mathcal{A}_2$ is a *quotient* of $\mathcal{A}_1$) if there is a surjective semilinear map ρ from $\mathbf{U}(\mathcal{A}_1)$ to $\mathbf{U}(\mathcal{A}_2)$ (thus if $q = 2$, ρ is just an $\mathbb{F}_2$-linear surjection) which maps $\mathcal{A}_1$ surjectively onto $\mathcal{A}_2$ and sends each member of $\mathcal{A}_1$ isomorphically to a member of $\mathcal{A}_2$. (In particular, ρ induces a bijective correspondence of the members of $\mathcal{A}_1$ with the members of $\mathcal{A}_2$.) When $\dim(\mathbf{U}(\mathcal{A}_1)) = \dim(\mathbf{U}(\mathcal{A}_2))$, we say that $\mathcal{A}_1$ is *isomorphic* to $\mathcal{A}_2$ if $\mathcal{A}_1$ covers $\mathcal{A}_2$. When $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, a map ρ on $\mathbf{U}(\mathcal{A})$ with this property is called an *automorphism* of $\mathcal{A}$. The set $\mathrm{Aut}(\mathcal{A})$ of all automorphisms of $\mathcal{A}$ forms a group with respect to the composition of maps, namely, $\mathrm{Aut}(\mathcal{A})$ is the group of all bijective semilinear transformations on $\mathbf{U}(\mathcal{A})$ which preserve $\mathcal{A}$. A DA $\mathcal{A}$ of rank n over $\mathbb{F}_q$ is called *simply connected* if any cover of $\mathcal{A}$ coincides with $\mathcal{A}$.

The notion of covers and quotients among DAs (specifically DHOs) naturally arises as an analogue to that of the universal covers and quotients of the associated incidence geometries, and has been used as one of the main tools to investigate DAs.

1.2. SubDAs and their disjoint unions

On the other hand, the notion of subDAs and their (disjoint) unions attracted less attention so far, although it naturally arises in the following manner.

Let $\mathcal{A}$ be a DHO of rank n over $\mathbb{F}_q$ with $n \geq 3$. Fix a positive integer n' satisfying $2 \leq n' < n$. From each member A of $\mathcal{A}$, we pick a subspace B of dimension n' and consider their collection $\mathcal{B}$. Needless to say, there are several choices for B inside each A, and accordingly, many possibilities for $\mathcal{B}$. However, any two distinct members of $\mathcal{B}$ do not contain a 2-dimensional subspace in common, because their intersection is contained in two distinct members of a DA $\mathcal{A}$. Furthermore, three mutually distinct members of $\mathcal{B}$ intersect at the zero space.

Thus we can decompose $\mathcal{B}$ into a disjoint union $\cup_{j=1}^{m}\mathcal{B}_j$, allowing some $\mathcal{B}_k$ with $|\mathcal{B}_k| = 1$, so that each $\mathcal{B}_j$ is a DA. Let $\mathcal{A}_j$ be the subset of $\mathcal{A}$ consisting of members containing some members of $\mathcal{B}_j$ $(j = 1, \ldots, m)$. As $n' \geq 2$, each member of $\mathcal{B}_j$ is contained in exactly one member of $\mathcal{A}_j$. Thus the above decomposition of $\mathcal{B} = \cup_{j=1}^{m}\mathcal{B}_j$ induces an expression of $\mathcal{A}$ as a disjoint union $\mathcal{A} = \cup_{j=1}^{m}\mathcal{A}_j$ of subsets $\mathcal{A}_j$ determined by 'sub DAs' $\mathcal{B}_j$ of $\mathcal{A}$ of rank n' $(j = 1, \ldots, m)$.

Now we formalize these terminologies and fix notation:

Definition 1.1. Let $\mathcal{A}$ be a DA of rank n over $\mathbb{F}_q$ with $n \geq 3$. Fix an integer n' with $2 \leq n' < n$. A DA $\mathcal{B}$ of rank n' over $\mathbb{F}_q$ is called a *subDA* of $\mathcal{A}$, if each member of $\mathcal{B}$ is contained in a member of $\mathcal{A}$. Throughout the paper, we use the following notation to denote the set of members of $\mathcal{A}$ containing members of a subDA $\mathcal{B}$:

$$\mathcal{A}(\mathcal{B}) := \{A \in \mathcal{A} \mid B \subset A \text{ for some } B \in \mathcal{B}\}. \tag{1}$$

Two subDAs $\mathcal{B}$ and $\mathcal{B}'$ of $\mathcal{A}$ are said to be *disjoint* if $\mathcal{A}(\mathcal{B}) \cap \mathcal{A}(\mathcal{B}')$, regarded as a subset of $\mathcal{A}$, is the empty set. We say that $\mathcal{A}$ is the *disjoint union* of subDAs $\mathcal{B}_j$ $(j = 1, \ldots, m)$ and denote $\mathcal{A} = \sqcup_{j=1}^{m}\mathcal{B}_j$, if $\mathcal{A} = \cup_{j=1}^{m}\mathcal{A}(\mathcal{B}_j)$ and $\mathcal{A}(\mathcal{B}_j) \cap \mathcal{A}(\mathcal{B}_k) = \emptyset$ for any distinct j, k in $\{1, \ldots, m\}$.

With the notation in Definition 1.1, we have $\mathbf{U}(\mathcal{B}) \subseteq \mathbf{U}(\mathcal{A}(\mathcal{B})) \subseteq \mathbf{U}(\mathcal{A})$. For subDAs $\mathcal{B}$ and $\mathcal{B}'$ of $\mathcal{A}$, if $\mathcal{B} \cap \mathcal{B}' \neq \emptyset$, then $\mathcal{A}(\mathcal{B}) \cap \mathcal{A}(\mathcal{B}') \neq \emptyset$. As $n' \geq 2$, every member of $\mathcal{B}$ is contained in a unique member of $\mathcal{A}(\mathcal{B})$.

On the other hand, if $n \geq 4$ and we choose an integer n' so that $n' < n \leq 2n' - 2$ (observe that such an integer always exists, e.g. $n' = n - 1$), then every member of $\mathcal{A}(\mathcal{B})$ contains exactly one member of $\mathcal{B}$: for otherwise, the subspace of dimension $2n' - 1$ spanned by two distinct members of $\mathcal{B}$ would be contained in a member of $\mathcal{A}$, a subspace of dimension n, which implies that $2n' - 1 \leq n$, contradicting the choice of n'. Thus, in this case, there is a bijective correspondence between $\mathcal{B}$ and $\mathcal{A}(\mathcal{B})$. This in particular implies that $|\mathcal{A}(\mathcal{B})| = |\mathcal{B}|$. The same argument as above shows that $\mathcal{B} \cap \mathcal{B}' = \emptyset$ if and only if $\mathcal{A}(\mathcal{B}) \cap \mathcal{A}(\mathcal{B}') = \emptyset$. Thus we have $\mathcal{A} = \sqcup_{j=1}^{m}\mathcal{B}_j$ if and only if $\mathcal{A} = \cup_{j=1}^{m}\mathcal{A}(\mathcal{B}_j)$ and $\mathcal{B}_j \cap \mathcal{B}_k = \emptyset$ for any distinct j, k in $\{1, \ldots, m\}$. Furthermore, two distinct members of $\mathcal{A}(\mathcal{B}_j)$ intersect at a 1-dimensional subspace lying in a member of $\mathcal{B}_j$. On the other hand, if j and k are distinct indexes in $\{1, \ldots, m\}$, we have $A_j \cap A_k \not\subset B_j \cup B_k$ for any member A_i of $\mathcal{A}(\mathcal{B}_i)$ containing $B_i \in \mathcal{B}_i$ $(i = j, k)$. These remarks will be frequently used later, sometimes without further reference.

With the terminologies in Definition 1.1, the observation made in the first paragraph in this subsection can be stated as follows:
Fix any natural number n' with $2 \leq n' < n$. An arbitrary DA $\mathcal{A}$ of rank n (≥ 3)

over $\mathbb{F}_q$ can be expressed as a disjoint union of several sub DAs $\mathcal{B}_j$ $(j = 1, \ldots, m)$ of rank n'; $\mathcal{A} = \sqcup_{j=1}^{m} \mathcal{B}_j$.

1.3. Assembling DAs into their disjoint union

It seems unlikely to derive much information about $\mathcal{A}$ from this expression $\mathcal{A} = \sqcup_{j=1}^{m} \mathcal{B}_j$, if all subDAs $\mathcal{B}_j$ consist of small members. In contrast, $\mathcal{A}$ is strongly restricted by subDAs $\mathcal{B}_j$ $(j = 1, \ldots, m)$ of *corank* 1 (namely of rank $n' = n - 1$), if one of the subDAs is in fact a DHO, because in this case we can show that $\mathbf{U}(\mathcal{A})$ is spanned by $\mathbf{U}(\mathcal{B}_j)$ for all $j = 1, \ldots, m$ together with a single 1-dimensional subspace (see Proposition 2.1(2)). Thus it is important to find an expression of a DA $\mathcal{A}$ as a disjoint union of some subDAs $\mathcal{B}_j$ $(j = 1, \ldots, m)$, including at least one DHO.

For that purpose, we shall start from a family $\{\mathcal{B}'_j\}_{j=1}^{m}$ of *any* DAs of constant rank n $(n \geq 3)$, in which $\mathcal{B}'_1$ is a DHO. We choose semilinear injections ι_j from $\mathbf{U}(\mathcal{B}'_j)$ into a common vector space U so that $(\mathcal{B}'_j)^{\iota_j} \cap (\mathcal{B}'_k)^{\iota_k} = \emptyset$ (as subsets of the set of n-dimensional subspaces of U) for any distinct j, k in $\{1, \ldots, m\}$. In this situation, if there is a DA $\mathcal{A}$ consisting of $(n+1)$-dimensional subspaces of U for which $\mathcal{A} = \sqcup_{j=1}^{m} (\mathcal{B}'_j)^{\iota_j}$, we obtain the desired expression.

This poses the following problem.

Problem 1. *Let $\mathcal{B}'_j$ $(j = 1, \ldots, m)$ be a family of DAs of rank n over $\mathbb{F}_q$ with $n \geq 3$, in which $\mathcal{B}'_1$ is a DHO. Choose a vector space U over $\mathbb{F}_q$ of dimension $> n$.*

Find a family of injective semilinear maps ι_j from $\mathbf{U}(\mathcal{B}'_j)$ to U for each $j \in \{1, \ldots, m\}$ which satisfies the following conditions, where we set $\mathcal{B}_j := (\mathcal{B}'_j)^{\iota_j}$:

(c0) *$\mathcal{B}_j \cap \mathcal{B}_k = \emptyset$ for any distinct j, k in $\{1, \ldots, m\}$.*

(c1) *there exists a DA $\mathcal{A}$ consisting of $(n+1)$-dimensional subspaces of U which has all $\mathcal{B}_j$'s $(j = 1, \ldots, m)$ as subDAs and $\mathcal{A} = \cup_{j=1}^{m} \mathcal{A}(\mathcal{B}_j)$.*

By Proposition 2.1(2), the ambient space $\mathbf{U}(\mathcal{A})$ is spanned by $\mathbf{U}(\mathcal{B}_j)$ $(j = 1, \ldots, m)$ together with a single 1-dimensional subspace $A \cap A'$ for some distinct $A, A' \in \mathcal{A}$. Hence, in order to find a subspace U in the above problem, it suffices to investigate a quotient vector space of the direct sum $E \oplus (\oplus_{j=1}^{m} \mathbf{U}(\mathcal{B}'_j))$ (with E a 1-dimensional space corresponding to $A \cap A'$) by a suitable surjective semilinear map ρ whose restriction onto each $\mathbf{U}(\mathcal{B}'_j)$ is bijective. If we succeed in constructing a DA in question inside such a quotient space then the desired ι_j is given as the restriction of ρ onto $\mathbf{U}(\mathcal{B}'_j)$ $(j = 1, \ldots, m)$. This implies that the solution for the above problem gives us not only a single DA but also its covers as well.

In particular, the solution may provide an efficient method for calculating universal covers of known family of DAs. It may provide a class of new (simply connected) DAs $\mathcal{A}$, starting from known (simply connected) DHOs $\mathcal{B}'_j$. The examples in the later sections will provide an evidence of these effects brought us by the solution of the above problem.

The above problem is a concrete version of the question posed as [6, Question 5]. Its origin was traced back to the attempt to characterize a DHO from the 'subDHO' realized as the 'centralizer' of a certain involutive automorphism [9, Section 3.2].

1.4. A DHO expressed as a disjoint union of its subDHOs

The decomposition $\mathcal{A} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_m$ seems to be most useful in the case where $\mathcal{A}$ is a DHO of rank $n+1$ and all $\mathcal{B}_j$'s are DHOs of rank n over the common finite field $\mathbb{F}_q$. However, as we shall show in Section 2.2, this gives a very strong restriction:

Proposition 1.2. *Let $\mathcal{A}$ be a DHO of rank $n+1$ over $\mathbb{F}_q$ with $n \geq 3$. If $\mathcal{A} = \sqcup_{j=1}^{m} \mathcal{B}_j$ for subDHOs $\mathcal{B}_j$ of rank n for all $j = 1, \ldots, m$, then the following hold:*

(1) *We have $m = 2$ and $q = 2$.*

(2) *$\mathbf{U}(\mathcal{A})$ is spanned by the members of $\mathcal{B}_1$ and a member A_2 of $\mathcal{A}(\mathcal{B}_2)$.*

(3) *If, furthermore, $\mathcal{B}_1$ is simply connected and $\dim \mathbf{U}(\mathcal{A}) = \dim \mathbf{U}(\mathcal{B}_1) + n + 1$, then $\mathcal{A}$ is simply connected.*

In the surviving case with $q = 2 = m$, a complete solution for the problem in Section 1.3 will be given in Theorem 1.3 when $\mathbf{U}(\mathcal{A}) = A_2 \oplus \mathbf{U}(\mathcal{B}_1)$ for $A_2 \in \mathcal{A}(\mathcal{B}_2)$ (see Proposition 1.2) and $\mathcal{B}_1$ 'splits over' Y_1 (see Section 1.5 for the precise definition). It will be given in a slightly general form: We take an arbitrary split DHO of rank n over $\mathbb{F}_2$ as $\mathcal{B}_1$ and a vector space U containing the direct sum $A_2 \oplus \mathbf{U}(\mathcal{B}_1)$. Then we give conditions for the existence of an DHO $\mathcal{A}$ of rank $n+1$ over $\mathbb{F}_2$ with $\mathbf{U}(\mathcal{A}) \subseteq U$ which has $\mathcal{B}_1$ as a subDHO and $\mathcal{A} = \mathcal{B}_1 \sqcup \mathcal{B}_2$ for a subDHO $\mathcal{B}_2$ with $A_2 \in \mathcal{A}(\mathcal{B}_2)$. Notice that the splitness of $\mathcal{B}_1$ may not give any restriction, in fact, in view of the fact that any known DHO over $\mathbb{F}_2$ is of split type.

1.5. Split DHOs over $\mathbb{F}_2$

We say that a DA $\mathcal{A}$ *splits* over a subspace Y of the ambient space $\mathbf{U}(\mathcal{A})$, if $X \oplus Y = \mathbf{U}(\mathcal{A})$ for every member X of $\mathcal{A}$. The subspace Y is called a *complement* to $\mathcal{A}$. A DA is said to be of *split* or *non-split* type according as there does or does

not exist a complement to it. As far as the author knows, all known DAs are of split type. DHOs over the two element field $\mathbb{F}_2$ of split type are specifically useful, because they are described by the collection of certain linear maps in the following way.

We fix a vector space X of dimension n over $\mathbb{F}_2$, and adopt X as an index set which parametrizes the members of a DHO $\mathcal{S}$ of rank n over $\mathbb{F}_2$; $\mathcal{S} = \{A(t) \mid t \in X\}$. Assume that Y is a complement to $\mathcal{S}$. Then each vector v of a member $A(t)$ ($t \in X$) is uniquely decomposed in the form $v = x + y$ with $x \in A(0)$ and $y \in Y$. As $Y \cap A(t) = \{0\}$, the vector y is uniquely determined by x, whence the map $L(t)$ sending x to y is a well-defined $\mathbb{F}_2$-linear map from $A(0)$ into Y; $A(t) = \{x + xL(t) \mid x \in A(0)\}$. (In this paper, we adopt the convention to denote by xL the image of a vector x by a linear map L, instead $L(x)$ or x^L, if it arises in an expression of a typical vector in a member of a DHO.) Thus we obtain a collection $\mathcal{L}_X(\mathcal{S}, Y) := \{L(t) \mid t \in X\}$ of linear maps from $A(0)$ to Y, parametrized by X. Notice that $\mathcal{L}_X(\mathcal{S}, Y)$ is uniquely determined by $\mathcal{S}$, Y and the index set X up to the permutations on the vectors in X fixing 0.

As $\mathcal{S}$ is a DHO, the collection $\mathcal{L}_X(\mathcal{S}, Y) = \{L(t) \mid t \in X\}$ has the following properties:

(L0) $L(0) = 0$, the zero map.

(L1) For any distinct vectors s and t in X, the kernel of the linear map $L(s) + L(t)$ has dimension 1. We denote by $\kappa(s, t)$ the unique nonzero vector of this kernel. For short, we write $\kappa_s = \kappa(0, s)$ for $s \in X$ if $s \neq 0$, with convention $\kappa_0 = 0$.

(L2) For each $t \in X$, the map sending $s \in X \setminus \{t\}$ to $\kappa(s, t)$ is a bijection from $X \setminus \{t\}$ to $A(0) \setminus \{0\}$.

Conversely, if a collection of linear maps $\{L(t) \mid t \in X\}$ from $A(0)$ to Y satisfying (L0)–(L2) exists, then we obtain a DHO $\mathcal{S} = \{A(t) \mid t \in X\}$ of rank n over $\mathbb{F}_2$ splitting over Y, with members parametrized by X, by defining $A(t) := \{x + xL(t) \mid x \in A(0)\}$ for each $t \in X$. For distinct s, t in X, we denote by $a\{s, t\}$ the unique nonzero vector contained in $A(s) \cap A(t)$. Then we have

$$a\{s, t\} = \kappa(s, t) + \kappa(s, t)L(s) = \kappa(s, t) + \kappa(s, t)L(t).$$

We refer to a collection $\mathcal{L}_X(\mathcal{S}, Y) = \{L(t) \mid t \in X\}$ of linear maps $A(0)$ from Y, parametrized by X, satisfying the above conditions (L0)–(L2) as *the linear system for $\mathcal{S}$ over Y*. Needless to say, any permutation of the index set X fixing 0 gives the same linear system for $\mathcal{S}$. On the other hand, if $\mathcal{S}$ splits over another subspace Y' of $\mathbf{U}(\mathcal{S})$, then the linear system $\mathcal{L}_X(\mathcal{S}, Y') = \{L'(t) \mid t \in X\}$ over Y' is related to $\mathcal{L}_X(\mathcal{S}, Y)$ as follows: for each $t \in X$ there exists a linear bijection σ_t from $A(0)$ to itself such that $L'(t) = \sigma_t L(t)$.

We say that a DHO $\mathcal{S}$ over $\mathbb{F}_2$ is *bilinear over* Y, if it splits over a subspace Y and there is an ordering on the vector space X parametrizing the members of $\mathcal{S}$ such that the linear system $\mathcal{L}_X(\mathcal{S}, Y) = \{L(t) \mid t \in X\}$ for $\mathcal{S}$ over Y satisfies the following equation:

$$L(s+t) = L(s) + L(t) \text{ for any } s, t \in X. \tag{2}$$

Notice that this notion *does* depend on the particular choice of a complement Y to $\mathcal{S}$. (See examples in Section 3.3.) On the other hand, a DHO $\mathcal{S}$ is called *bilinear* if there exists a complement over which $\mathcal{S}$ is bilinear. Needless to say, this notion depends only on $\mathcal{S}$.

1.6. Main results

Now we state our main result, which gives an almost complete solution to the problem in Section 1.3 for $q = 2 = m$. This gives a criterion for a DHO $\mathcal{B}_1$ of rank n over $\mathbb{F}_2$ to be a subDHO of a DHO $\mathcal{A}$ of rank $n+1$ over $\mathbb{F}_2$ with $\mathcal{A} = \mathcal{B}_1 \sqcup \mathcal{B}_2$ for some DHO $\mathcal{B}_2$ of rank n.

Theorem 1.3. *Let $\mathcal{B}_1$ be a DHO of rank n ($n \geq 3$) over $\mathbb{F}_2$ whose members are subspaces of a vector space U with $\dim(U) \geq \dim \mathbf{U}(\mathcal{B}_1) + n + 1$. Fix a member B_1 of $\mathcal{B}_1$ and take a subspace A_2 of U of dimension $n+1$ with $A_2 \cap \mathbf{U}(\mathcal{B}_1) = \{0\}$ as well as a hyperplane B_2 of A_2. We assume that $\mathcal{B}_1$ is of split type. Then the following hold.*

(i) *There exists a pair $(\mathcal{B}_2, \mathcal{A})$ of DHOs $\mathcal{B}_2$ and $\mathcal{A}$ of rank n and $n+1$, respectively, over $\mathbb{F}_2$ which satisfies the following conditions*

$$\mathbf{U}(\mathcal{A}) \subseteq U,\ B_2 \in \mathcal{B}_2,\ A_2 \in \mathcal{A},$$
$$\mathcal{B}_1 \text{ and } \mathcal{B}_2 \text{ are subDHOs of } \mathcal{A} \text{ with } \mathcal{A} = \mathcal{B}_1 \sqcup \mathcal{B}_2,$$

if and only if the members of $\mathcal{B}_1$ are parametrized by B_2 (so that each member of $\mathcal{B}_1$ is expressed as $B_1(c)$ for $c \in B_2$}) such that $B_1(0) = B_1$ and

$$b_1\{d + x_1 + x_2, x_1 + x_2\} = b_1\{d + x_1, x_1\} + b_1\{d + x_2, x_2\} + b_1\{d, 0\} \tag{3}$$

for all $d \in B_2^\times = B_2 \setminus \{0\}$ and all $x_1, x_2 \in B_2$, where $b_1\{c, d\}$ denotes the unique nonzero vector of $B_1(c) \cap B_1(d)$ for distinct $c, d \in B_2$.

(ii) *Fix a complement Y_1 to $\mathcal{B}_1$ and a vector a_{12} of $A_2 \setminus B_2$. If there is a parametrization of the members of $\mathcal{B}_1$ by B_2 such that $B_1(0) = B_1$ and equation (3) holds for all $d \in B_2^\times$ and all $x_1, x_2 \in B_2$, the pair $(\mathcal{B}_2, \mathcal{A})$ in Claim (i) with $A_1 := \langle a_{12}, B_1 \rangle \in \mathcal{A}$ is uniquely determined by $\mathcal{B}_1$ as follows: $\mathcal{A} = \{A_1(c), A_2(d) \mid c, d \in B_2\}$ and $\mathcal{B}_2 = \{B_2(c) \mid c \in B_2\}$ with*

$B_2(c) := A_2(c) \cap (B_2 + \mathbf{U}(\mathcal{B}_1))$ ($c \in B_2$), where $A_1(c)$ and $A_2(d)$ are subspaces of U defined by

$$A_1(c) := \{\varepsilon(a_{12} + c) + (x + xL_1(c)) \mid \varepsilon \in \mathbb{F}_2, x \in B_1\},$$
$$A_2(d) := \left\{ \begin{matrix} \varepsilon(a_{12} + \kappa_d) + x + (\kappa(x + d, x) + \kappa_d) \\ + \kappa(x + d, x)L_1(x) \end{matrix} \;\middle|\; \varepsilon \in \mathbb{F}_2, x \in B_2 \right\}.$$

In the above expressions, $L_1(c)$ denotes the linear map corresponding to $c \in B_2$ in the linear system $\mathcal{L}_{B_2}(\mathcal{B}_1, Y_1)$ for $\mathcal{B}_1$ over Y_1 with respect to the parametrization by B_2 satisfying the assumption. Furthermore, $\kappa(c, d)$ denotes the unique nonzero vector in the kernel of $L_1(c) + L_1(d)$ for distinct $c, d \in B_2$ with convention $\kappa(c, c) = 0$, and set $\kappa_d = \kappa(0, d)$ for $0 \neq d \in B_2$ with convention $\kappa_0 = 0$. (See Section 1.5.)

Defining a map $L_2(d)$ ($d \in B_2$) by $cL_2(d) := \kappa(d+c, c)+\kappa_d+\kappa(d+c, c)L_1(c)$ for $c \in B_2$, the DHO $\mathcal{B}_2$ splits over the subspace Y_2 of $B_1 \oplus Y_1$ spanned by $cL_2(d)$ for all $c, d \in B_2$, with the linear system $\mathcal{L}_{B_2}(\mathcal{B}_2, Y_2) = \{L_2(d) \mid d \in B_2\}$.

In the case when $\mathcal{B}_1$ is bilinear over Y_1 (see Section 1.5), it turns out that the criterion in Theorem 1.3(i) is satisfied, and thus, there always exists a pair $(\mathcal{B}_2, \mathcal{A})$ of DHOs satisfying the conditions there. The resulting DHO $\mathcal{A}$ has additional properties given in Corollary 1.4(ii), (iii) below.

Corollary 1.4. *Under the same setting as in Theorem 1.3, assume furthermore that $\mathcal{B}_1$ is bilinear over a complement Y_1. Fix subspaces B_1, A_2 and B_2 of U in the setting of Theorem 1.3 (ii) as well as the complement Y_1 and a vector a_{12} in $A_2 \setminus B_2$. Then the following hold.*

(i) *There exists a unique pair $(\mathcal{B}_2, \mathcal{A})$ which satisfies*

> *$\mathcal{B}_2$ is a DHO of rank n over $\mathbb{F}_2$, $B_2 \in \mathcal{B}_2$, $\mathcal{A}$ is a DHO of rank $n+1$ over $\mathbb{F}_2$, $\mathbf{U}(\mathcal{A}) \subseteq U$, $A_2 \in \mathcal{A}$, $A_1 := \langle a_{12}, B_1 \rangle \in \mathcal{A}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are subDHOs of $\mathcal{A}$ with $\mathcal{A} = \mathcal{B}_1 \sqcup \mathcal{B}_2$.*

Explicitly, $\mathcal{A} = \{A_1(c), A_2(d) \mid c, d \in B_2\}$ with

$$A_1(c) = \{\varepsilon(a_{12} + c) + (x + xL_1(c)) \mid \varepsilon \in \mathbb{F}_2, x \in B_1\},$$
$$A_2(d) = \{\varepsilon(a_{12} + \kappa_d) + x + \kappa_d L_1(x) \mid \varepsilon \in \mathbb{F}_2, x \in B_2\}.$$

The DHO $\mathcal{B}_2$ splits over the subspace $Y_1 = \mathbf{U}(\mathcal{B}_1) \cap \mathbf{U}(\mathcal{B}_2)$ with the linear system $\mathcal{L}_{B_2}(\mathcal{B}_2, Y_1) = \{L_2(d) \mid d \in B_2\}$, $cL_2(d) := \kappa_d L_1(c)$ ($c \in B_2$).

(ii) *The DHO $\mathcal{A}$ above splits over a subspace $Y_\rho := \{e + e\rho + y \mid e \in B_2, y \in Y_1\}$ of U, defined for any linear bijection ρ from B_2 onto B_1. The linear system*

$\mathcal{L}_{B_2\times\mathbb{F}_2}(\mathcal{A}, Y_\rho)$ for $\mathcal{A}$ over Y_ρ consists of linear maps $L(c,\delta)$ given below ($c \in B_2$, $\delta \in \mathbb{F}_2$): for $\varepsilon \in \mathbb{F}_2$ and $y \in B_1$,

$$(\varepsilon a_{12} + y)L(c,0) := \varepsilon c + \varepsilon c\rho + (y + \varepsilon c\rho)L_1(c),$$
$$(\varepsilon a_{12} + y)L(c,1) := (\varepsilon\kappa_c + y)\rho^{-1} + (\varepsilon\kappa_c + y) + \kappa_c L_1((\varepsilon\kappa_c + y)\rho^{-1}). \quad (4)$$

(iii) *If $\mathcal{A}$ is bilinear over Y_ρ, then we have*

$$(c)\rho L_1(d) = (d)\rho L_1(c) \text{ for any } c, d \in B_2. \quad (5)$$

If this equation holds, then we have $L(c_1,0) + L(c_2,0) = L(c_1 + c_2, 0)$, $L(c_1,1) + L(c_2,1) = L((\kappa_{c_1} + \kappa_{c_2})\rho^{-1}, 0)$ for any $c_1, c_2 \in B_2$, and the two subDHOs $\mathcal{B}_1$ and $\mathcal{B}_2$ are isomorphic.

So far, the author did not find any example of DHOs which satisfies the criterion given in Theorem 1.3(i) but is not bilinear over any complement. It is an interesting problem to prove or disprove the following statement: if the criterion given in Theorem 1.3(i) is satisfies for a DHO $\mathcal{B}_1$ of split type then $\mathcal{B}$ is bilinear over some complement.

1.7. Some remarks

We conclude this section with several remarks about Theorem 1.3 and Corollary 1.4.

First, observe that the criterion given in Theorem 1.3(i) (the requirement that equation (3) holds for all $d, x_1, x_2 \in B_2$ with $d \neq 0$) is a property for the given DHO $\mathcal{B}_1$, independent of the choice of the particular complement Y_1 to $\mathcal{B}_1$, but depending on the choice of a particular member B_1 (parametrized by $0 \in B_2$) and the ordering of $B_2^\times$.

On the other hand, the DHO $\mathcal{A}$ in Theorem 1.3(ii) is uniquely determined by the DHO $\mathcal{B}_1$, if we fix the following four subspaces of U and a vector a_{12}:
a member B_1 of $\mathcal{B}_1$, a complement Y_1 to $\mathcal{B}_1$, a subspace A_2 of dimension $n+1$ with $A_2 \cap \mathbf{U}(\mathcal{B}_1) = \{0\}$, a hyperplane B_2 of A_2 and $a_{12} \in A_2 \setminus B_2$.
Thus we can denote $\mathcal{A} = \mathcal{A}(B_1, Y_1; A_2, B_2, A_{12})$. It follows from the above remark that if $\mathcal{A}(B_1, Y_1; A_2, B_2, a_{12})$ exists then $\mathcal{A}(B_1, Y_1'; A_2, B_2, a_{12})$ exists for any complement Y_1' to $\mathcal{B}_1$.

Another choice (A_2', B_2', a_{12}') for subspaces U satisfying $\dim(A_2') = n+1$, $A_2' \cap \mathbf{U}(\mathcal{B}_1) = \{0\}$, $\dim(B_2') = n$, $B_2' \subset A_2'$ and a vector $a_{12}' \in A_2' \setminus B_2'$ gives a DHO $\mathcal{A}' = \mathcal{A}(B_1, Y_1; A_2', B_2', a_{12}')$, which is shown to be isomorphic to $\mathcal{A}$. More generally, if there is an automorphism ρ_1 of $\mathcal{B}_1$ with $B_1' := (B_1)\rho_1$, then $Y_1' := (Y_1)\rho_1$ is another complement to $\mathcal{B}_1$ and $\mathcal{A}(Y_1'; B_1', A_2', B_2', a_{12}')$ is isomorphic to the original $\mathcal{A}$, because for a linear bijection ρ_2 from A_2 onto A_2' with

$(B_2)\rho_2 = B_2'$ and $a_{12}\rho_2 = a_{12}'$, the map from $\mathbf{U}(\mathcal{B}_1) \oplus A_2$ onto $\mathbf{U}(\mathcal{B}_1) \oplus A_2'$ sending $x + a$ with $x \in \mathbf{U}(\mathcal{B}_1)$ and $a \in A_2$ to $(x)\rho_1 + (a)\rho_2$ gives an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$. (However, in general, the DHO $\mathcal{A}(B_1', Y_1'; A_2', B_2', a_{12}')$ may not be isomorphic to $\mathcal{A}$ for another $(B_1', Y_1'; A_2', B_2', a_{12}')$.) Thus, for a fixed pair (B_1, Y_1) of a member B_1 of $\mathcal{B}_1$ and a complement Y_1 to $\mathcal{B}_1$, if $(\mathcal{B}_1, Y_1)$ satisfies equation (3) in Theorem 1.3, we are allowed to refer to the isomorphism class of $\mathcal{A}$ as the *extension* of $\mathcal{B}_1$ with respect to (B_1, Y_1).

It may be an interesting problem to investigate when $\mathcal{A}(B_1, Y_1)$ is isomorphic to $\mathcal{A}(B_1, Y_1')$. The author suspects that they are not isomorphic in general. For example, we will see in Section 3.3 that a DHO $\mathcal{B}_1$, denoted $\mathcal{Y}^{\sigma,\tau}$ there, is bilinear over a complement Y, but not bilinear over another complement Y'. The DHO $\mathcal{A}(\mathcal{Y}^{\sigma,\tau}, Y)$ is shown to be bilinear if $\sigma\tau = \mathrm{id}_F$ or $\sigma = \tau$. On the other hand, for these pairs (σ, τ), the author has not yet succeeded in finding a complement over which $\mathcal{A}(\mathcal{Y}^{\sigma,\tau}, Y')$ is bilinear.

It should be mentioned that the notion of 'extension' introduced in [1, Section 5] corresponds to the extension in Corollary 1.4(iii) (with respect to a pair (B_1, Y_1) of a member of $\mathcal{B}_1$ and a complement to $\mathcal{B}_1$) in which $\mathcal{B}_1 = \mathcal{S}$ is determined by a collection of 'symmetric' bilinear maps. Namely, $\mathcal{S}$ is a split DHO determined by a collection of bilinear maps $\{L(t) \mid t \in X\}$ from X to itself, parametrized by a vector space X of dimension n over $\mathbb{F}_2$, which satisfies equation (2) as well as the symmetric property $xL(y) = yL(x)$ for all $x, y \in X$. Starting from such a DHO $\mathcal{S}$ of rank n over $\mathbb{F}_2$, the authors of [1] construct a split DHO $\overline{\mathcal{S}}$ of rank $n+1$ over $\mathbb{F}_2$. In their model, B_2 and B_1 in Corollary 1.4 are identified with X so that $\mathbf{U}(\overline{\mathcal{S}}) = \mathbb{F}_2 \oplus X \oplus X \oplus Y_1$, and Y_ρ in Corollary 1.4(ii) corresponds to the disjoint sum of the diagonal subspace of $X \oplus X$ and Y_1 so that ρ is the identity map. They are mainly interested in those $\overline{\mathcal{S}}$ which are bilinear over this subspace Y_{id_X}. Thus Corollary 1.4(iii) gives an account for their assumption that $\mathcal{S}$ is symmetric bilinear. Notice that the resulting $\overline{\mathcal{S}}$ may not be bilinear, even if $\mathcal{S}$ is symmetric bilinear. Even when this stronger assumption fails, we can obtain a split DHO (which is non-bilinear in general) as a disjoint union of two subDHOs in view of Corollary 1.4(i). This phenomenon has already been observed by Taniguchi for the Yoshiara DHOs, which are bilinear but not symmetric in general [4]. See also the examples in Section 3.

2. Proofs of results in the introduction

2.1. A preliminary result

Proposition 2.1. *Assume that $\mathcal{A}$ is a DA of rank $n+1$ over $\mathbb{F}_q$ with $n \geq 3$ which is expressed as a disjoint union $\mathcal{A} = \sqcup_{j=1}^{m} \mathcal{B}_j$ of at least two subDAs $\mathcal{B}_j$ ($j = 1, \ldots, m$,*

$m \geq 2$) of rank n in which $\mathcal{B}_1$ is a DHO. Then the following holds:

(1) *For an arbitrary member A_2 of $\mathcal{B}_2$, we have $\mathbf{U}(\mathcal{A}(\mathcal{B}_1)) \subseteq \mathbf{U}(\mathcal{B}_1) + A_2$.*

(2) *$\mathbf{U}(\mathcal{A})$ is spanned by $\mathbf{U}(\mathcal{B}_j)$ $(j = 1, \ldots, m)$ and a single 1-dimensional subspace $A_1 \cap A_2$ with $A_j \in \mathcal{A}(\mathcal{B}_j)$ $(j = 1, 2)$.*

Proof. (1) Take any member A of $\mathcal{A}(\mathcal{B}_1)$ and let B be the unique member of $\mathcal{B}_1$ contained in A. As $\mathcal{B}_1$ is a DHO, there is a bijective correspondence between $\mathcal{B}_1 \setminus \{B\}$ and the set of 1-dimensional subspaces of B given by sending C $(\in \mathcal{B}_1 \setminus \{B\})$ to $B \cap C$. Every member C of $\mathcal{B}_1 \setminus \{B\}$ is contained in the unique member $A(C)$ of $\mathcal{A}(\mathcal{B}_1) \setminus \{A\}$ (as $n \geq 3$), whence we have $A(C) \cap A = C \cap B$. As the number of 1-dimensional subspaces in B is $(q^n - 1)/(q - 1)$, which is equal to $|\mathcal{A}(\mathcal{B}_1) \setminus \{A\}|$, we conclude that every member of $\mathcal{A} \setminus \{A\}$ containing a 1-dimensional subspace of B lies in $\mathcal{A}(\mathcal{B}_1) \setminus \{A\}$. As $\mathcal{A}(\mathcal{B}_1) \cap \mathcal{A}(\mathcal{B}_2) = \emptyset$, this implies that if we fix a member A_2 of $\mathcal{A}(\mathcal{B}_2)$ then $A \cap A_2$ does not lie in the hyperplane B of A. Hence $A = \langle B, A \cap A_2 \rangle \subseteq \mathbf{U}(\mathcal{B}_1) + A_2$. As this holds for any $A \in \mathcal{A}(\mathcal{B}_1)$, we have $\mathbf{U}(\mathcal{A}(\mathcal{B}_1)) \subseteq \mathbf{U}(\mathcal{B}_1) + A_2$.

(2) Take any $j \in \{2, \ldots, m\}$. Pick a member A' from $\mathcal{A}(\mathcal{B}_j)$, and let B' be the unique member of $\mathcal{B}_j$ contained in A'. For each 1-dimensional subspace of B', there is at most one member of $\mathcal{A} \setminus \{A'\}$ containing it, because $\mathcal{A}$ is a DA. There are $(q^n - 1)/(q - 1)$ 1-dimensional subspaces of B', while $\mathcal{A}(\mathcal{B}_1)$ contains $|\mathcal{B}_1| = ((q^n - 1)/(q - 1)) + 1$ members, as $\mathcal{B}_1$ is a DHO of rank n over $\mathbb{F}_q$. Thus there is a member A of $\mathcal{A}(\mathcal{B}_1)$ such that $A \cap A'$ is not contained in B'. As B' is a hyperplane of A', we have $A' = \langle B', A \cap A' \rangle \subseteq \mathbf{U}(\mathcal{B}_j) + \mathbf{U}(\mathcal{A}(\mathcal{B}_1))$. Since this holds for every $j \in \{2 \ldots, m\}$ and all $A' \in \mathcal{A}(\mathcal{B}_j)$, we conclude that $\mathbf{U}(\mathcal{A})$, which is spanned by $\mathbf{U}(\mathcal{A}(\mathcal{B}_1))$ and $\mathbf{U}(\mathcal{A}(\mathcal{B}_j))$ for all $j \in \{2, \ldots, m\}$, lies in the sum of $\mathbf{U}(\mathcal{B}_j)$ $(j = 2, \ldots, m)$ and $\mathbf{U}(\mathcal{A}(\mathcal{B}_1))$.

Together with claim (1), we conclude that $\mathbf{U}(\mathcal{A})$ is spanned by $\mathbf{U}(\mathcal{B}_j)$ $(j = 1, \ldots, m)$ and a member A_2 of $\mathcal{A}(\mathcal{B}_2)$. Let B_2 be the unique member of $\mathcal{B}_2$ contained in A_2. Since there are only $(q^n - 1)/(q - 1)$ 1-dimensional subspaces in B_2, among $((q^n - 1)/(q - 1)) + 1$ members of $\mathcal{A}(\mathcal{B}_1)$, there exists a member A_1 such that $A_1 \cap A_2$ does not lie in B_2. Then $A_2 = \langle B_2, A_1 \cap A_2 \rangle$. Thus $\mathbf{U}(\mathcal{A})$ is spanned by all $\mathbf{U}(\mathcal{B}_j)$ for $j \in \{1, \ldots, m\}$ and a single 1-dimensional subspace $A_1 \cap A_2$ with $A_j \in \mathcal{A}(\mathcal{B}_j)$ $(j = 1, 2)$. □

2.2. Proof of Proposition 1.2

(1) As $\mathcal{A}$ is a disjoint union of $\mathcal{A}(\mathcal{B}_j)$ for $j = 1, \ldots, m$, it follows from the assumption that

$$\frac{q^{n+1}-1}{q-1} + 1 = |\mathcal{A}| = \sum_{j=1}^{m} |\mathcal{A}(\mathcal{B}_j)| = \sum_{j=1}^{m} |\mathcal{B}_j| = \sum_{j=1}^{m} \Big(\frac{q^n - 1}{q-1} + 1 \Big).$$

Thus $\sum_{k=1}^{n} q^k + 2 = m((\sum_{l=1}^{n-1} q^l) + 2)$, from which we have

$$q^n = (m-1)\Big((\sum_{l=1}^{n-1} q^l) + 2 \Big). \tag{6}$$

This implies that $(q^{n-1} + \cdots + q) + 2$ should divide q^n. As $q = p^e$ for a prime p and an integer $e \geq 1$, we have $(q^{n-1} + \cdots + q) + 2 = p^f$ for some $e < f \leq en$. If p is an odd prime, this equation implies that 2 is congruent to 0 modulo p, which is impossible. If $p = 2$ and $e \geq 2$, this equation implies that 2 is congruent to 0 modulo 4, which is impossible. Thus we should have $q = p = 2$. Then $q^n = 2^n$ and $((q^n - 1)/(q-1)) + 1 = 2^n$, and hence $m = 2$ by equation (6).

(2) By Proposition 2.1(2), the ambient space $\mathbf{U}(\mathcal{A})$ is spanned by $\mathbf{U}(\mathcal{B}_1)$, $\mathbf{U}(\mathcal{B}_2)$, and a single 1-dimensional subspace $A_1 \cap A_2$ for members A_j of $\mathcal{A}(\mathcal{B}_j)$ $(j = 1, 2)$. Let B_2 be the unique member of $\mathcal{B}_2$ contained in A_2. Take any member B' of $\mathcal{B}_2 \setminus \{B_2\}$. In order to establish the claim, it suffices to show that B' is contained in the space spanned by the members of $\mathcal{B}_1$ and A_2. Let A' be unique member of $\mathcal{A}$ containing B'. Since $\mathcal{B}_2$ is a DHO, any member of $\mathcal{A}$ containing a 1-dimensional subspace of B' lies in $\mathcal{A}(\mathcal{B}_2)$. Thus for every A of $\mathcal{A}(\mathcal{B}_1)$, the intersection $A \cap A'$ does not lie in B'. As $|\mathcal{A}(\mathcal{B}_1)| = 2^n$, the intersections $A \cap A'$ for $A \in \mathcal{A}(\mathcal{B}_1)$ exhaust all 1-dimensional subspaces of A' outside a hyperplane B'. In particular, they span A'. Thus A', and hence B' lies in $\mathbf{U}(\mathcal{A}(\mathcal{B}_1))$. The latter is contained in $\mathbf{U}(\mathcal{B}_1)$ and A_2 by Proposition 2.1(1). This established the claim.

(3) Let $\tilde{\mathcal{A}}$ be a DHO which covers $\mathcal{A}$. Then there exists a linear surjection ρ from $\mathbf{U}(\tilde{\mathcal{A}})$ onto $\mathbf{U}(\mathcal{A})$ which induces an isomorphism from each member $\tilde{A}$ of $\tilde{\mathcal{A}}$ to a member of $\mathcal{A}$. Notice that the inverse images of members of $\mathcal{B}_j$ form a subDHO, denoted $\tilde{\mathcal{B}}_j$, in $\tilde{\mathcal{A}}$ for each $j = 1, 2$, and that $\tilde{\mathcal{A}} = \tilde{\mathcal{B}}_1 \sqcup \tilde{\mathcal{B}}_2$. Thus we may apply claim (2) of Proposition 1.2 to conclude that $\mathbf{U}(\tilde{\mathcal{A}})$ is spanned by the members of $\tilde{\mathcal{B}}_1$ and the inverse image $\rho^{-1}(A_2)$ of A_2, which is isomorphic to A_2. As $\mathcal{B}_1$ is simply connected, the ambient space spanned by the members of $\tilde{\mathcal{B}}_1$ is identical to that of $\mathcal{B}_1$. Thus we have

$$\dim(\mathbf{U}(\tilde{\mathcal{A}})) \leq \dim(\mathbf{U}(\mathcal{B}_1)) + \dim(A_2) = \dim(\mathbf{U}(\mathcal{B}_1)) + n + 1.$$

As the last number is equal to the dimension of $\mathbf{U}(\mathcal{A})$ by the assumption, we conclude that $\mathbf{U}(\tilde{\mathcal{A}}) = \mathbf{U}(\mathcal{A})$. Thus $\mathcal{A}$ is simply connected. □

2.3. Proof of Theorem 1.3

We recall the setting in Theorem 1.3. Let n be a natural number with $n \geq 3$, and let $\mathcal{B}_1$ be an arbitrary DHO over $\mathbb{F}_2$ of rank n whose members are subspaces of a vector space U over $\mathbb{F}_2$ of dimension at least $\dim(\mathbf{U}(\mathcal{B}_1)) + n + 1$. Take a subspace A_2 of U of dimension $n+1$ with

(u) $A_2 \cap \mathbf{U}(\mathcal{B}_1) = \{0\}$.

Fix a hyperplane B_2 of A_2. We also assume that

(s1) $\mathcal{B}_1$ splits over a subspace Y_1 of $\mathbf{U}(\mathcal{B}_1)$ of codimension n.

We fix a member B_1 of $\mathcal{B}_1$. By assumption (s1), we have $\mathbf{U}(\mathcal{B}_1) = B_1 \oplus Y_1$. As B_2 is of dimension n over $\mathbb{F}_2$, we may adopt B_2 as a set to parametrize the members of a DHO $\mathcal{B}_1$ of rank n over $\mathbb{F}_2$ and so the linear system of a DHO $\mathcal{B}_1$ over Y_1 as well. (See Section 1.5.) We denote this system by $\mathcal{L}_{B_2}(\mathcal{B}_1, Y_1) := \{L_1(c) \mid c \in B_2\}$. We take any parametrization of the members of $\mathcal{B}_1$ by B_2 with $B_1(0) = B_1$ and $L_1(0) = 0$, and fix it in the sequel.

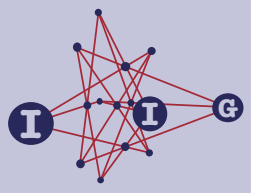
We shall first establish the "only if" part of Claim (i) as well as the uniqueness of Claim (ii). For these purposes, assume that there exists a pair $(\mathcal{B}_2, \mathcal{A})$ of a DHO $\mathcal{B}_2$ and a DHO $\mathcal{A}$ of rank n and $n+1$ respectively which satisfies

$\mathbf{U}(\mathcal{A}) \subseteq U$, $B_2 \in \mathcal{B}_2$, $A_2 \in \mathcal{A}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are subDHOs of $\mathcal{A}$ with $\mathcal{A} = \mathcal{B}_1 \sqcup \mathcal{B}_2$.

We also fix a complement Y_1 above and denote by A_1 the unique member of $\mathcal{A}$ containing B_1. Then $A_1 \cap A_2$ is a 1-dimensional subspace of A_2, which is not contained in B_2 nor B_1, because of the remarks in Section 1.2. We denote by a_{12} the unique nonzero vector in $A_1 \cap A_2$. Thus $A_j = \langle a_{12}, B_j \rangle$ for $j = 1$ and $j = 2$.

We shall show that we can parametrize the members of $\mathcal{A}(\mathcal{B}_1)$, $\mathcal{B}_2$ and $\mathcal{A}(\mathcal{B}_2)$ by B_2 so that each of them are uniquely described by the linear system $\mathcal{L}_{B_2}(\mathcal{B}_1, Y_1)$ (with given parametrization by B_2). This shows the uniqueness in Claim (ii). During this procedure, we shall also show that equation (3) holds for all $d \in B_2^\times$, $x_1, x_2 \in B_2$. This establishes the "only if" part of Claim (i).

From Proposition 1.2 (2), the ambient space $\mathbf{U}(\mathcal{A})$ of $\mathcal{A}$ is a sum of A_2 and $\mathbf{U}(\mathcal{B}_1)$. Then assumptions (u) and (s1) above implies the following direct sum



decomposition of $\mathbf{U}(\mathcal{A})$:

$$\mathbf{U}(\mathcal{A}) = A_2 + \mathbf{U}(\mathcal{B}_1) = A_2 \oplus B_1 \oplus Y_1 = \langle a_{12} \rangle \oplus B_2 \oplus B_1 \oplus Y_1,$$

in which $\langle A_1, A_2 \rangle$ coincides with the $(2n+1)$-dimensional subspace $\langle a_{12} \rangle \oplus B_2 \oplus B_1$. In particular,

$$\begin{aligned} &\varepsilon a_{12} + c + x + y = 0 \text{ with } \varepsilon \in \mathbb{F}_2,\ c \in B_2,\ x \in B_1 \text{ and } y \in Y_1 \\ &\text{if and only if } \varepsilon = 0,\ c = 0,\ x = 0, \text{ and } y = 0. \end{aligned}$$

This observation obtained from our assumptions will be frequently used later.

We first parametrize the members of $\mathcal{B}_2$ and $\mathcal{A}(\mathcal{B}_2)$ by the vectors of B_2. We set $B_2(0) = B_2$. For $0 \neq c \in B_2$, let $B_2(c)$ be the unique member of $\mathcal{B}_2 \setminus \{B_2\}$ such that $B_2(c) \cap B_2 = \langle c \rangle$. We denote by $A_2(c)$ the unique member of $\mathcal{A}(\mathcal{B}_2)$ containing $B_2(c)$. We have $\mathcal{B}_2 = \{B_2(c) \mid c \in B_2\}$ and $\mathcal{A}(\mathcal{B}_2) = \{A_2(c) \mid c \in B_2\}$.

Next we parametrize the members of $\mathcal{A}(\mathcal{B}_1)$ by the vectors of B_2. (Notice that we do use B_2, not B_1). Since A_2 is spanned by a_{12} and B_2, there are exactly 2^n vectors of the form $a_{12} + c$ with $c \in B_2$ in the complement $A_2 \setminus B_2$. Since $\mathcal{A}$ is a DHO, every such vector $a_{12} + c$ is contained in a unique member of $\mathcal{A} \setminus \{A_2\}$. As $\mathcal{B}_2$ is a DHO, every member of $\mathcal{A}(\mathcal{B}_2)$ intersects A_2 at a 1-dimensional subspace contained in B_2. As $\mathcal{A}$ is the disjoint union of $\mathcal{A}(\mathcal{B}_1)$ and $\mathcal{A}(\mathcal{B}_2)$, the member of $\mathcal{A} \setminus \{A_2\}$ containing $a_{12} + c$ lies in $\mathcal{A}(\mathcal{B}_1)$, which we shall denote $A_1(c)$. We also denote by $B_1(c)$ the unique member of $\mathcal{B}_1$ contained in $A_1(c)$. As $a_{12} + c \in A_2$ and $B_1(c)$ is a subspace of $\mathbf{U}(\mathcal{B}_1)$, the assumption (u) implies that $A_1(c) = \langle a_{12} + c, B_1(c) \rangle$ for every $c \in B_2$, and in particular $A_1(0) = \langle a_{12}, B_1 \rangle = A_1$. Furthermore, $\mathcal{B}_1 = \{B_1(c) \mid c \in B_2\}$ and $\mathcal{A}(\mathcal{B}_1) = \{A_1(c) \mid c \in B_2\}$.

Since $\mathcal{B}_1$ splits over Y_1, for each $c \in B_2$ there exists an $\mathbb{F}_2$-linear map $L_1(c)$ from $B_1 = B_1(0)$ into Y_1 such that

$$B_1(c) = \{x + xL_1(c) \mid x \in B_1\}.$$

Then $A_1(c)$ is given as follows:

$$A_1(c) = \{\varepsilon(a_{12} + c) + (x + xL_1(c)) \mid \varepsilon \in \mathbb{F}_2, x \in B_1\}, \tag{7}$$

in which $A_1(c) \setminus B_1(c)$ consists of 2^n vectors of the form

$$a_{12} + c + x + xL_1(c) \quad \text{with } x \in B_1.$$

Now we shall describe the vectors in $A_2(d)$ for each $d \in B_2$. For every vector v of $A_2(d) \setminus B_2(d)$, there exists a unique member A of $\mathcal{A} \setminus \{A_2(d)\}$ such that $A \cap A_2(d) = \langle v \rangle$. Since $\mathcal{B}_2$ is a DHO, every member of $\mathcal{A}(\mathcal{B}_2) \setminus \{A_2(d)\}$ intersects

$A_2(d)$ at a 1-dimensional subspace contained in $B_2(d)$. Thus A should lie in $\mathcal{A}(\mathcal{B}_1)$, so that there exists $c \in B_2$ such that $A_1(c) \cap A_2(d) = \langle v \rangle$. As there are 2^n vectors in $A_2(d) \setminus B_2(d)$, there is a bijective correspondence from $A_2(d) \setminus B_2(d)$ to B_2 sending v to $c \in B_2$ with $\langle v \rangle = A_1(c) \cap A_2(d)$. Since $\mathcal{B}_1$ is a DHO as well, the same argument as above shows that $A_1(c) \cap A_2(d)$ does not lie in $B_1(c)$. Thus the above remark on the shape of the vectors of $A_1(c) \setminus B_1(c)$ implies the following: for every pair (c, d) of vectors in B_2, there exists a vector $x(c, d)$ of B_1 uniquely determined by (c, d) such that

$$A_1(c) \cap A_2(d) = \langle a(c, d) \rangle , \quad a(c, d) := a_{12} + c + x(c, d) + x(c, d)L_1(c). \tag{8}$$

Thus we conclude that

$$A_2(d) \setminus B_2(d) = \{a_{12} + c + x(c, d) + x(c, d)L_1(c) \mid c \in B_2\}. \tag{9}$$

We have $a(0, d) = a_{12} + x(0, d)$ as $L_1(0) = 0$. As $a(0, d) \in A_2(d) \setminus B_2(d)$ and $a(c, d)$ ranges over $A_2(d) \setminus B_2(d)$ when c ranges over B_2, we conclude that

$$a(0, d) + a(c, d) = c + (x(0, d) + x(c, d)) + x(c, d)L_1(c)$$

exhaust the vectors of the hyperplane $B_2(d)$, when c ranges over B_2. Hence

$$B_2(d) = \{c + (x(0, d) + x(c, d)) + x(c, d)L_1(c) \mid c \in B_2\}. \tag{10}$$

$$A_2(d) = \left\{ \begin{matrix} \varepsilon a_{12} + c + ((\varepsilon + 1)x(0, d) + x(c, d)) \\ + x(c, d)L_1(c) \end{matrix} \;\middle|\; \varepsilon \in \mathbb{F}_2, c \in B_2 \right\}. \tag{11}$$

Notice that $c \in B_2$, $(\varepsilon + 1)x(0, d) + x(c, d) \in B_1$ and $x(c, d)L_1(c) \in Y_1$ in the above expression (11) of vectors in $A_2(d)$, whence expression (11) gives a direct sum decomposition of vectors of $A_2(d)$ in $\langle a_{12} \rangle \oplus B_2 \oplus B_1 \oplus Y_1$.

For each $d \in B_2$, we now define maps ϕ_d from B_2 to B_1 and ψ_d from B_2 to Y_1 by

$$c\phi_d := x(c, d) + x(0, d), \qquad c\psi_d := x(c, d)L_1(c) \quad (c \in B_2).$$

In the sequel, we shall show that the maps ϕ_d and ψ_d ($d \in B_2$) are linear maps with some properties.

We first claim that

> for each $c \in B_2$, the map $B_2 \ni d \mapsto x(c, d) \in B_1$ is bijective with $x(c, 0) = 0$.

Suppose $x(c, d_1) = x(c, d_2)$ for distinct vectors d_1 and d_2 of B_2. Then we have

$$\begin{aligned} a(c, d_1) &= a_{12} + c + x(c, d_1) + x(c, d_1)L_1(c) \\ &= a_{12} + c + x(c, d_2) + x(c, d_2)L_1(c) = a(c, d_2). \end{aligned}$$

However, this implies that $A_1(c) \cap A_2(d_1) = \langle a(c,d_1)\rangle = \langle a(c,d_2)\rangle = A_1(c) \cap A_2(d_2)$ by equation (8), which contradicts the condition (A3) for a DHO $\mathcal{A}$. Thus for each $c \in B_2$ the map sending $d \in B_2$ to $x(c,d)$ is injective and so bijective onto B_1. As for the latter property $x(c,0) = 0$, observe that $A_1(c) \cap A_2(0) = A_1(c) \cap A_2 = \langle a_{12} + c\rangle$. Then equation (8) implies that $a(c,0) = a_{12} + c + x(c,0) + x(c,0)L_1(c) = a_{12} + c$. Thus we have $x(c,0) + x(c,0)L_1(c) = 0$ with $x(c,0) \in B_1$ and $x(c,0)L_1(c) \in Y_1$. Then the direct sum decomposition $B_1 \oplus Y_1$ implies that $x(c,0) = 0$.

Next we claim that

$$x(0,d) = \kappa_d \text{ and } d\phi_d = 0 \text{ for each } d \in B_2.$$

For $0 \neq d \in B_2$, we have chosen $A_2(d) \in \mathcal{A}(\mathcal{B}_2) \setminus \{A_2\}$ so that $A_2 \cap A_2(d) = B_2 \cap B_2(d) = \langle d\rangle$. From equation (10), this implies $d = c + (x(0,d) + x(c,d)) + x(c,d)L_1(c)$ for some $c \in B_2$. As $c + d \in B_2$, $x(0,d) + x(c,d) \in B_1$ and $x(c,d)L_1(c) \in Y_1$, the direct sum decomposition $B_2 \oplus B_1 \oplus Y_1$ implies that $c = d$, $x(0,d) = x(d,d)$, and $x(d,d)L_1(d) = 0$. As $d \neq 0$, the kernel of the linear map $L_1(d)$ is a 1-dimensional subspace of B_1 generated by κ_d (see Section 1.5). By the conclusion in the above paragraph, $x(d,d) \neq x(0,0) = 0$ for $d \neq 0$. Thus we have $\kappa_d = x(d,d) = x(0,d)$ for every $0 \neq d \in B_2$. Moreover, $d\phi_d = x(d,d) + x(0,d) = 0$ for all $d \in B_2$.

In particular, we have

$$c\phi_d = x(c,d) + \kappa_d \text{ and } c\psi_d = (c\phi_d + \kappa_d)L_1(c) \text{ for all } c,d \in B_2.$$

Now we shall establish the linearity of ϕ_d and ψ_d for $d \in B_2$. We use the fact that $B_2(d)$ ($d \in B_2$) has a structure of a subspace. In view of the shapes of vectors in $B_2(d)$ given in (10), this implies that for any vectors c_1 and c_2 of B_2 there is a vector $c \in B_2$ satisfying the following equation:

$$\begin{aligned} &c_1 + (x(0,d) + x(c_1,d)) + x(c_1,d)L_1(c_1) \\ &+ c_2 + (x(0,d) + x(c_2,d)) + x(c_2,d)L_1(c_2) \\ &= c + (x(0,d) + x(c,d)) + x(c,d)L_1(c). \end{aligned}$$

Again the direct sum decomposition of $B_2 \oplus B_1 \oplus Y_1$ implies that

$$\begin{aligned} c &= c_1 + c_2, \\ (x(0,d) + x(c_1,d)) + (x(0,d) + x(c_2,d)) &= x(0,d) + x(c_1 + c_2, d), \text{ and} \\ x(c_1,d)L_1(c_1) + x(c_2,d)L_1(c_2) &= x(c_1 + c_2, d)L_1(c_1 + c_2). \end{aligned}$$

The second and the last equations above respectively show the linearity of ϕ_d and ψ_d ($d \in B_2$).

Finally we shall show equation (3). Notice that

$$d\psi_d = (d\phi_d + \kappa_d)L_1(d) = (0 + \kappa_d)L_1(d) = 0$$

by the above relation between ϕ_d and ψ_d, the property $d\phi_d = 0$, and the definition of κ_d. Take any $c \in B_2$. As ψ_d is a linear map with $d\psi_d = 0$, we have

$$\begin{aligned} 0 = d\psi_d &= (d + c + c)\psi_d = (d + c)\psi_d + c\psi_d \\ &= ((d + c)\phi_d + \kappa_d)L_1(d + c) + (c\phi_d + \kappa_d)L_1(c) \\ &= (c\phi_d + \kappa_d)(L_1(d + c) + L_1(c)) \end{aligned}$$

by the above relation between ϕ_d and ψ_d as well as the linearity of ϕ_d and the property $d\phi_d = 0$. As $c\phi_d + \kappa_d = x(c, d)$, the injectivity of the map $B_2 \ni d \mapsto c\phi_d + \kappa_d \in B_1$ ($c \in B_2$) we established as the first claim above implies that $c\phi_d + \kappa_d \neq c\phi_0 + \kappa_0 = x(c, 0) = 0$ if $d \neq 0$. Hence the above equation implies that $c\phi_d + \kappa_d = \kappa(d + c, c)$ for $d \neq 0$ by the definition of $\kappa(c + d, c)$ (see Section 1.5). Summarizing, we have

$$\begin{aligned} x(c, d) + x(0, d) = c\phi_d = \kappa(d + c, c) + \kappa_d \quad &\text{and} \\ x(c, d)L_1(c) = c\psi_d = \kappa(d + c, c)L_1(c) = \kappa(d + c, c)L_1(d + c) & \end{aligned}$$

for all $c \in B_2$ and $d \in B_2^\times$. Recall that $b_1\{a, b\} = \kappa(a, b) + \kappa(a, b)L_1(a)$ (see Section 1.5). Thus we have

$$b_1\{d + x, x\} = \kappa(d + x, x) + \kappa(d + x, x)L_1(x) = x\phi_d + \kappa_d + x\psi_d$$

for all $x \in B_2$. In particular, $b_1\{d, 0\} = \kappa_d$. Now the desired equation (3) follows from the linearity of ϕ_d and ψ_d.

Thus we established the "only if" part of Claim (i).

With the above notation, $\mathcal{B}_2$ splits over the subspace, say Y_2, of $B_1 \oplus Y_1$ spanned by $c\phi_d + c\psi_d = (\kappa(d + c, c) + \kappa_d) + \kappa(d + c, c)L_1(c)$ for all $c, d \in B_2$, because $Y_2 \subseteq B_1 \oplus Y_1$ and $B_2(d) \cap (B_1 \oplus Y_1) = \{0\}$ for all $d \in B_2$ in view of equation (10). Moreover the linear system for $\mathcal{B}_2$ over Y_2 is $\mathcal{L}_{B_2}(\mathcal{B}_2, Y_2) = \{L_2(d) \mid d \in B_2\}$, where $L_2(d)$ ($d \in B_2$) is given by

$$cL_2(d) := c\phi_d + c\psi_d = (\kappa(d + c, c) + \kappa_d) + \kappa(d + c, c)L_1(c) \quad (c \in B_2).$$

As $\kappa(d + c, c)$ for $c \in B_2$, $d \in B_2^\times$ are uniquely determined by the given linear system $\mathcal{L}_{B_2}(\mathcal{B}_1, Y_1)$ for $\mathcal{B}_1$ over Y_1, this implies that $\mathcal{B}_2$ is uniquely determined by $\mathcal{B}_1$ (under our choice of B_1, A_1, B_2, A_2 and a complement Y_1). Furthermore, $\mathcal{A}$ is uniquely determined by the given linear system $\mathcal{L}_{B_2}(\mathcal{B}_1, Y_1)$, as the members $A_2(d)$ are uniquely described by $x(c, d) + x(0, d) = c\phi_d = \kappa(c + d, c) + \kappa_d$ and $x(c, d)L_1(c) = c\psi_d = \kappa(c + d, c)L_1(c)$ ($c, d \in B_2$).

Hence we established the uniqueness of Claim (ii) as well as the splitness of the second DHO $\mathcal{B}_2$.

It remains to establish the "if" part of Claim (i). For that purpose, conversely, assume that equation (3) is satisfied for all $d \in B_2^\times$ and all $x_1, x_2 \in B_2$. We define the map ϕ_d for each $d \in B_2$ by $c\phi_d := \kappa(d+c,c)+\kappa_d$ if $d \neq 0$ and $\phi_0 := 0$. We also consider the map ψ_d sending $c \in B_2$ to $\kappa(d+c,c)L_1(c) = (c\phi_d+\kappa_d)L_1(c)$ $(\in Y_1)$. As $b_1\{x+d,x\} = \kappa(x+d,x)+\kappa(x+d,x)L_1(x) = (x\phi_d+\kappa_d)+x\psi_d$, in view of the B_1-part and the Y_1-part of equation (3), the assumption is equivalent to the linearity of both ϕ_d and ψ_d for all $d \in B_2^\times$. Moreover, $\phi_0 = 0 = \psi_0$ are linear.

The arguments for the uniqueness of Claim (ii) suggest that there is a unique candidate for $\mathcal{A}$ (and so $\mathcal{B}_2$). For $c, d \in B_2$, we set subsets $A_1(c)$ and $A_2(d)$ as follows (see equations (7) and (11)), where a_{12} is a nonzero vector in A_2 (satisfying assumption (u)) not contained in B_2:

$$A_1(c) := \{\varepsilon(a_{12}+c) + (x + xL_1(c)) \mid \varepsilon \in \mathbb{F}_2, x \in B_1\}, \tag{12}$$

$$A_2(d) := \{\varepsilon(a_{12}+\kappa_d) + x' + x'\phi_d + x'\psi_d \mid \varepsilon \in \mathbb{F}_2, x' \in B_2\}. \tag{13}$$

Observe that each of the above subspaces lies in the subspace $\langle a_{12}\rangle + B_2 + B_1 + Y_1$ of U, which is in fact a direct sum by assumptions (u) and (s1):

$$\langle a_{12}\rangle + B_2 + B_1 + Y_1 = \langle a\rangle \oplus B_2 \oplus B_1 \oplus Y_1. \tag{14}$$

We set $B_2(d) := A_2(d) \cap (B_2 \oplus B_1 \oplus Y_1) = \{x' + x'\phi_d + x'\psi_d \mid x' \in B_2\}$ for $d \in B_2$.

To establish the "if" part of Claim (i), thus it suffices to verify that the collections $\mathcal{A} := \{A_1(c), A_2(d) \mid c, d \in B_2\}$ and $\mathcal{B}_2 := \{B_2(d) \mid d \in B_2\}$ satisfy the axioms (A0)–(A2) for a DA, given in Section 1.1. The set $A_1(c)$ is a subspace of U of dimension $n+1$, because of the linearity of $L_1(c)$ and the direct sum decomposition (14). The set $A_2(d)$ forms a subspace of U of dimension $n+1$, from the linearity of ϕ_d and ψ_d and the direct sum decomposition (14). Thus (A0) is verified for $\mathcal{A}$. As $\mathcal{B}_2$ consists of intersections of some members of $\mathcal{A}$ with the subspace $B_2 \oplus B_1 \oplus Y_1$ of U, (A0) holds for $\mathcal{B}_2$ as well.

To show (A1) for $\mathcal{A}$ and $\mathcal{B}_2$, we shall first examine $A_1(c_1) \cap A_1(c_2)$ for distinct c_1, c_2 of B_2. Any vector v in $A_1(c_1) \cap A_1(c_2)$ is of the form

$$v = \varepsilon_j(a_{12} + c_j) + (x_j + x_j L_1(c_j))$$

for some $\varepsilon_j \in \mathbb{F}_2$, $c_j \in B_2$ and $x_j \in B_1$ for $j = 1, 2$. Then the direct sum decomposition (14) implies that

$$\varepsilon_1 = \varepsilon_2 =: \varepsilon; \quad \varepsilon(c_1 + c_2) = 0, \quad x_1 = x_2 =: x; \quad x(L_1(c_1) + L_1(c_2)) = 0.$$

As $c_1 \neq c_2$, this is equivalent to $\varepsilon = 0$ and x lies in the 1-dimensional kernel of $L_1(c_1) + L_1(c_2)$, spanned by $\kappa(c_1, c_2)$. Thus v lies in $B_1(c_1) \cap B_1(c_2)$, which is spanned by $\kappa(c_1, c_2) + \kappa(c_1, c_2)L_1(c_1)$. This verifies that $A_1(c_1) \cap A_1(c_2) = B_1(c_1) \cap B_1(c_2)$ is a 1-dimensional subspace spanned by $\kappa(c_1, c_2) + \kappa(c_1, c_2)L_1(c_1)$.

Next we shall examine $A_2(d_1) \cap A_2(d_2)$ for distinct d_1, d_2 of B_2 (this is the only case to examine for showing that $\mathcal{B}_2$ satisfies condition (A1)). Again any vector v in $A_2(d_1) \cap A_2(d_2)$ has the following expressions

$$\begin{aligned} v &= \varepsilon_1 a_{12} + c_1 + (\varepsilon_1 \kappa_{d_1} + c_1 \phi_{d_1}) + (c_1 \phi_{d_1} + \kappa_{d_1}) L_1(c_1) \\ &= \varepsilon_2 a_{12} + c_2 + (\varepsilon_2 \kappa_{d_2} + c_2 \phi_{d_2}) + (c_2 \phi_{d_2} + \kappa_{d_2}) L_1(c_2) \end{aligned}$$

for some $\varepsilon_j \in \mathbb{F}_2$ and $c_j \in B_2$ $(j = 1, 2)$. As $(\varepsilon_1 + \varepsilon_2)a_{12} \in \langle a_{12} \rangle$, $c_1 + c_2 \in B_2$, $\varepsilon_1 \kappa_{d_1} + c_1 \phi_{d_1} + \varepsilon_2 \kappa_{d_2} + c_2 \phi_{d_2} \in B_1$, and $(c_1 \phi_{d_1} + \kappa_{d_1}) L_1(c_1) + (c_2 \phi_{d_2} + \kappa_{d_2}) L_1(c_2) \in Y_1$, the direct sum decomposition (14) implies that we have

$$\varepsilon_1 = \varepsilon_2 =: \varepsilon; \quad c_1 = c_2 =: c; \quad \varepsilon \kappa_{d_1} + c \phi_{d_1} = \varepsilon \kappa_{d_2} + c \phi_{d_2}; \quad (c \phi_{d_1} + \kappa_{d_1}) L_1(c) = (c \phi_{d_2} + \kappa_{d_2}) L_1(c).$$

If $\varepsilon = 1$, the third equation reads $\kappa_{d_1} + c\phi_{d_1} = \kappa_{d_2} + c\phi_{d_2}$, which is equivalent to $\kappa(d_1 + c, c) = \kappa(d_2 + c, c)$ by the definition of ϕ_d for $d = d_1, d_2$. As $\mathcal{L}_{B_2}(\mathcal{B}_1, Y_1)$ is a linear system for $\mathcal{B}_1$ over Y_1, the map from $B_2 \setminus \{c\}$ to $B_1 \setminus \{0\}$ sending x to $\kappa(x, c)$ is injective (see property (L2) for a linear system, in Section 1.5). Then we have $d_1 = d_2$, which contradicts the choice of d_1 and d_2. Thus $\varepsilon = 0$ and $c\phi_{d_1} = c\phi_{d_2}$ from the third equation, and hence

$$(\kappa_{d_1} + \kappa_{d_2}) L_1(c) = 0,$$

from the last equation with the linearity of $L_1(c)$. This equation uniquely determines nonzero c as follows. By the above mentioned injectivity of the map from $B_2^\times$ to $B_1^\times$ sending x to $\kappa(x, 0) = \kappa_x$ (corresponding to property (L2) for a linear system, in Section 1.5), there exists a unique nonzero element d_3 of B_2 such that $\kappa_{d_1} + \kappa_{d_2} = \kappa_{d_3}$, and then the above equation is equivalent to $\kappa_{d_3} L_1(c) = 0$. (If $c = 0$, then we obtain the trivial solution $v = 0$.) If $c \neq 0$, it follows from the definition of κ_c that $\kappa_c = \kappa_{d_3}$. Thus $c = d_3$ from the injectivity of the map $x \mapsto \kappa_x$. Hence we conclude that $A_2(d_1) \cap A_2(d_2) = B_2(d_1) \cap B_2(d_2)$ is a 1-dimensional subspace spanned by $d_3 + d_3\phi_{d_1} + (d_3\phi_{d_1} + \kappa_{d_1}) L_1(d_3)$ for any distinct d_1, d_2 of B_2, where d_3 denotes the unique (nonzero) vector of B_2 with $\kappa_{d_1} + \kappa_{d_2} = \kappa_{d_3}$.

In particular, we have $B_2(d_1) \cap B_2(d_2) \cap B_2(d_2') = \{0\}$ if d_1, d_2, d_2' are pairwise distinct from the injectivity of the map $x \mapsto \kappa_x$. Thus we already verified that $\mathcal{B}_2$ satisfies condition (A2) for a DA as well, and therefore $\mathcal{B}_2$ is in fact a DHO.

Finally we examine $A_1(c) \cap A_2(d)$ for $c, d \in B_2$: any vector v of $A_1(c) \cap A_2(d)$ is expressed as

$$v = \varepsilon(a_{12} + c) + (x + xL_1(c)) = \varepsilon'(a_{12} + \kappa_d) + c' + c'\phi_d + (c'\phi_d + \kappa_d)L_1(c')$$

for some $\varepsilon, \varepsilon' \in \mathbb{F}_2$, $c, c' \in B_2$, and $x \in B_1$. The direct sum decomposition (14) implies that $\varepsilon = \varepsilon'$, $\varepsilon c = c'$, $x + \varepsilon\kappa_d + c'\phi_d = 0$ and $xL_1(c) = (c'\phi_d + \kappa_d)L_1(c')$. If $\varepsilon = 0$ then $c' = 0$ and $x = 0$, which yields $v = 0$. If $\varepsilon = 1$, the equation is equivalent to $c = c'$ and $x = \kappa_d + c\phi_d$. Thus

$$v = (a_{12} + c) + (\kappa_d + c\phi_d) + (\kappa_d + c\phi_d)L_1(c)$$

is a nonzero vector which is uniquely determined by c and d. Thus $A_1(c) \cap A_2(d)$ is a 1-dimensional subspace spanned by $(a_{12}+c)+(\kappa_d+c\phi_d)+(\kappa_d+c\phi_d)L_1(c)$, not contained in $B_1(c)$ nor $B_2(d)$.

This verifies that (A1) holds for any choice of two distinct members of $\mathcal{A} = \mathcal{A}(\mathcal{B}_1) \cup \mathcal{A}(\mathcal{B}_2)$. The verification of (A2) is straightforward now, because any mutually distinct three members of $\mathcal{A}$ is of the form $A_j(c)$, $A_j(c')$ and $A_{j'}(d)$ for some $j, j' \in \{1, 2\}$ and $c, c', d \in B_2$. If $j = j' = 1$ or 2, then the intersection $A_j(c) \cap A_j(c') \cap A_{j'}(d)$ is the same as $B_j(c) \cap B_j(c') \cap B_j(d)$, which is $\{0\}$ because $\mathcal{B}_j$ is a DHO. (For $j = 2$, we already verified this fact above. For $j = 1$, this is by definition.) While, if $j \neq j'$, the coefficient of a_{12} in the generator of $A_j(c) \cap A_{j'}(d)$ is 1, as we saw above, but that of $A_j(c) \cap A_j(c') = B_j(c) \cap B_j(c')$ is 0, whence $A_j(c) \cap A_j(c') \cap A_{j'}(d) = \{0\}$ in this case as well.

Hence we established the "if" part of Claim (i). □

2.4. Proof of Corollary 1.4

(i) We assume the setting in Theorem 1.3, and use the same notation. Furthermore, we assume that $\mathcal{B}_1$ is bilinear with respect to Y_1: namely, the linear system $\mathcal{L}_{B_2}(Y_1) = \{L_1(t) \mid t \in B_2\}$ for $\mathcal{B}_1$ satisfies the following equation with a suitable parametrization by B_2 (see equation (2)):

$$L_1(s + t) = L_1(s) + L_1(t) \text{ for any } s, t \in B_2. \tag{15}$$

In this case, for any distinct $a, b \in B_2$, the nonzero vector $\kappa(a, b)$ lying in the kernel of $L(a) + L(b) = L(a + b)$ coincides with $\kappa(a + b)$. Thus we have $\kappa(d + x, x) = \kappa_d$ for any $d \in B_2^\times$ and $x \in B_2$, and hence equation (3) is satisfied for all $d \in B_2^\times$, $x_1, x_2 \in B_2$. Thus Claim (i) follows from Theorem 1.3(i) and (ii). Notice that the maps ϕ_d and ψ_d ($d \in B_2$), appeared in the proof of Theorem 1.3, take values $c\phi_d = 0$ and $c\psi_d = \kappa_d L_1(c)$ at $c \in B_2$.

(ii) Fix an arbitrary linear bijection ρ from B_2 onto B_1. It is straightforward to verify that

$$Y = Y_\rho := \{e + (e)\rho + y \mid e \in B_2, y \in Y_1\}$$

is a subspace of $\mathbf{U}(\mathcal{A})$ satisfying $Y \cap A_1(c) = Y \cap A_2(c) = \{0\}$ for all $c \in B_2$, in view of the direct sum decomposition $\langle a_{12}\rangle \oplus B_2 \oplus B_1 \oplus Y_1$ of $\mathbf{U}(\mathcal{A})$ and the expressions

$$\begin{aligned} A_1(c) &= \{\varepsilon a_{12} + \varepsilon c + x + xL_1(c) \mid \varepsilon \in \mathbb{F}_2, x \in B_1\},\\ A_2(c) &= \{\varepsilon a_{12} + x' + \varepsilon\kappa_c + \kappa_c L_1(x') \mid \varepsilon \in \mathbb{F}_2, x' \in B_2\}. \end{aligned}$$

Fix a member $A_1(0) = \{\varepsilon a_{12} + x \mid x \in B_1\}$ of $\mathcal{A}$. The following gives the decomposition of a typical vector v of $A_1(c)$ or $A_2(c)$ in the form $v = v_1 + v_2$ with $v_1 \in A_1(0)$ and $v_2 \in Y$:

- for $v = \varepsilon a_{12} + \varepsilon c + x + xL_1(c)$ of $A_1(c)$ ($\varepsilon \in \mathbb{F}_2$, $x \in B_1$), we have $v_1 = \varepsilon a_{12} + (x + \varepsilon(c)\rho)$ and $v_2 = \varepsilon c + \varepsilon(c)\rho + xL_1(c)$;
- for $v = \varepsilon a_{12} + x' + \varepsilon\kappa_c + \kappa_c L_1(x')$ of $A_2(c)$ ($\varepsilon \in \mathbb{F}_2$, $x' \in B_2$), we have $v_1 = \varepsilon a_{12} + (\varepsilon\kappa_c + (x')\rho)$ and $v_2 = x' + (x')\rho + \kappa_c L_1(x')$.

Then, setting $y = x + \varepsilon(c)\rho \in B_1$ (resp. $y = \varepsilon\kappa_c + (x')\rho \in B_1$) in the above expression of v_2 for $v \in A_1(c)$ (resp. $A_2(c)$), the linear maps $L(c, 0)$ and $L(c, 1)$ sending v_1 to v_2 can be described as in equation (4).

(iii) Assume that $\mathcal{A}$ is bilinear with respect to Y_ρ for a linear bijection ρ from B_2 onto B_1. Then there is a permutation π on $B_2 \times \mathbb{F}_2 = \{(c, \delta) \mid c \in B_2, \delta \in \mathbb{F}_2\}$ (with componentwise addition) such that

$$L((c_1, \delta_1)^\pi) + L((c_2, \delta_2)^\pi) = L((c_1 + c_2, \delta_1 + \delta_2)^\pi) \tag{16}$$

for all (c_1, δ_1), $(c_2, \delta_2) \in B_2 \times \mathbb{F}_2$. (We should mention the permutation π here, because we already fix an ordering on B_2 so that equation (15) is satisfied for $\mathcal{L}_{B_2}(\mathcal{B}_1, Y_1)$.) Then for any c_1, c_2 of B_2 we have

$$\begin{aligned} L((c_1, 0)) + L((c_2, 0)) &= L(((c_1, 0)^{\pi^{-1}})^\pi) + L(((c_2, 0)^{\pi^{-1}})^\pi)\\ &= L(((c_1, 0)^{\pi^{-1}} + (c_2, 0)^{\pi^{-1}})^\pi)). \end{aligned} \tag{17}$$

We set $(c, \delta) := ((c_1, 0)^{\pi^{-1}} + (c_2, 0)^{\pi^{-1}})^\pi$. Take any $\varepsilon a_{12} + y$ of $A_1(0)$ with $\varepsilon \in \mathbb{F}_2$, $y \in B_1$. It follows from equation (17) together with (4) that

$$\begin{aligned} &(\varepsilon a_{12} + y)(L(c_1, 0) + L(c_2, 0))\\ &\quad = \varepsilon(c_1 + c_2) + \varepsilon(c_1 + c_2)\rho\\ &\qquad + \{y(L_1(c_1) + L_1(c_2)) + \varepsilon\left((c_1)\rho L_1(c_1) + (c_2)\rho L_1(c_2)\right)\}. \end{aligned}$$

This equals to $(\varepsilon a_{12} + y)L(c, \delta)$ for all $\varepsilon \in \mathbb{F}_2$, $y \in B_1$. Suppose $\delta = 1$. Putting $\varepsilon = 0$ and then comparing the component in B_2 of $(y)(L(c_1, 0) + L(c_2, 0))$ obtained above with that of $(y)L(c, 1)$ in equation (4), we have $0 = (y)\rho^{-1}$ for all $y \in B_1$, which is a contradiction. Thus $\delta = 0$. Then, putting $\varepsilon = 1$, we have $c = c_1 + c_2$ by comparing the component in B_2 of $(a_{12}+y)L(c, 0)$ in equation (4) with that of $(a_{12}+y)(L(c_1, 0)+L(c_2, 0))$ in the above equation. Hence we established $L(c_1, 0)+L(c_2, 0) = L(c_1+c_2, 0)$ for all $c_1, c_2 \in B_2$.

Using the bilinearity of $\{L_1(t) \mid t \in B_2\}$, we have the following equation for any $\varepsilon \in \mathbb{F}_2$ and $y \in B_1$ from equation (4):

$$\begin{aligned}(\varepsilon a_{12} + y)&(L(c_1 + c_2, 0) + L(c_1, 0) + L(c_2, 0)) \\ &= y\{L_1(c_1 + c_2) + L_1(c_1) + L_1(c_2)\} \\ &\qquad + \varepsilon\{(c_1 + c_2)\rho L_1(c_1 + c_2) + (c_1)\rho L_1(c_1) + (c_2)\rho L_1(c_2)\} \\ &= \varepsilon\{(c_1)\rho L_1(c_2) + (c_2)\rho L_1(c_1)\}.\end{aligned}$$

As $L(c_1 + c_2, 0) = L(c_1, 0) + L(c_2, 0)$, we have equation (5) by putting $\varepsilon = 1$.

Assume that equation (5) is satisfied. Then the above calculation shows that $L(c_1, 0) + L(c_2, 0) = L(c_1 + c_2, 0)$ for all $c_1, c_2 \in B_2$. We shall verify that the following equation holds for any c_1, c_2 of B_2:

$$L(c_1, 1) + L(c_2, 1) = L((\kappa_{c_1} + \kappa_{c_2})\rho^{-1}, 0). \tag{18}$$

We obtain the following equation for $\varepsilon \in \mathbb{F}_2$, $y \in B_1$, using equation (4), the linearity of ρ and the bilinearity of $L_1(c)$ $(c \in B_2)$:

$$\begin{aligned}(\varepsilon a_{12} + y)&(L(c_1, 1) + L(c_2, 1)) \\ &= \{(\varepsilon\kappa_{c_1} + y) + (\varepsilon\kappa_{c_2} + y)\}\rho^{-1} + \varepsilon(\kappa_{c_1} + \kappa_{c_2}) \\ &\qquad + \varepsilon\{\kappa_{c_1} L_1(\kappa_{c_1}\rho^{-1}) + \kappa_{c_2} L_1(\kappa_{c_2}\rho^{-1})\} \\ &\qquad\qquad + \{\kappa_{c_1} L_1(y\rho^{-1}) + \kappa_{c_2} L_1(y\rho^{-1})\}.\end{aligned}$$

At the last line in the above equation, observe that

$$(\kappa_{c_1} + \kappa_{c_2})L_1((\kappa_{c_1} + \kappa_{c_2})\rho^{-1}) = \kappa_{c_1} L_1(\kappa_{c_1}\rho^{-1}) + \kappa_{c_2} L_1(\kappa_{c_2}\rho^{-1})$$

by the linearity of ρ^{-1}, the bilinearity of $L_1(c)$ $(c \in B_2)$ and equation (5) applied to $(c, d) = (\kappa_{c_2}, \kappa_{c_1}\rho^{-1})$. Similarly, at the last line in the above equation we have

$$\begin{aligned}\kappa_{c_1} L_1(y\rho^{-1}) + \kappa_{c_2} L_1(y\rho^{-1}) &= yL_1(\kappa_{c_1}\rho^{-1}) + yL_1(\kappa_{c_2}\rho^{-1}) \\ &= yL_1((\kappa_{c_1} + \kappa_{c_2})\rho^{-1}).\end{aligned}$$

Then it follows from equation (4) that we have

$$(\varepsilon a_{12} + y)(L(c_1, 1) + L(c_2, 1) + L((\kappa_{c_1} + \kappa_{c_2})\rho^{-1}, 0)) = 0$$

for any $\varepsilon \in \mathbb{F}_2$ and $y \in B_2$, as desired.

It remains to show that $\mathcal{B}_1$ is isomorphic to $\mathcal{B}_2$, if equation (5) is satisfied. Recall that $\mathcal{B}_1$ consists of $B_1(c) = \{x + xL_1(c) \mid x \in B_1\}$ for $c \in B_2$, while $\mathcal{B}_2$ consists of $B_2(d) = \{c + \kappa_d L_1(c) \mid c \in B_2\}$ for $d \in B_2$. Assume that equation (5) holds for a linear bijection ρ from B_2 to B_1. Define a linear map $\overline{\rho}$ from $\mathbf{U}(\mathcal{B}_2) = B_2 \oplus Y_1$ to $\mathbf{U}(\mathcal{B}_1) = B_1 \oplus Y_1$ by

$$(c + y)\overline{\rho} := (c\rho) + y, \quad \text{for } c \in B_2, y \in Y_1. \tag{19}$$

Then a typical vector $c + \kappa_d L_1(c)$ ($c \in B_2$) of $B_2(d)$ is mapped by $\overline{\rho}$ to $(c)\rho + \kappa_d L_1(c)$, which is equal to

$$(c)\rho + \kappa_d L_1(c) = (c)\rho + (c)\rho L_1((\kappa_d)\rho^{-1})$$

by equation (5). As c runs over B_2, this vector runs over the member $B_1((\kappa_d)\rho^{-1})$. Thus $B_2(d)\overline{\rho} = B_1(\kappa_d\rho^{-1})$ for every $d \in B_2$, and hence $\overline{\rho}$ induces an isomorphism from $\mathcal{B}_2$ to $\mathcal{B}_1$. □

3. Some examples

In this section, we discuss several examples of Theorem 1.3, specifically those of Corollary 1.4.

First, we treat four known infinite families $\mathcal{H}_n(\mathbb{F}_2)$, $\mathcal{D}_n(\mathbb{F}_2)$, $\mathcal{V}_n(\mathbb{F}_2)$ and $\mathcal{T}_n(\mathbb{F}_2)$ of DHOs over $\mathbb{F}_2$, defined for each $n \geq 3$, with ambient spaces of dimension $n(n+1)/2$. (See [7] for a uniform description of them.) The former two are bilinear over some complements, but the latter two are not. It is immediate to verify whether each of the latter two families satisfies the criterion in Theorem 1.3(i), as the intersection of two members are already calculated. It turns out that none of them cannot be extended to a DHO with conditions in Theorem 1.3(i). See Section 3.1. We shall show that each of the former two can be extended to a DHO in the same family. See Section 3.2.

Then we discuss a family of DHOs $\mathcal{Y}^{\sigma,\tau}$ of rank n over $\mathbb{F}_2$ with ambient space of dimension $2n$ or $2n-1$ parametrized by a pair (σ, τ) of generators σ and τ of the Galois group of $\mathbb{F}_{2^n}$. It is extended to be a DHO $\mathcal{A}$ with conditions in Theorem 1.3(i), as it is bilinear over a complement. We determine the complements to the extension $\mathcal{A}$ over which $\mathcal{A}$ is bilinear, if they are of shape Y_ρ with automorphisms ρ of the underlying vector space of $\mathbb{F}_{2^n}$. See Section 3.3.

This may gives us many examples of DHOs which are not bilinear. In the last section 3.4, we treat a family $\mathcal{S}^{\sigma,\phi}$ indexed by a generator σ of the Galois group of $\mathbb{F}_{2^n}$ and an o-polynomial φ on $\mathbb{F}_{2^n}$, which contains the last family as special cases when φ is linear. The criterion in Theorem 1.3(i) can be interpreted in terms of polynomial φ and σ, which implies that φ is quadratic when σ is the square map. However, it seems unlikely to find an example of $\mathcal{S}^{\sigma,\phi}$ which can be extended to an DHO satisfying the conditions in Theorem 1.3(i), unless φ is linear.

3.1. Veronesean DHOs and Taniguchi DHOs

The Veronesean DHO $\mathcal{V}_n(\mathbb{F}_2)$ and the Taniguchi DHO $\mathcal{T}_n(\mathcal{F}_2)$ are simply connected DHOs with the common ambient space $S^2(V)$, the symmetric tensor product of a vector space V of dimension n with a specified basis $\{e_i\}_{i=0}^d$ ($d := n-1$). See [7] for the precise description. We may take V as a set B_2 in Theorem 1.3 to parametrize the members of $\mathcal{V}_n(\mathbb{F}_2)$ and $\mathcal{T}_n(\mathbb{F}_2)$ in which 0 parametrizes the subspace $\{\Delta(x) \mid x \in V\}$, where we set $\Delta(x) = x \otimes x$.

For distinct $a, b \in V$, we shall write $v\{a,b\}$ or $t\{a,b\}$ the unique nonzero vector in the intersection of two members of $\mathcal{V}_n(\mathbb{F}_2)$ or $\mathcal{T}_n(\mathbb{F}_2)$, respectively, indexed by a and b. The intersection of the members indexed by 0 and d ($\in V^\times$) is spanned by $\Delta(d)$ ($d \in V$) in both $\mathcal{V}_n(\mathbb{F}_2)$ and $\mathcal{T}_n(\mathbb{F}_2)$. It is immediate to see that $v\{a,b\} = a \otimes b$ for $a, b \in V^\times$. By the last equation in [7, Section 2.3],

$$t\{a,b\} = a \otimes b + \xi\{a,b\}(a \cup b) \otimes (a \cup b + e_0)$$

for $a, b \in V^\times$, where $a \cup b = a + b + (a \cap b)$ with $a \cap b := \sum_{i=0}^d a_i b_i e_i$ for $a = \sum_{i=0}^d a_i e_i$, $b = \sum_{i=0}^d b_i e_i$ with $a_i, b_i \in \mathbb{F}_2$, and $\xi\{a,b\}$ is 1 or 0 according as $\{a, b, a+b\} \cap \{0, e_0\}$ is equal to $\emptyset$ or not. In particular, $v\{d+x, x\} = \Delta(x) + d \otimes x$, and therefore

$$v\{d + x_1 + x_2, x_1 + x_2\} + v\{d + x_1, x_1\} + v\{d + x_2, x_2\} + \Delta(d) = \Delta(d) \neq 0,$$

for any $x_1, x_2 \in V$ and $d \in V^\times$. For $d = e_1$ and $\{x_1, x_2, x_1 + x_2\} = \{e_2, e_3, e_2 + e_3\}$, we have $t\{d+x_i, x_i\} = \Delta(x_i) + d \otimes x_i + \Delta(x_i + d) + e_0 \otimes (x_i + d)$ ($i = 1, 2$) and $t\{d+x_1+x_2, x_1+x_2\} = \Delta(x_1+x_2) + d \otimes (x_1+x_2) + \Delta(x_1+x_2+d) + e_0 \otimes (x_1+x_2+d)$, and therefore

$$t\{d + x_1 + x_2, x_1 + x_2\} + t\{d + x_1, x_1\} + t\{d + x_2, x_2\} + \Delta(d) = e_0 \otimes d \neq 0.$$

Thus it follows from Theorem 1.3 that there is no DHO $\mathcal{A}$ of rank $n+1$ over $\mathbb{F}_2$ such that $\mathcal{A} = \mathcal{B}_1 \sqcup \mathcal{B}_2$ for some DHO $\mathcal{B}_2$ of rank n, if $\mathcal{B}_1 = \mathcal{V}_n(\mathbb{F}_2)$ or $\mathcal{T}_n(\mathbb{F}_2)$. (On the other hand, it is immediate to see that $\mathcal{V}_n(\mathbb{F}_2)$ and $\mathcal{T}_n(\mathbb{F}_2)$ are subDHOs of $\mathcal{V}_{n+1}(\mathbb{F}_2)$ and $\mathcal{T}_{n+1}(\mathbb{F}_2)$ respectively, via the natural embedding of $V \ni x \mapsto x \in \langle V, e_n \rangle$.)

3.2. Huybrechts DHOs and Buratti-Del Fra DHOs

The Huybrechts DHO $\mathcal{H}_n(\mathbb{F}_2)$ has the ambient space $F \oplus K = \{(x, y) \mid x \in F, y \in K\}$, where F is the underlying vector space of $\mathbb{F}_{2^n}$ and $K = F \wedge F$, the alternating square tensor product of F. Notice that $F \wedge F$ is identical to the factor space $(F \otimes F)/W$, where W is a subspace of the tensor product $F \otimes F$ spanned by $x \otimes y + y \otimes x$ and $x \otimes x$ for all $x, y \in F$. The image $(x \otimes y) + W$ of $x \otimes y$ in $(F \otimes F)/W = F \wedge F$ is denoted $x \wedge y$ for short. The DHO $\mathcal{H}_n(\mathbb{F}_2)$ splits over $Y_1 := \{(0, y) \mid y \in K\}$. We adopt F to parametrize the members of $\mathcal{B}_1 := \mathcal{H}_n(\mathbb{F}_2)$ with $B_1(0) = B_1 := \{(x, 0) \mid x \in F\}$. The linear system $\mathcal{L}_F(\mathcal{H}_n(\mathbb{F}_2), K)$ is given as the collection of linear maps $L_1(t)$ $(t \in F)$ sending each vector $(x, 0)$ of the member $B_1(0)$ to $(x, 0)L_1(t) = (0, x \wedge t)$. In particular, $\mathcal{B}_1$ is bilinear over Y_1.

Thus we can apply Corollary 1.4(i) to $\mathcal{B}_1 = \mathcal{H}_n(\mathbb{F}_2)$ to conclude that there is a unique DHO $\mathcal{A}$ (called the extension $\mathcal{A}(B_1, Y_1)$ in Section 1.7) of rank $n + 1$ over $\mathbb{F}_2$ as well as its subDHO $\mathcal{B}_2$ of corank 1 such that the following holds:

> $\mathbf{U}(\mathcal{A}) = \mathbb{F}_2 \oplus F \oplus F \oplus K = \{(\varepsilon; z, x, y) \mid \varepsilon \in \mathbb{F}_2, x, z \in F, y \in K\}$, where $\mathbf{U}(\mathcal{B}_1)$ corresponds to $\{(0; 0, x, y) \mid x \in F, y \in K\}$, where $B_1 = \{(0; 0, x, 0) \mid x \in F\}$ and $B_2 := \{(0; z, 0, 0) \mid z \in F\} \in \mathcal{B}_2$, $A_1 := \langle a_{12}, B_1 \rangle \in \mathcal{A}$ and $A_2 := \langle a_{12}, B_2 \rangle \in \mathcal{A}$ for $a_{12} := (1; 0, 0, 0)$, and $\mathcal{A}$ is a disjoint union of subDHOs $\mathcal{B}_1$ and $\mathcal{B}_2$.

Explicitly, $\mathcal{A} = \{A_1(t), A_2(t) \mid t \in F\}$, where $A_1(t) = \{(\varepsilon, \varepsilon t, x, x \wedge t) \mid \varepsilon \in \mathbb{F}_2, x \in F\}$ and $A_2(t) = \{(\varepsilon, z, \varepsilon t, z \wedge t) \mid \varepsilon \in \mathbb{F}_2, z \in F\}$. Moreover, $\mathcal{B}_2 = \{B_2(t) \mid t \in F\}$ with $B_2(t) = \{(0; z, 0, z \wedge t) \mid z \in F\}$. It follows from Proposition 1.2(3) that the DHO $\mathcal{A}$ is simply connected, as $\mathcal{B}_1 = \mathcal{H}_n(\mathbb{F}_2)$ is simply connected. The dimension of the ambient space of $\mathcal{A}$ is $(n(n + 1)/2) + (n + 1)$.

Observe that $(0; 0, x, 0)L_1(x) = (0; 0, 0, x \wedge x) = 0$ and thus $\kappa_x = (0; 0, x, 0)$ for all $x \in F$. Hence L_1, regarded as a bilinear map L_1 from $B_1 \times F$ to Y_1, is alternating. The functions $L_2(t)$ $(t \in F)$ in Corollary 1.4(i) for $\mathcal{B}_2$ over Y_1 sends $(0; z, 0, 0) \in B_2$ to $(0; 0, t, 0)L_1(z) = (0; 0, 0, z \wedge t)$. Identifying elements $(0; 0, x, 0) \in B_1$ and $(0; z, 0, 0) \in B_2$ with elements z and x in F, respectively, this implies that the maps $L_2(t)$ and $L_1(t)$ $(t \in F)$ are identical to the map sending $x \in F$ to $x \wedge t \in K$. Thus, under the above identifications, the second DHO $\mathcal{B}_2$ is identical to $\mathcal{B}_1 = \mathcal{H}_n(\mathbb{F}_2)$. The DHO $\mathcal{A}$ corresponds to the extension $\overline{\mathcal{S}}$ of $\mathcal{H}_n(\mathbb{F}_2) = \mathcal{S}_\beta$ for the symmetric bilinear map $\beta(x, t) = x \wedge t$ $(x, t \in F)$ in the sense of [1, Section 5].

We shall verify that $\mathcal{A}$ is in fact isomorphic to $\mathcal{H}_{n+1}(\mathbb{F}_2)$ (with ambient space of dimension $(n + 1)(n + 2)/2$). As $\mathcal{A}$ is a unique DHO satisfying the conditions given in Corollary 1.4(i), it suffices to verify that $\mathcal{H}_{n+1}(\mathbb{F}_2)$ satisfies these

conditions. We shall work in the model of $\mathcal{H}_{n+1}(\mathbb{F}_2)$, which is obtained by replacing F in the model of $\mathcal{H}_n(\mathbb{F}_2)$ above by a vector space $\overline{F}$ containing F as a hyperplane (with basis $e_0, \ldots, e_{n-1}$). Take e_n from $\overline{F} \setminus F$. We denote a vector $v = x + \varepsilon e_n$ of $\overline{F}$ by (x, ε) ($x \in F$, $\varepsilon \in \mathbb{F}_2$). We denote $\overline{K} = \overline{F} \wedge \overline{F}$. The DHO $\mathcal{H}_{n+1}(\mathbb{F}_2)$ splits over a subspace $\overline{Y} := \{(0, y) \mid y \in \overline{K}\}$ of the ambient space $\overline{F} \oplus \overline{K}$. We identify $\overline{Y}$ with $\overline{K}$ via the correspondence $(0, y) \mapsto y$. Adopting $\overline{F}$ as a set of parametrization, the linear system $\mathcal{L}_{\overline{F}}(\mathcal{H}_{n+1}(\mathbb{F}_2), \overline{K})$ of $\mathcal{H}_{n+1}(\mathbb{F}_2)$ over $\overline{Y}$ (identified with $\overline{K}$) consists of linear maps $L(c, \delta)$ ($(c, \delta) \in \overline{F}$) from $\overline{F}$ (identified with a member $A((0, 0))$ of $\mathcal{H}_{n+1}(\mathbb{F}_2)$) to $\overline{K}$ given by

$$(x, \varepsilon)L(c, \delta) := (x + \varepsilon e_n) \wedge (c + \delta e_n) = x \wedge c + (\delta x + \varepsilon c) \wedge e_n$$

for $x \in F$, $\varepsilon \in \mathbb{F}_2$. For every member

$$A((c, \delta)) = \{(x, \varepsilon) + (x, \varepsilon) \wedge (c, \delta) \mid x \in F, \varepsilon \in \mathbb{F}_2\},$$

define $B((c, \delta))$ to be $A((c, \delta)) \cap (F \oplus (\overline{F} \wedge \overline{F}))$; namely

$$B((c, \delta)) := \{x + \delta(x \wedge e_n) + x \wedge c \mid x \in F\}.$$

Then the collection $\mathcal{B}_1$ of $B_1(c) := B((c, 0)) = \{x + x \wedge c \mid x \in F\}$ for $c \in F$ is a subDHO of $\mathcal{H}_{n+1}(\mathbb{F}_2)$, which is identical to $\mathcal{H}_n(\mathbb{F}_2)$, parametrized by F. This subDHO $\mathcal{B}_1$ splits over a subspace $Y_1 := \{x \wedge y \mid x, y \in F\} = F \wedge F$ of the ambient space $F \oplus (F \wedge F)$. Take $A_2 := A((0, 1)) = \{x + (x \wedge e_n) \mid x \in F\}$. Then $A_2 \cap (F \oplus (F \wedge F)) = \{0\}$. Defining $B_2(c) := A((c, 1)) \cap (F \oplus (\overline{F} \wedge \overline{F})) = \{x + (x \wedge (c + e_n)) \mid x \in F\}$ for each $c \in F$, we have the second subDHO $\mathcal{B}_2 := \{B_2(c) \mid c \in F\}$ of $\mathcal{H}_{n+1}(\mathbb{F}_2)$, because $A((c, 1)) \cap A((c', 1))$ is spanned by $c + c' + (c + c') \wedge (c + e_n)$, and thus $A((c, 1)) \cap A((c', 1)) = B_2(c) \cap B_2(c')$ is 1-dimensional for all distinct $c, c' \in F$. By definitions, $\mathcal{A}$ is a disjoint union of $\mathcal{A}_1 = \{A(c, 0) \mid c \in F\}$ and $\mathcal{A}_2 = \{A(c, 1) \mid c \in F\}$, and hence $\mathcal{A} = \mathcal{B}_1 \sqcup \mathcal{B}_2$. Thus we verified that $\mathcal{H}_{n+1}(\mathbb{F}_2)$ satisfies the conditions in Corollary 1.4(i), and hence $\mathcal{A} = \mathcal{H}_{n+1}(\mathbb{F}_2)$ by the uniqueness of $\mathcal{A}$ in Corollary 1.4(i). In particular, $\mathcal{A}$ is bilinear over the subspace $\overline{K}$.

The situation is similar to the Buratti-Del Fra DHO $\mathcal{D}_n(\mathbb{F}_2)$, because $\mathcal{D}_n(\mathbb{F}_2)$ can be constructed as follows by slightly modifying the above construction of $\mathcal{H}_n(\mathbb{F}_2)$ (see [6] for the detail). The ambient space of $\mathcal{D}_n(\mathbb{F}_2)$ is given as $F \oplus K'$ for $K' = (F \otimes F)/W'$, where W' is the subspace spanned by $x \otimes y + y \otimes x$ and $x \otimes x + \xi(x)x \otimes e_0$ for all $x, y \in F$, with the characteristic function ξ for $F \setminus \{0, e_0\}$. We denote the natural image $(x \otimes y) + W'$ of $x \otimes y \in F \otimes F$ in $(F \otimes F)/W'$ by $x \wedge' y$. We have $x \wedge' y = y \wedge' x$ but $x \wedge' x = \xi(x)e_0 \wedge' x$. The DHO $\mathcal{D}_n(\mathbb{F}_2)$ splits over $Y' := \{(0, y) \mid y \in K'\}$ (identified with K'), which gives the linear system $\mathcal{L}_F(\mathcal{D}_n(\mathbb{F}_2), K')$ consisting of linear maps $L_1(t)$ ($t \in F$) sending $x \in F$, identified with $(x, 0)$ in the member B_1 of $\mathcal{D}_n(\mathbb{F}_2)$, to $x \wedge' t \in K'$

(identified with $(0, x \wedge' t) \in Y'$). In particular, $\mathcal{B}_1 = \mathcal{D}_n(\mathbb{F}_2)$ is bilinear over K'. Hence Corollary 1.4 implies that there is a unique DHO $\mathcal{A}$ of rank $n+1$ over $\mathbb{F}_2$ satisfying the conditions in Corollary 1.4. As $xL_1(c) = x \wedge' c = c \wedge' x = cL_1(x)$, the second subDHO $\mathcal{B}_2$ is identical to $\mathcal{B}_1$ modulo an idetification of $(0; 0, x, 0) \in B_1$ with $(0; x, 0, 0) \in B_2$. It follows from Proposition 1.2(3) that the DHO $\mathcal{A}$ is simply connected, as $\mathcal{B}_1 = \mathcal{D}_n(\mathbb{F}_2)$ is simply connected. The dimension of the ambient space of $\mathcal{A}$ is $(n(n+1)/2) + (n+1)$.

It is almost verbatim as well to show that $\mathcal{A} = \mathcal{D}_{n+1}(\mathbb{F}_2)$ by verifying that the conditions in Corollary 1.4(i) are satisfied by $\mathcal{D}_{n+1}(\mathbb{F}_2)$. We omit the details. In particular, $\mathcal{A}$ is bilinear over $\overline{K'} = \overline{F} \otimes \overline{F}/\overline{W'}$, which corresponds to K' above with $\overline{F}$ containing F as a hyperplane.

3.3. Yoshiara DHOs

Let σ and τ be generators of the Galois group of $F \cong \mathbb{F}_{2^n}$ over $\mathbb{F}_2$, and define a bilinear map $L = L^{(\sigma,\tau)}$ from $F \times F$ to F by

$$L(x, y) := x^\sigma y + xy^\tau \quad \text{for } x, y \in F. \tag{20}$$

The collection $\mathcal{Y}^{\sigma,\tau}$ of subspaces $X(t) := \{(x, L(x, t)) \mid x \in F\}$ of $F \oplus F$ ($t \in F$) is a DHO of rank n over $\mathbb{F}_2$ constructed first in [8]. It splits over $Y := \{(0, y) \mid y \in F\} \cap \mathbf{U}(\mathcal{Y}^{\sigma,\tau})$ and the linear system $\mathcal{L}_F(\mathcal{Y}^{\sigma,\tau}, Y)$ for $\mathcal{Y}^{\sigma,\tau}$ over Y consists of linear maps $L(t)$ on F given by $xL(t) = L(x, t)$. If $\tau \neq \sigma^{-1}$ (resp. $\tau = \sigma^{-1}$), the ambient space of $\mathcal{Y}^{\sigma,\tau}$ is $F \oplus F$ (resp. a hyperplane of $F \oplus F$). Furthermore, we have $\mathcal{Y}^{\sigma,\tau} \cong \mathcal{Y}^{\sigma',\tau'}$ if and only if either (1) or (2) below:

(1) $\tau = \sigma^{-1}$ and $\tau' = \sigma'^{-1}$.

(2) $\tau \neq \sigma^{-1}$ and $\tau' \neq \sigma'^{-1}$, and (σ', τ') is equal to either (σ, τ) or (σ^{-1}, τ^{-1}).

In many cases, this DHO is known to be simply connected (see [2, 3]): $\mathcal{Y}^{\sigma,\sigma^{-1}}$ is always simply connected, and it is conjectured that if $\sigma \neq \tau$ and the subfield of F fixed by $\sigma\tau$ coincides with $\mathbb{F}_2$ then $\mathcal{Y}^{\sigma,\tau}$ is simply connected (verified up to $n \leq 14$).

The DHO $\mathcal{Y}^{\sigma,\tau}$ is bilinear over Y, as L is a bilinear map. In fact, Y is the unique complement over which $\mathcal{Y}^{\sigma,\tau}$ is bilinear. Furthermore, there is a complement over which $\mathcal{Y}^{\sigma,\tau}$ is not bilinear.

These obsevations follow from the following general remarks. Let $\mathcal{S}$ be any bilinear DHO of rank $n \geq 4$ with ambient space U. A complement W to $\mathcal{S}$ is called *bilinear* or *non-bilinear*, according as $\mathcal{S}$ is bilinear over W or not. Every bilinear complement W to $\mathcal{S}$ determines the 'standard' translation group T_W, which is a translation group in the sense of [1, Section 2] with $C_U(T_W) = W$. In particular, if $\mathrm{Aut}(\mathcal{S})$ has a unique translation group (especially when $\dim(U) \leq$

$3n-4$ by [1, Theorem 4.10]), there is a unique bilinear complement to $\mathcal{S}$. In general, $\mathrm{Aut}(\mathcal{S})$ is transitive on the set of all bilinear complements, because the set of all translation groups form a single conjugacy class under $\mathrm{Aut}(\mathcal{S})$ by [1, Theorem 3.11].

Applying these remarks to the bilinear DHO $\mathcal{Y}^{\sigma,\tau}$ of rank n with ambient space of dimension $2n$ or $2n-1$, we conclude that Y is the unique bilinear complement to $\mathcal{Y}^{\sigma,\tau}$. (We do not need [1, Theorem 4.10], whose proof requires some deep results in finite group theory. In view of the structure of $\mathrm{Aut}(\mathcal{Y}^{\sigma,\tau})$ [8], the largest normal 2-subgroup T of $\mathrm{Aut}(\mathcal{Y}^{\sigma,\tau})$ is a translation group, and therefore T is the unique transalation group in $\mathrm{Aut}(\mathcal{Y}^{\sigma,\tau})$, because all translation subgroups form a conjugacy class under $\mathrm{Aut}(\mathcal{Y}^{\sigma,\tau})$.)

On the other hand, we can verify that $Y' := \{(\mathrm{tr}(y), y) \mid y \in F\}$ is a complement to $\mathcal{Y}^{\sigma,\tau}$, where tr denotes the absolute trace function. In fact, any vector v in $X(c) \cap Y'$ ($c \in F$) is of the form $v = (x, x^\sigma c + xc^\tau) = (\mathrm{tr}(y), y)$ for some $x, y \in F$, whence $x = \mathrm{tr}(x^\sigma c + xc^\tau)$ is either 0 or 1. However, $x = 1$ does not satisfy this condition, as $\mathrm{tr}(c + c^\tau) = 0$. Then $v = 0$ and $X(c) \cap Y' = \{(0,0)\}$ for all $c \in F$. Thus Y' is a complement to $\mathcal{Y}^{\sigma,\tau}$, which is distinct from $Y = \{(0, y) \mid y \in F\}$. Hence $\mathcal{Y}^{\sigma,\tau}$ is not bilinear over Y'.

We shall investigate a DHO $\mathcal{A} = \mathcal{B}_1 \cup \mathcal{B}_2$ of rank $n+1$ obtained by applying Corollary 1.4(i) to a bilinear DHO $\mathcal{B}_1 = \mathcal{Y}^{\sigma,\tau}$ over Y, which is the extension of $\mathcal{B}_1$ in the sense of Section 1.7. Explicitly, $\mathcal{A}$ is uniquely determined as follows, if we fix a member B_1 of $\mathcal{B}_1$, a complement Y_1 to $\mathcal{B}_1$, a vector space A_2 of dimension $n+1$ with $A_2 \cap \mathbf{U}(\mathcal{B}_1) = \{0\}$, a hyperplane B_2 of A_2, and a vector a_{12} in $A_2 \setminus B_2$. The ambient space of $\mathcal{A}$ is contained in $U := \{(\varepsilon; y, x, z) \mid \varepsilon \in \mathbb{F}_2, x, y, z \in F\}$, in which the member $B_1(c)$ of $\mathcal{B}_1$ indexed by $c \in F$, the subspace A_2, the hyperplane B_2 of A_2 and the vector a_{12} are respectively identified with:

$$B_1(c) = \{(0; 0, x, xL(c)) \mid x \in F\}, \quad A_2 = \{(\varepsilon; z, 0, 0) \mid z \in F\},$$
$$B_2 = \{(0; z, 0, 0) \mid z \in F\}, \quad a_{12} = (1; 0, 0, 0).$$

Thus we identify the original DHO $\mathcal{B}_1$ with the collection of the above subspaces $B_1(c)$ ($c \in F$) of $\{(0; 0, x, y) \mid x, y \in F\}$. We choose $B_1 := B_1(0) = \{(0; 0, x, 0) \mid x \in F\}$ and take a complement $Y_1 := \{(0; 0, 0, y) \mid y \in F\} \cap \mathbf{U}(\mathcal{B}_1)$ to $\mathcal{B}_1$, corresponding to the above complement Y to $\mathcal{Y}^{\sigma,\tau}$. We also identify F with B_2, the set adopted for parametrizing the members of $\mathcal{B}_i$ ($i = 1, 2$) in Corollary 1.4(i), via the map $F \ni c \mapsto (0; c, 0, 0) \in B_2$. Under these identification, $\mathcal{A}$ is given as the collection of the following subspaces $A_1(c)$ and $A_2'(d)$ for $c, d \in F$ by

Corollary 1.4(i):

$$A_1(c) := \{(\varepsilon; \varepsilon c, x, xL(c)) \mid \varepsilon \in \mathbb{F}_2, x \in F\},$$
$$A_2'(d) := \{(\varepsilon; z, \varepsilon\kappa_d, \kappa_d L(z)) \mid \varepsilon \in \mathbb{F}_2, z \in F\}.$$

The second subDHO $\mathcal{B}_2$ is a collection of hyperplanes $B_2'(d) := \{(0; z, 0, \kappa_d L(z)) \mid z \in F\}$ of $A_2'(d)$, $d \in F$.

For convenience, we now change the parametrization to the members $A_2'(d)$ ($d \in F$). Recall that the map $\kappa : F \ni x \mapsto \kappa_x \in F$ is a bijection with $\kappa_0 = 0$ (see Section 1.5). Thus we may rearrange $A_2(d) := A_2'(\kappa^{-1}(d))$ and accordingly $B_2(d) := B_2'(\kappa^{-1}(d))$ ($d \in F$). Then for $d \in F$

$$A_2(d) = \{(\varepsilon; z, \varepsilon d, dL(z)) \mid \varepsilon \in \mathbb{F}_2, z \in F\} \text{ and}$$
$$B_2(d) = \{(0; z, 0, dL(z)) \mid z \in F\}.$$

The members $B_1(c) = \{(0; 0, x, xL(c)) \mid x \in F\}$ ($c \in F$) of $\mathcal{B}_1$ are copies of the members $X(c) = \{(x, xL(c)) = (x, x^\sigma c + xc^\tau) \mid x \in F\}$ ($c \in F$) of $\mathcal{Y}^{\sigma,\tau}$, while the members $B_2'(c) = \{(0; x, 0, cL(x)) \mid x \in F\}$ of $\mathcal{B}_2$ are copies of the members $X'(c) = \{(x, xL^{(\tau,\sigma)}(c)) \mid x \in F\}$ of $\mathcal{Y}^{\tau,\sigma}$, because $xL^{(\tau,\sigma)}(c) = c^\sigma x + cx^\tau = cL^{(\sigma,\tau)}(x)$ for $x \in F$. Thus we conclude that $\mathcal{A}$ is a disjoint union of $\mathcal{B}_1 \cong \mathcal{Y}^{\sigma,\tau}$ and $\mathcal{B}_2 \cong \mathcal{Y}^{\tau,\sigma}$.

If $\tau \neq \sigma^{-1}$ (resp. $\tau = \sigma^{-1}$), we have $\mathbf{U}(\mathcal{A}) = \mathbb{F}_2 \oplus F \oplus F \oplus F$ of dimension $3n + 1$ (resp. $\mathbf{U}(\mathcal{A})$ is a hyperplane of $\mathbb{F}_2 \oplus F \oplus F \oplus F$). If, furthermore, $\mathcal{Y}^{\sigma,\tau}$ is simply connected, it follows from Proposition 1.2(3) that $\mathcal{A}$ is simply connected as well. This DHO $\mathcal{A}$, the extension of $\mathcal{B}_1 = \mathcal{Y}^{\sigma,\tau}$ in the sense of Section 1.7, was constructed first in [1, Section 5] for the cases $\tau = \sigma$ and $\tau = \sigma^{-1}$, and generalized by Taniguchi in [4] for the other cases. However, the authors of [1] and [4] do not work in the broader context of disjoint unions of subDHOs nor mention the uniqueness in Corollary 1.4.

Notice that for any linear bijection λ on F, $\mathcal{A}$ splits over $Y_\lambda := \{(0; x, x^\lambda, y) \mid x, y \in F\}$. (This corresponds to the complement Y_ρ in Corollary 1.4(ii),(iii) with $\rho : B_2 \ni (0; x, 0, 0) \mapsto (0; 0, x^\lambda, 0) \in B_1$. We shall determine all λ for which $\mathcal{A}$ is bilinear over Y_λ. (The author does not know whether $\mathcal{A}$ is bilinear over another complement $Y_\lambda' := \{(\mathrm{tr}(y); x, x, y) \mid x, y \in F\}$. We can verify that Y_λ' is a complement to $\mathcal{A}$ by the arguments similar to those in the observations about complements of $\mathcal{Y}^{\sigma,\tau}$.)

Assume that the linear system $\{L(c, \delta) \mid c \in F, \delta \in \mathbb{F}_2\}$ for $\mathcal{A}$ over $Y = Y_\lambda$ (given in Corollary 1.4(iii)) is bilinear. Then we have $(c^\lambda)L(x) = (x^\lambda)L(c)$ for any $c, x \in F$ by equation (5) in Corollary 1.4(ii). Furthermore, it follows from the last remark in Corollary 1.4(iii) that $\mathcal{B}_1 \cong \mathcal{Y}^{\sigma,\tau}$ is isomorphic to $\mathcal{B}_2 \cong \mathcal{Y}^{\tau,\sigma}$.

As we reviewed in the above paragraph, this happens only when $\tau = \sigma^{-1}$ or $\tau = \sigma$. Correspondingly, the above equation $(c^\lambda)L(x) = (x^\lambda)L(c)$ reads

$$(c^\lambda)^\sigma x + c^\lambda x^{\sigma^{-1}} = (x^\lambda)^\sigma c + x^\lambda c^{\sigma^{-1}} \quad \text{if } \tau = \sigma^{-1}, \tag{21}$$

$$(c^\lambda)^\sigma x + c^\lambda x^\sigma = (x^\lambda)^\sigma c + x^\lambda c^\sigma \quad \text{if } \tau = \sigma. \tag{22}$$

We consider the former case where $\sigma^{-1} = \tau$. We first claim that any bijective linear map λ on F satisfying equation (21) for all $x, c \in F$ is given by $x^\lambda = kx^{\sigma^{-1}}$ $(x \in F)$ for a nonzero element k of F.

This claim is verified as follows. Equation (21) implies that $c^{\lambda\sigma}x + x^{\lambda\sigma}c = (c^{\lambda\sigma}x + x^{\lambda\sigma}c)^{\sigma^{-1}}$, whence $\epsilon(c, x) := c^{\lambda\sigma}x + x^{\lambda\sigma}c$ lies in $\mathbb{F}_2$, as σ is a generator of $\mathrm{Gal}(F/\mathbb{F}_2)$. By the bijectivity of λ, there is an element $c_0 \in F$ with $c_0^\lambda = 1$. Then $c_0 \neq 0$ and $k := c_0^{-\sigma^{-1}}$ satisfies $x^\lambda = k(x^{\sigma^{-1}} + \epsilon(x))$ for all $x \in F$, where we denote $\epsilon(x) := \epsilon(c_0, x)$ $(\in \mathbb{F}_2)$. As λ, σ^{-1} and the multiplication by k are linear maps, the map ϵ sending $x \in F$ to $\epsilon(x) \in \mathbb{F}_2$ is linear. In particular, the kernel of ϵ is of dimension at least $n-1 \geq 2$. On the other hand, putting $y^\lambda = k(y^{\sigma^{-1}} + \epsilon(y))$ for $y = c, x$ into equation (21), we obtain $\epsilon(c)(k^\sigma x + kx^{\sigma^{-1}}) = \epsilon(x)(k^\sigma c + kc^{\sigma^{-1}})$ for all $c, x \in F$. Suppose there is $c \in F$ with $\epsilon(c) = 1$. Then any x in the kernel of ϵ satisfies $k^\sigma x + kx^{\sigma^{-1}} = 0$, whence $x = 0$ or $x = k^{-\sigma}$ by the injectivity of $\sigma - \mathrm{id}_F$. This contradicts the above remark on the dimension of the kernel of ϵ. Hence we have $x^\lambda = kx^{\sigma^{-1}}$ for all $x \in F$. Conversely, for any nonzero element k of F, the bijective linear map λ defined by $x^\lambda = kx^{\sigma^{-1}}$ $(x \in F)$ satisfies equation (21) for all $x, c \in F$.

Now we fix a nonzero element k of F. We denote by λ a bijective linear map on F defined by $x^\lambda = kx^{\sigma^{-1}}$ $(x \in F)$. With some calculations, we can verify that equations (4) in Corollary 1.4(ii) read

$$(\varepsilon; 0, y, 0)L(c, 0) = \varepsilon(0; c, kc^{\sigma^{-1}}, k^\sigma c^2 + (k^\sigma c^2)^{\sigma^{-1}}) + (0; 0, 0, cy^\sigma + (cy^\sigma)^{\sigma^{-1}})$$

$$\begin{aligned}(\varepsilon; 0, y, 0)L(c, 1) = {} & \varepsilon(0; k^{-\sigma}(1/c), (1/c)^{\sigma^{-1}}, k^{-\sigma}(1/c)^2 + (k^{-\sigma}(1/c)^2)^{\sigma^{-1}}) \\ & + (0; k^{-\sigma}y^\sigma, y, k^{-\sigma}(1/c)y^\sigma + (k^{-\sigma}(1/c)y^\sigma)^{\sigma^{-1}}),\end{aligned}$$

where we understand $1/c = 0$ in the expression of $L(c, 1)$ for $c = 0$. (Note that $(0; x, 0, 0)^\rho = (0; 0, kx^{\sigma^{-1}}, 0)$ and $\kappa_c = (1/c)^{\sigma^{-1}}$ for $c \in F$ with $c \neq 0$ and $\kappa_0 = 0$.) Thus applying the permutation π on $F \times \mathbb{F}_2$ given by $(c, 0)^\pi = (c, 0)$, $(0, 1)^\pi = (0, 1)$ and $(c, 1)^\pi = (k^{-\sigma}(1/c), 1)$, we have

$$\begin{aligned}(\varepsilon; 0, y, 0)L((c, \delta)^\pi) = {} & \varepsilon(0; c, kc^{\sigma^{-1}}, k^\sigma c^2 + (k^\sigma c^2)^{\sigma^{-1}}) \\ & + \delta(0; k^{-\sigma}y^\sigma, y, 0) + (0; 0, 0, cy^\sigma + (cy^\sigma)^{\sigma^{-1}})\end{aligned}$$

for any $\varepsilon \in \mathbb{F}_2$, $y \in F$ and $(c, \delta) \in B_2 \times \mathbb{F}_2$. From this equation, we can easily conclude that

$$L((c_1, \delta_1)^\pi) + L((c_2, \delta_2)^\pi) = L((c_1 + c_2, \delta_1 + \delta_2)^\pi)$$

for all (c_1, δ_1), (c_2, δ_2) of $B_2 \times \mathbb{F}_2$. Thus in this case $\mathcal{A}$ is in fact bilinear over Y_λ for any λ given by $x^\lambda = kx^{\sigma^{-1}}$ $(x \in F)$. This DHO $\mathcal{A}$ is simply connected by Proposition 1.2(3), because $\mathcal{B}_1 \cong \mathcal{Y}^{\sigma,\sigma^{-1}}$ is simply connected. The ambient space of $\mathcal{A}$ is of dimension $(n+1)+(2n-1) = 3n$.

As we saw above, the form $\beta(x,y) := kx^{\sigma^{-1}} L_1(y) = (k^\sigma xy) + (k^\sigma xy)^{\sigma^{-1}}$ is a symmetric bilinear form on F. We may also apply [1, Theorem 5.1] to this form to obtain $\mathcal{A}$ as the extension $\overline{\mathcal{S}}$ of $\mathcal{S} = \mathcal{S}_\beta \cong \mathcal{B}_1$ in the sense of [1, Section 5]. This is given as the first example there.

We denote Y_λ by $Y_{(k)} = \{(0; x, kx^{\sigma^{-1}}, y) \mid x, y \in F\}$, if $x^\lambda = kx^{\sigma^{-1}}$ for a nonzero element k in F. We also denote a linear map $L((c,\delta)\pi)$ (from $A(0;0) = \{(\varepsilon; 0, y, 0) \mid \varepsilon \in \mathbb{F}_2, y \in F\}$ into $Y_{(k)}$) in the linear system $\mathcal{L}_F(\mathcal{A}, Y_\lambda)$ by $L^{(k)}((c,\delta)\pi)$. In the observations given before investigating $\mathcal{A} = \mathcal{B}_1 \cup \mathcal{B}_2$, we saw that $\mathrm{Aut}(\mathcal{A})$ for a bilinear DHO $\mathcal{A}$ acts transitively on the set of bilinear complements. Thus there is an automorphism α of $\mathcal{A}$ which sends a bilinear complement $Y_{(1)}$ to a bilinear complement $Y_{(k)}$. The following is an explicit automorphism with this property: Define a linear bijection α on $U = \{(\varepsilon; x, y, z) \mid \varepsilon \in \mathbb{F}_2, x, y, z \in F\}$ by $(\varepsilon; x, y, z)\alpha := (\varepsilon; k^{-\sigma/2}x, k^{1/2}y, z)$. Clearly $Y_{(1)}^\alpha = Y_{(k)}$. We can verify that $L^{(1)}((c,\delta)\pi)\alpha = m_{k^{1/2}} L^{(k)}((k^{-\sigma/2}c, \delta)\pi)$, where $m_{k^{1/2}}$ denotes the bijection on $A(0;0)$ sending $(\varepsilon; 0, y, 0)$ to $(\varepsilon; 0, k^{1/2}y, 0)$. Thus α sends a typical vector $(\varepsilon; 0, y, 0) + (\varepsilon; 0, y)L^{(1)}((c,\delta)\pi)$ in a member $X((c,\delta)\pi)$ of $\mathcal{A}$ to a vector $(\varepsilon; 0, k^{1/2}y, 0) + (\varepsilon; 0, k^{1/2}y, 0)L^{(k)}((k^{-\sigma/2}c, \delta)\pi)$, which lies in a member $X((k^{-\sigma/2}c, \delta)\pi)$ of $\mathcal{A}$. Hence $\alpha \in \mathrm{Aut}(\mathcal{A})$.

Now we move to the remaining case where $\sigma = \tau$. We first claim that $\lambda = \mathrm{id}_F$ is the unique solution satisfying (22) for all $x, c \in F$. Take $c_0 \in F$ with $c_0^\lambda = 1$. Then we have $x + x^\sigma = x^{\lambda\sigma} c_0 + x^\lambda c_0^\sigma$ for all $x \in F$. Taking the absolute trace of the both sides of this equation, we conclude that $0 = \mathrm{tr}(x^\lambda(c_0^{\sigma^{-1}} + c_0^\sigma))$ for all $x \in F$. Thus $c_0^{\sigma^2} = c_0$, whence c_0 is a nonzero element in the subfield $\mathbb{F}_{2^{(2,n)}}$ of F. In particular, $c_0^\sigma = c_0^2$ and $c_0^3 = 1$. Then the above equation $x + x^\sigma = x^{\lambda\sigma} c_0 + x^\lambda c_0^\sigma$ reads $(x + x^\lambda c_0^\sigma)^\sigma = x + x^\lambda c_0^\sigma$, whence $x^\lambda c_0^2 + x =: \epsilon(x) \in \mathbb{F}_2$. By the linearity of λ, the map ϵ sending $x \in F$ to $\epsilon(x) \in \mathbb{F}_2$ is linear with $x^\lambda = c_0(x + \epsilon(x))$ for all $x \in F$. Then the kernel of ϵ has dimension at least $n - 1 \geq 2$. Thus there are distinct nonzero elements x and c of F with $x^\lambda = c_0 x$ and $c^\lambda = c_0 c$. To these elements, equation (22) reads $c_0^2 c^\sigma x + c_0 c x^\sigma = c_0^2 x^\sigma c + c_0 x c^\sigma$, which is equivalent to $(c_0^2 + c_0)(xc^\sigma + cx^\sigma) = 0$. This implies that $c_0^2 = c_0$ by our choice for x and c, because $xc^\sigma + cx^\sigma = 0$ for nonzero x and c in F implies that $x = c$. Thus $c_0 = 1$ and $x^\lambda = x + \epsilon(x)$ for all $x \in F$. Then equation (22) reads $\epsilon(c)(x + x^\sigma) = \epsilon(x)(c + c^\sigma)$ for all $c, x \in F$. If there is $c \in F$ with $\epsilon(c) = 1$, this implies that $x = x^\sigma \in \mathbb{F}_2$ for all x in the kernel of ϵ, which contradicts the above remark on the dimension of the kernel of ϵ. Hence $\epsilon(x) = 0$ and $x^\lambda = x$ for all $x \in F$. Conversely, $\lambda = \mathrm{id}_F$ satisfies $c^\sigma x + cx^\tau = x^\sigma c + xc^\tau$ for any $c, x \in F$.

In this case, we have $\kappa_c = c$ and the form $\beta(x,y) = xL_1(y) = x^\sigma y + xy^\sigma$ is alternating. We can apply [1, Corollary 5.3(a)] to this form to conclude that $\mathcal{A}$ (this is again the same as the extension of $\mathcal{S} = \mathcal{S}_\beta \cong \mathcal{B}_1$ in the sense of [1, Section 5]) is bilinear. We can show this fact more directly by the similar calculation to the above:

$$(\varepsilon;0,y,0)L(c,\delta) = \varepsilon(0;c,c,0) + \delta(0;y,y,0) + (0;0,0,y^\sigma c + yc^\sigma)$$

for all $\varepsilon \in \mathbb{F}_2$, $y \in F$, $(c,\delta) \in F \times \mathbb{F}_2$, and therefore $L((c_1,\delta_1)) + L((c_2,\delta_2)) = L((c_1+c_2,\delta_1+\delta_2))$ for all $(c_1,\delta_1), (c_2,\delta_2) \in F \times \mathbb{F}_2$.

Summarizing, the extension $\mathcal{A}$ of $\mathcal{B}_1 \cong \mathcal{Y}^{\sigma,\tau}$ is bilinear over Y_λ if and only if $(\lambda,\sigma,\tau) = (\mathrm{id}_F,\sigma,\sigma)$ or $(m_k\sigma^{-1},\sigma,\sigma^{-1})$ for a nonzero element k in F, where m_k denotes the multiplication by k.

Recall that $\mathcal{B}_1 \cong \mathcal{Y}^{\sigma,\sigma}$ is a DHO obtained from the Gold function, the most standard quadratic APN function on F. It is not simply connected but covered by the Huybrechts DHO $\mathcal{H}_n(\mathbb{F}_2)$. The ambient space of the extension $\mathcal{A}$ of $\mathcal{B}_1$ has dimension $(n+1)+2n = 3n+1$, and $\mathcal{A}$ is covered by the extension of $\mathcal{H}_n(\mathbb{F}_2)$ (in the sense of Section 1.7). As we saw in Section 3.2, t he extension of $\mathcal{H}_n(\mathbb{F}_2)$ is isomorphic to $\mathcal{H}_{n+1}(\mathbb{F}_2)$. (Recall that there is a non-bilinear complement Y' to $\mathcal{Y}^{\sigma,\sigma}$. We can verify that the inverse image by a covering map from $\mathcal{H}_n(\mathbb{F}_2)$ onto $\mathcal{Y}^{\sigma,\sigma}$ is a non-bilinear complement to $\mathcal{H}_n(\mathbb{F}_2)$.)

It should also be mentioned that, if $\tau \neq \sigma$ and $\tau \neq \sigma^{-1}$, the extension $\mathcal{A}$ ($\cong \mathcal{Y}^{\sigma,\tau} \sqcup \mathcal{Y}^{\tau,\sigma}$) of $\mathcal{Y}^{\sigma,\tau}$ is not bilinear over any complement. This provides many new examples of non-bilinear, simply connected DHOs of rank n with ambient space of dimension $3n+1$.

3.4. Yoshiara-Taniguchi DHO

The DHOs $\mathcal{Y}^{\sigma,\tau}$ in Section 3.3 form a subfamily of a wider class of DHOs, denoted $\mathcal{S}^{\sigma,\varphi}$, where φ is an o-polynomial on $F = \mathbb{F}_{2^n}$ (see [9, Section 5.5]). Recall that a polynomial $f(X) \in F[X]$ is called *permutation polynomial* if the map sending $x \in F$ to $f(x) \in F$ is a bijection. A permutation polynomial $f(X) \in F[X]$ is called an *o-polynomial* if $f(0) = 0$, $f(1) = 1$ and for each $s \in F$, the polynomial $f_s(X) \in F[X]$ given by $f_s(X) = (f(X+s)+f(s))/X$ is a permutation polynomial with $f_s(0) = 0$. For a generator τ of the Galois group $\mathrm{Gal}(F/\mathbb{F}_2)$, if $x^\tau = x^{2^m}$ with $(n,m) = 1$, the polynomial $f(X) = X^{2^m}$ is an o-polynomial, denoted f_τ or f_{2^m}. Thus $\mathcal{Y}^{\sigma,\tau}$ can be regarded as $\mathcal{S}^{\sigma,f_\tau}$.

Now we recall the construction of $\mathcal{S}^{\sigma,\varphi}$. See [5] for the details. We assume that $\varphi \neq f_{\sigma^{-1}}$. (The case $\varphi = f_{\sigma^{-1}}$ has been already treated in Section 3.3.) The ambient space is $F \oplus F$. The members are parametrized by $F = \mathbb{F}_{2^n}$ and the member $S(c)$ corresponding to $c \in F$ is given by $S(c) = \{(x, x^\sigma c + x\varphi(c)) \mid$

$x \in F\}$. We identify a specific member $B_1 := S(0) = \{(x,0) \mid x \in F\}$ with F via the map sending $(x,0)$ to x. The DHO $\mathcal{S}^{\sigma,\varphi} := \{S(c) \mid c \in F\}$ splits over $Y := \{(0,y) \mid y \in F\}$. The linear system $\mathcal{L}_F(\mathcal{S}^{\sigma,\varphi}, Y) = \{L(c) \mid c \in F\}$ is given by the linear maps $xL(c) = x^\sigma c + x\varphi(c)$ for $x \in F$. This system gives a bilinear DHO if and only if φ is linear, which is known to equivalent to the condition that $\varphi = f_\tau$ for some Galois group generator τ. As this case has already been treated in Section 3.3, we shall assume that φ is not of the form f_τ for a generator τ of the Galois group $\mathrm{Gal}(F/\mathbb{F}_2)$.

We are concerning whether there exists a DHO $\mathcal{A}$ of rank $n+1$ over $\mathbb{F}_2$ satisfying conditions in Theorem 1.3(i) for $\mathcal{B}_1 = \mathcal{S}^{\sigma,\varphi}$ splitting over Y. To apply the criterion in Theorem 1.3(i) we need to find $s\{c,d\} = \kappa(c,d) + \kappa(c,d)L(c)$, the unique nonzero vector in $S(c) \cap S(d)$ for distinct $c, d \in F$. This criterion reads that the following two equations hold for all $d \in F^\times$, $x_1, x_2 \in F$:

$$0 = \kappa(d + x_1 + x_2, x_1 + x_2) + \kappa(d + x_1, x_1) + \kappa(d + x_2, x_2) + \kappa_d, \tag{23}$$

$$0 = \kappa(d + x_1 + x_2, x_1 + x_2)L(x_1 + x_2) + \kappa(d + x_1, x_1)L(x_1) + \kappa(d + x_2, x_2)L(x_2). \tag{24}$$

It is straightforward to show that $\kappa(c,d) = ((\varphi(c) + \varphi(d))/(c+d))^{1/(\sigma-1)}$, where $1/(\sigma-1)$ denotes the inverse map of the map $F^\times \ni x \mapsto x^{\sigma-1} = x^\sigma/x \in F^\times$. (As σ is a generator of the absolute Galois group $\mathrm{Gal}(F/\mathbb{F}_2)$, this is injective.) Then for $d \in F^\times$ we have

$$\kappa(d + x, x) = \left(\frac{\varphi(d+x) + \varphi(x)}{d}\right)^{1/(\sigma-1)}. \tag{25}$$

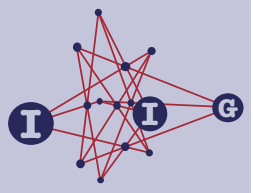
At the present stage, the author does not know whether the above two requirements (23) and (24) for an o-polynomial φ with equation (25) implies that φ is linear, and hence $\varphi = f_\tau$ for a generator τ of the Galois group $\mathrm{Gal}(F/\mathbb{F}_2)$.

If σ is the square map (and so $\sigma - 1$ is just the identity map), equation (23) for $\kappa(d+x,x)$ given in equation (25) reads

$$\varphi(d + x_1 + x_2) + \varphi(d + x_1) + \varphi(d + x_2) + \varphi(x_1 + x_2) + \varphi(x_1) + \varphi(x_2) + \varphi(d) = 0$$

for all $d, x_1, x_2 \in F$. Recall that a map φ on F with $\varphi(0) = 0$ is called *quadratic* if it satisfies this condition. Hence if σ is the square map, there exists a DHO $\mathcal{A}$ satisfying the conditions given in Theorem 1.3 for $\mathcal{B}_1 = \mathcal{S}^{\sigma,\varphi}$ exists if and only if φ is a quadratic o-polynomial satisfying equation (24).

As far as the author knows, the following two are the only known families of quadratic monomial o-polynomials on $F = \mathbb{F}_{2^n}$: $\phi(x) = x^6$ on $F = \mathbb{F}_{2^n}$ with n odd, $\phi(x) = x^{2^{2m+1}+2^{3m+1}}$ on $F = \mathbb{F}_{2^n}$ with $n = 4m+1$. However, it is not difficult to verify that none of them satisfies equation (24). We omit the details.



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