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An alternative existence proof of the geometry of Ivanov–Shpectorov for O’Nan’s sporadic group

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In honor of J. A. Thas’s 70th birthday

Abstract

We provide an existence proof of the Ivanov–Shpectorov rank 5 diagram geometry together with its boolean lattice of parabolic subgroups and establish the structure of hyperlines.

Keywords: incidence geometry, diagram geometry, Buekenhout diagrams, O’Nan’s sporadic group
MSC 2010: 51E24, 20D08, 20B99

1 Introduction

We start essentially but not exclusively from:

- Leemans [26] giving the complete partially ordered set $\Lambda_{\mathrm{O'N}}$ of conjugacy classes of subgroups of the O’Nan group $\mathrm{O'N}$. This includes 581 classes and provides a structure name common for all subgroups in a given class;
- the Ivanov–Shpectorov [24] rank 5 diagram geometry for the group $\mathrm{O'N}$, especially its diagram Δ as in Figure 1;
- the rank 3 diagram geometry Γ_{Co} for $\mathrm{O'N}$ due to Connor [15];
- detailed data on the diagram geometries for the groups M_{11} and J_1 [13, 6, 27].

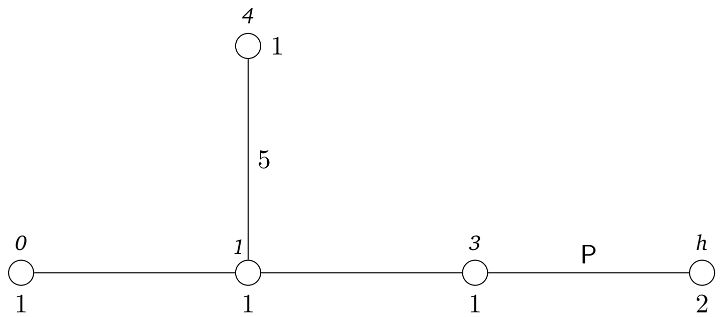


Figure 1: The diagram Δ_{IvSh} of the geometry Γ_{IvSh}

Our results are the following:

- we get the Connor geometry Γ_{Co} as a truncation of the Ivanov–Shpectorov geometry Γ_{IvSh} (see Theorem 7.1);
- using the paper of Ivanov and Shpectorov [24], we establish the full structure of the boolean lattice L_{IvSh} of their geometry as in Figure 17 (See Section 8);
- conversely, within $\Lambda_{\mathrm{O'N}}$ we prove the existence and uniqueness up to fusion in $\mathrm{Aut}(\mathrm{O'N})$ of a boolean lattice isomorphic to L_{IvSh}. This step is independent of [24] (see Theorem 8.1);
- we prove that this boolean lattice yields a rank 5 diagram geometry Γ_{IvSh} and so establish the existence of the Ivanov–Shpectorov geometry (see Theorem 9.4);
- using the 0-elements of Γ_{IvSh} as in Figure 1 and calling them *points*, we show that every h-element deserves the name *hyperline* (see Theorem 10.5).

In 1986 Ivanov and Shpectorov gave the construction of a geometry Γ_{IvSh} of rank 5 on which the O'Nan group acts flag-transitively [24]. It belongs to the diagram given in Figure 1 in which our set of types is $\{0, 1, 3, 4, h\}$. Motivation for this seemingly strange set of types will be given in Section 5. Their work provided the existence proof of two conjugacy classes of subgroups isomorphic to M_{11} in $\mathrm{O'N}$. The paper is not easy to read and it does not seem to have given rise to more detailed versions. In several steps we provide a new approach to this geometry which is broadly independent from the original paper. In a recent paper Connor [15] constructed a new coset geometry Γ_{Co} for the O'Nan sporadic simple group which is of rank 3 over the diagram of Figure 2. The construction is based on a convenient amalgam of known rank 2 coset geometries for the sporadic groups J_1 and M_{11}. His finding was based on the subgroup lattice for $O'N$ as provided by Leemans [26]. Connor's idea was to extend the boolean lattices of the rank 2 geometries for J_1 and M_{11} in a rank 3 boolean lattice of subgroups of $O'N$. It turned out that there is a unique solution to this problem up to conjugacy. Applying a theorem due to Aschbacher [1], he got the existence of this new flag-transitive coset geometry and additional properties.

Using the diagram of Γ_{IvSh} alone we show that Γ_{Co} is a truncation of Γ_{IvSh} (Theorem 7.1). It matters to state that Γ_{Co} was constructed and studied without making use of Γ_{IvSh}. From this fact it is conceivable to extend the Connor geometry with elements so as to produce the Ivanov–Shpectorov geometry. In order to do so we show that Γ_{Co} and the diagram of Figure 1 determine uniquely Γ_{IvSh} (Theorems 8.1 and 9.4 combined). A major step of this characterization makes use of Leemans's subgroup lattice of $O'N$ [26]. We use it to show that the diagram of Figure 1 and the boolean lattice of Γ_{Co} determine a unique boolean lattice of rank 5 in $O'N$. The final step is to use this boolean lattice in order to construct a geometry whose diagram is exactly the one of Figure 1. We provide all of this in full detail. In this way we produce an alternative existence proof for the Ivanov–Shpectorov geometry. Our main result is stated as follows.

Theorem 1.1. *Up to conjugacy in* $\mathrm{Aut}(O'N)$*, there exists a unique boolean lattice of rank 5 in the subgroup lattice of* $O'N$ *as in Figure 17. This boolean lattice defines a unique firm, residually connected, flag-transitive geometry of rank 5 over the diagram of Figure 1.*

The first part of this theorem is Theorem 8.1 and the second part is Theorem 9.4.

The paper is organized as follows. In Section 2 we provide the definitions needed to understand this paper. In Section 4 we give the constructions of two rank 4 geometries that are residues of Γ_{IvSh}. In Section 7 we prove that the Connor geometry Γ_{Co} is a truncation of the Ivanov–Shpectorov geometry Γ_{IvSh}. In Section 8 we extend uniquely the boolean lattice of Γ_{Co} to a rank 5 boolean lattice in the subgroup lattice of $O'N$. In Section 9 we show that this extended boolean lattice provides a unique flag-transitive geometry that is Γ_{IvSh}. In Section 10 we thoroughly detail structural properties of the h-elements of Γ_{IvSh} that we call *hyperlines*.

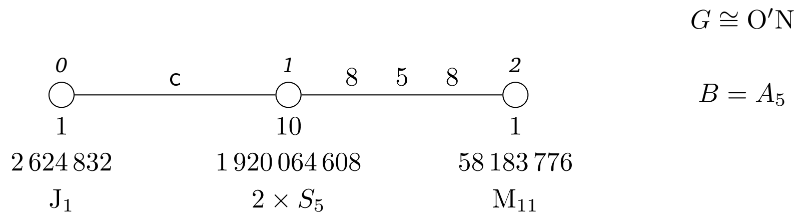


Figure 2: The diagram of the geometry Γ_{Co}

2 Definitions and notations about graphs and incidence (resp. coset, diagram) geometries

In this section we define the notions that are needed to understand our paper. It includes explanations about the meaning of diagrams and their symbols. Our reference for this section is [10].

2.1 Graphs

We formalize the notion of a graph and we fix notation as well as terminology.

A *graph* $\mathcal{G}$ is a pair (V, E) where V is a set whose elements are called *vertices* and E is a set of pairs of distinct elements of V. Elements of E are called *edges*.

A graph in this sense is commonly called simple, i.e. with no loop nor multi-edge. However we only deal with simple graphs and there is no possible ambiguity by omitting the adjective 'simple'.

We call V the *vertex-set* of $\mathcal{G}$ and E the *edge-set* of $\mathcal{G}$. The edge-set E of a graph $\mathcal{G}$ defines a symmetric relation on $\sim : V \times V$. We call such a relation an *adjacency* relation. The data of V and E to define $\mathcal{G}$ is equivalent to the data of V together with the adjacency relation $\sim$. For the sake of simplification, we make the following abuse of terminology: we say that a vertex v of V *belongs* to $\mathcal{G}$, and we write $v \in \mathcal{G}$ for that.

Given a vertex $v \in \mathcal{G}$, we call the set $\mathrm{N}_{\mathcal{G}}(v) := \{w \in \mathcal{G} \mid v \sim w\}$ the *neighborhood* of v (where the subscript $\mathcal{G}$ can be omitted if there is no possible confusion). The elements of this set are the *neighbors* of v. We also denote $\mathrm{N}_{\mathcal{G}}(v)$ by $v^{\perp}$ and we may write $v \perp w$ to denote that v and w are adjacent.

A *subgraph* of a graph $\mathcal{G} = (V, E)$ is a graph $\mathcal{H} = (W, F)$ such that $W \subseteq V$ and $F \subseteq E$. We call $\mathcal{H}$ *induced* if for any $v, w \in W$ the following holds: $\{v, w\}$ is an edge of $\mathcal{H}$ if and only if $\{v, w\}$ is an edge of $\mathcal{G}$. In that case F is completely determined by W and E.

A *path* in $\mathcal{G}$ is a sequence of vertices in which any two consecutive vertices are adjacent. The length of a path is the number of elements in the sequence minus one. We call $\mathcal{G}$ *connected* provided that, given any two vertices v and w of $\mathcal{G}$, there exists a path from v to w. Define a binary relation $\equiv$ on V as follows: $v \equiv w$ if and only if v and w are connected by a path. The relation $\equiv$ is obviously an equivalence relation. The *connected components* of $\mathcal{G}$ are the equivalence classes under the relation $\equiv$. The notion of path yields a natural notion of distance on $\mathcal{G}$: given v, w two vertices of $\mathcal{G}$, the *distance* $d_{\mathcal{G}}(v, w) = d(v, w)$ of v and w is the shortest integer d such that there exists a path of length d from v to w. If v and w belong to different connected components of $\mathcal{G}$ we say that $d(v, w) = \infty$. The *diameter of $\mathcal{G}$ from a vertex* v is the largest distance from v to a vertex of $\mathcal{G}$. The *diameter* of $\mathcal{G}$ (without further reference to a vertex) is the largest diameter of $\mathcal{G}$ from some vertex.

An *s-arc* of $\mathcal{G}$ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ such that $\{v_{i-1}, v_i\}$ is an edge of $\mathcal{G}$ for all $i \in \{1, 2, \ldots, s\}$ and $v_{j-1} \neq v_{j+1}$ for all $j \in \{1, \ldots, s-1\}$.

A *circuit* in a graph $\mathcal{G}$ is a path $c = (v_0, \ldots, v_n, v_0)$ from a vertex to itself. The *girth* of $\mathcal{G}$ is the smallest integer g such that there exists a circuit of length g in $\mathcal{G}$. If there is no circuit in $\mathcal{G}$, we say that the girth of $\mathcal{G}$ is ∞.

We say that $\mathcal{G} = (V, E)$ is a *complete graph* provided that E is the set of all pairs of distinct elements of V. It is common to denote a complete graph of n vertices with $\mathcal{K}_n$. A synonym of complete graph is *clique*. If $\mathcal{K}$ is a subgraph of $\mathcal{G}$ and $\mathcal{K}$ is a complete graph, we prefer to say that $\mathcal{K}$ is a clique of $\mathcal{G}$ rather than a complete subgraph of $\mathcal{G}$.

We say that a property P of a graph $\mathcal{G}$ is *local* provided that the induced subgraph on the neighborhood of every vertex has the property P. In particular, if $\mathcal{H}$ is a graph, we say that $\mathcal{G}$ is *locally* $\mathcal{H}$ if the subgraph induced on the neighborhood of any vertex of $\mathcal{G}$ is isomorphic to $\mathcal{H}$.

Let $\mathcal{G} = (V, E)$ be a graph and let G be a group acting on V. This action induces an action of G on pairs of elements of V given by $g(\{v, w\}) = \{g(v), g(w)\}$. We say that G acts on $\mathcal{G}$ provided that G preserves the adjacency relation of $\mathcal{G}$, i.e. $g(e) \in E$ for any $e \in E$. In that case, a permutation of G is called an *automorphism* of $\mathcal{G}$. The set of all automorphisms of $\mathcal{G}$ is endowed with a group structure and is called the *automorphism group* of $\mathcal{G}$, denoted by $\mathrm{Aut}(\mathcal{G})$. We say that $\mathcal{G}$ is *vertex-transitive* provided that $\mathrm{Aut}(\mathcal{G})$ has one orbit on the set of vertices of $\mathcal{G}$. We call $\mathcal{G}$ *edge-transitive* provided that $\mathrm{Aut}(\mathcal{G})$ acts transitively on the set of edges of $\mathcal{G}$. Observe however that vertex-transitivity and edge-transitivity do not imply each other.

Suppose that $\mathrm{Aut}(\mathcal{G})$ acts 2-transitively on the set of vertices of $\mathcal{G}$. Then obviously E is empty or is the set of all pairs of distinct elements of V. In the latter case $\mathcal{G}$ is a complete graph. Hence n-transitivity is a concept that is not well suited to graphs. The concept of arc-transitivity may be preferred.

Let G be a group of automorphisms of a graph $\mathcal{G}$ and let $s \geq 1$ be an integer. We say that $\mathcal{G}$ is (G, s)-*arc-transitive* provided that G acts transitively on the arcs of $\mathcal{G}$ of length s. We say that $\mathcal{G}$ is s-*arc-transitive* provided that it is $(\mathrm{Aut}(\mathcal{G}), s)$-arc-transitive.

A 1-arc-transitive graph is also called *arc-transitive* for the sake of brevity. Observe that arc-transitivity does not force the graph to be empty nor complete.

2.2 Incidence geometries

Let I be a finite set whose elements are called *types* while I itself is called a *set of types*. A triple $\Gamma = (X, *, \tau)$ is called a *pregeometry* over I if

1. X is a set whose elements are called *elements* of Γ;
2. $*$ is a symmetric and reflexive relation on X called the *incidence relation* of Γ;
3. τ is a mapping from X to I, called the *type function* of Γ, such that distinct elements $x, y \in X$ with $x * y$ satisfy $\tau(x) \neq \tau(y)$.

If τ is surjective and if any maximal set of mutually incident elements of Γ contains one element of each type then Γ is called a *geometry*. The *rank* of Γ is the cardinality of the set of types I.

A set of mutually incident elements is called a *flag*; if it contains one element of each type, it is called a *chamber*. A geometry is called *flag-transitive* if its automorphism group is transitive on the set of its chambers. The *type* of a flag is the set of types of the elements of the flag.

Let Γ denote a geometry. The *residue* of a flag F of Γ is the set of all elements in $X \setminus F$ incident to every element in F together with the incidence relation. It is an easy exercise to check that the residue of F is a geometry.

For every flag F of Γ, the J-*shadow* of F is the set of all elements of type in $J \subseteq I$ that are incident with F. It inherits the incidence of Γ.

The *truncation* of Γ on the set of types $J \subseteq I$ is the pregeometry ${}_J\Gamma = ({}_JX, {}_J*, {}_J\tau)$ over J such that ${}_JX$ is the preimage $\tau^{-1}(J)$, and such that ${}_J*$ (resp. ${}_J\tau$) is the restriction of $*$ (resp. τ) to ${}_JX$. In other words, ${}_J\Gamma$ is the restriction of Γ on the elements of type is in J.

A geometry is called *residually connected* provided that the incidence graph of the residue of any flag of rank at most $n-2$ is connected. A geometry is called *firm* provided that any flag of rank $n-1$ is contained in at least two chambers.

The *incidence graph* $(X,*)$ of Γ is the graph whose vertices are elements of X and adjacency is provided by incidence. By convention, if the graph is drawn, loops are deleted.

2.3 Coset geometries

According to Tits [34], given a group G and a family of its subgroups $\{G_0,\ldots,G_{n-1}\}$, we define a pregeometry $\Gamma(G,\{G_0,\ldots,G_{n-1}\})$ as follows. The type set of Γ is the set $I=\{0,\ldots,n-1\}$; the elements of Γ are the right cosets G_ig for $i=0,\ldots,n-1$ and $g\in G$; incidence is defined by nonempty intersection, i.e. $G_ig*G_jh \iff G_ig\cap G_jh\neq\emptyset$. If the pregeometry Γ is flag-transitive then it is a geometry. The subgroups $G_0,\ldots,G_{n-1}$ are called *maximal parabolic subgroups* of Γ. Any intersection of maximal parabolic subgroups $\cap_{j\in J}G_j$ with $J\subset I$ is called a *parabolic subgroup*; the subgroups $\cap_{j\in I\setminus\{i\}}G_j$ are called *minimal parabolic subgroups*; the subgroup $\cap_{i\in I}G_i$ is called the *Borel subgroup* of Γ. A geometry arising in this way is called a *coset geometry*. A coset geometry Γ is residually connected provided that each nonminimal parabolic subgroup of Γ is generated by its proper parabolic subgroups.

2.4 Diagram geometries

A diagram of a (pre)geometry is a labelled graph providing information on its residues of rank 2. Let $\Gamma=(X,*,t)$ be a pregeometry over some set of types I. The *digon diagram* associated to Γ consists of a nonoriented graph of $|I|$ vertices named after the elements of I. Two vertices corresponding to the elements $i,j\in I$ are joined by an edge in the graph if the following two conditions hold:

1. there exists a flag F of type $I\setminus\{i,j\}$ in Γ; and
2. there exist two elements of type i and j in Γ_F which are not incident in Γ_F.

We say that a pregeometry *belongs* to its diagram.

Suppose from now on that $\Gamma=(X,*,\tau)$ is a firm, residually connected, flag-transitive geometry of rank 2 over the set of type $I=\{i,j\}$. It is possible to refine the digon diagram of Γ with some parameters that provides further information about the residues and the flags of Γ. We say that $x,y\in X$ are at distance k if they are at distance k in the incidence graph $(X,*)$. For $j\in I$, the

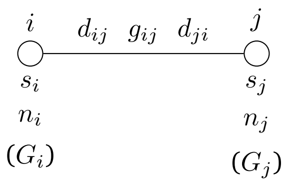


Figure 3: The Buekenhout diagram of a (coset) geometry

j-diameter d_j of Γ is the greatest number occuring as a diameter of $(X,*)$ for some element of type j. The *diameter* of Γ is defined as $d := \max\{d_i \mid i \in I\}$. The *gonality* is the smallest $\infty \geq g > 0$ such that $(X,*)$ has a circuit of length $2g$. Observe that every circuit in $(X,*)$ has even length. The *order* of an element x of type j is $s_j := |\Gamma_x| - 1$. That information is summed up in the diagram of Figure 3 where n_k is the number of elements of type k with $k = i, j$.

The *Buekenhout diagram* of Γ is its digon diagram together with the parameters given in Figure 3. If Γ is a coset geometry, we moreover provide stabilizers G_i for all $i \in I$. More generally, if Γ is a firm, residually connected, flag-transitive geometry of arbitrary rank, its Buekenhout diagram is its digon diagram together with parameters given as in Figure 3.

Sometimes G_i acts with a kernel $K_i \lhd G_i$, in which case we write $G_i = \boxed{K_i}.(G_i/K_i)$ to emphasize the kernel of this action.

We use the following conventions: in case the labels of an edge are $d_i = g = d_j = n$, we write only n above the edge for any $i, j \in I$; in case $n = 2$, we do not draw any edge at all; if $n = 3$, we draw the edge without any label. More conventions are useful in some circumstances. Let us mention only two more. If $(d_{ij}, g_{ij}, d_{ji}) = (3, 3, 4)$, $s_i = 1$ and $s_j = n \geq 2$ then we write c over the stroke. In this case Γ is a complete graph of $n + 2$ vertices. The notation c is used to suggest the word 'circle' or 'complete graph'. If $(d_{ij}, g_{ij}, d_{ji}) = (5, 5, 6)$ and $(s_i, s_j) = (1, 2)$, we write P over the stroke instead because the corresponding residue is isomorphic to the rank 2 geometry of vertices and edges of the Petersen graph.

In an Atlas [27], Leemans provides diagram geometries for the nine smallest sporadic simple groups. In particular, he gives rank 4 geometries for the groups M_{11} and J_1 over the diagrams of Figures 4 and 5. We prove in Theorem 9.4 that these two geometries are rank 4 residues of Γ_{IvSh}.

Convention. When the context is not ambiguous, it is useful to adopt the convention that 'geometry' means 'firm, residually connected geometry'.

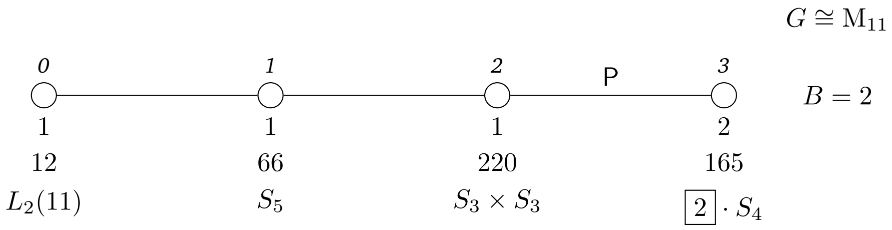


Figure 4: The diagram of the geometry $\Gamma_{\mathrm{M}_{11}}$

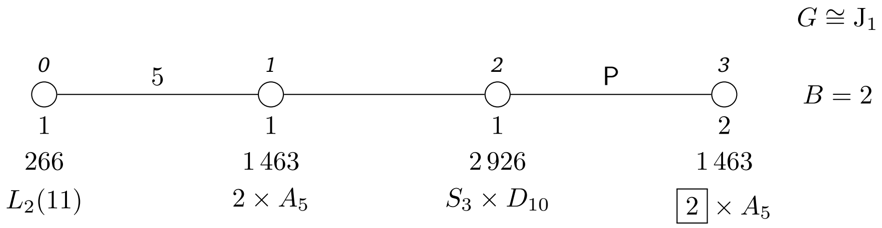


Figure 5: The diagram of Γ_{J_1}

3 Lattices of subgroups

In our paper, we make abundant use of lattices of subgroups in order to produce and describe coset geometries. Let $\Gamma = \Gamma(G, (G_i)_{i \in I})$ be a firm, residually connected, flag-transitive coset geometry. The Borel subgroup $B = \cap_{i \in I} G_i$ of Γ is the stabilizer of some chamber C of Γ. The maximal parabolic subgroup G_i, $i \in I$ of Γ is the stabilizer of the i-element of C. Accordingly, G_J, $J \subseteq I$ is the stabilizer of the flag $F \subseteq C$ of type J. The collection of all parabolic subgroups G_J, $J \subseteq I$ of Γ structured by inclusion yields a boolean lattice L of subgroups of G. However, notice that not every boolean lattice of subgroups of a group G yields a geometry. Lemma 3.1 provides a characterization of residual connectedness in coset geometries.

Lemma 3.1. *A coset geometry Γ is residually connected if and only if every nonminimal parabolic subgroup of Γ is generated by the parabolic subgroups it contains properly.*

Proof. See [10, Lemma 1.8.9]. □

Lemma 3.1 motivates the introduction of the following terminology. A boolean lattice of subgroups L of some group G is called *generating* provided that every nonminimal subgroup in L is generated by the subgroups in L that it contains properly. Obviously a residually connected coset geometry yields a generating boolean lattice of parabolic subgroups.

As a matter of fact, we rely deeply on the maximal subgroups of O′N. The classification of those is due to Wilson [36], Yoshiara [37] (computer-free proof) and Ivanov, Tsaranov & Shpectorov [25] independently. See also Soicher [32].

In [26], Leemans built an algorithm that computes the subgroup lattice of a permutation group G based on the following ideas. Given a permutation group (G, Ω) and a construction of the classes of the maximal subgroups of G with representatives $M_1, M_2, \ldots, M_n$, Leemans looks for a faithful permutation representation of minimal degree for every M_i, $i \in \{1, \ldots, n\}$ which is an ordered pair (M_i, M_{ij}) where M_{ij} is a proper subgroup of M_i of largest order containing no normal proper subgroup of M_i. From this on it is assumed (and all right for $G = \mathrm{O'N}$) that the maximal subgroups of each maximal subgroup of G are computable. Afterwards a delicate analysis of overlappings between Σ_i and every other Σ_j occurs.

Leemans implemented this algorithm in MAGMA and managed to compute the subgroup lattice of O′N. This paper has been extended in a work by Connor and Leemans which now contains subgroup lattices of many finite almost simple groups, including O′N and O′N : 2 [16].

Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
1	A_5	60	1	2 (5), 3 (6), 4 (10)	
2	A_4	12	5	6, 7 (4)	1
3	D_{10}	10	6	5, 8 (5)	1
4	S_3	6	10	7, 8 (3)	1
5	5	5	6	9	3
6	2^2	4	5	8 (3)	2
7	3	3	10	9	2 (2), 4
8	2	2	15	9	3 (2), 4 (2), 6
9	1	1	1		5 (6), 7 (10), 8 (15)

Table 1: The subgroup lattice of A_5

Table 1 gives an example for the alternating group A_5. Each conjugacy class of subgroups is listed with a number c, a structure describing the subgroups of the class, the order of the groups in this class, the length of the class and the labels of the conjugacy classes of subgroups where the maximal subgroups and minimal overgroups are. In the antepenultimate (resp. last) column, when a class number x is followed by a number between parentheses, this number n means that there are n subgroups (resp. overgroups) of conjugacy class $\#x$ contained (resp. containing) a given group of the class $\#c$. For instance, class labelled #6 contains 5 Klein 4-groups in Table 1. Each Klein 4-group of that class has exactly 3 maximal subgroups in class #8, that are cyclic groups of order 2. Moreover, each of these cyclic groups of order 2 is a maximal subgroup of two groups of class #3, two groups of class #4 and one group of class #6.

The proof of our main result is based on the identification of boolean lattices in the subgroup lattice of various groups, including M_{11}, J_1 and $\mathrm{O'N}$, that yield coset geometries, as well as on observations in the subgroup lattice of $\mathrm{O'N}:2$. We denote the subgroup lattice available in [26] of $\mathrm{O'N}$ by $\Lambda_{\mathrm{O'N}}$; the subgroup lattice of $\mathrm{O'N}:2$ available in [16] is $\Lambda_{\mathrm{O'N}:2}$. When we refer to 'a subgroup of class $\#x$ in $\Lambda_{\mathrm{O'N}:2}$', we mean some subgroup of the conjugacy class of subgroups labelled $\#x$, as given in our references. We always make clear whether we deal with $\Lambda_{\mathrm{O'N}}$ or $\Lambda_{\mathrm{O'N}:2}$. We often draw boolean lattices or partial subgroup lattices that are extracted from $\Lambda_{\mathrm{O'N}}$ or $\Lambda_{\mathrm{O'N}:2}$ in order to help the reader. We do not prove their correctness because it consists only of an observation of the lattices $\Lambda_{\mathrm{O'N}}$ and $\Lambda_{\mathrm{O'N}:2}$.

4 Two geometries for M_{11} and J_1

Among the maximal subgroups of $\mathrm{O'N}$ there are two sporadic groups (up to automorphism) namely the Janko group J_1 and the Mathieu group M_{11}. All subgroups of $\mathrm{O'N}$ isomorphic to J_1 are conjugate while there are two conjugacy classes of subgroups isomorphic to M_{11}. Those two classes are fused in $\mathrm{Aut}(\mathrm{O'N})$. These groups and the related geometries of rank 4 that we are about to describe are crucial for the rest of this paper. Both of them possess a residue which is a rank 3 geometry with $L_2(11)$ acting flag-transitively. We start with the description of the latter geometry.

4.1 A locally Petersen geometry with $L_2(11)$ acting flag-transitively

The classification of locally Petersen graphs is due to Jonathan Hall [20]. There are exactly three such graphs. They are flag-transitive and actually distance-transitive. We describe them briefly with a further reference to Weisstein [35] and to Brouwer–Cohen–Neumaier [4].

1. The Conway-Smith graph of 63 vertices and 315 edges whose group of automorphisms is $3 \cdot S_7$. It is a triple cover of the next graph.

2. The graph of the 21 pairs of elements in $\{1, \dots, 7\}$ such that two distinct pairs are adjacent provided they have nonempty intersection. It is

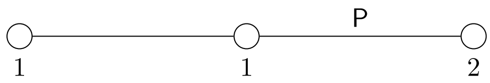


Figure 6: Diagram $\mathsf{A}_2\mathsf{P}$ of a locally Petersen geometry

the unique distance regular graph with 21 vertices and intersection array $[10, 6; 1, 6]$. Its automorphism group is S_7. It is the $(7, 2)$-Kneser graph.

3. The Doro–Hall graph of 65 vertices and 325 edges whose automorphism group is $\mathrm{P\Sigma L}(2, 25)$.

It is immediate to see that each of those three graphs provides a geometry over the diagram $\mathsf{A}_2\mathsf{P}$ of Figure 6 whose 2-elements (planes) are the 3-cliques in the graph.

Conversely, Shpectorov [31] proved that there are exactly four flag-transitive geometries that belong to the diagram of Figure 6: the three locally Petersen graphs and a geometry built from the action of $L_2(11)$ on the short Galois line (see also [7, Section 9, geometry 3]). This terminology was used in [9] to denote the 2-transitive action of $L_2(11)$ on 11 points, as observed by Galois. We denote this geometry with $\Gamma_{L_2(11)}$ and we describe it further. Let us call elements of $\Gamma_{L_2(11)}$ points, lines and planes, by reading the diagram of Figure 6 from left to right. The diagram shows that there are 10 lines through each point and 2 points on each line. Therefore the truncation of $\Gamma_{L_2(11)}$ on its points and lines is a complete graph. The group $L_2(11)$ has two orbits on the trios of points: one of length 55 and the other of length 110. The planes of $\Gamma_{L_2(11)}$ are the trios of the smallest orbit. Incidence is provided by inclusion.

4.2 A rank 4 geometry for M_{11}

Let us consider the problem of determining the flag-transitive geometries that are 'locally $\Gamma_{L_2(11)}$'. In other words, what are the firm, residually connected, flag-transitive geometries over the diagram $\mathsf{A}_3\mathsf{P}$ where we assume that the stabilizer of a point is isomorphic to $L_2(11)$ (see Figure 7)? Let Υ be such a geometry with flag-transitive automorphism group G. We call elements of Υ points, lines, planes and hyperplanes by reading the diagram from left to right. The residue in Υ of every point is isomorphic to the geometry $\Gamma_{L_2(11)}$ described in Section 4.1, i.e. every point is on 11 lines. The diagram $\mathsf{A}_3\mathsf{P}$ shows that every line contains 2 points, hence the truncation of Υ on its points and lines is a graph $\mathcal{G}$ in which every point has 11 neighbors. The stabilizer of a point acts 2-transitively on those 11 points, by assumption. Therefore the induced subgraph

Figure 7: Diagram A_3P with a residue isomorphic to $\Gamma_{L_2(11)}$

on the neighborhood $p^\perp$ of a point p in $\mathcal{G}$ is either a complete graph or an empty graph. Moreover we read from the diagram that the residue of a plane of Υ contains 3 points and 3 lines forming a triangle. Since there are cycles of length 3 in $\mathcal{G}$, we conclude that $p^\perp$ is a complete graph. Therefore a point p of $\mathcal{G}$ together with its neighborhood $p^\perp$ is a clique of 12 points (in which we observe trivially that every point has 11 neighbors). By residual connectedness of Υ, $\mathcal{G}$ cannot be a disjoint union of 12-cliques. In conclusion Υ has 12 points on which G must act transitively and on which the stabilizer of a point acts 2-transitively, i.e. G is 3-transitive. This group must be of order $12 \times 660 = 7\,920$, i.e. G is a primitive group on 12 points of order $7\,920$. We conclude that $G \cong \mathrm{M}_{11}$ (see [17] and [18, Chapter 6]).

We will denote the geometry Υ by $\Gamma_{\mathrm{M}_{11}}$ and we describe it further. Define the triple $(X_{\mathrm{M}_{11}}, *_{\mathrm{M}_{11}}, \tau_{\mathrm{M}_{11}})$ as follows. The set $X_{\mathrm{M}_{11}}$ of varieties is the union of four sets of subsets of vertices of a 12-clique K_{12}: the set of vertices of K_{12}, the set of pairs of vertices of K_{12}, the set of trios of vertices of K_{12} and a set of 4-cliques of K_{12}. The latter set consists of one of the two orbits of 4-cliques occuring under the action of M_{11} on K_{12}, namely the orbit of cardinality 165 (while the second orbit has cardinality 330). The incidence $*_{\mathrm{M}_{11}}$ is defined by symmetrized inclusion in K_{12} and the type function $\tau_{\mathrm{M}_{11}}$ associates the type $0, 1, 2, 3$ to a vertex, pair of vertices, triple of vertices and 4-cliques of $X_{\mathrm{M}_{11}}$ respectively. Then $\Gamma_{\mathrm{M}_{11}} = (X_{\mathrm{M}_{11}}, *_{\mathrm{M}_{11}}, \tau_{\mathrm{M}_{11}})$ is a firm, residually connected, flag-transitive geometry over $\{0, 1, 2, 3\}$ under the action of M_{11}.

This proves existence and uniqueness of a firm, residually connected, flag-transitive geometry over the diagram A_3P with point stabilizer $L_2(11)$.

The geometry $\Gamma_{\mathrm{M}_{11}}$ is well known: it corresponds to Geometry 89 in [6] and Geometry 4.3 in [27]. We give its Buekenhout diagram in Figure 4.

4.3 A rank 4 geometry for J_1

In this section we provide an existence and uniqueness proof of a flag-transitive geometry with automorphism group isomorphic to J_1 over the diagram of Figure 8. We call this diagram H_3^*P because it is the linear amalgam of a

Figure 8: Diagram $\mathsf{H}_3^*\mathsf{P}$ with a residue isomorphic to $\Gamma_{L_2(11)}$

reversed Coxeter diagram H_3 and the Petersen diagram P.

Let us briefly consider the Livingstone graph $\mathcal{L} = (V_{\mathcal{L}}, E_{\mathcal{L}})$ defined in [28] and further studied for instance in [4, 22, 35]. It is a distance regular graph of degree 11 with 266 vertices and $1\,463$ edges. The automorphism group of $\mathcal{L}$ is isomorphic to J_1. The stabilizer in J_1 of a vertex is isomorphic to $L_2(11)$ and the stabilizer of an edge is isomorphic to $2 \times A_5$. We recall without proof some properties of the Livingstone graph. The first is proven in [4], the remaining ones are proven in [22].

Lemma 4.1 ([4, 22]). *The Livingstone graph $\mathcal{L}$ is endowed with the following properties.*

- *$\mathcal{L}$ is distance-transitive.*
- *There are two orbits of pentagons in $\mathcal{L}$, say an orbit of white pentagons and an orbit of orange pentagons. The orbit of white pentagons is of size $2\,926$ and the orbit of orange pentagons is of size $8\,778$. The stabilizer of a white pentagon is isomorphic to $S_3 \times D_{10}$ and the stabilizer of an orange pentagon is isomorphic to $2 \times D_{10}$.*
- *The stabilizer of a Petersen subgraph of $\mathcal{L}$ is isomorphic to $2 \times A_5$ and J_1 has one orbit on the set of Petersen subgraphs of $\mathcal{L}$.*
- *Each white pentagon P belongs to three Petersen subgraphs of $\mathcal{L}$. Each orange pentagon belongs to one Petersen subgraph of $\mathcal{L}$.*
- *The pointwise stabilizer of a white pentagon P is isomorphic to S_3 and it permutes transitively the three Petersen subgraphs containing P.*

We make use of Lemma 4.1 to define a geometry $\Gamma_{\mathrm{J}_1} = (X_{\mathrm{J}_1}, *_{\mathrm{J}_1}, \tau_{\mathrm{J}_1})$ over the set of types $I_{\mathrm{J}_1} = \{0, 1, 2, 3\}$. In Section 9.3, these types are rather named $4, 1, 3, h$ respectively where h stands for 'hyperline' as defined in Section 10. The 0-elements are the vertices of $\mathcal{L}$, the 1-elements are the edges of $\mathcal{L}$, the 2-elements are the white pentagons of $\mathcal{L}$ and the 3-elements are the Petersen subgraphs of $\mathcal{L}$. The incidence relation is induced by inclusion.

Theorem 4.2. *There exists a unique flag-transitive geometry over the diagram of Figure 8 with automorphism group isomorphic to* J_1*. It is geometry* Γ_{J_1}*.*

Proof. We divide the proof of this Theorem in two parts: existence is proven in Section 4.3.1 and uniqueness in Section 4.3.2. □

This geometry was first observed by Buekenhout in 1980 but he did not publish it. In [24], Ivanov and Shpectorov implicitly assert the existence of this geometry. It is coming as a residue of their rank 5 geometry for the group $\mathrm{O'N}$. The geometry Γ_{J_1} is Geometry 4.1 for J_1 in Leemans [27]. Moreover Leemans and Gottschalk obtained the uniqueness of that geometry under the further hypotheses that the geometry is residually weakly primitive [19]. Notice furthermore that Perkel [30] used a rank 3 geometry obtained by Buekenhout in [5] to characterize the Livingstone graph and the Janko group J_1. The geometry he used is Geometry 28 in [6] and is a truncation of Γ_{J_1} on the set of types $\{0, 1, 2\}$.

4.3.1 Existence proof

The triple $\Gamma_{\mathrm{J}_1} = (X_{\mathrm{J}_1}, *_{\mathrm{J}_1}, \tau_{\mathrm{J}_1})$ is a pregeometry according to Section 2.2. It is actually a geometry because any maximal flag contains one element of each type, by construction. Moreover J_1 acts as a group of automorphisms on Γ_{J_1}. We now prove that this action is flag-transitive. Let $F = \{v, e, p, h\}$ and $F' = \{v', e', p', h'\}$ be two chambers of Γ_{J_1}, where v and v' are vertices, e and e' are edges, p and p' are pentagons and h and h' are Petersen subgraphs.

Since $\mathcal{L}$ is distance-transitive, there exists an element γ of J_1 mapping $\{v, e\}$ onto $\{\gamma(v), \gamma(e)\} = \{v', e'\}$. The elements $\gamma(p), \gamma(h)$ are incident and are incident with both v' and e'. It remains to show that there exists $\delta \in \mathrm{Stab}_{\mathrm{J}_1}\{v', e'\}$ mapping $\{\gamma(p), \gamma(h)\}$ onto $\{\delta(\gamma(p)), \delta(\gamma(h))\} = \{p', h'\}$. The stabilizer of $\{v', e'\}$ is the pointwise stabilizer of two adjacent vertices v', w in $\mathcal{L}$ (where $e' = \{v', w\}$). The 2-arc-transitivity of $\mathcal{L}$ implies that this stabilizer acts transitively on the neighbors of w that are at distance 2 from v', hence it acts transitively on the set of white pentagons containing v' and w. Those pentagons are precisely the white pentagons incident to $\{v', e'\}$. There are now three Petersen subgraphs incident with $\{v', e', p'\}$. By Lemma 4.1, $\mathrm{Stab}_{\mathrm{J}_1}\{v', e', p'\} \cong S_3$ permutes transitively those three Petersen subgraphs. This achieves the proof that J_1 acts flag-transitively on Γ_{J_1}: we showed that given any two chambers F and F' there exists an element of J_1 mapping F onto F'.

Firmness is obvious from the construction. In view of the fact that $\mathcal{L}$ is a connected graph and that this also holds for the incidence graph of Γ_{J_1}, there readily follows that Γ_{J_1} is residually connected. It is also an easy exercise to show that the Buekenhout diagram of Γ_{J_1} is as depicted in Figure 5: it is enough to observe that the residue of a point is the geometry of Section 4.1 and that the residue of a Petersen subgraph is the geometry of a hemidodecahedron (quotient of a dodecahedron by the central symmetry).

4.3.2 Uniqueness proof

As for uniqueness, assume that Γ is a flag-transitive geometry over the diagram $\mathsf{H}_3^*\mathsf{P}$ admitting a flag-transitive action of J_1. That geometry must have $266 = |\mathrm{J}_1|/|L_2(11)|$ points. The diagram shows that the residue of a line has 2 points. Since we also know that the residue of a point is the flag-transitive $\mathsf{A}_2\mathsf{P}$ geometry described in Section 4.1, every point is incident to 11 lines. Hence there are $1\,463 = \frac{266\times 11}{2}$ lines. The truncation of Γ on the 0- and 1-elements is thus a graph $\mathcal{G}$, with incidence provided by inclusion. Since we assumed that J_1 acts on Γ, this is already enough to conclude that this truncation is the Livingstone graph $\mathcal{L}$. Then we can count the number of 2-elements: there are 55 2-elements in the residue of a point and there are 5 points in the residue of a 2-element, that we now call pentagons. Therefore there are $\frac{266\times 55}{5} = 2\,926$ pentagons in Γ. Now the diagram of the residue of a 3-element is the diagram of only two flag-transitive geometries, necessarily finite: they are the geometries of a dodecahedron and a hemidodecahedron. If it were a dodecahedron, there would be $\frac{266\times 55}{20} = \frac{1463}{2}$ 3-elements, a contradiction. Hence it is a hemidodecahedron and there are $1\,463$ hemidodecahedra in Γ. There are now several ways to conclude. Recall that the truncation on vertices and edges of a hemidodecahedron is a Petersen graph. Hence hemidodecahedra of Γ are Petersen subgraphs of $\mathcal{L}$. Since there are exactly $1\,463$ Petersen subgraphs in $\mathcal{L}$ (see for instance [22, Section 3.1]), they are the 3-elements of Γ. The same methods show that pentagons of Γ are the white pentagons of Lemma 4.1. This amounts to the construction of Γ_{J_1} and finishes the uniqueness proof.

4.4 Boolean lattices of $\Gamma_{\mathrm{M}_{11}}$ and Γ_{J_1}

In Sections 4.2 and 4.3 we established existence and uniqueness of two flag-transitive geometries. Both of them come with a group and can be seen as coset geometries as defined in Section 2.3. The purpose of this section is to compute the boolean lattice of parabolic subgroups of each of them. Let us start with $\Gamma_{\mathrm{M}_{11}}$.

The flag-transitive geometry $\Gamma_{\mathrm{M}_{11}}$ constructed in Section 4.2 can be seen as a coset geometry $\Gamma(\mathrm{M}_{11}, \{G_0, G_1, G_2, G_3\})$. We now identify the boolean lattice of subgroups of $\Gamma_{\mathrm{M}_{11}}$ in the subgroup lattice $\Lambda_{\mathrm{M}_{11}}$ of M_{11} given in Table 4 (this lattice is borrowed from [16]). We do this by applying and describing

Tits's algorithm. We provide much detail because it is a description of a general method that permits to 'see' a geometry inside a lattice of subgroups.

Theorem 4.3. *The boolean lattice $L_{\mathrm{M}_{11}}$ of the geometry $\Gamma_{\mathrm{M}_{11}}$ is depicted in Figure 9. Moreover there exists a unique such boolean lattice of subgroups in M_{11} up to conjugacy.*

Proof. By construction, the stabilizer of a point is isomorphic to $L_2(11)$. The 12 subgroups of conjugacy class #3 are in bijection with the points of $\Gamma_{\mathrm{M}_{11}}$. Since all subgroups of M_{11} isomorphic to $L_2(11)$ are conjugate, all choices of a particular subgroup $L_2(11)$ are equivalent. Let $G_0 \cong L_2(11)$ be a subgroup of class #3. The stabilizer of an edge G_1 is isomorphic to S_5 because the residue of an edge in $\Gamma_{\mathrm{M}_{11}}$ is the direct sum of a Petersen graph and a rank 1 geometry of 2 points. The 120 subgroups of conjugacy class #6 are also in bijection with the edges of $\Gamma_{\mathrm{M}_{11}}$. Notice however that not all choices of subgroups of class #6 are equivalent at this stage. Indeed a point is incident to 11 edges and an edge is incident to two points. The stabilizer of a flag point-edge is of order $60 = \frac{660}{11} = \frac{120}{2}$. Thus $G_{01} = G_0 \cap G_1$ is a subgroup of class #10: the lattice $\Lambda_{\mathrm{M}_{11}}$ of Table 4 shows that every subgroup of class #10 (isomorphic to A_5) sits in two subgroups of class #3 (stabilizers of points) and one subgroup of #6 (stabilizers of edges). On the other hand, every subgroup of class #3 contains 11 subgroups of class #10 and every subgroup of class #6 contains one subgroup of class #10. It should be clear at this point that G_1 can be chosen inside class #6 in such a way that $G_0 \cap G_1$ is a subgroup G_{01} of class #10. Moreover, notice that the coset geometry $\Gamma(\mathrm{M}_{11}, \{G_0, G_1\})$ is the geometry of a complete graph on 12 points, i.e. the truncation of $\Gamma_{\mathrm{M}_{11}}$ on the set of types $\{0, 1\}$.

Every flag point-edge can be extended in 10 flags point-edge-trio. The parabolic subgroup G_{01} (of class #10) contains 10 subgroups of class #31. Hence we see that $G_0 \cap G_1 \cap G_2 = G_{012}$ is a subgroup of class #31. Now G_{02} and G_{12} must be two distinct subgroups of class #23 (isomorphic to $D_{12} \cong 2 \times S_3$). Indeed the diagram shows that every flag point-trio lies in two flags point-edge-trio and that every flag edge-trio lies in two flags point-edge-trio. Hence G_{02} and G_{12} have to be twice larger than G_{012}. The lattice $\Lambda_{\mathrm{M}_{11}}$ shows that the only choice for G_{02} and G_{12} is to take distinct subgroups of class #23, which is possible because every subgroup of class #31 sits in three subgroups of class #23. Notice furthermore that G_{02} and G_{12} must be distinct for otherwise $G_{02} \cap G_{12} = G_{02} = G_{12}$ and the geometry would not be firm. The parabolic subgroup G_2 must contain subgroups of class #23 and there must be 110 subgroups conjugate to G_2. Indeed, there are 110 trios in $\Gamma_{\mathrm{M}_{11}}$ and M_{11} permutes them transitively. Quick inspection of the overgroups of a subgroup of class #23 shows that G_2 is a subgroup of class #16, isomorphic to $S_3 \times S_3$.

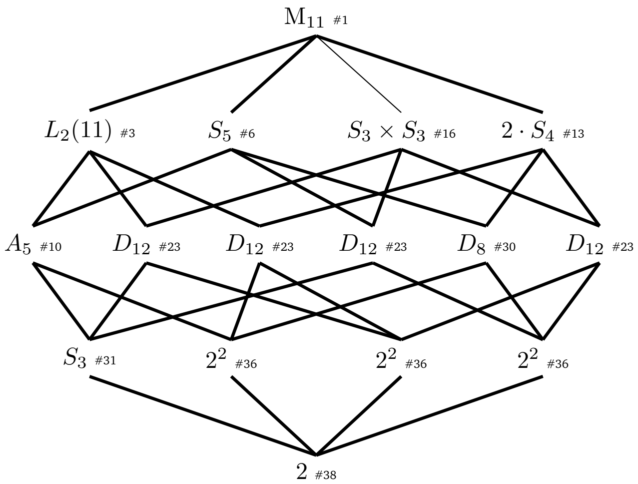


Figure 9: The boolean lattice of $\Gamma_{\mathrm{M}_{11}}$ identified in $\Lambda_{\mathrm{M}_{11}}$

The residue of a flag point-edge is isomorphic to the geometry of a Petersen graph. Moreover $G_{01} \cong A_5$ acts flag-transitively on this residue. The stabilizer of a point in this action is isomorphic to S_3, the stabilizer of an edge is isomorphic to 2^2, and the Borel subgroup (i.e. the stabilizer of a chamber) is a cyclic group of order 2 (see for instance [11]). Hence $G_{012} \cong S_3$ must be a subgroup of class $\#31$ (which we already know) and $G_{013} \cong 2^2$ must be a subgroup of class $\#36$, the only conjugacy class of subgroups isomorphic to 2^2 in M_{11}. The Borel subgroup of $\Gamma_{\mathrm{M}_{11}}$ is a subgroup of class $\#39$.

At last, we come to G_3. The residue of a 3-element of $\Gamma_{\mathrm{M}_{11}}$ is the geometry of a tetrahedron. Hence $|G_3|$ has order divisible by 24. There are 55 elements of type 3 through every point, and there are 4 points through every 3-element. Therefore there are $165 = \frac{12\times 55}{4}$ elements of type 3 in $\Gamma_{\mathrm{M}_{11}}$. Now $\frac{7920}{165} = 48$ is the order of G_3. It is then clear that G_3 belongs to class $\#13$. This parabolic subgroup acts with a kernel of size 2.

It is now an easy exercise to determine the remaining unknowns. □

We provide the boolean lattice of $\Gamma_{\mathrm{M}_{11}}$ in Figure 9 where thick lines represent maximal intersection. We also provide the name of parabolic subgroups with the label of the conjugacy class to which the corresponding parabolic subgroup belongs in $\Lambda_{\mathrm{M}_{11}}$.

Theorem 4.4. *The boolean lattice L_{J_1} of the geometry Γ_{J_1} is depicted in Figure 10. Moreover there exists a unique such boolean lattice of subgroups in J_1 up to conjugacy.*

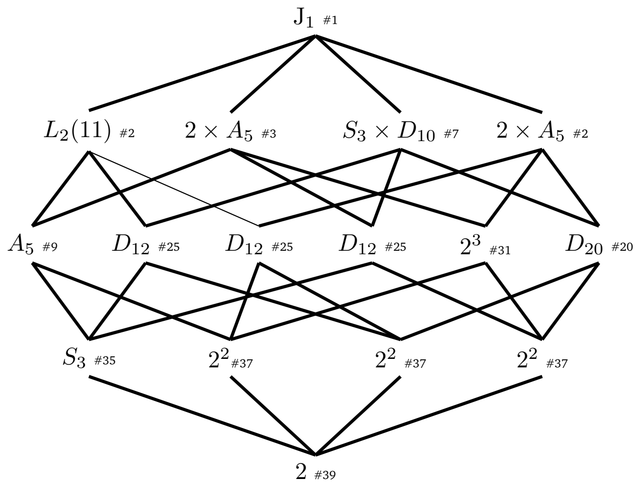


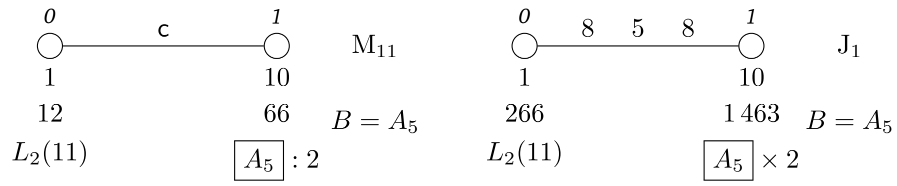
Figure 10: The boolean lattice of Γ_{J_1} identified in Λ_{J_1}

Proof. The very same techniques described in the case of $\Gamma_{\mathrm{M}_{11}}$ can be used to compute and identify the boolean lattice of Γ_{J_1} in the subgroup lattice Λ_{J_1} of J_1 given in Table 5 (borrowed from [16]). We provide the boolean lattice in Figure 10 (see also [19]). □

5 The rank 5 geometry of Ivanov–Shpectorov for O′N

Ivanov and Shpectorov give evidence for the existence of a geometry of rank 5 with $\mathrm{O'N}$ acting in [24]. Their starting point is the original paper of O'Nan that predicts and describes $\mathrm{O'N}$ [29]. Here is a sketch of their construction.

The starting point is the permutation representation of $\mathrm{O'N}$ on the (left) cosets of J_1. Using the fact that J_1 centralizes an involution of $\mathrm{Aut}(\mathrm{O'N}) \setminus \mathrm{O'N}$, they build a vertex-transitive graph $\mathcal{G}_{\mathrm{IvSh}}$ of valency $1\,463$ whose points are the cosets of J_1 in $\mathrm{O'N}$ and in which the stabilizer of an edge is isomorphic to $2 \times S_5$. Vertices (resp. edges) of $\mathcal{G}_{\mathrm{IvSh}}$ are the 0-elements (resp. 1-elements) of the geometry (with respect to our notation as in Figure 1). By looking at normalizers in $\mathrm{O'N}$ of subgroups isomorphic to S_3 lying in J_1, they exhibit cliques of 6 vertices stabilized by subgroups of $\mathrm{O'N}$ isomorphic to $S_3 \times A_5$. Those subgraphs are the 3-elements. Then they consider the action of $L_2(11) < \mathrm{J}_1$ on $\mathcal{G}_{\mathrm{IvSh}}$ and after a few observations, they show that subgroups of $\mathrm{O'N}$ isomorphic to M_{11}



(a) M_{11} on 12 points (b) The Livingstone graph

Figure 11: Rank 2 coset geometries for M_{11} and J_1

stabilize cliques on 12-points in $\mathcal{G}_{\mathrm{IvSh}}$. Those cliques are the 4-elements. Finally they look at involutions of $\mathrm{O'N}$ lying in subgroups isomorphic to $S_3 < \mathrm{J}_1$. They consider the action of the centralizer of an involution in $\mathrm{O'N}$ on $\mathcal{G}_{\mathrm{IvSh}}$. This allows them to identify subgraphs of size 16 which in turn provide them with subgraphs that become the h-elements of the geometry. However the stabilizer of an h-element in $\mathrm{O'N}$ is not made explicit.

Ivanov and Shpectorov conclude their paper with the assertion that the elements they have produced together with maximal intersection yields a geometry over the diagram of Figure 1 that admits a flag-transitive action of $\mathrm{O'N}$. There is no proof of this assertion in [24]. To the best of our knowledge we provide the first proof of Ivanov–Shpectorov's assertion.

The reason behind our names for types is the following. Points are 0-elements and edges are 1-elements. The cliques of 12-points are 4-elements because the residue of such an element is a geometry of rank 4 with a connected diagram. Residues of the 6-cliques are geometries with a disconnected diagram: the direct sum of an icosahedron and a rank one geometry. It is then intrinsically a geometry of rank 3. The last elements are of type h to remind the word 'hyperline'. As we will detail in Section 10, those elements are endowed with remarkable properties. The name 'hyperline' is an analogy of the French word *hyperdroite* used by Tits to describe objects in geometries of exceptional type E_n, $n = 6, 7, 8$. The first occurence of that terminology can be found in [33, Page 25].

6 A rank 3 geometry for O′N due to Connor

In [15] the second author provides a construction of a rank 3 coset geometry $\Gamma(\mathrm{O'N})$ for O'Nan's sporadic simple group $\mathrm{O'N}$ over the set of types $I = \{0, 1, 2\}$. In the present work we denote $\Gamma(\mathrm{O'N})$ by Γ_{Co}. The geometry Γ_{Co} belongs to the diagram of Figure 2.

Connor starts with coset geometries of rank 2 for M_{11} and J_1 whose dia-

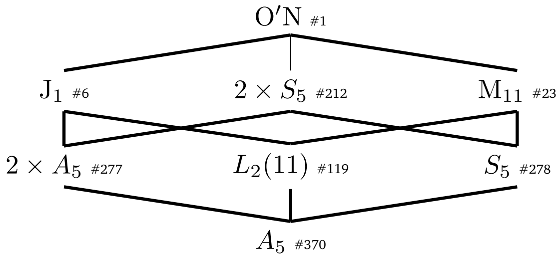


Figure 12: An identification of the boolean lattice of Γ_{Co} in $\Lambda_{\mathrm{O'N}}$

grams are given in Figure 11 [27]. Both geometries have a maximal parabolic subgroup isomorphic to $L_2(11)$ and a Borel subgroup isomorphic to A_5. This observation led him to consider the amalgam of the two diagram geometries in the group $\mathrm{O'N}$ in which M_{11} and J_1 are maximal subgroups. The boolean lattices of the two geometries are given by their Buekenhout diagram. Connor shows that the amalgam of the boolean lattices can be done in a unique way up to conjugacy in $\mathrm{Aut(O'N)}$. The resulting boolean lattice of rank 3 is given in Figure 12. Using a theorem due to Aschbacher [1], Connor proves that this boolean lattice yields a firm, residually connected, flag-transitive coset geometry Γ_{Co} over the diagram of Figure 2. Then he proves that Γ_{Co} is residually weakly primitive, locally 2-transitive and that it satisfies the intersection property in rank 2 residues. Finally he proves that the automorphism group of the geometry Γ_{Co} is $\mathrm{O'N}$, using a study of the truncation of Γ_{Co} on its elements of type in $\{0, 1\}$.

The construction of Γ_{Co} in [15] is independent from [24].

7 From the Ivanov–Shpectorov geometry to the Connor geometry

So far we have explained the various starting concepts and results as mentionned in the abstract and the introduction. We now turn to new results.

From now on and throughout the remaining of this paper, let G denote a copy of $\mathrm{O'N}$. Let us start with the diagram of the Ivanov–Shpectorov geometry $\Gamma_{\mathrm{IvSh}} = \Gamma(G, \{G_0, G_1, G_3, G_4, G_h\})$ as in Figure 1. As we observed in Section 5, the parabolic subgroup G_0 is isomorphic to J_1 and the parabolic subgroup G_4 is isomorphic to M_{11}. The residue Γ_0 of Γ_{IvSh} is the diagram geometry described in Section 4.3 (see also Figure 5) and the residue Γ_4 is the diagram geometry described in Section 4.2 (see also Figure 4). Thanks to Section 4, we know that M_{11} and J_1 have unique flag-transitive geometries with those Γ_{IvSh} residue diagrams. Hence we know that the residues Γ_0 and Γ_4 of the geometry Γ_{IvSh} are isomorphic to Γ_{J_1} and $\Gamma_{\mathrm{M}_{11}}$ respectively. The observations made in Section 5 show that G_1 is isomorphic to $2 \times S_5$. This leads us to ask whether the truncation of Γ_{IvSh} on the elements of type in $\{0,1,4\}$ could be the Connor geometry Γ_{Co} (see Figure 2). Then the use of the subgroup lattice $\Lambda_{\mathrm{O'N}}$ of $\mathrm{O'N}$ readily shows that these data provide indeed the needed diagram geometry as is shown in Theorem 7.1. This provides an alternative construction of the Connor geometry from the knowledge provided in [24].

Theorem 7.1. *The Connor geometry* Γ_{Co} *is a truncation of the Ivanov–Shpectorov geometry* Γ_{IvSh} *on the set of types* $\{0,1,4\}$.

Proof. The strategy of the proof is to show that the truncation of Γ_{IvSh} on its elements of type in $\{0,1,4\}$ is a coset geometry with the same boolean lattice as Γ_{Co}. Then we make use of results in [15] stating that such a boolean lattice yields a unique flag-transitive geometry with automorphism group isomorphic to $\mathrm{O'N}$.

By the developments of Section 5, we know that $G_0 \cong \mathrm{J}_1$, $G_1 \cong 2 \times S_5$, $G_4 \cong \mathrm{M}_{11}$, $G_{04} \cong L_2(11)$ and $G_{01} \cong 2 \times A_5$. Consider a flag F of types in $\{0,1,4\}$. By Assertion 1 in [24], the residue of F is isomorphic to the Petersen graph and it is flag-transitive. Since it has $10 \times 3 = 30$ flags, the group G_{014} must have order divisible by 30. Moreover this group must be isomorphic to a subgroup of $2 \times S_5$, $L_2(11)$ and M_{11} (see Tables 6, 7 and 4). By looking at the subgroup lattices of each of these groups, it is easily seen that $G_{014} \cong A_5$. Moreover G_{14} must strictly contain a subgroup isomorphic to A_5 and must be strictly contained in a subgroup isomorphic to $2 \times S_5$. Consequently, $G_{14} \cong 2 \times A_5$ or $G_{14} \cong S_5$. Since M_{11} does not have a subgroup isomorphic to $2 \times A_5$, we get $G_{14} \cong S_5$. We just showed that the boolean lattice of Figure 12 is a rank 3 sublattice of the boolean lattice of Γ_{IvSh}. By Theorem 4.1 of [15], there are two such boolean lattices of rank 3 up to conjugacy in $\mathrm{O'N}$ and by Theorem 4.2 in [15] each of them determines a geometry isomorphic to Γ_{Co}. Those two families of rank 3 boolean lattices are fused in $\mathrm{Aut}(\mathrm{O'N}) \cong \mathrm{O'N} : 2$. □

Theorem 7.1 is the first result announced in our abstract.

8 Identification of L_{IvSh} in $\Lambda_{\mathrm{O'N}}$

The group $G \cong \mathrm{O'N}$ acts flag-transitively on the geometry Γ_{IvSh}. Using the Tits construction described in Section 2.3, we can see Γ_{IvSh} as a coset geometry

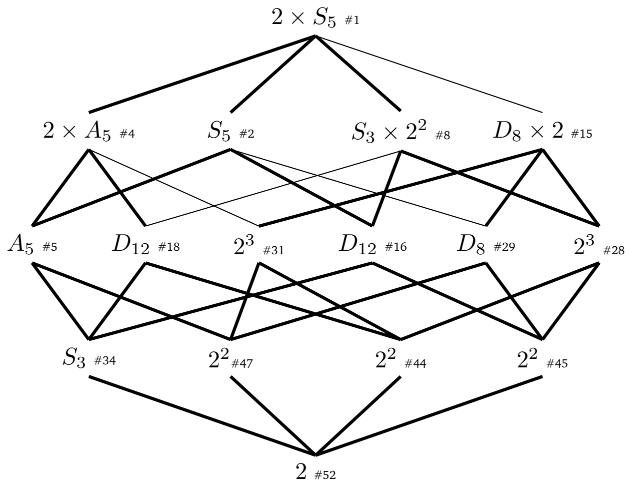


Figure 13: The boolean lattice of $\Gamma_{2\times S_5}$ identified in $\Lambda_{2\times S_5}$

$\Gamma(G, \{G_0, G_1, G_3, G_4, G_h\})$. In this section we determine the boolean lattice L_{IvSh} of parabolic subgroups of this coset geometry on the basis of Section 5.

As a matter of fact, we know four maximal parabolic subgroups: $G_0 \cong \mathrm{J}_1$, $G_1 \cong 2 \times S_5$, $G_4 \cong \mathrm{M}_{11}$, and $G_3 \cong A_5 \times S_3$ (see Section 5). Moreover we proved in Section 4 that J_1 and M_{11} have unique flag-transitive geometries over the residue diagrams provided by the diagram of Γ_{IvSh} and we computed the boolean lattice of each of them in Section 4.4.

The residue Γ_1 (resp. Γ_3) is a geometry with $2 \times S_5$ (resp. $A_5 \times S_3$) acting flag-transitively. We denote this geometry by $\Gamma_{2\times S_5}$ (resp. $\Gamma_{A_5\times S_3}$) to emphasize the group action.

We now further analyze the residue $\Gamma_1 \cong \Gamma_{2\times S_5}$. The stabilizer of a point in the residue $\Gamma_0 \cong \Gamma_{\mathrm{J}_1}$ is also a maximal parabolic subgroup of $\Gamma_1 \cong \Gamma_{2\times S_5}$. Hence a maximal parabolic subgroup of $\Gamma_{2\times S_5}$ is isomorphic to $2 \times A_5$. This subgroup acts on the direct sum of the rank 2 geometry of a Petersen graph and a rank 1 geometry of 2 points. A parabolic subgroup isomorphic to A_5 acts flag-transitively on the Petersen graph. Table 6 shows that this subgroup belongs to conjugacy class #5 in $\Lambda_{2\times S_5}$. Moreover a subgroup of class #5 has exactly three overgroups in $2 \times S_5$: an overgroup isomorphic to $2 \times A_5$ belonging to class #4 and two overgroups isomorphic to S_5 belonging to classes #3 and #2 respectively. Consequently a maximal parabolic subgroup of $\Gamma_{2\times S_5}$ is isomorphic to S_5 and acts on a direct sum of a Petersen graph and a rank 1 geometry of 2 points. This coset geometry of rank 3 is already known (see for instance [11, page 36]). At this point, knowing already two maximal parabolic subgroups, it is an easy

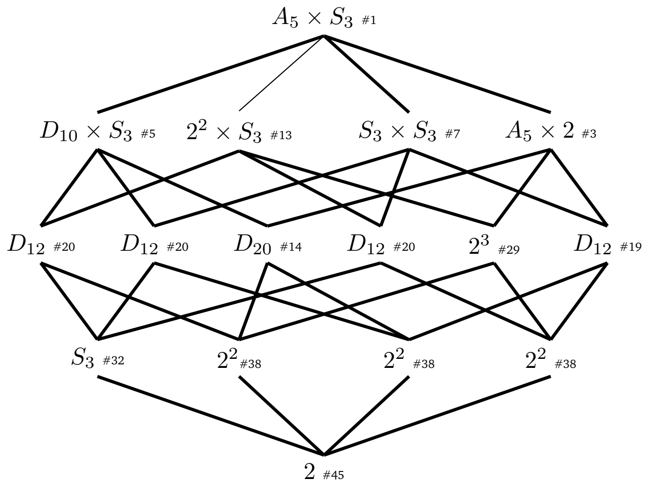


Figure 14: The boolean lattice of $\Gamma_{A_5\times S_3}$ identified in $\Lambda_{A_5\times S_3}$

task to compute the whole boolean lattice $L_{2\times S_5}$ of $\Gamma_{2\times S_5}$ as in Figure 13.

Let us mention further that geometries for $2 \times S_5$ have been studied by Cara and Leemans [14]. They classified the residually weakly primitive geometries for this group and so they knew geometry Γ_1.

Let us now focus on $\Gamma_3 \cong \Gamma_{A_5\times S_3}$. In this case too we know already some parabolic subgroups thanks to our conclusions on the residues $\Gamma_0 \cong \Gamma_{\mathrm{J}_1}$, $\Gamma_1 \cong \Gamma_{2\times S_5}$ and $\Gamma_4 \cong \Gamma_{\mathrm{M}_{11}}$. We leave the reader checking for himself that the boolean lattice $L_{A_5\times S_3}$ of the coset geometry Γ_3 is as depicted in Figure 14 by making use of the subgroup lattice $\Lambda_{A_5\times S_3}$ given in Table 8.

The boolean lattices L_{J_1}, $L_{\mathrm{M}_{11}}$, $L_{2\times S_5}$ and $L_{A_5\times S_3}$ of the four coset geometries Γ_{J_1}, $\Gamma_{2\times S_5}$, $\Gamma_{\mathrm{M}_{11}}$ and $\Gamma_{A_5\times S_3}$ occur as sublattices in the boolean lattice L_{IvSh} of Γ_{IvSh}. Our strategy to compute L_{IvSh} is to use that knowledge by filling in a general rank 5 boolean lattice with 32 unknowns with those four rank 4 boolean lattices (see Figures 15 and 16). We prove that this can be done in a unique way up to conjugacy in $\mathrm{Aut}(\mathrm{O'N})$. There remains an unknown in this boolean lattice, namely G_h, and we prove that it is in fact uniquely determined in Theorem 8.1 under the assumption that the resulting boolean lattice is generating (see proof of Lemma 3.1). This will finish the computation of the boolean lattice of Γ_{IvSh}, the second result stated in our abstract.

Afterwards we start with the boolean lattice of subgroups of L_{IvSh} identified in the subgroup lattice of $\mathrm{O'N}$ and we prove in Theorem 9.4 that it yields a flag-transitive, residually connected, firm geometry. This is a new construction

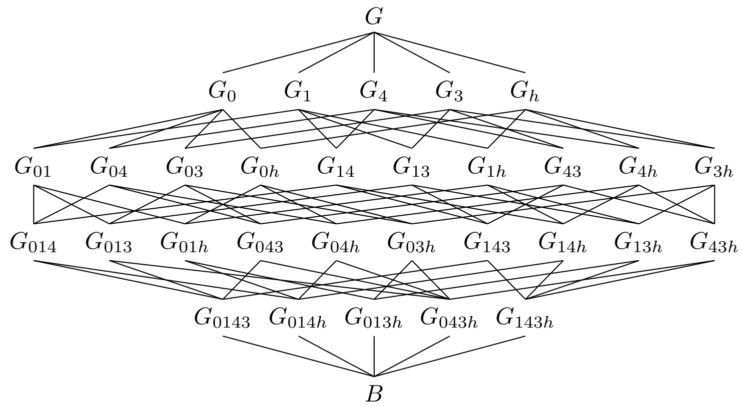


Figure 15: The general boolean lattice of a rank 5 coset geometry with set of types according to our choice for Γ_{IvSh}

of Γ_{IvSh} from the subgroup lattice of $\mathrm{O'N}$.

8.1 Existence and uniqueness of Γ_{IvSh}'s boolean lattice in $\mathrm{O'N}$

We prove the following theorem on the basis of the developments of Section 5.

Theorem 8.1. *Up to conjugacy, there are exactly two partial boolean lattices as in Figure 16 in the subgroup lattice of* $\mathrm{O'N}$. *Each of them extends uniquely to a generating boolean lattice of subgroups of* $\mathrm{O'N}$ *as in Figure 17. Those two families of lattices are fused in* $\mathrm{Aut(O'N)} \cong \mathrm{O'N} : 2$.

8.2 Proof of Theorem 8.1

Table 9 contains carefully chosen subgroups of $\mathrm{O'N}$ extracted from $\Lambda_{\mathrm{O'N}}$ in order to help the reader during the process of the upcoming proof. Labels that matter in that proof are bold to ease the reading.

There is exactly one conjugacy class of subgroups isomorphic to J_1 in $\mathrm{O'N}$. Thanks to Theorem 4.4 it is quite straightforward to identify the boolean lattice of Γ_{J_1} in $\Lambda_{\mathrm{O'N}}$ depicted in Figure 18. Since our aim is now to extend this boolean lattice of rank 4 to a boolean lattice of rank 5, it is convenient to provide parabolic subgroups of Γ_{J_1} embedded in $\mathrm{O'N}$ with names suited to our purpose. This is done by the natural comparison between Figure 15 and Figure 16. Namely $G_0 \cong \mathrm{J}_1$, $G_{01} \cong 2 \times A_5$, $G_{04} \cong L_2(11)$, $G_{03} \cong S_3 \times D_{10}$,

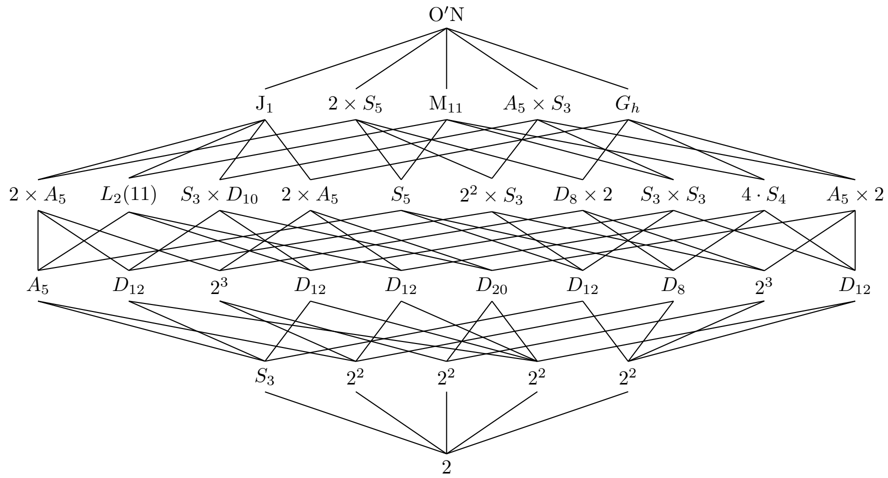


Figure 16: The boolean lattices of Γ_{J_1}, $\Gamma_{2\times S_5}$, $\Gamma_{\mathrm{M}_{11}}$ and $\Gamma_{A_5\times S_3}$ amalgamated in a rank 5 boolean lattice with a unique unknown

$G_{0h} \cong 2 \times A_5$, $G_{014} \cong A_5$, $G_{013} \cong D_{12}$, $G_{01h} \cong 2^3$, $G_{043} \cong D_{12}$, $G_{04h} \cong D_{12}$, $G_{03h} \cong D_{20}$, $G_{0143} \cong S_3$, $G_{014h} \cong G_{013h} \cong G_{043h} \cong 2^2$ and $B \cong 2$.

Notice that a subgroup isomorphic to $L_2(11)$ (necessarily in class #119) is a maximal subgroup of two subgroups of $\mathrm{O'N}$ isomorphic to M_{11} (in classes #23 and #24) and one subgroup isomorphic to J_1. Hence we can choose the subgroups $G_0 \cong \mathrm{J}_1$ and $G_4 \cong \mathrm{M}_{11}$ in $\mathrm{O'N}$ in classes #6 and #23 respectively such that $G_{04} = G_0 \cap G_4$ is a subgroup isomorphic to $L_2(11)$ of class #119. The parabolic subgroup $G_{014} \cong A_5$ belongs to class #370. By construction, $G_{014} \cong A_5$ (in class #370) is a subgroup of $G_4 \cong \mathrm{M}_{11}$. Table 9 shows that a subgroup of class #370 sits in exactly two overgroups of class #278, isomorphic to S_5. On the other hand M_{11} has a unique class of (maximal) subgroups isomorphic to S_5. In the case of G_4, that class of S_5 is #278. Therefore we can choose a subgroup $G_{14} \cong S_5$ in class #278 contained in G_4 and containing G_{014}. Now $\langle G_{01}, G_{14}\rangle$ is by definition the smallest common overgroup of G_{01} and G_{14}. In this situation, it should be clear that the group they generate is a subgroup of $\mathrm{O'N}$ of class #212, isomorphic to $2 \times S_5$ (compare with Table 6). Observe at this point that we just proved Theorem 4 stated in [15].

By Theorem 4.3 we may choose subgroups of $\mathrm{M}_{11} \leq \mathrm{O'N}$ such as in Figure 9. This provides us with $G_{43} \cong S_3 \times S_3$ in class #407, $G_{4h} \cong 2 \cdot S_4$ in class #382, $G_{143} \cong D_{12}$ in class #537, $G_{14h} \cong D_8$ in class #564 and $G_{143h} \cong 2^2$ in class #578. At this stage, we have already extracted 25 subgroups of $\mathrm{O'N}$ that intersect pairwise in such a way that they form a partial boolean lattice. The

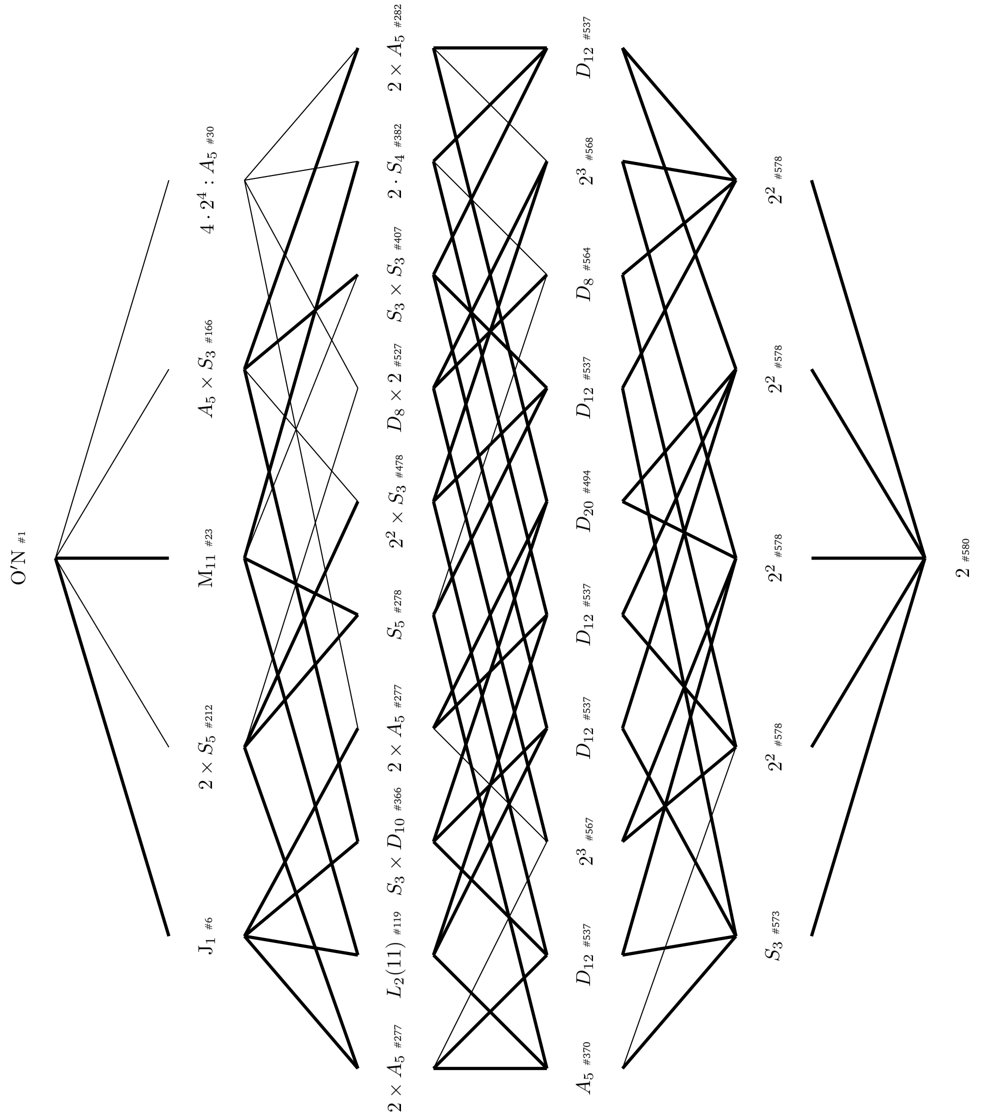


Figure 17: The boolean lattice of Γ_{IvSh} identified in $\Lambda_{\mathrm{O'N}}$

seven remaining unknowns are G_3, G_h, G_{13}, G_{1h}, G_{3h}, G_{13h} and G_{14h}.

In Section 8 we extracted a boolean lattice of subgroups $\Gamma_{2\times S_5}$ from the subgroup lattice of $2 \times S_5$. We apply this construction here to $G_1 \cong 2 \times S_5$ and we obtain $G_{13} \cong 2^2 \times S_3$ in class #478, $G_{1h} \cong Q_8 : 2$ in class #527, $G_{13h} \cong 2^3$ in class #568 and $G_{14h} \cong D_{12}$ in class #537. This leaves us with three unknowns namely G_3, G_h and G_{3h}.

$G_{03} \cong S_3 \times D_{10}$ has exactly one overgroup isomorphic to $A_5 \times S_3$ in class #166. A subgroup of O′N of class #166 contains subgroups of class #407. Therefore we can assume that the overgroup of G_{03} contains $G_{43} \cong S_3 \times S_3$ in class #407, for otherwise we choose conjugate subgroups to G_{43} in G_4. Similarly, we can assume that it contains $G_{13} \cong 2^2 \times S_3$ in class #478. Hence we can assume that subgroups G_{03}, G_{13} and G_{43} generate $G_3 \cong A_5 \times S_3$ in class #166. Now we apply the construction of $\Gamma_{A_5 \times S_3}$ given in Section 8 which allows us to choose G_{3h} as a subgroup isomorphic to $2 \times A_5$ which must belong to class #282.

For the last unknown G_h, careful observation of the respective overgroups of G_{0h}, G_{1h}, G_{4h}, G_{3h} shows that their least possible common overgroup in O′N is a subgroup of class #30 isomorphic to $4 \cdot 2^4 : A_5$ or one of its overgroups. However a subgroup of class #30 is quasimaximal, i.e. it has a unique chain of overgroups with maximal element O′N. Hence the least common overgroup of G_{0h}, G_{1h}, G_{4h}, G_{3h} is either a subgroup of class #30 or G itself. If it were G, then Γ_{IvSh} would be degenerate.

The configuration of 32 subgroups of O′N as we just produced yields a boolean lattice of subgroups. Observe moreover that it is generating by construction.

9 Construction of the geometry Γ_{IvSh}

We prove that the Γ_{IvSh} boolean lattice described in Section 8 is the boolean lattice of a unique firm, residually connected, flag-transitive geometry for O′N. Then we prove that this geometry is the geometry Γ_{IvSh} built in [24].

9.1 Graphs for O′N and O′N : 2

The truncation of Γ_{IvSh} on the set of types $\{0, 1\}$ is a graph $\mathcal{G}_{\mathrm{IvSh}}$ of which objects of type $3, 4$ and h are subgraphs. We construct and describe this graph in this section. We first define the 2B–commuting involution graph of O′N : 2. Then we describe some of the action of O′N $<$ O′N : 2 on that graph in order to produce the graph $\mathcal{G}_{\mathrm{IvSh}}$.

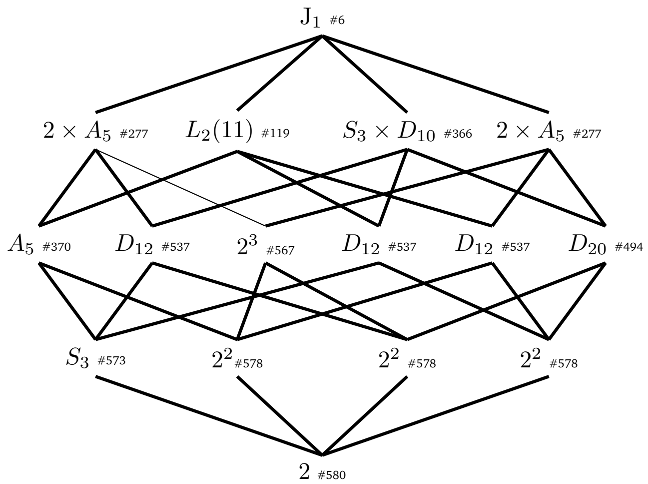


Figure 18: The boolean lattice of Γ_{J_1} identified in $\Lambda_{\mathrm{O'N}}$

9.1.1 The commuting 2B-involution graph of O′N : 2

The group $\mathrm{Aut}(G) \cong \mathrm{O'N} : 2$ has two classes of involutions namely $2\mathrm{A}$ and $2\mathrm{B}$ according to the notation of the ATLAS [17]. Let ρ be a $2\mathrm{B}$-involution (an outer automorphism of G). The centralizer $C := \mathrm{C}_{\mathrm{Aut}(G)}(\rho)$ of ρ in $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{J}_1 \times 2$. We define the graph $\mathcal{C} = (V_{\mathcal{C}}, E_{\mathcal{C}})$ as follows. The set $V_{\mathcal{C}}$ of vertices of $\mathcal{C}$ is the set of $2\mathrm{B}$-involutions of $\mathrm{O'N} : 2$. The set $E_{\mathcal{C}}$ of edges of $\mathcal{C}$ consists of the pairs of $2\mathrm{B}$-involutions that commute. Such a graph is called a *commuting involution graph* [2]. The graph $\mathcal{C}$ has $2\,624\,832$ vertices and $1\,920\,064\,608$ edges. It is arc-transitive of valency $1\,463$.

Lemma 9.1. *The centralizer C has an orbit Σ of size $2\,926$ on the set of vertices of $\mathcal{C}$.*

Proof. We work in the subgroup lattice Λ_2 of $\mathrm{Aut}(G) \cong \mathrm{O'N} : 2$ and we consider the conjugacy class of $2\mathrm{B}$-involutions of $\mathrm{Aut}(G)$ that correspond to class $\#732$ in Λ_2.

Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
712	D_8	8	960032304	**727** (2), 728	430 (6), 493 (20), 531 (6), 532 (30), 620 (10), 625 (10), 656 (10), 662 (15)
713	D_8	8	14400484560	**727** (2), 726	431 (2), 440 (4), 533 (4), 626 (4), 654 (2), 662, 666 (2)
716	2^3	4	4800161520	730, **727** (6)	529 (12), 610 (12), 622 (4), 662 (3), 666 (3), 667
727	2^2	4	1 920 064 608	733, **732** (2)	412 (40), 520 (24), 605 (20), 606 (40), 633 (60), 635 (20), 638 (6), 639 (30), 685 (60), 686 (10), 687 (12), 692 (10), **712**, **713** (15), **716** (15)
732	2	2	2 624 832	734	479 (11704), 535 (29260), 536 (1540), 628 (1596), 629 (17556), 676 (4180), 680 (29260), 681 (4180), 696 (2926), 699 (14630), 721 (2926), 724 (2926), **727** (1463)

Table 2: Overgroups of order 8 of a $2\mathrm{B}$-involution in $\mathrm{O'N}:2$

Table 2 is extracted from Λ_2 and shows sublattices of subgroups of $\mathrm{Aut}(G)$ of order 8 that contain a $2\mathrm{B}$-involution. Two of them are dihedral groups of order 8, namely subgroups of classes $\#712$ and $\#713$. The dihedral group D_8 contains 5 involutions. Seeing D_8 as the automorphism group of a square S, we can give a geometric interpretation of those 5 involutions: they are the central symmetry of S, two symmetries around axes through opposite vertices of S and two symmetries around axes through midpoints of opposite edges of S. Considering Table 2, we see that dihedral subgroups of class $\#712$ have four $2\mathrm{B}$-involutions and one (central) $2\mathrm{A}$-involution. Out of the four $2\mathrm{B}$-involutions, one is ρ and another one is an involution ρ' that commutes with ρ. The involution ρ' belongs to the orbit of size $1\,463$ of $\mathcal{C}$ under the action of C since ρ' is a neighbor of ρ in $\mathcal{C}$. The remaining two $2\mathrm{B}$-involutions σ and σ' commute together but do not commute with either ρ or ρ'. Moreover we read from Table 2 that ρ belongs to $1\,463$ dihedral subgroups of class $\#712$ which yield $1\,463$ $2\mathrm{B}$-involutions that commute with ρ. Thus any dihedral subgroup of class $\#712$ also yields two more $2\mathrm{B}$-involutions. There are $2 \times 1\,463 = 2\,926$ $2\mathrm{B}$-involutions arising in this way.

Let us call Σ the set of those $2\,926$ 2B-involutions. Since C is transitive on the involutions commuting with ρ and since a pair (ρ, ρ') of commuting involutions yields a dihedral subgroup of class $\#712$, we conclude that Σ is an orbit under the action of $C = \mathrm{Stab}_{\mathrm{Aut}(G)}(\rho) = C_{\mathrm{Aut}(G)}(\rho)$. □

We determine with MAGMA that the full orbit distribution of C on $2\,624\,832$ points is

$$1 + 1\,463 + 2\,926 + 5\,852 + 25\,080 + 29\,260^3 + 43\,890 + 58\,520^3 + 87\,780^2 + 175\,560^{10} + 351\,120.$$

9.1.2 The Ivanov–Shpectorov graph for O′N

Lemma 9.2. *The index* 2 *subgroup* $J \cong \mathrm{J}_1$ *of* C *has two orbits on the set* Σ*, each of size* $1\,463$*.*

Proof. The stabilizer in J of a point of Σ is isomorphic to $2 \times A_5$ (the centralizer of an involution in J_1), which is a maximal subgroup of index $1\,463$ of J. Therefore J acts intransitively on Σ with two orbits of size 1463 each. □

Consider the action of $J < C = C_{\mathrm{Aut}(\mathrm{O'N})}(\rho)$ on the set of vertices of $\mathcal{C}$. It is now clear that J has three orbits O^1_{1463}, O^2_{1463}, O^3_{1463} of size 1463, namely the orbit O^1_{1463} of 2B-involutions commuting with ρ and two orbits O^2_{1463} and O^3_{1463}, each of size 1463 such that $O^2_{1463} \sqcup O^3_{1463} = \Sigma$. Let $\alpha \in O^1_{1463}$, $\beta \in O^2_{1463}$ and $\gamma \in O^3_{1463}$.

Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
1	O′N	460815505920	1	2(122760), 3(122760), 6(2 624 832), 7(2857239), 11(17778376), 12(17778376), 15(30968784), 16(30968784), 21(42858585), 23(58 183 776), 24(58 183 776), 43(182863296), 44(182863296)	

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Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
6	J_1	175560	2 624 832	119(266), 237(1045), 277(1463), 285(1540), 286(1596), 366(2926), 397(4180)	1
7	$4 \cdot L_3(4) : 2$	161280	2857239	8, 32(56), 33(56), 34(56), 55(105), 67(120), 65(120), 66(120), 140(280), 143(336)	1
8	$4 \cdot L_3(4)$	80640	2857239	30(42), 57(56), 58(56), 59(56), 112(120), 113(120), 114(120), 194(280)	7
30	$4 \cdot 2^4 : A_5$	3840	120004038	97(5), 133(6), 156(10), 208(16), 209(16), 210(16), 211(16)	8
143	$A_5 : D_8$	480	960032304	212, 213, 210, 299(5), 312(6), 386(10)	7
210	$4 \times A_5$	240	960032304	277, 391(5), 403(6), 481(10)	30(2), 143
212	$2 \times S_5$	240	960032304	278(2), 277, 393(5), 402(6), 478(10)	143
213	$2 \times S_5$	240	960032304	279(2), 277, 394(5), 401(6), 479(10)	143
277	$2 \times A_5$	120	960032304	370, 482(5), 494(6), 537(10)	**6(4)**, 210, 212, 213

Table 3: Overgroups of a subgroup of class #277 in $\Lambda_{\mathrm{O'N}}$

Lemma 9.3. *We have* $\mathrm{Stab}_G\{\rho,\alpha\} \cong 4 \times A_5$*,* $\mathrm{Stab}_G\{\rho,\beta\} \cong \mathrm{Stab}_G\ \{\rho,\gamma\} \cong 2 \times S_5$*. Moreover* $\mathrm{Stab}_G\{\rho,\beta\}$ *and* $\mathrm{Stab}_G\ \{\rho,\gamma\}$ *are not conjugate in* G*.*

Proof. Consider a subgroup $2 \times A_5 \cong A < J$. It belongs to class #277 in $\Lambda_{\mathrm{O'N}}$. The lattice of overgroups of A extracted from $\Lambda_{\mathrm{O'N}}$ is pictured in Table 3. We see that A is in exactly four distinct subgroups of G isomorphic to J_1. Therefore A fixes four points of $\mathcal{C}$. The normalizer $N := \mathrm{N}_G(A) \cong A_5 : D_8 \cong (A_5 \times 2) \cdot 2^2$ stabilizes those four points and acts transitively on them. We see in Table 3 that there are exactly three subgroups between N and A. They are subgroups of classes #212, #213 and #210. Since N acts transitively on 4 points with kernel $2 \times A_5$, each of its subgroups $2 \times S_5^A$ and $2 \times S_5^B$ fixes two points and exchanges the remaining two while $4 \times A_5$ has no fixed point. Moreover $\mathrm{Aut}(G)$ fuses the

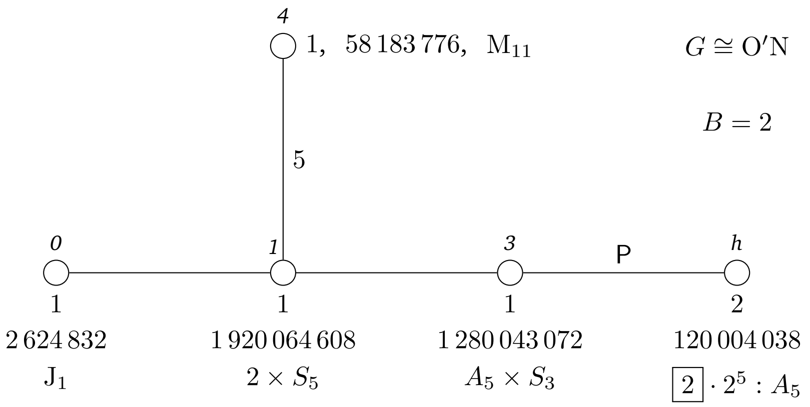

Figure 19: The Buekenhout diagram of Γ_{IvSh}

two conjugacy classes #212 and #213. Therefore we have $\mathrm{Stab}_G\{\rho,\alpha\} \cong 4\times A_5$, $\mathrm{Stab}_G\{\rho,\beta\} \cong 2\times S_5^A$ and $\mathrm{Stab}_G\{\rho,\gamma\} \cong 2\times S_5^B$. □

We are now able to define a graph $\mathcal{G}_{\mathrm{IvSh}} = (V_{\mathrm{IvSh}}, E_{\mathrm{IvSh}})$ as follows. The vertex set of $\mathcal{G}_{\mathrm{IvSh}}$ is the set of vertices of $\mathcal{C}$. The set of edges E_{IvSh} is the set of all the pairs of vertices whose stabilizer in O′N is a subgroup of class #212. Definition of $\mathcal{G}_{\mathrm{IvSh}}$ corresponds obviously to the construction of the graph defined in [24]. With MAGMA we compute the distance distribution map of $\mathcal{G}_{\mathrm{IvSh}}$ given in Tables 10, 11 and 12.

9.2 Existence and uniqueness of a geometry from L_{IvSh}

We prove the following theorem.

Theorem 9.4. *The boolean lattice of Figure 17 is the lattice of parabolic subgroups of a unique firm, residually connected, flag-transitive coset geometry over the Buekenhout diagram in Figure 19.*

9.3 Proof of Theorem 9.4

First we define a set of types $I = \{0,1,3,4,h\}$ as suggested by Figure 1. Next we provide the construction of a geometry $\Gamma = (X,*,\tau)$ over the set of types I. We define the elements of each type as subsets of vertices of $\mathcal{G}_{\mathrm{IvSh}}$ stabilized by parabolic subgroups of the boolean lattice L_{IvSh}. Then we provide an incidence relation and we prove that we obtain a firm, residually connected, flag-transitive geometry.

9.3.1 Elements of types $0, 1, 3$ and 4

Elements of type 0 are the left cosets of $G_0 \cong \mathrm{J}_1$ in $G \cong \mathrm{O'N}$. They are the vertices of the graph $\mathcal{G}_{\mathrm{IvSh}}$ defined in Section 9.1.2. We also call elements of type 0 *points*. The stabilizer of any point is isomorphic to J_1 by construction. Call p the vertex stabilized by G_0 in $\mathcal{G}_{\mathrm{IvSh}}$. The group G acts primitively on the set X_0 of points by left translation. (Notice it is also the case for $\mathrm{O'N}:2 \cong \mathrm{Aut}(\mathrm{O'N})$.)

The elements of type 1 are edges of $\mathcal{G}_{\mathrm{IvSh}}$. By Lemma 9.3 the stabilizer of an edge in $\mathcal{G}_{\mathrm{IvSh}}$ is isomorphic to $2 \times S_5$ and conjugate to G_1.

The boolean lattice of subgroups of $G_4 \cong \mathrm{M}_{11}$ yields the flag-transitive geometry $\Gamma_{\mathrm{M}_{11}}$ constructed in Section 4.2 which has 12 points. Since M_{11} acts 3-transitively on those 12 points, G_4 yields a clique of 12 vertices in $\mathcal{G}_{\mathrm{IvSh}}$. Each left coset of G_4 in G provides such a 12-clique. They are the 4-elements of Γ.

The boolean lattice of subgroups of $G_3 \cong A_5 \times S_3$ yields the flag-transitive geometry $\Gamma_{A_5 \times S_3}$ constructed in Section 8 which has 6 points forming a clique. Each left coset of G_3 in G provides such a 6-clique. They are the 3-elements of Γ.

9.3.2 h-Elements

By Lemma 9.3 the maximal parabolic subgroup $G_0 \cong \mathrm{J}_1$ acts on three orbits of size 1463 in the same way it acts on its set of involutions. Let p be the point fixed by G_0. The subgroup $G_{01} \cong 2 \times A_5$ is the pointwise stabilizer $\mathrm{Stab}_G[p,x]$ of a pair $\{p,x\}$, $x \in O^2_{1463}$. Obviously $\mathrm{Stab}_G\{p,x\}$ is conjugate to G_1 in G. Moreover $\mathrm{Stab}_G[p,x]$ has two orbits of size 15 in O^2_{1463} on which it acts in the same way it acts on its 2-Sylow subgroups isomorphic to 2^3. In particular, $G_{0h} \cong 2 \times A_5$ fixes a point q in O^2_{1463} and has two such orbits O^1_{15} and O^2_{15}. One of them, say O^1_{15}, has a nontrivial intersection with the subgraphs stabilized by G_3 and G_4 detailed in the previous section. On the other hand, G_0 has two orbits of size $21\,945$ on the set of vertices V_{IvSh}, namely O^1_{21945} and O^2_{21945}, on which it acts in the same way it acts on its 2-Sylow subgroups 2^3. Therefore G_{0h} has at least one orbit of size 15 in each of those two G_0-orbits. Using the distance distribution map provided in Tables 11 and 12, we see that a point of O^2_{1463} has neighbors in O^1_{21945} and none in O^2_{21945}.

As we saw in the proof of Lemma 9.3, $\mathrm{Stab}_G[p,q]$ fixes two more points, say $r \in O^1_{1463}$ and $s \in O^3_{1463}$. By our previous developments, r corresponds to the $2B$-involution commuting with p in $\mathrm{Aut}(G)$. Let o be the vertex in O^1_{21945} fixed by $G_{01h} \cong 2^3$. Since G_{01h} is one of the 15 2-Sylow subgroups of G_{0h}, it belongs to a G_{0h}-orbit of size 15, say O_{15}^2. At this point, we have identified 4 orbits under the action of G_{0h}: p, r, O_{15}^1 and O_{15}^2. Notice that G_{1h} swaps o and p. Let us now consider the subgroup $H := \langle \mathrm{Stab}_G[p,r], \mathrm{Stab}_G\{p,o\}\rangle$. This subgroup acts transitively on those 32 points. By looking at the subgroup lattice of G, we see that common overgroups of subgroups of classes #277 ($\mathrm{Stab}_G[p,r]$) and #527 ($\mathrm{Stab}_G\{p,o\}$) could be in classes #1, #7, #8, #30, #143, #212 or #213. Since H acts transitively on 32 points, H cannot be a subgroup of classes #143, #212 and #213. It cannot be G itself since H is not transitive on all points. Since a subgroup of class #30 is quasimaximal in G, it follows that H is a subgroup of that class and $H = G_h$. We now define the h-element stabilized by G_h to be the subgraph of the H-orbit of size 32 that we emphasized.

9.3.3 Incidence relation

We define an incidence relation $*$ on X induced by maximal intersection, i.e. inclusion in all but two cases. More precisely, a 3-element x is incident to a 4-element y if and only if $|x \cap y| = 3$, and a 4-element y is incident to an h-element z if and only if $|y \cap z| = 4$.

According to Section 4, the 3-tuple $(X, *, \tau)$ defines a pregeometry Γ of rank 5 over I.

9.3.4 Firmness and residual connectedness

The boolean lattice L_{IvSh} is generating, hence Γ is residually connected by virtue of Lemma 3.1. Moreover Γ is firm since all of its residues are.

9.3.5 Flag-transitivity

The group G is transitive on the 0-elements of the pregeometry Γ. The stabilizer J in G of a 0-element is isomorphic to J_1. The residue of a 0-element is isomorphic to the pregeometry Γ_{J_1} with J acting flag-transitively on it. Let C and C' be two chambers of Γ. By transitivity on the 0-elements, the 0-element p of C can be brought onto the 0-element q of C'. Now the stabilizer of q in G is transitive on the flags incident to q. Hence there exists an element in G that maps C onto C'. Therefore Γ is a flag-transitive pregeometry, thus a flag-transitive geometry according to Section 4. By uniqueness of the Γ_{IvSh} boolean lattices, Γ is the Γ_{IvSh} geometry.

This technique was used by Buekenhout to prove flag-transitivity in most of the geometries he got in his catalogue [6].

9.4 Properties of Γ

We just proved that Γ is a firm, residually connected, flag-transitive geometry. Moreover G acts primitively on the 0-elements and the 4-elements of Γ since the stabilizer of an element of each of those types is a maximal subgroup of G. Hence Γ is weakly primitive, in the terminology of [8]. However Γ is not residually weakly primitive since there exists a residue of Γ that is not weakly primitive itself, namely the residue of a hyperline of Γ. This last property was proven by Buekenhout and Leemans in [12].

9.4.1 The Buekenhout diagram of Γ

Since the geometry Γ is firm, residually connected and flag-transitive, we associate to it a Buekenhout diagram (see [10], Chapter 2, §3). This diagram is an amalgam of the diagrams of the residues of Γ. The orders can be computed by taking the index of the stabilizer of some element of each type in G. The resulting diagram is shown in Figure 19.

10 More on h-elements

In this section we provide a detailed study of the residue of an h-element H in Γ_{IvSh}. The diagram Δ_{IvSh} together with the boolean lattice L_{IvSh} of Γ_{IvSh} yields the Buekenhout diagram of the residue Γ_H of H depicted in Figure 20. Since Γ_{IvSh} is a flag-transitive, residually connected geometry, so is Γ_H. The truncation of this geometry on its elements of types in $\{0,1\}$ is a graph, that we call the *underlying graph* of Γ_H and that we denote with ${}_{\{0,1\}}\Gamma_H$. A 0-element is now called a point and a 1-element an edge. The residue of a 2-element is obviously a hemi-icosahedron and the residue of a 3-element is a tetrahedron.

Let us describe the structure of ${}_{\{0,1\}}\Gamma_H$. By flag-transitivity of Γ_H, the truncation ${}_{\{0,1\}}\Gamma_H$ is flag-transitive, i.e. arc-transitive in the terminology of graph theory. Every vertex has degree 15, there are 32 vertices and 240 edges. Since Γ_H is residually connected, ${}_{\{0,1\}}\Gamma_H$ is connected. The group $4 \cdot 2^4 : A_5$ acts flag-transitively on ${}_{\{0,1\}}\Gamma_H$ with a kernel of size 2, whose quotient is $2^5 : A_5$. This quotient has a central involution which induces a pairing of the vertices of ${}_{\{0,1\}}\Gamma_H$. The vertex paired with p through the central involution is called the *opposite* of p and is denoted by p^{op}. Hence the distance distribution map of the graph ${}_{\{0,1\}}\Gamma_H$ is endowed with a central symmetry and can be depicted as in Figure 21. There remains to determine the unknowns a and b.

Lemma 10.1. *The distance distribution map of ${}_{\{0,1\}}\Gamma_H$ is as depicted in Figure 22.*

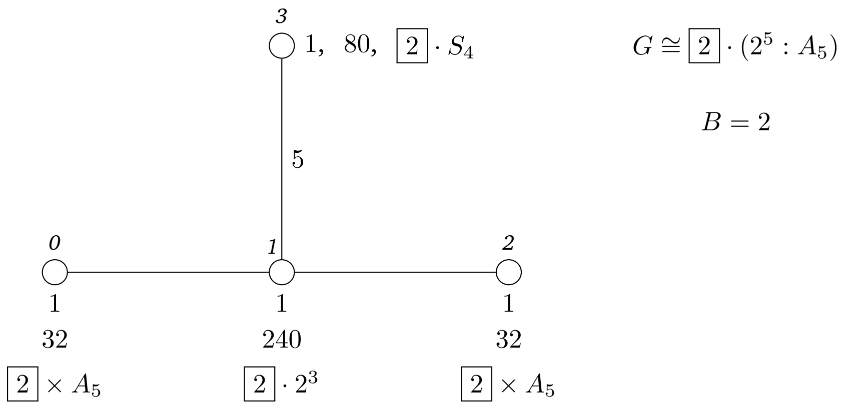

Figure 20: The residue of a hyperline in Γ_{IvSh}

Figure 21: The distance distribution map of the underlying graph of a hyperline

Proof. Since the residue of a 2-element is a hemi-icosahedron, the $\{0,1\}$-shadow of a 2-element in Γ_H is a clique K_6 of 6 vertices. The diagram shows that there are 6 elements of type 2 incident with each point. Therefore every vertex of ${}_{\{0,1\}}\Gamma_H$ lies in 6 cliques K_6. Hence the neighborhood $p^\perp$ of any vertex p of ${}_{\{0,1\}}\Gamma_H$ contains 6 cliques of size 5. By flag-transitivity, every vertex of $p^\perp$ is contained in the same number x of cliques K_5. By double counting, we obtain

$$5 \times 6 = 15x \iff x = 2.$$

We deduce that $a \geq 8$ and thus $b \leq 6$. Every edge of ${}_{\{0,1\}}\Gamma_H$ is contained in exactly 2 cliques of size 6. Given two adjacent vertices $v \in p^\perp$ and $w \in p^{op\perp}$, we conclude that they must have at least 8 common neighbors, and thus $b \geq \frac{8}{2} + 1 = 5$. Finally, we observe that $p, w \in v^\perp$ are at distance 2 in ${}_{\{0,1\}}\Gamma_H$. By flag-transitivity, we conlude that there exists a vertex in $p^\perp$ adjacent to w but not adjacent to v. Hence w has a sixth neighbor in $p^\perp$ and $b = 6$, $a = 8$. □

Now we proceed with the recognition of the graph ${}_{\{0,1\}}\Gamma_H$. First of all, let us introduce some terminology and a surprising situation. The *n-halved cube* is a graph of 2^{n-1} vertices consisting of one of the two connected components of the graph of vertices at distance 2 in the n-cube graph. In [23], Imrich, Klavzar and Vesel provide a characterization of halved cube graphs as follows.

Lemma 10.2 ([23])**.** *Let $n \geq 5$. The n-halved cube graph $\mathcal{Q}_n$ is the only connected, $\binom{n}{2}$-regular graph on 2^{n-1} vertices in which every edge is in two n-cliques and no two n-cliques intersect in a vertex.*

We use Lemma 10.2 to characterize ${}_{\{0,1\}}\Gamma_H$ in Theorem 10.4. During that process, we also make use of Lemma 10.3 due to Harary [21].

Lemma 10.3 ([21, Theorem 8.4])**.** *A graph is a line graph if and only if its edges can be partitioned into complete subgraphs in such a way that no vertex lies in more than two of the subgraphs.*

Given a line graph $\mathcal{L}(\mathcal{G})$ and a partition $\mathcal{P}$ as in Lemma 10.3, we recover the original graph G as follows: the vertex set of G is the set $\mathcal{P}$; two vertices of G are now joined by an edge if and only if the corresponding cliques share a vertex in $\mathcal{L}(\mathcal{G})$.

Theorem 10.4. *The underlying graph ${}_{\{0,1\}}\Gamma_H$ of a hyperline H is isomorphic to the 6-halved cube.*

Proof. The purpose of the proof is to show that ${}_{\{0,1\}}\Gamma_H$ satisfies the hypotheses of Lemma 10.2. Notice that ${}_{\{0,1\}}\Gamma_H$ has $2^5 = 2^{6-1}$ vertices and that it is regular of valency $\binom{6}{2} = 15$. Let p be any vertex of ${}_{\{0,1\}}\Gamma_H$ and let $p^\perp$ be its neighborhood. As we observe in the proof of Lemma 10.1, every vertex of $p^\perp$ sits in exactly two 5-cliques. We now apply Lemma 10.3 to conclude that $p^\perp$ is the line graph of some graph. We recover that graph by applying the construction described after Lemma 10.3 and we see that the induced subgraph on $p^\perp$ is isomorphic to the line graph of a 6-clique. Moreover every edge of ${}_{\{0,1\}}\Gamma_H$ sits in exactly two 6-cliques. It is also obvious that two 6-cliques cannot intersect in a single vertex. We finish the proof by applying Lemma 10.2. □

The full automorphism group of ${}_{\{0,1\}}\Gamma_H \cong \mathcal{Q}_6$ is isomorphic to $2^5 : S_6$. Let us mention that the neighborhood of any point in ${}_{\{0,1\}}\Gamma_H$ is isomorphic to the line graph of a complete graph of 6 vertices with S_6 acting. Observe moreover that this graph of 15 vertices is also the complement of the collinearity graph of the generalized quadrangle of order $(2,2)$ with $Sp_4(2) \cong S_6$ acting. It is also the graph of hyperbolic lines of that generalized quadrangle.

Finally Theorem 10.5 shows that our h-elements deserve the name hyperline.

Theorem 10.5. *Through any pair of opposite points, there is a unique hyperline.*

Proof. Let p be any vertex of $\mathcal{G}_{\mathrm{IvSh}}$. Then there are 1463 opposites to p and there are 1463 hyperlines through p. Since there is one opposite to p in each hyperline, any pair of opposite points provides a unique hyperline. □

Figure 22: The distance distribution map of the underlying graph of a hyperline

11 Computational resources

Generators for each maximal parabolic subgroup of Γ_{IvSh} are available upon request to the authors. They are given as permutations of $\mathrm{O'N}$ in its representation on $122\,760$ points. They were obtained following the process of the proof of Theorem 8.1 in MAGMA [3].

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13 Appendix

We provide subgroup lattices of various groups appearing in this work as well as the distance distribution map of the graph $\mathcal{G}_{\mathrm{IvSh}}$.

Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
1	M_{11}	7920	1	2 (11), 3 (12), 5 (55), 6 (66), 13 (165)	
2	M_{10}	720	11	4, 7 (10), 19 (36), 22 (45)	1
3	$L_2(11)$	660	12	10 (11), 11 (11), 12 (12), 23 (55)	1
4	A_6	360	11	11 (12), 14 (10), 17 (30)	2
5	$\mathrm{M}_9 : 2$	144	55	8, 7, 9, 22 (9)	1
6	S_5	120	66	10, 17 (5), 19 (6), 23 (10)	1
7	M_9	72	55	14 (2), 15, 28 (9)	2 (2), 5
8	$3 : S_3 \cdot 2 \cdot 2$	72	55	15, 29 (9)	5
9	$3 : S_3 \cdot 2 : 2$	72	55	16 (2), 15, 30 (9)	5
10	A_5	60	66	24 (5), 26 (6), 31 (10)	3 (2), 6
11	A_5	60	132	24 (5), 26 (6), 32 (10)	3, 4
12	$11 : 5$	55	144	25, 34 (11)	3
13	$Q_8 : S_3$	48	165	18, 22 (3), 23 (4)	1

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Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
14	$3 : S_3 \cdot 2$	36	110	20, 35 (9)	4, 7
15	$3 : S_3 \cdot 2$	36	55	20, 35 (9)	7, 8, 9
16	$S_3 \times S_3$	36	110	20, 21 (2), 23 (6)	9
17	$2^2 : S_3$	24	330	24, 30 (3), 32 (4)	4, 6
18	$Q_8 : 3$	24	165	28, 33 (4)	13
19	$D_{10} \cdot 2$	20	396	26, 35 (5)	2, 6
20	$3 : S_3$	18	55	27, 32 (12)	14 (2), 15, 16 (2)
21	$S_3 \times 3$	18	220	27, 31, 33 (3)	16
22	$Q_8 : 2$	16	495	30, 29, 28	2, 5, 13
23	D_{12}	12	660	32, 33, 31, 36 (3)	3, 6, 13, 16
24	A_4	12	330	36, 37 (4)	10, 11 (2), 17
25	11	11	144	39	12
26	D_{10}	10	396	34, 38 (5)	10, 11 (2), 19
27	3^2	9	55	37 (4)	20, 21 (4)
28	Q_8	8	165	35 (3)	7 (3), 18, 22 (3)
29	8	8	495	35	8, 22
30	D_8	8	495	36 (2), 35	9, 17 (2), 22
31	S_3	6	220	37, 38 (3)	10 (3), 21, 23 (3)
32	S_3	6	660	37, 38 (3)	11 (2), 17 (2), 20, 23
33	6	6	660	37, 38	18, 21, 23
34	5	5	396	39	12 (4), 26
35	4	4	495	38	14 (2), 15, 19 (4), 28, 29, 30
36	2^2	4	330	38 (3)	23 (6), 24, 30 (3)
37	3	3	220	39	24 (6), 27, 31, 32 (3), 33 (3)
38	2	2	165	39	26 (12), 31 (4), 32 (12), 33 (4), 35 (3), 36 (6)
39	1	1	1		25 (144), 34 (396), 37 (220), 38 (165)

Table 4: Subgroup lattice of M_{11}

Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
1	J_1	175560	1	2 (266), 3 (1045), 4 (1463), 5 (1540), 6 (1596), 7 (2926), 13 (4180)	
2	$L_2(11)$	660	266	8 (11), 9 (11), 12 (12), 25 (55)	1
3	$2^3 : 7 : 3$	168	1045	11, 18 (7), 20 (8)	1
4	$A_5 \times 2$	120	1463	9, 18 (5), 21 (6), 25 (10)	1
5	$19 : 3 : 2$	114	1540	10, 14, 33 (19)	1
6	$11 : 5 : 2$	110	1596	12, 19, 28 (11)	1
7	$D_{10} \times S_3$	60	2926	15, 16, 17, 21 (3), 25 (5)	1
8	A_5	60	2926	26 (5), 29 (6), 34 (10)	2
9	A_5	60	1463	26 (5), 29 (6), 35 (10)	2 (2), 4
10	$19 : 3$	57	1540	22, 38 (19)	5
11	$2^3 : 7$	56	1045	31, 32 (8)	3
12	$11 : 5$	55	1596	27, 36 (11)	2 (2), 6
13	$7 : 3 : 2$	42	4180	20, 24, 33 (7)	1
14	D_{38}	38	1540	22, 39 (19)	5
15	$S_3 \times 5$	30	2926	23, 28 (3), 35	7
16	$15 : 2$	30	2926	23, 30, 33 (5)	7
17	D_{30}	30	2926	23, 29 (3), 34 (5)	7
18	$A_4 \times 2$	24	7315	26, 31, 33 (4)	3, 4
19	D_{22}	22	1596	27, 39 (11)	6
20	$7 : 3$	21	4180	32, 38 (7)	3 (2), 13
21	D_{20}	20	8778	29, 28, 30, 37 (5)	4, 7
22	19	19	1540	40	10, 14
23	15	15	2926	36, 38	15, 16, 17
24	D_{14}	14	4180	32, 39 (7)	13
25	D_{12}	12	14630	35, 33, 34, 37 (3)	2, 4, 7
26	A_4	12	7315	37, 38 (4)	8 (2), 9, 18
27	11	11	1596	40	12, 19
28	10	10	8778	36, 39	6 (2), 15, 21
29	D_{10}	10	8778	36, 39 (5)	8 (2), 9, 17, 21
30	D_{10}	10	2926	36, 39 (5)	16, 21 (3)
31	2^3	8	1045	37 (7)	11, 18 (7)
32	7	7	4180	40	11 (2), 20, 24
33	6	6	14630	38, 39	5 (2), 13 (2), 16, 18 (2), 25
34	S_3	6	14630	38, 39 (3)	8 (2), 17, 25
35	S_3	6	2926	38, 39 (3)	9 (5), 15, 25 (5)
36	5	5	2926	40	12 (6), 23, 28 (3), 29 (3), 30
37	2^2	4	7315	39 (3)	21 (6), 25 (6), 26, 31
38	3	3	2926	40	10 (10), 20 (10), 23, 26 (10), 33 (5), 34 (5), 35

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Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
39	2	2	1463	40	14 (20), 19 (12), 24 (20), 28 (6), 29 (30), 30 (10), 33 (10), 34 (30), 35 (6), 37 (15)
40	1	1	1		22 (1540), 27 (1596), 32 (4180), 36 (2926), 38 (2926), 39 (1463)

Table 5: Subgroup lattice of J_1

Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
1	$S_5 \times 2$	240	1	2, 3, 4, 6 (5), 7 (6), 8 (10)	
2	S_5	120	1	5, 9 (5), 12 (6), 16 (10)	1
3	S_5	120	1	5, 10 (5), 13 (6), 17 (10)	1
4	$A_5 \times 2$	120	1	5, 11 (5), 14 (6), 18 (10)	1
5	A_5	60	1	19 (5), 24 (6), 34 (10)	2, 3, 4
6	$S_4 \times 2$	48	5	10, 9, 11, 15 (3), 20 (4)	1
7	$D_{10} \cdot 2 \times 2$	40	6	13, 14, 12, 27 (5)	1
8	$S_3 \times 2^2$	24	10	21, 20, 22, 16, 17, 23, 18, 28 (3)	1
9	S_4	24	5	19, 29 (3), 35 (4)	2, 6
10	S_4	24	5	19, 30 (3), 36 (4)	3, 6
11	$A_4 \times 2$	24	5	19, 31, 37 (4)	4, 6
12	$D_{10} \cdot 2$	20	6	24, 42 (5)	2, 7
13	$D_{10} \cdot 2$	20	6	24, 43 (5)	3, 7
14	D_{20}	20	6	25, 26, 24, 44 (5)	4, 7
15	$D_8 \times 2$	16	15	29, 32, 28, 31, 33, 27, 30	6
16	D_{12}	12	10	34, 35, 38, 45 (3)	2, 8
17	D_{12}	12	10	39, 36, 34, 46 (3)	3, 8
18	D_{12}	12	10	40, 37, 34, 44 (3)	4, 8
19	A_4	12	5	47, 51 (4)	5, 9, 10, 11
20	D_{12}	12	10	37, 36, 35, 48 (3)	6 (2), 8
21	D_{12}	12	10	40, 39, 35, 49 (3)	8
22	D_{12}	12	10	36, 40, 38, 49 (3)	8
23	2×6	12	10	37, 39, 38, 48	8
24	D_{10}	10	6	41, 52 (5)	5, 12, 13, 14
25	D_{10}	10	6	41, 53 (5)	14
26	10	10	6	41, 54	14
27	2×4	8	15	42, 43, 44	7 (2), 15
28	2^3	8	15	45, 44, 48 (2), 49 (2), 46	8 (2), 15
29	D_8	8	15	45, 42, 47	9, 15
30	D_8	8	15	46, 43, 47	10, 15
31	2^3	8	5	44 (3), 50 (3), 47	11, 15 (3)
32	D_8	8	15	46, 42, 50	15
33	D_8	8	15	45, 50, 43	15
34	S_3	6	10	51, 52 (3)	5, 16, 17, 18
35	S_3	6	10	51, 55 (3)	9 (2), 16, 20, 21
36	S_3	6	10	51, 56 (3)	10 (2), 17, 20, 22
37	6	6	10	51, 54	11 (2), 18, 20, 23
38	6	6	10	51, 55	16, 22, 23
39	6	6	10	51, 56	17, 21, 23
40	S_3	6	10	51, 53 (3)	18, 21, 22
41	5	5	6	57	24, 25, 26
42	4	4	15	52	12 (2), 27, 29, 32
43	4	4	15	52	13 (2), 27, 30, 33
44	2^2	4	15	52, 53, 54	14 (2), 18 (2), 27, 28, 31
45	2^2	4	15	55 (2), 52	16 (2), 28, 29, 33
46	2^2	4	15	52, 56 (2)	17 (2), 28, 30, 32
47	2^2	4	5	52 (3)	19, 29 (3), 30 (3), 31
48	2^2	4	10	55, 54, 56	20 (3), 23, 28 (3)
49	2^2	4	30	55, 53, 56	21, 22, 28
50	2^2	4	15	53 (2), 52	31, 32, 33
51	3	3	10	57	19 (2), 34, 35, 36, 37, 38, 39, 40
52	2	2	15	57	24 (2), 34 (2), 42, 43, 44, 45, 46, 47, 50
53	2	2	15	57	25 (2), 40 (2), 44, 49 (2), 50 (2)
54	2	2	1	57	26 (6), 37 (10), 44 (15), 48 (10)
55	2	2	10	57	35 (3), 38, 45 (3), 48, 49 (3)
56	2	2	10	57	36 (3), 39, 46 (3), 48, 49 (3)
57	1	1	1		41 (6), 51 (10), 52 (15), 53 (15), 54, 55 (10), 56 (10)

Table 6: Subgroup lattice of $2 \times S_5$

Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
1	$L_2(11)$	660	1	2 (11), 3 (11), 4 (12), 5 (55)	
2	A_5	60	11	6 (5), 8 (6), 9 (10)	1
3	A_5	60	11	6 (5), 8 (6), 10 (10)	1
4	$11:5$	55	12	7, 12 (11)	1
5	D_{12}	12	55	11, 9, 10, 13 (3)	1
6	A_4	12	55	13, 14 (4)	2, 3
7	11	11	12	16	4
8	D_{10}	10	66	12, 15 (5)	2, 3
9	S_3	6	55	14, 15 (3)	2 (2), 5
10	S_3	6	55	14, 15 (3)	3 (2), 5
11	6	6	55	14, 15	5
12	5	5	66	16	4 (2), 8
13	2^2	4	55	15 (3)	5 (3), 6
14	3	3	55	16	6 (4), 9, 10, 11
15	2	2	55	16	8 (6), 9 (3), 10 (3), 11, 13 (3)
16	1	1	1		7 (12), 12 (66), 14 (55), 15 (55)

Table 7: Subgroup lattice of $L_2(11)$

Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
1	$A_5 \times S_3$	360	1	2, 3 (3), 4 (5), 5 (6), 7 (10)	
2	$A_5 \times 3$	180	1	6, 8 (5), 9 (6), 15 (10)	1
3	$A_5 \times 2$	120	3	6, 12 (5), 14 (6), 19 (10)	1
4	$A_4 \times S_3$	72	5	8, 13, 12 (3), 16 (4)	1
5	$D_{10} \times S_3$	60	6	9, 10, 11, 14 (3), 20 (5)	1
6	A_5	60	1	21 (5), 25 (6), 30 (10)	2, 3 (3)
7	$S_3 \times S_3$	36	10	17, 16, 15, 19 (3), 20 (3)	1
8	$A_4 \times 3$	36	5	22 (2), 21, 23, 28 (4)	2, 4
9	$15:2$	30	6	18, 25, 31 (5)	2, 5
10	$S_3 \times 5$	30	6	18, 26 (3), 32	5
11	D_{30}	30	6	18, 27 (3), 33 (5)	5
12	$A_4 \times 2$	24	15	21, 29, 34 (4)	3, 4
13	$S_3 \times 2^2$	24	5	20 (3), 24 (3), 23, 29 (3)	4
14	D_{20}	20	18	27, 25, 26, 38 (5)	3, 5
15	$S_3 \times 3$	18	10	28, 30, 31 (3)	2, 7
16	$S_3 \times 3$	18	10	28, 32, 34 (3)	4 (2), 7
17	$3:S_3$	18	10	28, 35 (3), 36 (6), 33 (3)	7
18	15	15	6	37, 41	9, 10, 11
19	D_{12}	12	30	30, 35, 34, 38 (3)	3, 7
20	D_{12}	12	15	33, 31, 32, 38 (3)	5 (2), 7 (2), 13
21	A_4	12	5	39, 42 (4)	6, 8, 12 (3)
22	A_4	12	10	39, 43 (4)	8
23	2×6	12	5	31 (3), 39	8, 13
24	D_{12}	12	15	33 (2), 31, 40 (3)	13
25	D_{10}	10	6	37, 44 (5)	6, 9, 14 (3)
26	10	10	18	37, 45	10, 14
27	D_{10}	10	18	37, 46 (5)	11, 14
28	3^2	9	10	41, 43 (2), 42	8 (2), 15, 16, 17
29	2^3	8	15	38 (3), 40 (3), 39	12, 13
30	S_3	6	10	42, 44 (3)	6, 15, 19 (3)
31	6	6	15	41, 44	9 (2), 15 (2), 20, 23, 24
32	S_3	6	1	41, 45 (3)	10 (6), 16 (10), 20 (15)
33	S_3	6	15	41, 46 (3)	11 (2), 17 (2), 20, 24 (2)
34	6	6	30	42, 45	12 (2), 16, 19
35	S_3	6	30	42, 46 (3)	17, 19
36	S_3	6	60	43, 46 (3)	17
37	5	5	6	47	18, 25, 26 (3), 27 (3)
38	2^2	4	45	45, 46, 44	14 (2), 19 (2), 20, 29
39	2^2	4	5	44 (3)	21, 22 (2), 23, 29 (3)
40	2^2	4	45	46 (2), 44	24, 29
41	3	3	1	47	18 (6), 28 (10), 31 (15), 32, 33 (15)
42	3	3	10	47	21 (2), 28, 30, 34 (3), 35 (3)
43	3	3	20	47	22 (2), 28, 36 (3)
44	2	2	15	47	25 (2), 30 (2), 31, 38 (3), 39, 40 (3)
45	2	2	3	47	26 (6), 32, 34 (10), 38 (15)
46	2	2	45	47	27 (2), 33, 35 (2), 36 (4), 38, 40 (2)
47	1	1	1		37 (6), 41, 42 (10), 43 (20), 44 (15), 45 (3), 46 (45)

Table 8: Subgroup lattice of $A_5 \times S_3$

Nr.	Structure	Order	Length	Maximal Subgroups	Minimal Overgroups
1	O′N	460815505920	1	2(122760), 3(122760), **6**(2 624 832), 7(2857239), 11(17778376), 12(17778376), 15(30968784), 16(30968784), 21(42858585), **23**(58 183 776), 24(58 183 776), 43(182863296), 44(182863296)	
6	J_1	175560	2 624 832	**119**(266), 237(1045), 277(1463), 285(1540), 286(1596), **366**(2926), 397(4180)	**1**
23	M_{11}	7920	58 183 776	102(11), **119**(12), 252(55), **278**(66), 382(165)	**1**
30	$4 \cdot 2^4 : A_5$	3840	120004038	97(5), 133(6), 156(10), 208(16), 209(16), 210(16), 211(16)	8
119	$L_2(11)$	660	698205312	368(11), **370**(11), 375(12), **537**(55)	**6**, **23**, 24
166	$A_5 \times S_3$	360	1280043072	232, **282**(3), 335(5), **366**(6), 407(10)	48, 86
212	$2 \times S_5$	240	960032304	**278**(2), **277**, 393(5), 402(6), 478(10)	143
277	$2 \times A_5$	120	960032304	**370**, 482(5), 494(6), **537**(10)	**6**(4), 210, **212**, 213
278	S_5	120	1 920 064 608	**370**, 472(5), 491(6), **537**(10)	**23**(2), **212**
282	$2 \times A_5$	120	1 920 064 608	371, 485(5), 494(6), **537**(10)	104, 165(2), **166**(2), 209
366	$S_3 \times D_{10}$	60	7680258432	464, 465, 463, 494(3), **537**(5)	**6**, 165, **166**, 230
370	A_5	60	960032304	545(5), 550(6), 573(10)	**119**(8), **277**, **278(2)**, 279(2)
382	$2 \cdot S_4$	48	4800161520	477, 514(3), **537**(4)	**23**(2), 229, 296
407	$S_3 \times S_3$	36	6400215360	497(2), 502, **537**(6)	**166**(2), 253(2), 287(2), 330
478	$3 : 2^3$	24	3200107680	**537**(3), 539(3), 542, 568(3)	110(3), **212**(3), 335, 337, 386(3)
527	$Q_8 : 2$	16	3600121140	**567**, 562, **564**(2), 565(2), **568**	393(4), 394(4), 447(2), 449(2), 455
537	D_{12}	12	9600323040	573, 572, 571, **578**(3)	**119**(4), **277**, 278(2), 279(2), 282(2), **366**(4), 382(2), 383(2), 387(2), 406(4), 407(4), 418, **478**, 479, 481
564	D_8	8	3600121140	**578**(2), 575	330(8), 472(8), 504, 505(2), 514(2), 526, 527(2), 530(2), 531
567	2^3	8	42858585	**578**(7)	374(64), 482(112), 525(7), 527(84)
568	2^3	8	1200040380	**578**(3), 577(4)	**478**(8), 479(8), 485(4), 521, 522(6), 523(6), 527(3), 531(3)
573	S_3	6	213340512	579, **580**(3)	**370**(45), 371(10), 465(36), 480(10), 496(20), 497(20), 500, **537**(45)
578	2^2	4	300010095	**580**(3)	**494**(96), **537**(96), 545(16), 561(3), 562(12), 563(3), **564**(24), 565(24), **567**, **568**(12)
580	2	2	2857239	581	404(26880), 487(16128), 533(1920), 534(960), 535(1920), 536(3840), 549(2016), 550(10080), 551(1120), 571(10080), 572(1120), **573**(224), 575(315), 576, 577(560), **578**(315)

Table 9: Subgroups occuring in L_{IvSh} extracted from $\Lambda_{\mathrm{O'N}}$

Orbit	1	2	3	4	5	6	7
Size	1	1463	1463	1463	5852	12540	12540
Name	$\{p\}$	O^2_{1463}	O^1_{1463}	O^3_{1463}			

Orbit	8	9	10	11	12	13	14
Size	21945	21945	29260	29260	29260	29260	29260
Name	O^2_{21945}	O^1_{21945}					

Orbit	15	16	17	18	19	20	21
Size	58520	58520	87780	87780	87780	87780	87780

Orbit	22	23	24	25	26	27	28
Size	87780	87780	87780	87780	87780	87780	87780

Orbit	29	30	31	32	33	34	35
Size	175560	175560	175560	175560	175560	175560	175560

Table 10: Orbit sizes and names of J_1 on 2 624 832 points (as referred to in the text)

Orbits	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	0	1463	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	40	0	0	12	0	0	0	90	60	0	60	0	120	0	120	120
3	0	0	0	13	0	0	60	15	75	20	0	20	0	0	0	0	60
4	0	0	13	0	0	0	60	15	15	20	0	20	0	0	0	0	0
5	0	3	0	0	0	30	0	30	0	30	5	0	15	0	0	60	75
6	0	0	0	0	14	42	0	7	28	0	21	14	28	0	0	42	77
7	0	0	7	7	0	0	28	28	21	14	14	28	0	21	14	28	7
8	0	0	1	1	8	4	16	26	19	28	16	20	20	32	8	56	64
9	0	6	5	1	0	16	12	19	16	12	12	28	8	12	40	24	56
10	0	3	1	1	6	0	6	21	9	4	6	20	12	33	30	30	57
11	0	0	0	0	1	9	6	12	9	6	16	15	30	9	48	24	60
12	0	3	1	1	0	6	12	15	21	20	15	22	12	48	18	36	66
13	0	0	0	0	3	12	0	15	6	12	30	12	12	1	20	26	48
14	0	6	0	0	0	0	9	24	9	33	9	48	1	66	20	32	60
15	0	0	0	0	0	0	3	3	15	15	24	9	10	10	37	41	54
16	0	3	0	0	6	9	6	21	9	15	12	18	13	16	41	40	45
17	0	2	1	0	5	11	1	16	14	19	20	22	16	20	36	30	50
18	0	1	0	0	3	7	5	6	10	14	14	16	16	10	34	36	44
19	0	0	0	3	8	13	2	9	1	20	6	6	14	9	26	44	36
20	0	1	2	2	0	4	6	16	14	13	14	14	19	14	34	34	42
21	0	0	1	2	2	8	4	11	11	17	23	14	18	20	36	28	52
22	0	4	1	1	6	4	8	13	12	20	16	17	19	10	38	30	49
23	0	1	2	2	2	6	8	16	20	19	16	11	19	26	36	34	46
24	0	0	1	2	2	4	16	13	9	10	20	17	18	7	30	30	33
25	0	0	1	1	2	12	9	11	14	10	27	22	24	4	26	24	56
26	0	0	1	0	1	5	12	8	6	12	16	9	22	2	32	24	30
27	0	0	3	1	2	4	7	5	18	21	22	10	9	6	32	30	62
28	0	1	0	0	5	9	7	6	16	16	16	17	16	22	36	36	56
29	0	0	1	1	5	4	8	14	7	22	13	16	17	17	27	26	50
30	0	1	0	0	5	10	6	11	13	18	12	21	13	25	31	34	52
31	0	0	0	0	1	9	8	14	11	13	16	18	21	19	31	32	38
32	0	1	1	1	4	9	3	10	13	17	13	11	17	15	36	32	53
33	0	0	1	1	1	4	14	15	14	12	23	20	14	16	45	38	46
34	0	1	1	1	3	0	6	14	14	17	18	16	15	16	37	34	54
35	0	0	0	0	5	11	2	8	12	19	13	16	17	18	27	32	54

Table 11: The distance distribution map of $\mathcal{G}_{\mathrm{IvSh}}$ (1/2)

Orbits	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	60	0	60	0	240	60	0	0	0	0	60	0	120	0	120	0	120	0
3	0	0	120	60	60	120	60	60	60	180	0	120	0	0	120	120	120	0
4	0	180	120	120	60	120	120	60	0	60	0	120	0	0	120	120	120	0
5	45	120	0	30	90	30	30	30	15	30	75	150	150	30	120	30	90	150
6	49	91	28	56	28	42	28	84	35	28	63	56	140	126	126	56	0	154
7	35	14	42	28	56	56	112	63	84	49	49	112	84	112	42	196	84	28
8	24	36	64	44	52	64	52	44	32	20	24	112	88	112	80	120	112	64
9	40	4	56	44	48	80	36	56	24	72	64	56	104	88	104	112	112	96
10	42	60	39	51	60	57	30	30	36	63	48	132	108	78	102	72	102	114
11	42	18	42	69	48	48	60	81	48	66	48	78	72	96	78	138	108	78
12	48	18	42	42	51	33	51	66	27	30	51	96	126	108	66	120	96	96
13	48	42	57	54	57	57	54	72	66	27	48	102	78	126	102	84	90	102
14	30	27	42	60	30	78	21	12	6	18	66	102	150	114	90	96	96	108
15	51	39	51	54	57	54	45	39	48	48	54	81	93	93	108	135	111	81
16	54	66	51	42	45	51	45	36	36	45	54	78	102	96	96	114	102	96
17	44	36	42	52	49	46	33	56	30	62	56	100	104	76	106	92	108	108
18	52	60	46	37	56	35	49	58	50	50	46	94	112	112	94	90	100	106
19	60	86	51	60	41	39	56	46	52	49	48	104	98	78	118	80	88	112
20	46	51	60	44	46	54	47	57	58	40	35	92	84	112	108	100	118	82
21	37	60	44	54	49	60	54	50	43	48	49	90	96	90	106	96	92	98
22	56	41	46	49	32	48	50	45	34	64	50	116	88	78	88	102	112	116
23	35	39	54	60	48	58	48	33	56	48	44	92	88	104	104	102	92	94
24	49	56	47	54	50	48	58	54	66	59	54	90	96	94	82	108	96	90
25	58	46	57	50	45	33	54	64	43	58	34	102	88	110	84	102	104	88
26	50	52	58	43	34	56	66	43	70	53	56	114	78	118	104	94	82	112
27	50	49	40	48	64	48	59	58	53	52	50	94	100	86	108	92	98	82
28	46	48	35	49	50	44	54	34	56	50	50	94	116	88	98	92	82	118
29	47	52	46	45	58	46	45	51	57	47	47	107	97	95	103	83	101	108
30	56	49	42	48	44	44	48	44	39	50	58	97	98	98	98	97	95	106
31	56	39	56	45	39	52	47	55	59	43	44	95	98	111	92	107	93	101
32	47	59	54	53	44	52	41	42	52	54	49	103	98	92	96	85	99	107
33	45	40	50	48	51	51	54	51	47	46	46	83	97	107	85	119	107	72
34	50	44	59	46	56	46	48	52	41	49	41	101	95	93	99	107	103	86
35	53	56	41	49	58	47	45	44	56	41	59	108	106	101	107	72	86	100

Table 12: The distance distribution map of $\mathcal{G}_{\mathrm{IvSh}}$ (2/2)

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