Erratum to “Buildings with isolated subspaces and relatively hyperbolic Coxeter groups”

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The goal of this note is to correct two independent errors in [4], respectively in Theorems A and B from *loc. cit.* I am indebted to Alessandro Sisto, who pointed them out to me. Those corrections affect neither the characterization of toral relatively hyperbolic Coxeter groups (Corollaries D and E from [4]), nor the other intermediate results from the original paper.

We keep the notation and terminology from *loc. cit.* Moreover all Coxeter groups under consideration are assumed to be finitely generated. The first correction concerns Theorem A: its assertions (ii), (iii), (iv) are indeed equivalent, but a third condition (RH3) has to be added to (RH1) and (RH2) in assertion (i), as in the following reformulation.

Theorem A′. *Let (W, S) be a Coxeter system and $\mathscr{T}$ be a set of subsets of the Coxeter generating set S. Then W is hyperbolic relative to $\mathscr{P} = \{W_J \mid J \in \mathscr{T}\}$ if and only if the following three conditions hold:*

(RH1) *For each irreducible affine subset $J \subset S$ of cardinality ≥ 3, there exists $K \in \mathscr{T}$ such that $J \subset K$. Similarly, for each pair of irreducible non-spherical subsets $J_1, J_2 \subset S$ with $[J_1, J_2] = 1$, there exists $K \in \mathscr{T}$ such that $J_1 \cup J_2 \subset K$.*

(RH2) *For all $K_1, K_2 \in \mathscr{T}$ with $K_1 \neq K_2$, the intersection $K_1 \cap K_2$ is spherical.*

(RH3) *For each $K \in \mathscr{T}$ and each irreducible non-spherical $J \subset K$, we have $J^\perp \subset K$.*

Proof. The necessity of (RH1) and (RH2) is established in [4]. The condition (RH3) is also necessary, as pointed out by Alessandro Sisto: if there is a reflection $s \in S$ and a set $K \in \mathscr{T}$ such that $s \notin K$ and s commutes with an irreducible non-spherical subset $J \subset K$, then the cosets W_K and sW_K of the parabolic subgroup W_K are distinct, but the intersection of their respective 1-neighbourhoods in the Cayley graph of (W, S) is unbounded, since it contains W_J. This contradicts the fact that W is hyperbolic relative to $\mathscr{P}$.

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Assume conversely that (RH1), (RH2) and (RH3) hold. As in [4], we need to show that the set $\mathscr{F}$, consisting of all residues of the Davis complex of (W, S) whose type belongs to $\mathscr{T}$, satisfies the isolation conditions (A) and (B) from *loc. cit.* The arguments given there show that (RH1) is sufficient to ensure that (A) holds. Moreover it is shown that if $\mathscr{F}$ does not satisfy (B), then there exists two distinct residues $F, F' \in \mathscr{F}$ whose respective stabilisers P, P', which are parabolic subgroups of W, share a common infinite dihedral reflection subgroup. The mistake in [4] lies in the sentence: 'By (RH2), this implies that P and P' coincide.' The corrected argument, which requires also invoking (RH3), goes as follows. We may write $P = gW_Kg^{-1}$ and $P' = g'W_{K'}(g')^{-1}$ for some $K, K' \in \mathscr{T}$ and $g, g' \in W$. Since $P \cap P'$ contains an infinite dihedral reflection subgroup, it also contains the parabolic closure of that subgroup, say Q, which is of irreducible non-spherical type by [4, Lemma 2.1]. Therefore there is an irreducible non-spherical subset $J \subset K$ (resp. $J' \subset K'$) such that Q is conjugate to gW_Jg^{-1} in P (resp. to $g'W_{J'}(g')^{-1}$ in P'). It follows that W_J is conjugate to $W_{J'}$ and, hence, that J and J' are conjugate in W. By [5, Proposition 5.5], it follows that $J = J'$, so that $K = K'$ by (RH2). In particular P and P' are conjugate. Let $p \in P$ be an element which conjugates gW_Jg^{-1} to Q. Upon replacing g by pg, we may assume that $Q = gW_Jg^{-1}$. Similarly we may assume that $Q = g'W_J(g')^{-1}$. It follows that $g^{-1}g'$ normalises W_J. By [5, Proposition 5.5], the normaliser of W_J coincides with $W_{J \cup J^\perp}$, and is thus contained in W_K by (RH3). Hence $g^{-1}g'$ normalizes W_K, so that $P = P'$. Condition (RH3) together with [3, Proposition 2.1] and [5, Proposition 5.5] also implies that P is self-normalising, which implies that there is a unique residue in the Davis complex, whose full stabiliser is P. We deduce that $F = F'$, a contradiction. This confirms that (B) holds. $\square$

We next remark that Corollaries D and E from [4] are not affected by the above correction: indeed, in the respective settings of those corollaries, the condition (RH3) holds automatically. In Corollary C, for all three conditions (RH1)–(RH3) to be satisfied, the definition of $\mathscr{T}$ has to be adapted as follows:

$$\mathscr{T} = \big\{S \setminus \{s_0\}\big\} \cup \big\{J \cup J^\perp \mid J \text{ is irreducible affine of cardinality} \geq 3 \\ \text{and contains } s_0\big\}.$$

We now turn to the second error, which lies in Theorem B from [4]. The purpose of that statement was to answer the following question:

> *Assuming that W is hyperbolic with respect to some peripheral subgroups $H_1, \dots, H_m$, can one relate those peripheral subgroups to the parabolic subgroups of W (in the usual Coxeter group theoretic sense)?*

Theorem B asserted that those peripheral subgroups are always parabolic in the Coxeter group theoretic sense. This is not true in general: indeed, any Gromov hyperbolic group is also relatively hyperbolic with respect to any malnormal collection of quasi-convex subgroups, see [2, Theorem 7.11]. Therefore, even if W is Gromov hyperbolic, one can always make it relatively hyperbolic by adding maximal self-normalising cyclic subgroups as peripheral subgroups, and those are not parabolic in the Coxeter sense. The correct statement can be phrased as follows:

> *If W is relatively hyperbolic with respect to some peripheral subgroups $H_1, \ldots, H_m$, then it is also relatively hyperbolic with respect to a (possibly empty) collection of Coxeter-parabolic subgroups $P_1, \ldots, P_k$, and moreover, each P_i is conjugate to a subgroup of some H_j.*

In particular every Coxeter group admits a canonical, minimal, relatively hyperbolic structure, whose peripheral subgroups are indeed parabolic in the Coxeter group theoretic sense. The latter result has been obtained in a joint work with Jason Behrstock, Mark Hagen and Alessandro Sisto. In that work, we also provide various characterizations of the canonical parabolic subgroups $P_1, \ldots, P_k$, and describe necessary and sufficient conditions on a Coxeter presentation of W ensuring that W is not relatively hyperbolic with respect to any collection of proper subgroups. Those results appear in the Appendix to [1].

References

[1] **J. Behrstock**, **M. Hagen** and **A. Sisto**, Thickness, relative hyperbolicity, and randomness in Coxeter groups (2013), available at `arxiv.org/abs/1312.4789`. Preprint (with an appendix jointly written with P-E. Caprace).

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[4] **P-E. Caprace**, Buildings with isolated subspaces and relatively hyperbolic Coxeter groups, *Innov. Incidence Geom.* **10** (2009), 15–31.

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