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α-Flokki and partial α-flokki

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Abstract

Connections are made between deficiency one α-flokki and Baer groups of associated α-flokki translation planes, extending the theory of Johnson and Payne–Thas. The full collineation group of an α-flokki is completely determined. Many of the ideas are extended to the infinite case.

Keywords: α-flokki, Baer subgroup, translation plane.

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1 Introduction

In this article, we are interested in extending the ideas of deficiency one flocks of quadratic cones in $\mathrm{PG}(3, q)$ and the associated Baer group theory of translation planes with spreads in $\mathrm{PG}(3, q)$ to their generalizations, α-flokki of cones $\mathcal{C}_\alpha$, where $\mathcal{C}_\alpha = \{(x_0, x_1, x_2, x_3) \mid x_0^\alpha x_1 = x_2^{\alpha+1}\}$ is a cone in $\mathrm{PG}(3, q)$ with vertex $(0, 0, 0, 1)$ with $\alpha \in \mathrm{Aut}(K)$. An α-flokki is a 'flock' of this cone, that is, a set of planes of $\mathrm{PG}(3, q)$ which partition the points of $\mathcal{C}_\alpha$ except for the vertex. There are corresponding translation planes here, as in the quadratic cone case, which we will call *α-flokki* planes.

First, we mention the Payne–Thas extension theorem for partial flocks of a quadratic cone.

Theorem 1.1 (Payne–Thas [13])**.** *A partial flock of a quadratic cone of deficiency one in $\mathrm{PG}(3, q)$ has a unique extension to a flock of a quadratic cone.*

The theory of Baer groups of Johnson [7] connects such partial flocks with translation planes admitting Baer groups.

Theorem 1.2 (Johnson [7])**.** *Translation planes with spread in $\mathrm{PG}(3, q)$ that admit Baer groups of order q are equivalent to deficiency one partial flocks of a quadratic cone.*

In this setting, we have $q-1$ planes of a partial flock of deficiency one of a quadratic cone, which may be extended to a flock by the theorem of Payne–Thas.

Therefore, we have:

Theorem 1.3. *Let π be a translation plane of order q^2 with spread in $\mathrm{PG}(3,q)$. If π admits a Baer group of order q then the partial spread of degree $q+1$, whose components are off the Baer axis, is a regulus partial spread. Derivation of this spread constructs a translation plane of order q^2 with spread in $\mathrm{PG}(3,q)$, admitting an elation group of order q whose orbits together with the elation axis are reguli; a conical translation plane.*

Therefore, translation planes of order q^2 with spread in $\mathrm{PG}(3,q)$ admitting Baer groups of order q are equivalent to flocks of quadratic cones in $\mathrm{PG}(3,q)$.

The Payne–Thas result in the odd order case involves the idea of derivation of a conical flock, whereas the proof in the even order case used ideas from extensions of k-arcs. However, a proof of this result by Sziklai [14] is independent of order. Here, it is realized that this proof may be adapted to prove the same theorem for α-flokki. The Baer group theory that applies is then an extension of the work of Johnson [7].

We also consider the cones $\mathcal{C}_q$ in $\mathrm{PG}(3,q^2)$ and algebraically lifting the spreads in $\mathrm{PG}(3,q)$. Such lifted spreads automatically give rise to *q-flokki* of the cones $\mathcal{C}_q$ (also see Kantor and Penttila [12]). A *bilinear* flock is a flock in which each plane passes through at least one of two distinct lines of $\mathrm{PG}(3,q)$; these lines (called *supporting lines*) may either meet or be skew. A result of Thas [15] shows that flocks of quadratic cones whose planes share a point must be linear in the even characteristic case and either linear or Knuth–Kantor in odd characteristic and hence bilinear flocks of quadratic cones with intersecting supporting lines do not exist in the finite case. Also, no bilinear flocks of quadratic cones with skew supporting lines are known in the finite case (see [4] for a more thorough study of finite bilinear flocks). However, Biliotti and Johnson [1], show that bilinear flocks can exist in $\mathrm{PG}(3,K)$, where K is a infinite field. In fact, the situation is much more complex for infinite flocks, for example, there are n-linear flocks for any positive integer n (the planes share exactly n lines). Recently, Cherowitzo and Holder [4], found an extremely interesting bilinear q-flokki using ideas from blocking sets. It might be suspected that the Cherowitzo–Holder q-flokki might be algebraically lifted from a translation plane, and indeed, in this article, we show that this is, in fact, the case.

We also show how work on extensions and Baer groups gives results that allow a reverse procedure that identifies certain translation planes of order q^4 that admit Baer groups of orders q^2 and $q+1$ as lifted and derived spreads.

2 Elation groups and flokki planes

Let π denote a translation plane of order q^2 with spread in $\mathrm{PG}(3,q)$ that admits an elation group E such that some orbit Γ union the axis is a derivable partial spread. We know from Johnson [8], that the derivable partial spread may be represented in the form

$$\left\{x=0,\ y=x\begin{bmatrix}u & 0\\ 0 & u^{\alpha}\end{bmatrix};\ u\in \mathrm{GF}(q)\right\},$$

where α is an automorphism of $\mathrm{GF}(q)$, and we have chosen the axis of E to be $x=0$ and Γ to contain $y=0$ and $y=x$. Here, as usual, $x=(x_1,x_2)$, $y=(y_1,y_2)$, for $x_i,y_i\in \mathrm{GF}(q)$, $i=1,2$ and vectors in the 4-dimensional vector space over $\mathrm{GF}(q)$ are (x_1,x_2,y_1,y_2).

Since Γ is an orbit, this means that E has the form

$$E=\left\{\begin{bmatrix}1 & 0 & u & 0\\ 0 & 1 & 0 & u^{\alpha}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};\ u\in \mathrm{GF}(q)\right\}.$$

Let $y=x\begin{bmatrix}g(t) & f(t)\\ t & 0\end{bmatrix}$ be a typical component of the spread of π for $t\in \mathrm{GF}(q)$ and the $(2,2)$-entry equal to zero, where g and f are functions on $\mathrm{GF}(q)$, with $g(0)=f(0)=0$. This is always possible by a basis change allowing that $y=0$ and $y=x$ represent components of π. Hence, the action of E on $y=x\begin{bmatrix}g(t) & f(t)\\ t & 0\end{bmatrix}$ produces components

$$y=x\begin{bmatrix}u+g(t) & f(t)\\ t & u^{\alpha}\end{bmatrix};\ t,u\in \mathrm{GF}(q).$$

Now assume that π is a translation plane with spread in $\mathrm{PG}(3,K)$, where K is an infinite field, and π admits an elation group such that the axis and some orbit Γ is a derivable partial spread. In this case, by Jha and Johnson [6], a derivable partial spread has the form

$$\left\{x=0,\ y=x\begin{bmatrix}u & A(u)\\ 0 & u^{\alpha}\end{bmatrix};\ u\in \mathrm{GF}(q)\right\},$$

where α is an automorphism of K and such that $\left\{\begin{bmatrix}u & A(u)\\ 0 & u^{\alpha}\end{bmatrix};\ u\in K\right\}$ is a field. Also, if there are at least two Baer subplanes that are K-subspaces, then $A\equiv 0$.

Definition 2.1. A translation plane π with spread in $\mathrm{PG}(3, K)$, where K is a field, is said to be an *α-flokki plane* if and only if there are functions g and f on K so that $f(0) = g(0) = 0$, and

$$\mathcal{S} = \left\{ x = 0,\ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u^{\alpha} \end{bmatrix} ;\ t, u \in K \right\} \tag{1}$$

is the spread for π, and α is an automorphism of K.

We will also say that $\mathcal{S}$ is an α-spread.

Lemma 2.2. *Let K be an infinite field. If there is a representation* (1) *of a partial spread in* $\mathrm{PG}(3, K)$*, then this partial spread is a maximal partial spread.*

Proof. To see that such a partial spread is maximal in $\mathrm{PG}(3, K)$, we assume not, then there is a matrix M so that $y = xM$ is mutually disjoint from the other components, which means, since $x = 0$ and $y = 0$ are components that $\begin{bmatrix} u + g(t) & f(t) \\ t & u^{\alpha} \end{bmatrix} - M$ is non-singular for all t, u in K. Letting $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, choose $t = c$ and $u^{\alpha} = d$, this forces $a = b = 0$. □

A maximal partial spread as described in Lemma 2.2 will be said say to be *injective but not bijective*. We will also mention α, as in *α-partial spreads*, when reference to the automorphism α is needed.

Remark 2.3. The previous lemma cannot be extended to the case when K is finite, as a set of matrices represented as in (1) would never yield a *partial* spread.

The next theorem follows immediately from Lemma 2.2 and the discussion before Definition 2.1.

Theorem 2.4. *A translation plane π with spread in* $\mathrm{PG}(3, K)$*, for K a field, is an α-flokki plane if and only if there is an elation group E one of whose orbits is a derivable partial spread containing at least two Baer subplanes that are K-subspaces.*

Remark 2.5. It makes sense to believe that linear α-flokki should correspond to Desarguesian α-flokki spreads. But this is not necessarily the case. For instance, let $\alpha(x) = x^q$ and consider the α-flokki spread in $\mathrm{PG}(3, q^2)$ given by,

$$\left\{ x = 0,\ y = x \begin{bmatrix} u & \gamma s^q \\ s & u^q \end{bmatrix} ;\ s, u \in \mathrm{GF}(q^2) \right\},$$

q odd and γ a non-square. Then the associated α-flokki is given by

$$\rho_t :\ x_0 t - x_1 \gamma^q t - x_3 = 0$$

for all $t \in K$. This is a linear α-flokki with an associated translation plane that is a semifield plane.

Example 2.6. Let K be any ordered (infinite) field and let α be an automorphism of K. Consider the α-partial flokki spread in $\mathrm{PG}(3,K)$:

$$\mathcal{S} = \left\{ x = 0,\ y = x \begin{bmatrix} u & -t^{3\alpha^{-1}} \\ t & u^{\alpha} \end{bmatrix};\ t, u \in K \right\}.$$

We first check that $\phi_u : t \to u^{\alpha+1}t^3$, is injective. Now

$$(t-s) + u^{\alpha+1}(t^3 - s^3) = 0,\ \text{for } t \neq s,$$

if and only if

$$1 + u^{\alpha+1}(t^2 + st + s^2) = 0,\ \text{for } t \neq s.$$

Consider the quadratic in t, $t^2+st+s^2+u^{-(\alpha+1)} = 0$, the discriminant of which is $s^2 - 4(s^2 + u^{-(\alpha+1)}) = -3s^2 - u^{-(\alpha+1)}$. But in any ordered field, $u^{\alpha+1} > 1$. Thus, the discriminant is negative, which is never a square. Hence, $\mathcal{S}$ is an α-partial spread, which because of Lemma 2.2 must be maximal.

Given any element r of K, the question is whether there is a solution to $t + u^{\alpha+1}t^3 = r$. Suppose that K is a subfield of the reals that does not contain all cube roots of elements of K. Then by Cardano's equations the roots will involve cube roots of elements of K and so the $t + u^{\alpha+1}t^3$ will not be surjective. Hence, there are subfields K of the field of real numbers for which the maximal partial spread is not a spread.

3 Maximal partial spreads and α-flokki

In this section, we connect α-flokki translation planes and maximal α-partial spreads (which are injective but not bijective) with flocks of the cone $\mathcal{C}_\alpha$. The ideas presented here originate from Cherowitzo–Holder [4], and Kantor–Penttila [12], in the finite case.

Definition 3.1. Let K be any field and let $\alpha \in \mathrm{Aut}(K)$. Considering homogeneous coordinates (x_0, x_1, x_2, x_3) of $\mathrm{PG}(3,K)$, we define the *α-cone* $\mathcal{C}_\alpha$ as $x_0^\alpha x_1 = x_2^{\alpha+1}$, with vertex $v_0 = (0,0,0,1)$.

A set of planes of $\mathrm{PG}(3,K)$ which partition the non-vertex points of $\mathcal{C}_\alpha$ will be called an *α-flokki*. The intersections are called *α-conics*.

The name *α-flokki* was coined (for K finite) by Kantor and Penttila [12].

Now we show that there are maximal partial spreads in $\mathrm{PG}(3,K)$, associated with α-flokki, which are called *α-partial spreads* or, if the context is clear, *flokki partial spreads*. When K is finite, these partial spreads are the spreads arising from α-flokki.

Theorem 3.2. *Let K be any field, and f and g be functions from K to K such that $f(0)=g(0)=0$. Then*

(1)(a)
$$\left\{x=0,\ y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u^{\alpha}\end{bmatrix};\ t,u\in K\right\}$$
is a maximal α-partial spread if and only if
$$\phi_u:\ t\to t-u^{\alpha+1}f(t)^{\alpha}+ug(t)^{\alpha},$$
is injective for all $u\in K$.

(1)(b)
$$\left\{x=0,\ y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u^{\alpha}\end{bmatrix};\ t,u\in K\right\}$$
is injective if and only if
$$\left\{x=0,\ y=x\begin{bmatrix} u+g(t) & t \\ f(t) & u^{\alpha}\end{bmatrix};\ t,u\in K\right\}$$
is injective.

(2)(a) *An injective maximal α-partial spread is equivalent to a partial α-flokki of $\mathcal{C}_{\alpha}$, having defining equations for the planes as follows:*
$$\rho_t:\ x_0t-x_1f(t)^{\alpha}+x_2g(t)^{\alpha}-x_3=0$$
for all $t\in K$.

(2)(b) *The two sets of functions*
$$\mathcal{F}=\{\phi_u\mid\phi_u:\ t\to t-u^{\alpha+1}f(t)^{\alpha}+ug(t)^{\alpha},\text{ for all }u\in K\}$$
$$\mathcal{F}^{\perp}=\{\phi_u^{\perp}\mid\phi_u^{\perp}:\ t\to f(t)-u^{\alpha+1}t^{\alpha}+ug(t)^{\alpha},\text{ for all }u\in K\}$$
both consist of injective functions if and only if either set consists of injective functions.

(3) *An α-flokki of $\mathcal{C}_{\alpha}$ is obtained if and only if ϕ_u is bijective for all $u\in K$.*

(4) *When K is finite, the set of α-flokki planes is equivalent to the set of α-flokki of $\mathcal{C}_{\alpha}$.*

(5) *If $\alpha^2=1$, $g\equiv 0$, and ϕ_u is bijective then, for any field K, this subset of α-flokki planes is equivalent to the corresponding set of α-flokki of $\mathcal{C}_{\alpha}$.*

Proof. Consider $\Gamma_u(t) = tu^{\alpha+1} - f(t)^\alpha + u^\alpha g(t)^\alpha$, and note that $u^{\alpha+1}\Gamma_{u^{-1}} = \phi_u$. It is immediate to check that Γ_u is injective if and only if ϕ_u is injective. So, we first show that Γ_u is injective for all u, if and only if

$$\mathcal{S} = \left\{ \begin{bmatrix} u + g(t) & f(t) \\ t & u^\alpha \end{bmatrix};\ u, t \in K \right\},$$

is a set of non-singular matrices or identically zero, whose distinct differences are also non-singular. This will show that the injectivity of the functions ϕ_u will prove that there are associated injective maximal partial α-flokki.

If we let $tu = w^\alpha$, we get

$$\begin{aligned} \Gamma_u(t) &= tu^{\alpha+1} - f(t)^\alpha + u^\alpha g(t)^\alpha \\ &= t^{-\alpha}(w^{\alpha+1} - tf(t) + w^\alpha g(t))^\alpha \\ &= t^{-\alpha}\left(\det \begin{bmatrix} w + g(t) & f(t) \\ t & w^\alpha \end{bmatrix}\right)^\alpha. \end{aligned}$$

Since u may be varied, and by looking at $\Gamma_u(t) - \Gamma_u(s)$, we see that the matrices in $\mathcal{S}$ are non-singular (or zero) and the differences of distinct pairs of matrices are non-singular if and only if Γ_u is injective. This proves (1)(a).

Part (1)(b) is immediate as the determinants of a matrix and its transpose are equal. Also, using an argument similar to that proving (1)(a), but now with $\Gamma_u^\perp(t) = f(t)u^{\alpha+1} - t^\alpha + u^\alpha g(t)^\alpha$ and $\phi_u^\perp = f(t) - u^{\alpha+1}t^\alpha + ug(t)^\alpha$ one obtains a 'transpose' analogue to (1)(a). This proves (2)(b).

Now let $P = (x_0, x_1, x_2, x_3)$ (homogeneous coordinates) be a point on $\mathcal{C}_\alpha \cap \rho_t$. If $x_0 = 0$, as $x_0^\alpha x_1 = x_2^{\alpha+1}$, then so is x_2. So, we get $P = (0, 1, 0, -f(t)^\alpha)$. If $x_0 \neq 0$ then we get

$$0 = t - x_2^{\alpha+1} f(t)^\alpha + x_2 g(t)^\alpha - x_3 = \phi_{x_2}(t) - x_3$$

and so $P = (1, x_1, x_2, \phi_{x_2}(t))$. It follows that ϕ_u being injective, for all u, is equivalent to all the intersections $\mathcal{C}_\alpha \cap \rho_t$ being disjoint. This proves (2)(a).

Now take a point $(u^{\alpha+1}, 1, u^\alpha, 0)$ in $\mathcal{C}_\alpha$ and form the line through $(0, 0, 0, 1)$. The point where this line intersects ρ_t is $(u^{\alpha+1}, 1, u^\alpha, tu^{\alpha+1} - f(t)^\alpha + u^\alpha g(t)^\alpha)$. Therefore, $tu^{\alpha+1} - f(t)^\alpha + u^\alpha g(t)^\alpha$ is bijective, for each u in K, if and only if we have an α-flokki of $\mathcal{C}_\alpha$, thus proving part (3). Since finite injective functions are bijective, we also have the proof of part (4).

Finally, consider $\alpha^2 = 1$ and $g(t) = 0$ for all t. Then, we consider for $x_1 x_2 \neq 0$, (x_1, x_2, y_1, y_2) is on $y = x \begin{bmatrix} u & f(t) \\ t & u^\alpha \end{bmatrix}$, if and only if

$$x_1 u + x_2 t = y_1 \qquad \text{and} \qquad x_1 f(t) + x_2 u^\alpha = y_2.$$

We now multiply the first equation by x_2^α, and apply the automorphism α to the second followed by a multiplication times x_1. We get,

$$x_1 x_2^\alpha u + x_2^{\alpha+1} t = y_1 x_2^\alpha \qquad \text{and} \qquad x_1^{\alpha+1} f(t)^\alpha + x_1 x_2^\alpha u = y_2^\alpha x_1.$$

Therefore, subtracting and multiplying by $x_1^{-(\alpha+1)}$,

$$(x_1^{-1}x_2)^{\alpha+1} t - f(t)^\alpha = (y_1 x_2^\alpha - y_2^\alpha x_1) x_1^{-(\alpha+1)},$$

which may be rewritten as $\Gamma_{x_1^{-1}x_2}(t) = (y_1 x_2^\alpha - y_2^\alpha x_1)x_1^{-(\alpha+1)}$. It now follows that if the functions Γ_u are all bijective, we obtain a spread for an α-flokki plane. The converse is similar and left to the reader. This proves all parts of the theorem. □

Remark 3.3. Because of the previous theorem, when all the functions in $\mathcal{F}$ are injective, we will say that $\mathcal{F}$ *is* an injective α-partial flokki.

We have seen that the set of functions $\mathcal{F}$ produces a maximal partial α-flokki, but to ensure that these objects are equivalent we need the concept of a dual spread. Given a spread $\mathcal{S}$ in $\mathrm{PG}(3,K)$, for K a field, applying a polarity $\perp$ to $\mathrm{PG}(3,K)$ transforms $\mathcal{S}$ to a set of lines $\mathcal{S}^\perp$, with the property that each plane of $\mathrm{PG}(3,K)$ contains exactly one line of $\mathcal{S}^\perp$, which is the definition of a *dual spread*. In the finite case, dual spreads are also spreads, which may be seen by an easy counting argument. However, when K is infinite, there are spreads that are not dual spreads and dual spreads that are not spreads.

Definition 3.4. Let K be a field and α an automorphism of K. Choose functions f and g on K, with $f(0) = g(0) = 0$, and consider the set of functions

$$\mathcal{F} = \{\phi_u \mid \phi_u : \ t \to t - u^{\alpha+1} f(t)^\alpha + u g(t)^\alpha, \ \text{for all } u \in K\}.$$

Then we define the *dual of* $\mathcal{F}$, $\mathcal{F}^\perp$, as follows:

$$\mathcal{F}^\perp = \{\phi_u^\perp \mid \phi_u^\perp : \ t \to f(t) - u^{\alpha+1} t^\alpha + u g(t)^\alpha, \ \text{for all } u \in K\}.$$

Assume that both $\mathcal{F}$ and $\mathcal{F}^\perp$ consist of bijective functions, then there are corresponding α-flokki by Theorem 3.2. In this case, because of Remark 3.3, we shall say that $\mathcal{F}$ and $\mathcal{F}^\perp$ are α*-flocks*, and furthermore use the terminology that the 'dual of an α-flokki is an α-flokki'.

Remark 3.5. In the proof of Theorem 3.2, we used that $u^{\alpha+1}\Gamma_{u^{-1}} = \phi_u$ to get that

$$\{\phi_u \mid \phi_u : \ t \to t - u^{\alpha+1} f(t)^\alpha + u g(t)^\alpha, \ \text{for all } u \in K\}$$

is a set of bijective functions if and only if

$$\{\Gamma_u \mid \Gamma_u : \ t \to t u^{\alpha+1} - f(t)^\alpha + u^\alpha g(t)^\alpha, \ \text{for all } u \in K\}$$

is a set of bijective functions. Similarly, one may prove that

$$\{\phi_u^\perp \mid \phi_u^\perp : \ t \to f(t) - u^{\alpha+1}t^\alpha + ug(t)^\alpha, \text{ for all } u \in K\}$$

is a set of bijective functions if and only if

$$\{\Gamma_u^\perp \mid \Gamma_u^\perp : \ t \to f(t)u^{\alpha+1} - t^\alpha + u^\alpha g(t)^\alpha, \text{ for all } u \in K\}$$

is a set of bijective functions.

Applying a polarity to a spread in $\mathrm{PG}(3, K)$ will produce a dual spread, which in the α-flokki case, may not be a spread. It turns out that the dual spread may be coordinatized by the transpose of matrices defining the spread. In other words, if

$$\pi = \left\{ x = 0, \ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u^\alpha \end{bmatrix}; \ u, t \in K \right\}$$

is an α-flokki spread, then the dual spread $\pi^\perp$ is (isomorphic to)

$$\pi^\perp = \left\{ x = 0, \ \begin{bmatrix} u + g(t) & t \\ f(t) & u^\alpha \end{bmatrix}; \ u, t \in K \right\}.$$

We call $\pi^\perp$ the *transposed spread* of π to avoid confusion with the dual of a projective plane. The connections are as follows.

Theorem 3.6. *Let K be a field, $\mathcal{F}$ be an injective partial α-flokki, and $\pi_{\mathcal{F}}$ be the associated maximal partial α-flokki plane. Then, $\mathcal{F}$ is bijective and the dual flokki is bijective if and only if $\pi_{\mathcal{F}}$ is a spread and the transposed spread $\pi_{\mathcal{F}}^\perp$ is a spread.*

Proof. Assume that $\mathcal{F}$ is bijective and the dual flokki is bijective. We first show that

$$\mathcal{S} = \left\{ x = 0, \ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u^\alpha \end{bmatrix}; \ t, u \in K \right\}$$

is a spread. Consider a vector (x_1, x_2, y_1, y_2). If $x_1 = 0 = x_2$ then $x = 0$ is the unique 2-space containing the vector. If x_1 is not zero but $x_2 = 0$, then there are unique t and u so that $x_1 t = y_1$ and $x_2 u^\alpha = y_2$. Hence, we may assume that x_2 is non-zero. Therefore, we need to solve the following simultaneous equations uniquely for t and u.

$$x_1(u + g(t)) + x_2 t = y_1, \qquad x_1 f(t) + x_2 u^\alpha = y_2.$$

Taking the α-automorphism of the first equation, multiplying the resulting equation by x_2, multiplying the second equation by x_1^α and subtracting the two resulting equations we obtain the following:

$$x_2 x_1^\alpha g(t)^\alpha + x_2^{\alpha+1} t^\alpha - x_1^{\alpha+1} f(t) = y_1^\alpha x_2 - x_1^\alpha y_2.$$

Since $x_2 \neq 0$, divide by $x_2^{\alpha+1}$, to transform this equation into

$$\left(x_1x_2^{-1}\right)^{\alpha} g(t)^{\alpha} + t^{\alpha} - \left(x_1x_2^{-1}\right)^{\alpha+1} f(t) = y_1^{\alpha}x_2 - x_1^{\alpha}y_2. \tag{2}$$

Notice that

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}u+g(t) & f(t)\\ t & u^{\alpha}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}u+g(t) & -f(t)\\ -t & u^{\alpha}\end{bmatrix}.$$

This means that we may use this transformed version of the maximal partial α-flokki, which, letting $v = x_1x_2^{-1}$, turns equation (2) into

$$v^{\alpha+1}f(t) - t^{\alpha} + v^{\alpha}g(t)^{\alpha} = y_1^{\alpha}x_2 - x_1^{\alpha}y_2. \tag{3}$$

Recall that we are assuming the four sets in Remark 3.5 are all sets of bijective functions. Therefore, there is a unique t so that equation (3) has a solution. It is now easily verified that returning to our set of simultaneous equations, there is a unique t and a unique u that solve these equations. This proves that $\mathcal{S}$ is an α-flokki spread.

Now if we repeat the argument for the transposed maximal partial α-flokki plane $\pi_{\mathcal{F}}^{\perp}$, the assumptions on the α-flokki and its dual show that the transposed maximal partial α-flokki plane is also an α-flokki plane.

Now assume that both $\pi_{\mathcal{F}}$ and $\pi_{\mathcal{F}}^{\perp}$ are both α-flokki planes. Then by rereading the previous argument, it is immediate that the fact that $\pi_{\mathcal{F}}$ is an α-flokki plane implies that the dual $\mathcal{F}^{\perp}$ of the injective partial α-flokki $\mathcal{F}$ is an α-flokki. Since $\mathcal{F}^{\perp\perp} = \mathcal{F}$, and $\pi_{\mathcal{F}}^{\perp}$ is an α-flokki plane then also $\mathcal{F}$ is an α-flokki. This completes the proof of the theorem. □

Corollary 3.7. *Every α-flokki plane is isomorphic to an α^{-1}-flokki plane. In particular,*

$$\begin{bmatrix}u+g(t) & f(t)\\ t & u^{\alpha}\end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix}u+g(t)^{\alpha} & t\\ f(t) & u^{\alpha^{-1}}\end{bmatrix}$$

give isomorphic flokki planes.

Proof. Note that

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}u+g(t) & f(t)\\ t & u^{\alpha}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}u^{\alpha} & t\\ f(t) & u+g(t)\end{bmatrix}.$$

Now let $u+g(t) = v^{-1}$, so that $u^{\alpha} = v - g(t)^{\alpha}$, and obtain

$$\begin{bmatrix}u^{\alpha} & t\\ f(t) & u+g(t)\end{bmatrix} = \begin{bmatrix}v-g(t)^{\alpha} & t\\ f(t) & v^{-1}\end{bmatrix}.$$

A basis change by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v - g(t)^\alpha & t \\ f(t) & v^{-1} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

followed by replacing $-v$ by w, gives

$$\begin{bmatrix} w + g(t)^\alpha & t \\ f(t) & w^{\alpha^{-1}} \end{bmatrix}$$

which has the desired form. □

For a given cone, a *partial flock* is a set of planes which do not intersect on this cone. Any finite flock is a partial flock of q planes. A partial flock of $q-1$ planes is said to be of *deficiency one*. The Payne–Thas theorem (Theorem 1.1) states that for a quadratic cone, any deficiency one partial flock can be uniquely extended to a flock. We extend this theorem to α-flokki, for when K is finite, by noting that the proof of the result due to Sziklai [14] is valid in this situation as well.

Theorem 3.8. *Let $K \cong \mathrm{GF}(q)$. A deficiency one α-flokki may be extended to a unique α-flokki.*

Proof. Let $\alpha \in \mathrm{Aut}(K)$, and $\mathcal{C}_\alpha$ be the α-cone with vertex $v_0 = (0,0,0,1)$ given by $x_0^\alpha x_1 = x_2^{\alpha+1}$. Consider a partial α-flokki of $\mathcal{C}_\alpha$ of deficiency one consisting of $q-1$ planes of the following form:

$$\rho_t : \; x_0 t - x_1 f(t)^\alpha + x_2 g(t)^\alpha - x_3 = 0,$$

for t in a subset λ of K of cardinality $q-1$. So, the function

$$\phi_u : \; t \to t - u^{\alpha+1} f(t)^\alpha + u g(t)^\alpha$$

is injective in K, for each u in K. Therefore, we have $q-1$ of the elements of K as images for each ϕ_u. Note that the point on the generator $\left\langle v_0, (u^{\alpha+1}, 1, u^\alpha, 0) \right\rangle$ on ρ_t is $(u^{\alpha+1}, 1, u^\alpha, u^{\alpha+1}t - f(t)^\alpha + u^\alpha g(t)^\alpha)$. For $q > 2$, we see that $-\sum_{t\in\lambda} t$ is the missing element from K in λ. We note that the point

$$\left(u^{\alpha+1}, 1, u^\alpha, -\sum_{t\in\lambda} (u^{\alpha+1} t - f(t)^\alpha + u^\alpha g(t)^\alpha) \right)$$

is the missing point on each generator other than $\langle (0,0,0,1), (1,0,0,0) \rangle$. The points $(1,0,0,t)$ on ρ_t are on the generator $\langle (0,0,0,1), (1,0,0,0) \rangle$ and the missing point is $\left(1,0,0,-\sum_{t\in\lambda} t\right)$. Now all of these points lie on the plane

$$x_0 \left(\sum_\lambda t \right) + x_1 \left(-\sum_\lambda f(t)^\alpha \right) + x_2 \left(\sum_\lambda g(t)^\alpha \right) + x_3 = 0,$$

so the missing plane is determined. □

4 The 2^{nd}-cone

An α-flokki is a flock of the cone $\mathcal{C}_\alpha$. There is a projectively equivalent cone $\mathcal{C}'_\alpha = \{(x_0, x_1, x_2, x_3) \mid x_0 x_1^\alpha = x_2^{\alpha+1}\}$ with vertex $(0,0,0,1)$ and it is of interest to ask if an α-flokki can simultaneously be a flock of this '2^{nd}-cone'. Let an α-flokki be given by the planes $\pi_t: \ x_0 t - f(t)^\alpha x_1 + g(t)^\alpha x_2 - x_3 = 0$, hence $\phi_u : t \to u^{\alpha+1}t - f(t)^\alpha + u^\alpha g(t)^\alpha$ is a bijective function. Note that for this set of planes to be a flock of $\mathcal{C}'_\alpha$, we must have that $\rho_u : t \to t - u^{\alpha+1} f(t)^\alpha + u^\alpha g(t)^\alpha$ is a bijective function. By changing all signs in this function we obtain the bijective function $\tau_u: \ t \to u^{\alpha+1} f(t)^\alpha - (t^{\alpha^{-1}})^\alpha + u^\alpha(-g(t))^\alpha$. This means that the associated α-flokki spread or maximal partial spread relative to the second cone is given by

$$\left\{ x = 0,\ y = x \begin{bmatrix} u - g(t) & t^{\alpha^{-1}} \\ f(t)^\alpha & u^\alpha \end{bmatrix};\ t, u \in \mathrm{GF}(q) \right\}.$$

This argument may be 'reversed' to prove that a given α-flokki plane produces a flock in both cones as long as τ_u is bijective.

Note that when $g = 0$, if $\phi_u: \ t \to u^{\alpha+1}t - f(t)^\alpha$ is a bijective function for all elements $u \in K$, then for u non-zero, we can factor out $u^{\alpha+1}$ to obtain that $\rho_{u^{-1}=v}: \ t \to t - v^{\alpha+1} f(t)^\alpha$ is bijective.

Definition 4.1. Given an α-flokki plane relative to the functions $(g(t), f(t), t)$, the set $(g(t^\alpha), f(t^\alpha)^\alpha, t)$ is called the '2^{nd}-cone triple'.

Theorem 4.2. (1) *Given an α-flokki plane relative to $(g(t), f(t), t)$, the 2^{nd}-cone triple $(g(t^\alpha), f(t^\alpha)^\alpha, t)$ corresponds to an α-flokki plane if and only if*

$$\phi_u: \ t \to u^{\alpha+1}t - f(t)^\alpha + u^\alpha g(t)^\alpha$$

for all u is a bijective function implies

$$\tau_u: \ t \to t - u^{\alpha+1} f(t)^\alpha + u^\alpha g(t)^\alpha$$

is a bijective function, for all $u \in K$.

(2) *If $g = 0$ then any α-flokki of $\mathcal{C}_\alpha$ is a flock of $\mathcal{C}'_\alpha$.*

We leave it as an open problem to show that if g is not zero then an associated α-flokki of $\mathcal{C}_\alpha$ is never a flock of $\mathcal{C}'_\alpha$.

5 Baer groups

The Baer group theory for translation planes with spreads in $\mathrm{PG}(3, q)$ of Johnson [7], shows that Baer groups of order q produce partial flocks of quadratic cones of deficiency one. Given any conical flock plane, there is an elation group E, whose orbits union the axis form reguli in $\mathrm{PG}(3,q)$. Derivation of one of these regulus nets, produces a translation plane with spread in $\mathrm{PG}(3,q)$ admitting a Baer group of order q. Now consider any given α-flokki and corresponding α-flokki plane. Again, there is an elation group E (see Section 2), whose orbits union the axis form derivable partial spreads. Derivation of one of the derivable nets produces a translation plane admitting a Baer group of order q, but now the spread for this plane π^* is no longer in $\mathrm{PG}(3,q)$ (for $\alpha \neq 1$). The components not in the derivable net are still subspaces in $\mathrm{PG}(3,q)$, as are the Baer subplanes of the derivable net of π^*. Therefore, we would expect that Baer groups of order q in such translation planes might also produce deficiency one partial α-flokki.

Theorem 5.1. *Let π be a translation plane of order q^2 that admits a Baer group B of order q. Assume that the components of π and the Baer axis of B are lines of $\mathrm{PG}(3,q)$.*

(1) *Then π corresponds to a partial α-flokki of deficiency one.*

(2) *Therefore, the Baer partial spread defined by the Baer group is derivable and the Baer subplanes incident with the zero vector are also lines in $\mathrm{PG}(3,q)$. The derived plane is the unique α-flokki plane associated with the extended α-flokki.*

Proof. Let $q = p^r$, for p a prime. Let (x_1, x_2, y_1, y_2) represent points, where x_i are r-vectors over the prime field. We may also represent x_i as an element of $\mathrm{GF}(q)$. Choose the Baer axis to be $(0, x_2, 0, y_2)$, and obtain the group in the form

$$\left\langle \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \right\rangle,$$

where $A = \begin{bmatrix} I & C \\ 0 & I \end{bmatrix}$ and C is in a field K. The Baer group of order q is elementary abelian and corresponds to a field by fundamental results of Foulser [5]. It follows that the Baer axis together with any orbit of length q forms a derivable partial spread. If the Baer axis and the components not on the Baer net are lines of $\mathrm{PG}(3,q)$ and we choose the Baer axis to be $x = 0$, then by Johnson [7], we have that the group B may be represented in the form

$$\left\{ \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 0 & 0 & u^{\alpha} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;\ u \in \mathrm{GF}(q) \right\},$$

where α is an automorphism of $\mathrm{GF}(q)$.

It follows that the components not on the Baer net have the form

$$x = 0,\ y = x \begin{bmatrix} u + a & b \\ c & u^{\alpha} \end{bmatrix},$$

for $q - 1$ values of c in $\mathrm{GF}(q)$. Let $c = t^{\alpha}$, then a and b are functions of c, say $g(t)$ and $f(t)$, respectively. Then we have a partial spread of the form

$$\left\{ x = 0,\ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u^{\alpha} \end{bmatrix};\ t \in \lambda,\ u \in \mathrm{GF}(q) \right\},$$

where $\lambda \subseteq \mathrm{GF}(q)$ has cardinality $q - 1$. Therefore, we have a partial α-flokki of deficiency one with planes

$$x_0 t - x_1 f(t)^{\alpha} + x_2 g(t)^{\alpha} - x_3 = 0$$

for all $t \in \lambda^{\alpha^{-1}}$.

Since any partial α-flokki of deficiency one can be extended to a unique α-flokki, we see that the Baer net must be derivable, the Baer group is the elation group of the derived plane and the Baer subplanes of the Baer net are lines of $\mathrm{PG}(3, q)$. This completes the proof of the theorem. □

6 q-Flokki and algebraic lifting

When α is the automorphism $x \mapsto x^q$ of $\mathrm{GF}(q^2)$, an α-flokki will be referred to as a *q-flokki*. We have that a q-flokki in $\mathrm{PG}(3, q^2)$ has an associated flokki plane with spread

$$\left\{ x = 0,\ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u^{q} \end{bmatrix};\ t, u \in \mathrm{GF}(q^2) \right\}.$$

The following theorem tells us when an α-flokki plane has been algebraically lifted from a plane with spread in $\mathrm{PG}(3, q)$.

Theorem 6.1 (Cherowitzo and Johnson [3]). *A translation plane with spread*

$$\left\{ x = 0,\ y = x \begin{bmatrix} u + G(s) & F(s) \\ s & u^{\alpha} \end{bmatrix}; s, u \in \mathrm{GF}(q^2) \right\},$$

is algebraically lifted if and only if there is a coordinate change so that $G(s) = 0$ and $\alpha = q$.

The bilinear q-Flokki of Cherowitzo and Holder [4] yield a flokki plane with spread

$$\left\{x=0,\ y=x\begin{bmatrix}u & \gamma(s^q)^{(q^2+1)/2}\\ s & u^q\end{bmatrix};\ s,u\in \mathrm{GF}(q^2)\right\},$$

where q is odd and γ is a non-square in $\mathrm{GF}(q^2)$ such that γ^2 is a non-square in $\mathrm{GF}(q)$. The flokki planes of Cherowitzo of order q^4 may be lifted from regular nearfield planes of order q^2 (see [3]).

Remark 6.2. The semifield q-flokki plane in Remark 2.5 is lifted from a Desarguesian plane.

Now we consider an algebraically lifted translation plane of order q^4. Because of Theorem 6.1, we know that it has a spread of the form

$$\left\{x=0,\ y=x\begin{bmatrix}u & F(t)\\ t & u^q\end{bmatrix};\ t,u\in \mathrm{GF}(q^2)\right\}.$$

Since

$$\begin{bmatrix}1 & 0\\ 0 & e^q\end{bmatrix}\begin{bmatrix}u & 0\\ 0 & u^q\end{bmatrix}\begin{bmatrix}e & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}ue & 0\\ 0 & (ue)^q\end{bmatrix},$$

the Baer group of order $q+1$,

$$B=\left\langle\begin{bmatrix}1&0&0&0\\0&e&0&0\\0&0&e&0\\0&0&0&1\end{bmatrix};e^{q+1}=1\right\rangle$$

maps $y=x\begin{bmatrix}u & 0\\ 0 & u^q\end{bmatrix}$ to $y=x\begin{bmatrix}ue & 0\\ 0 & (ue)^q\end{bmatrix}$. If we now derive the associated net, the Baer group B is still a Baer group of order $q+1$, but the elation group of order q^2 is now a Baer group E of order q^2 (see e.g. Johnson, Jha and Biliotti [10, Theorem 35.18] for more details).

Theorem 6.3. *Let π be a translation plane of order q^4 that admits a Baer group E of order q^2 and a non-trivial Baer group B such that $[E,B]\neq 1$. If the axis of E and its non-trivial orbits are lines of $\mathrm{PG}(3,q^2)$ then π is a derived q-flokki translation plane; a translation plane that has been algebraically lifted and then derived.*

Proof. The plane is derivable since it corresponds to a partial α-flokki. Hence, the derived plane has the following spread set

$$\left\{x=0,\ y=x\begin{bmatrix}u+G(t) & F(t)\\ t & u^\alpha\end{bmatrix};\ t,u\in K\right\}.$$

The group E is now an elation group E and since B and E do not centralize each other, it follows, as noted in Theorem 6.1, that the plane is an algebraically lifted plane. Hence, $\alpha = q$ and $G(t) = 0$, that is, once we know that there is a non-trivial Baer group, this group is forced to have order $q+1$. □

7 Collineations and isomorphisms of α-flokki planes

We now want to study the full collineation group of a finite α-flokki plane π with spread

$$\left\{x = 0,\ y = x\begin{bmatrix} u + g(t) & f(t) \\ t & u^{\alpha}\end{bmatrix};\ t, u \in \mathrm{GF}(q)\right\}.$$

We will start by looking at the elation group

$$E = \left\{\begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u^{\alpha} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix};\ u \in \mathrm{GF}(q)\right\}.$$

Let E' be any elation group of π, different from E. Assume that E' does not have axis $x = 0$. Then, by the Hering–Ostrom theorem (see [10]), $\langle E, E'\rangle$ is isomorphic to $\mathrm{SL}(2, p^t)$, $\mathrm{Sz}(2^{2e+1})$ and q is even, or $\mathrm{SL}(2, 5)$ and 3 divides q. In the $\mathrm{SL}(2, p^t)$ case by Johnson [9], the group is either $\mathrm{SL}(2, q)$ or $\mathrm{SL}(2, q^2)$. In the former case, the planes are Desarguesian, Ott–Schaeffer, Hering, Dempwolff of order 16 or Walker of order 25. Since 2^{2e+1} is at least q, the $\mathrm{Sz}(2^{2e+1})$ case does not occur by Büttner [2]. Assume that $q > 4$, then $\mathrm{SL}(2, 5)$ cannot occur. The Dempwolff plane is not in $\mathrm{PG}(3, q)$. Hence, this leaves the Hering and Ott–Schaeffer planes. The Hering planes are not derivable and the Ott–Schaeffer planes admit reguli, so the plane is Desarguesian.

Therefore, assume that E' does have axis $x = 0$. We now assume $E' \cap E \neq \langle 1\rangle$, and get derivable nets sharing $y = 0$, and $y = x\begin{bmatrix} u_0 & 0 \\ 0 & u_0^{\alpha}\end{bmatrix}$, for some $u_0 \in \mathrm{GF}(q)$. Note that a change of basis that allows us to represent this last component by $y = x$ does not change the general form of the derivable net

$$\left\{x = 0,\ y = x\begin{bmatrix} u & 0 \\ 0 & u^{\alpha}\end{bmatrix};\ u \in \mathrm{GF}(q)\right\}. \tag{4}$$

Hence, by Johnson [8], we have two distinct derivable nets of π: the one in (4) and

$$\left\{x = 0,\ y = x\begin{bmatrix} v & 0 \\ 0 & v^{\sigma}\end{bmatrix};\ v \in \mathrm{GF}(q)\right\}, \tag{5}$$

where $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$. But, in this case, there are differences of matrices that are non-singular, a contradiction. Hence, $E' \cap E = \langle 1 \rangle$.

First notice that if E is normal in $\mathrm{Aut}(\pi)$ then the α-derivable nets are permuted by $\mathrm{Aut}(\pi)$.

Now assume that E is not normal in $\mathrm{Aut}(\pi)$. Thus, there exists an element $\phi \in \mathrm{Aut}(\pi)$ such that the elation group $\overline{E} = \phi E \phi^{-1}$ is different from E. Clearly ϕ must fix $x = 0$, and thus representing a generic element of $\overline{E}$ by

$$\begin{bmatrix} E_{11} & E_{12} \\ 0 & I \end{bmatrix}$$

we obtain $E_{11} = I$. It follows that $\overline{E}$ and E commute with each other and belong to an elation group with axis $x = 0$. So, the group $\langle \overline{E}, E \rangle$ contains the set product $\overline{E}E$, of order q^2, which means that the α-flokki plane is a semifield plane.

Now, having that $\overline{E}E$ has order q^2 allows us to assume that ϕ fixes both $x = 0$ and $y = 0$, which then has the general form

$$(x, y) \to (x^\rho, y^\rho) \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where $A, B \in \mathrm{GL}(2, q)$ and $\rho \in \mathrm{Aut}(\mathrm{GF}(q))$. Then ϕ maps $y = x$ to $y = xA^{-1}B$. Change basis by $(x, y) \to (x, yB^{-1}A)$ to find a derivable net with components $x = 0,\ y = 0$, and $y = x$. This net must be of the general form in equation (5) by Johnson [8]. But the image of

$$y = x \begin{bmatrix} u & 0 \\ 0 & u^\alpha \end{bmatrix},$$

for $u \in \mathrm{GF}(q)$, under ϕ, and the posterior change of basis $(x, y) \to (x, yB^{-1}A)$, is

$$y = xA^{-1} \begin{bmatrix} u^\rho & 0 \\ 0 & u^{\rho\alpha} \end{bmatrix} A.$$

So, we get that either $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, $v = u^\rho$ and $\sigma = \alpha$, or $A = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$, $v = u^{\rho\alpha}$ and $\sigma = \alpha^{-1}$.

Similarly, since ϕ maps $y = x$ to $y = xA^{-1}B$ we change basis using the transformation $(x, y) \to (xA^{-1}B, y)$ to find a derivable net with components $x = 0,\ y = 0$, and $y = x$. Then again, this net must be of the general form in equation (5) by Johnson [8]. But the image of

$$y = x \begin{bmatrix} u & 0 \\ 0 & u^\alpha \end{bmatrix},$$

for $u \in \mathrm{GF}(q)$, under ϕ, and the posterior change of basis $(x, y) \to (xA^{-1}B, y)$, is

$$y = xB^{-1} \begin{bmatrix} u^{\rho} & 0 \\ 0 & u^{\rho\alpha} \end{bmatrix} B.$$

So, we get that either $B = \begin{bmatrix} \overline{a} & 0 \\ 0 & \overline{d} \end{bmatrix}$, $v = u^{\rho}$ and $\sigma = \alpha$, or $B = \begin{bmatrix} 0 & \overline{b} \\ \overline{c} & 0 \end{bmatrix}$, $v = u^{\rho\alpha}$ and $\sigma = \alpha^{-1}$.

It follows that $A^{-1}B$ is either a diagonal or an anti-diagonal matrix. Since $A^{-1}B$ being diagonal would yield a contradiction with ϕ not normalizing E we get that

$$B^{-1}A = \begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix},$$

for some $m, n \in \mathrm{GF}(q)^{*}$.

We now make the change of basis $(x, y) \to (x, yB^{-1}A)$, to get a derivable net in π of the form

$$\left\{ x = 0,\ y = x \begin{bmatrix} 0 & vm \\ v^{\sigma} n & 0 \end{bmatrix};\ v \in \mathrm{GF}(q) \right\}.$$

Let $t = v$, and get the orbits of this net under the group E to obtain:

$$\left\{ x = 0,\ y = x \begin{bmatrix} u & tm \\ t^{\sigma} n & u^{\alpha} \end{bmatrix},\ t, u \in \mathrm{GF}(q) \right\}.$$

Use $t^{\sigma} n = s^{\sigma}$ to transform the spread set into the form

$$\left\{ x = 0,\ y = x \begin{bmatrix} u & fs \\ s^{\sigma} & u^{\alpha} \end{bmatrix},\ s, u \in \mathrm{GF}(q) \right\}.$$

where f is a constant, and $\sigma = \alpha^{\pm 1}$.

If $\sigma = \alpha$ then this is a Hughes–Kleinfeld semifield plane (see [10, 91.22]), and if $\sigma = \alpha^{-1}$ then this is a dual/transposed Hughes–Kleinfeld semifield plane.

Since there are α-flokki spreads of this form, exactly when the flock is linear, we obtain the following theorem. Note that the case where $x = 0$ is not left invariant leads to the Desarguesian plane and $\alpha = 1$.

Theorem 7.1. *Let π be an α-flokki plane of order q^2, where $q > 5$. Then one of the following occurs:*

(1) *the full collineation group permutes the q α-derivable nets,*

(2) *$\alpha = 1$ and the plane is Desarguesian or*

(3) *$\alpha \neq 1$ and the α-flokki plane is a Hughes–Kleinfeld, or transposed Hughes–Kleinfeld, semifield plane that corresponds to a linear flokki.*

Now assume that π_1 and π_2 are two isomorphic α-flokki planes of order q^2, for $q > 5$, neither of which are Hughes–Kleinfeld (transposed or not) or Desarguesian. Let σ be an isomorphism from π_1 to π_2, so we may assume that $x = 0$ is left invariant and that the elation group E of order q is normalized by σ. Initially, assume that π_i has spread

$$\left\{x = 0,\ y = x \begin{bmatrix} u + g_i(t) & f_i(t) \\ t & u^\alpha \end{bmatrix};\ t, u \in \mathrm{GF}(q)\right\},$$

for $i = 1, 2$.

By following with an element of E, if necessary, we may assume that σ maps $y = x$ to $y = x \begin{bmatrix} 1 + g_2(t_0) & f_2(t_0) \\ t_0 & 1 \end{bmatrix}$. Then, by changing the basis using the map $\begin{bmatrix} 1 & 0 & -g_2(t_0) & -f_2(t_0) \\ 0 & 1 & -t_0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, the resulting matrix spread set has the following form:

$$\left\{x = 0,\ y = x \begin{bmatrix} u + g_2(t) - g_2(t_0) & f_2(t) - f_2(t_0) \\ t - t_0 & u^\alpha \end{bmatrix};\ t, u \in \mathrm{GF}(q)\right\},$$

for $i = 1, 2$. Note that this spread, after a suitable selection of f_2 and g_2, maintains the form of the original matrix spread set.

By this recoordinatization we may assume that σ fixes $x = 0$ and $y = x$, leaving invariant

$$\left\{x = 0,\ y = x \begin{bmatrix} u & 0 \\ 0 & u^\alpha \end{bmatrix};\ u \in \mathrm{GF}(q)\right\}. \tag{6}$$

Since E is transitive on each derivable α-net, it follows that σ fixes $x = 0$ and $y = 0$ (though not necessarily fixing $y = x$ anymore), and is of the general form:

$$(x, y) \to (x^\beta, y^\beta) \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where $\beta \in \mathrm{Aut}(\mathrm{GF}(q))$, and the matrices $A, B \in \mathrm{GL}(2, q)$ with $A^{-1}B = \begin{bmatrix} u_0 & 0 \\ 0 & u_0^\alpha \end{bmatrix}$, for some $u_0 \in \mathrm{GF}(q)^*$. Then, we must have

$$\begin{aligned} \begin{bmatrix} v + g_2(s) & f_2(s) \\ s & v^\alpha \end{bmatrix} &= A^{-1} \begin{bmatrix} u^\beta + g_1(t)^\beta & f_1(t)^\beta \\ t^\beta & u^{\alpha\beta} \end{bmatrix} B \\ &= A^{-1} \begin{bmatrix} u^\beta + g_1(t)^\beta & f_1(t)^\beta \\ t^\beta & u^{\alpha\beta} \end{bmatrix} A \begin{bmatrix} u_0 & 0 \\ 0 & u_0^\alpha \end{bmatrix}. \end{aligned}$$

We let $s = 0$ and use that (6) is fixed by σ to get that $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ or $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$. Thus, $B = A \begin{bmatrix} u_0 & 0 \\ 0 & u_0^\alpha \end{bmatrix} = \begin{bmatrix} au_0 & 0 \\ 0 & du_0^\alpha \end{bmatrix}$ or $\begin{bmatrix} 0 & cu_0^\sigma \\ bu_0 & 0 \end{bmatrix}$, respectively. Moreover, by applying a scalar mapping, we may assume that $a = 1$ or $b = 1$. In the first situation,

$$\begin{aligned} \begin{bmatrix} v + g_2(s) & f_2(s) \\ s & v^\alpha \end{bmatrix} &= A^{-1} \begin{bmatrix} u^\beta + g_1(t)^\beta & f_1(t)^\beta \\ t^\beta & u^{\alpha\beta} \end{bmatrix} A \begin{bmatrix} u_0 & 0 \\ 0 & u_0^\alpha \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & d^{-1} \end{bmatrix} \begin{bmatrix} u^\beta + g_1(t)^\beta & f_1(t)^\beta \\ t^\beta & u^{\alpha\beta} \end{bmatrix} \begin{bmatrix} u_0 & 0 \\ 0 & du_0^\alpha \end{bmatrix} \\ &= \begin{bmatrix} (u^\beta + g_1(t)^\beta)u_0 & f_1(t)^\beta du_0^\alpha \\ t^\beta d^{-1} & u^{\alpha\beta} u_0^\alpha \end{bmatrix}. \end{aligned}$$

Let $u_0 = w_0^\beta$. Then for $v = (uw_0)^\beta$, $s = t^\beta d^{-1}$, we have

$$f_2(s) = f_1(s^{\beta^{-1}} d^{\beta^{-1}})^\beta du_0^\alpha \qquad \text{and} \qquad g_2(s) = g_1(s^{\beta^{-1}} d^{\beta^{-1}})^\beta u_0.$$

In the second situation, we have

$$\begin{aligned} \begin{bmatrix} v + g_2(s) & f_2(s) \\ s & v^\alpha \end{bmatrix} &= A^{-1} \begin{bmatrix} u^\beta + g_1(t)^\beta & f_1(t)^\beta \\ t^\beta & u^{\alpha\beta} \end{bmatrix} A \begin{bmatrix} u_0 & 0 \\ 0 & u_0^\alpha \end{bmatrix} \\ &= \begin{bmatrix} 0 & c^{-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u^\beta + g_1(t)^\beta & f_1(t)^\beta \\ t^\beta & u^{\alpha\beta} \end{bmatrix} \begin{bmatrix} 0 & cu_0^\alpha \\ u_0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} c^{-1} u^{\alpha\beta} u_0 & t^\beta u_0^\alpha \\ f_1(t)^\beta u_0 & (u^\beta + g_1(t)^\beta) cu_0^\alpha \end{bmatrix}. \end{aligned}$$

We may assume that $g_i(0) = f_i(0) = 0$, for $i = 1, 2$ (this is not affected by all the manipulations done to get that σ fixes $y = 0$). Therefore, taking $s = 0$, we see that $t = 0$, implying that $v = (c^{-1} u^{\alpha\beta} u_0)$, and $v^\alpha = (u^\beta cu_0^\alpha)$, which implies that $c^{-\alpha} u^{\alpha^2 \beta} = cu^\beta$, for all u, thus $\alpha^2 = 1$ and $c^{\alpha+1} = 1$. Letting $v = 0$ implies $u = 0$, so that $g_2(s) = 0$ and $f_2(s) = t^\beta u_0^\alpha$, where $s = f_1(t)^\beta u_0$. Therefore, $t = f_1^{-1}((su_0^{-1})^{\beta^{-1}})$, so that $f_2(s) = f_1^{-1}((su_0^{-1})^{\beta^{-1}})^\beta u_0^\alpha$. Moreover, it also follows that $g_1(t) = 0$. Assume that q is a non-square then $\alpha = 1$, a contradiction. Hence, $q = h^2$, and $\alpha = h$. Therefore, we have the spreads

$$\left\{ x = 0,\ y = x \begin{bmatrix} u & f_i(t) \\ t & u^h \end{bmatrix};\ t, u \in \mathrm{GF}(h^2) \right\},$$

for $i = 1, 2$, where $f_2(t) = f_1^{-1}((tu_0^{-1})^{\beta^{-1}})^\beta u_0^\alpha$.

We now have the following theorem:

Theorem 7.2. *Two finite α-flokki planes, π_i, for $\alpha \neq 1$ and not Hughes–Kleinfeld (transposed or not), with spreads*

$$\left\{ x = 0,\ y = x \begin{bmatrix} u + g_i(t) & f_i(t) \\ t & u^\alpha \end{bmatrix};\ t, u \in \mathrm{GF}(q) \right\},$$

for $i = 1, 2$, are isomorphic if and only if one of the following occurs:

(1) *There is an automorphism β, a constant t_0 and a constant u_0 so that*

$$f_2(s + t_0) - f_2(t_0) = f_1(s^{\beta^{-1}} d^{\beta^{-1}})^\beta d u_0^\alpha,$$
$$g_2(s + t_0) - g_2(t_0) = g_1(s^{\beta^{-1}} d^{\beta^{-1}})^\beta u_0.$$

(2) *There is an automorphism β, a constant t_0 and a constant u_0 so that*

$$f_2(s + t_0) - f_2(t_0) = f_1^{-1}((t u_0^{-1})^{\beta^{-1}})^\beta u_0^\alpha,$$
$$g_2(s + t_0) - g_2(t_0) = 0.$$

Furthermore, $q = h^2$ and $\alpha = h$.

Corollary 7.3. *If $\alpha \neq 1$ and π is a finite α-flokki plane that is not Hughes–Kleinfeld (transposed or not), then the transposed α-flokki plane of π is isomorphic to π if and only if there is an automorphism β and constants t_0 and u_0 so that*

(1)

$$f_1^{-1}(s + t_0) - f_1^{-1}(t_0) = f_1(s^{\beta^{-1}} d^{\beta^{-1}})^\beta d u_0^\alpha,$$
$$g_1 f_1^{-1}(s + t_0) - g_1 f_1^{-1}(t_0) = g_1(s^{\beta^{-1}} d^{\beta^{-1}})^\beta u_0, \qquad \textit{or}$$

(2)

$$f_1^{-1}(s + t_0) - f_1^{-1}(t_0) = f_1^{-1}((t u_0^{-1})^{\beta^{-1}})^\beta u_0^\alpha,$$
$$g_1(s + t_0) - g_1(t_0) = 0,$$

where $q = h^2$ and $\alpha = h$.

Proof. Note that $g_2 = g_1 f_1^{-1}$ and $f_2 = f_1^{-1}$, in the transposed plane. The parts of the functions involving t_0 simply ensure that an isomorphism leaves the derivable net in (6) invariant. □

We finish this section by taking up the problem of determining when a finite α-flokki plane π_1 is isomorphic to a δ-flokki plane π_2. Since we know (Corollary 3.7) that this is always the case when $\delta = \alpha^{-1}$, for $g_2(t) = g_1(t)^\alpha$, and $f_2(t) = f_1(t)$, assume that δ is not α^{-1}, that neither plane is Hughes–Kleinfeld (transposed or not), and that α and δ are both not equal to 1. The analogous equations in the first situation (see proof of Theorem 7.2) are

$$\begin{aligned}\begin{bmatrix} v+g_2(s) & f_2(s) \\ s & v^\delta \end{bmatrix} &= A^{-1}\begin{bmatrix} u^\beta + g_1(t)^\beta & f_1(t)^\beta \\ t^\beta & u^{\alpha\beta} \end{bmatrix} A \begin{bmatrix} u_0 & 0 \\ 0 & u_0^\delta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & d^{-1} \end{bmatrix}\begin{bmatrix} u^\beta + g_1(t)^\beta & f_1(t)^\beta \\ t^\beta & u^{\alpha\beta} \end{bmatrix}\begin{bmatrix} u_0 & 0 \\ 0 & du_0^\delta \end{bmatrix} \\ &= \begin{bmatrix} (u^\beta + g_1(t)^\beta)u_0 & f_1(t)^\beta du_0^\delta \\ t^\beta d^{-1} & u^{\alpha\beta}u_0^\delta \end{bmatrix}.\end{aligned}$$

However, this implies that $v^\delta = u^{\alpha\beta}u_0^\delta$, $v = u^\beta u_0$, so that $v^\delta = u^{\delta\beta}u_0^\delta = u^{\alpha\beta}u_0^\delta$, for all v, so that $\delta = \alpha$. In the second situation, the analogous equations are

$$\begin{aligned}\begin{bmatrix} v+g_2(s) & f_2(s) \\ s & v^\delta \end{bmatrix} &= A^{-1}\begin{bmatrix} u^\beta + g_1(t)^\beta & f_1(t)^\beta \\ t^\beta & u^{\alpha\beta} \end{bmatrix} A \begin{bmatrix} u_0 & 0 \\ 0 & u_0^\delta \end{bmatrix} \\ &= \begin{bmatrix} 0 & c^{-1} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u^\beta + g_1(t)^\beta & f_1(t)^\beta \\ t^\beta & u^{\alpha\beta} \end{bmatrix}\begin{bmatrix} 0 & cu_0^\delta \\ u_0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} c^{-1}u^{\alpha\beta}u_0 & t^\beta u_0^\delta \\ f_1(t)^\beta u_0 & (u^\beta + g_1(t)^\beta)cu_0^\delta \end{bmatrix}.\end{aligned}$$

Then, similar to the previous arguments, $v^\delta = u^\beta cu_0^\delta$, and $v = c^{-1}u^{\alpha\beta}u_0$, implying that $v = u^{\delta^{-1}\beta}c^{\delta^{-1}}u_0 = c^{-1}u^{\alpha\beta}u_0$. This implies that $c^{\delta^{-1}} = c^{-1}$ and $\delta = \alpha^{-1}$, contrary to our assumptions. Hence, we have the following theorem.

Theorem 7.4. *Let π_1 and π_2 be α-flokki and δ-flokki planes respectively, of order q^2. If π_1 and π_2 are isomorphic then $\delta = \alpha^{\pm 1}$.*

Proof. If π_1 and π_2 are isomorphic and one is a Hughes–Kleinfeld (or transposed) semifield plane then the other is also a Hughes–Kleinfeld semifield plane and the result follows (for example, from Johnson and Liu [11]). The other case follows from our previous remarks. □

8 Comments on the open problems of Kantor and Penttila

In Kantor and Penttila [12], there are eight open problems listed. We are able to shed some light on a few of them.

Problem (1) asks if the orbits of the group, leaving the cone $\mathcal{C}_\alpha$ invariant, of a linear flock determines the isomorphism classes of the associated Hughes–Kleinfeld flokki planes. More generally, one could ask how to connect the group of an α-flokki plane with the group of of the associated α-flokki.

If $\alpha = 1$, the points of the flock correspond to the Baer subplanes of the q-regulus nets of the associated conical flock plane. A collineation of a non-Desarguesian conical flock plane of order q^2, $q > 5$, must permute these q-regulus nets and therefore, the set of Baer subplanes corresponding to the quadratic cone are also permuted. Therefore, it is possible to connect groups of conical flock planes with groups of the quadratic cone. However, when $\alpha \neq 1$, the set of Baer subplanes is not a set of $\mathrm{GF}(q)$-subspaces and therefore, the associated permutation of the Baer subplanes might not indicate how to define a point to point collineation mapping of the cone $\mathcal{C}_\alpha$. There is a permutation of the associated planes, but without knowledge that any permutation of the associated planes somehow is related to a collineation of the α-flokki plane, at present the theory cannot be developed.

Problem (2) is answered in the negative by the in- and out-star bilinear flokki of Cherowitzo and Holder [4] (e.g. Examples 1 and 2).

Problem (3) has also a negative answer in the case that $g(t)^2 = g(t)$. By Corollary 3.7, and using $\alpha = 1/2$, the following matrices

$$\begin{bmatrix} u + g(t)^2 & f(t) \\ t & u^{1/2} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} u + g(t) & t \\ f(t) & u^2 \end{bmatrix}$$

yield isomorphic flokki planes.

Problem (4). In Kantor and Penttila, it is noted that if g, f give a flokki plane then gf^{-1}, f^{-1} also give a flokki plane. The question is when this second pair produces an isomorphic flokki plane.
We notice that this latter pair gives the transposed plane of the original flokki. So, the answer is yes if and only if the original flock and the transpose are not isomorphic.

For example, the 2-flokki plane of Kantor and Penttila (Theorem 11 in [12]) with spread

$$\left\{ x = 0,\ y = x \begin{bmatrix} u + t^5 & t^{14} \\ t & u^2 \end{bmatrix};\ t, u \in \mathrm{GF}(2^e) \right\},$$

where 3 does not divide e, has transpose isomorphic to (using Corollary 3.7)

$$\left\{ x = 0,\ y = x \begin{bmatrix} u^2 + t^{10} & t^{14} \\ t & u \end{bmatrix};\ t, u \in \mathrm{GF}(2^e) \right\},$$

where 3 does not divide e. These planes are isomorphic if and only if

$$f_1^{-1}(s+t_0) - f_1^{-1}(t_0) = f_1(s^{\beta^{-1}} d^{\beta^{-1}})^\beta d u_0^\alpha,$$
$$g_1 f_1^{-1}(s+t_0) - g_1 f_1^{-1}(t_0) = g_1(s^{\beta^{-1}} d^{\beta^{-1}})^\beta u_0,$$

where $f_1(t) = t^{14}$ and $g_1(t) = t^5$. This leads to the following equation:

$$(t^{14}a + b)^5 = t^5 c + d, \text{ for all } t \text{ in } \mathrm{GF}(2^e),$$

for constants a, b, c, d, where a, c are both non-zero. This, clearly, cannot hold for $2^e > 70$.

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