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A geometric proof of Wilbrink's characterization of even order classical unitals

Alice M. W. Hui

Abstract

Using geometric methods and without invoking deep results from group theory, we prove that a classical unital of even order $n \geq 4$ is characterized by two conditions (I) and (II): (I) is the absence of O'Nan configurations of four distinct lines intersecting in exactly six distinct points; (II) is a notion of parallelism. This was previously proven by Wilbrink (1983), where the proof depends on the classification of finite groups with a split BN-pair of rank 1.

Keywords: unital, classical unital, Hermitian curve, spread

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1 Introduction

A unital of order $n > 2$ is a design with parameters 2–$(n^3+1, n+1, 1)$ (see [12, 4]). If π is a projective plane of order m, i.e. a 2–$(m^2+m+1, m+1, 1)$ design, and if a unital U is an induced substructure of π, then we call U an *embedded unital*. Some embedded unitals U of order n are the incidence structure formed from the absolute points and non-absolute lines of a unitary polarity in a projective plane π of order n^2. Any unital which is isomorphic to such a unital U as a design is called a *polar unital*. Further if the ambient plane is PG$(2, n^2)$, then the unital is called *classical*. The set of absolute points of a unitary polarity in PG$(2, n^2)$ is called the *Hermitian curve* (see [15, 17]).

In 1972 [21], O'Nan showed that the classical unital does not contain a configuration of four lines meeting in six points (an *O'Nan configuration*). Piper

(1981) [23] conjectured that this property characterizes the classical unital. Wilbrink (1983) [25] characterized the classical unital by three conditions (I), (II) and (III). His proof depends on a result in the classification of finite groups with a split BN-pair of rank 1. Wilbrink [25] further proved that when the order of unital is even, (III) is a necessary condition of (I) and (II).

Let $\mathcal{U}$ be a unital of even order $n \geq 4$ satisfying Wilbrink's conditions (I) and (II). In this article, we give an alternative proof that $\mathcal{U}$ is classical without invoking deep results from group theory, as follows. We construct from $\mathcal{U}$ a hyperbolic Buekenhout unital $\mathcal{U}'$ [9] in PG$(2, n^2)$, via the Bruck-Bose construction of projective plane [6, 7]. Then we prove that $\mathcal{U}$ is isomorphic to $\mathcal{U}'$, and hence is classical by a result of Barwick [3]. To construct $\mathcal{U}'$, we shall consider some inversive planes and a generalized quadrangle derived from $\mathcal{U}$ (Wilbrink [25]), and the special spreads of $\mathcal{U}$ (Hui and Wong [18]). We also need a theorem of Cameron and Knarr [10] on how to build a regular spread of PG$(3, q)$ from a tube in PG$(3, q)$, and a theorem of Hui [19] on when two inversive planes are identical.

In Section 2, we follow Wilbrink's [25] construction of the inversive planes $\mathcal{I}(x)$ at each point x of $\mathcal{U}$. Then following the work in [18], we construct a special spread $\mathcal{S}_L$ for each line L of $\mathcal{U}$ using these inversive planes. As a consequence, $\mathcal{U}$ can be embedded in a projective plane π as a polar unital [18]. This enables us to define self-polar triangles with respect to $\mathcal{U}$ intrinsically in terms of Wilbrink's x-parallelism (Theorem 2.2).

In Section 3, by studying the inversive planes $\mathcal{I}(x)$ for various x's, we prove that the set of points of $\mathcal{U}$ is partitioned into a self-polar triangle, and $n - 2$ subsets of $(n+1)^2$ points triply ruled by lines through the vertices of the triangle (Theorem 3.6). This describes how unital lines in π through distinct non-unital points intersect.

In Section 4, we fix one line L of $\mathcal{U}$ and consider the generalized quadrangle $GQ(L)$ as in Wilbrink [25]. Through $GQ(L)$, we associate $\mathcal{U}$ with $Q(4, n)$ formed by the set of points and lines of a parabolic quadric $\mathcal{P}$ in PG$(4, n)$ [22]. Wilbrink's construction gives naturally a 3-dimensional subspace Σ of PG$(4, n)$. We find a spread $\mathscr{S}$ in Σ by studying the special spread $\mathcal{S}_L$. We then prove that $\mathscr{S}$ is regular (Theorem 4.4) using a result on tubes in PG$(3, n)$ (Cameron and Knarr [10]) and properties of self-polar triangles with respect to $\mathcal{U}$.

In Section 5, we prove that the partition of $\mathcal{U}$ into a self-polar triangle and triply ruled sets corresponds to a pencil of quadrics in Σ of two lines and $n - 1$ hyperbolic quadrics (Theorem 5.5). In particular, this gives a correspondence between the structure of $\mathcal{U}$ and that of Σ. The regularity of $\mathscr{S}$ is essential for proving Theorem 5.5, because we have to describe the reguli of $\mathscr{S}$ in terms by geometry of $\mathcal{U}$ by applying a result of Hui [18] to the Miquelian inversive plane formed by the lines and reguli of $\mathscr{S}$ (Bruck [5]).

In Section 6, by considering the spread $\mathscr{S}$ of Σ in PG$(4, n)$, we construct a projective plane $\overline{\pi(\mathscr{S})}$ by the Bruck-Bose construction [6]. Since $\mathscr{S}$ is regular, $\overline{\pi(\mathscr{S})}$ is PG$(2, n^2)$ [7]. By Buekenhout [9], $\mathcal{P}$ defines a hyperbolic Buekenhout unital $\mathcal{U}'$ in $\overline{\pi(\mathscr{S})}$ [9] (also known as a nonsingular Buekenhout unital). By Barwick [3], since $\overline{\pi(\mathscr{S})} \cong$ PG$(2, n^2)$, $\mathcal{U}'$ is the classical unital. With the help of Theorem 5.5, we write down an isomorphism between $\mathcal{U}$ and the classical unital $\mathcal{U}'$ (Theorem 6.2).

2 Self-polar triangles and Parallelism

Let $\mathcal{U}$ be a unital of even order $n \geq 4$, satisfying Wilbrink's first two conditions (I) and (II) [25]:

- (I) $\mathcal{U}$ contains no O'Nan configurations.
- (II) Let x be a point, L be a line through x, and M be a line missing x, such that L and M meets. For any point $y' \in L \setminus \{x\}$, there is a line M' through y' but not x meeting all lines from x which meet M.

Following Wilbrink [25], we introduce *x-parallelism* in $\mathcal{U}$ [25]:

> Let x be a point, and M, M' be two lines missing x. M, M' are said to be x-parallel if M, M' intersect the same lines through x. We write $M \|_x M'$.

The relation $\|_x$ defines an equivalence relation on the set of all lines missing x [25]. We denote the equivalence class of a line M under $\|_x$ by $\overline{M}^x$, or simply $\overline{M}$ if there is no confusion.

Further following Wilbrink [25], we introduce an inversive plane $\mathcal{I}(x)$ of order n for every point x in $\mathcal{U}$ ([25, Lemmas 1, 2 and Corollary 3]; see also [18]). The points of $\mathcal{I}(x)$ are the lines of $\mathcal{U}$ through x together with a symbol ∞_x. The circle set of $\mathcal{I}(x)$ is $\mathcal{C}^x \cup \mathcal{C}_x$. Here $\mathcal{C}^x$, $\mathcal{C}_x$ are given by: $\mathcal{C}^x$ is the set of $\|_x$-equivalence classes on the set of lines of $\mathcal{U}$ missing x; $\mathcal{C}_x$ consists of blocks of the form $C_x(L, L') \cup \{\infty_x\}$, where for any lines L, L' on x, $C_x(L, L')$ = $\{L, L'\} \cup \{L'' \mid L''$ is a line through x such that no line of through x meets L, L' and $L''\}$. The incidence in $\mathcal{I}(x)$ is defined as follows: Whenever L is a line of $\mathcal{U}$ through x and M is a line of $\mathcal{U}$ missing x, L is incident with $\overline{M}$ in $\mathcal{I}(x)$ if and only if L meets M in $\mathcal{U}$; whenever L, L', L'' are lines of $\mathcal{U}$ through x, L is incident with $C_x(L', L'')$ in $\mathcal{I}(x)$ if and only if $L \in C_x(L', L'')$; the point ∞_x is incident with all circles in $\mathcal{C}^x$ but none in $\mathcal{C}_x$.

According to Dembowski [11], $\mathcal{I}(x)$ is egglike. By Thas [24], every flock in $\mathcal{I}(x)$ is uniquely determined by its carriers. We denote the unique flock in $\mathcal{I}(x)$ with carriers p_1 and p_2 by $\mathcal{F}(p_1, p_2)$.

With the help of flocks of the form $\mathcal{F}(L, \infty_x)$, $\mathcal{U}$ can be shown to satisfy condition (P) [18, Theorem 1.6], formulated in terms of *special spreads* in $\mathcal{U}$.

Let $\mathcal{S}$ be a spread of $\mathcal{U}$, with $L \in \mathcal{S}$. Then $\mathcal{S}$ is *special with respect to* L if the following condition is satisfied:

> for any point x on L, $\mathcal{S} \setminus \{L\}$ can be partitioned into $n-1$ subsets $\mathcal{L}_1^x, \ldots, \mathcal{L}_{n-1}^x$, each of cardinality n, and the set of lines on x, except L, can be partitioned into $n-1$ subsets $\mathcal{K}_x^1, \ldots, \mathcal{K}_x^{n-1}$, each of cardinality $n+1$, such that whenever $L' \in \mathcal{L}_i^x$ and $K \in \mathcal{K}_x^j$, L' and K intersect if and only if $i = j$.

Now, [18, Theorem 1.6] says that $\mathcal{U}$ satisfies condition (P), which is a strengthened version of condition (p):

(p) Let $L_1, L_2, \ldots, L_{n^4-n^3+n^2}$ be the lines of $\mathcal{U}$. There exists a family of lines $\mathscr{F} = \{\mathcal{S}_{L_1}, \mathcal{S}_{L_2}, \ldots, \mathcal{S}_{L_{n^4-n^3+n^2}}\}$ such that:
- (i) For $i = 1, 2, \ldots, n^4 - n^3 + n^2$, $\mathcal{S}_{L_i}$ is a spread containing L_i.
- (ii) For $i \neq j$, $L_i \in \mathcal{S}_{L_j} \setminus \{L_j\}$ if and only if $L_j \in \mathcal{S}_{L_i} \setminus \{L_i\}$.
- (iii) For any two lines L_i and L_j missing each other, there exists a line L_k such that $L_k \in \mathcal{S}_{L_i} \setminus \{L_i\}$ and $L_k \in \mathcal{S}_{L_j} \setminus \{L_j\}$.

(P) (p) holds such that for $i = 1, 2, \ldots, n^4 - n^3 + n^2$, $\mathcal{S}_{L_i}$ is a special spread with respect to L_i.

In the above statement, the set $\mathcal{S}_L$ is given explicitly by this construction:

> Pick some point x on L. $\mathcal{S}_L$ is given by $\overline{L_1} \cup \overline{L_2} \cup \ldots \cup \overline{L_{n-1}} \cup \{L\}$, where $\overline{L_1}$, $\overline{L_2}$, $\ldots$, $\overline{L_{n-1}}$ are the circles of $\mathcal{F}(L, \infty_x)$ in $\mathcal{I}(x)$.

The set $\mathcal{S}_L$ is shown to be independent of x [18, Lemma 5.4].

For each line L of $\mathcal{U}$, denote by $\mathcal{S}_L^*$ the set $\mathcal{S}_L \setminus \{L\}$.

By [18, Theorem 1.1], since $\mathcal{U}$ satisfies (p), $\mathcal{U}$ can be embedded in a projective plane π as a polar unital, so that in π, for each unital line J, the unital lines of π through the pole of J are exactly the lines in $\mathcal{S}_J^*$. Since two distinct points in π determine a unique line, $\mathcal{S}_J^* \cap \mathcal{S}_{J'}^*$ contains at most one (unital) line for any distinct unital lines J, J'. Thus there are at most three lines in $\mathcal{S}_J \cap \mathcal{S}_{J'}$. Whenever L, M, N are three distinct lines of $\mathcal{U}$ satisfying $\{L, M, N\} = \mathcal{S}_L \cap \mathcal{S}_M = \mathcal{S}_L \cap \mathcal{S}_N = \mathcal{S}_M \cap \mathcal{S}_N$, we say that L, M, N form a *self-polar triangle with respect to* $\mathcal{U}$.

Lemma 2.1. *Let L, M be disjoint lines of $\mathcal{U}$. If $M \in \mathcal{S}_L^*$ (or equivalently $L \in \mathcal{S}_M^*$), then there exists a unique line N of $\mathcal{U}$ such that L, M, N form a self-polar triangle with respect to $\mathcal{U}$.*

Proof. In π, L and M meet in a unique non-unital point a. Let N be the polar line of a. By construction of π, $L, M \in \mathcal{S}_N^*$ and $N \neq L, M$. Since $\mathcal{U}$ satisfies (p), $N \in \mathcal{S}_L \cap \mathcal{S}_M$ by (p)(ii). By (p)(i), $L \in \mathcal{S}_L$, $M \in \mathcal{S}_M$ and $N \in \mathcal{S}_N$. The result follows from the fact the there are at most three lines in $\mathcal{S}_L \cap \mathcal{S}_M$, $\mathcal{S}_L \cap \mathcal{S}_N$ and $\mathcal{S}_M \cap \mathcal{S}_N$. □

Theorem 2.2. *Let L, M, N be disjoint lines of $\mathcal{U}$. Then the following are equivalent.*

(a) *L, M, N form a self-polar triangle with respect to $\mathcal{U}$.*
(b) *$L \|_z M$ for any point $z \in N$.*
(c) *Any line meeting two of L, M, N meet all of L, M, N.*
(d) *$M \in \mathcal{S}_L^*$, and there exist distinct points $z_1, z_2 \in N$ such that $L \|_{z_1} M$ and $L \|_{z_2} M$.*

Remark 2.3. Since lines in a self-polar triangle play the same role, in Theorem 2.2, statement (a) is also equivalent to (b)$'$: $M \|_z N$ for any point $z \in L$, or other statement obtained by permuting L, M, N in (b) and (d).

Proof of Theorem 2.2. Let x be a point on L, and $K_1, K_2, \ldots, K_{n+1}$ be the lines through x meeting M.

(a) $\Rightarrow$ (b): Since $M \in \mathcal{S}_L^*$, there is a point $y_i \in K_i$ such that $L \|_{y_i} M$ for each $i = 1, 2, \ldots, n+1$, by Lemmas 5.3 and 5.4 of [18]. Hence, for each i, $\overline{L} = \overline{M}$ in $\mathcal{I}(y_i)$. Then there is a unique point N_i such that $\overline{L} \in \mathcal{F}(N_i, \infty_{y_i})$. This N_i is a line of $\mathcal{U}$ through y_i such that $L \in \mathcal{S}_{N_i}^*$ and $M \in \mathcal{S}_{N_i}^*$. By condition (p), $N_i \in \mathcal{S}_L^* \cap \mathcal{S}_M^*$. Since there is at least one line in $\mathcal{S}_L^* \cap \mathcal{S}_M^*$, we have $N_1 = N_2 = \cdots = N_{n+1} = N$.

(b) $\Rightarrow$ (c): For each point $z \in N$, since $L \|_z M$, there are $n+1$ lines through z meeting L and M. Thus there are $(n+1)^2$ lines meeting L, M, N. They are the lines meeting at least two of L, M, N.

(c) $\Rightarrow$ (b) follows from the definition of z-parallelism.

(c) $\Rightarrow$ (d): It suffices to show $M \in \mathcal{S}_L^*$. Let $z' \in N$. By (c), K_i meets N for each $i = 1, 2, \ldots, n+1$. Let w_i be the intersection points of K_i and N. Then w_i is a point on K_i such that $L \|_{w_i} M$. By Lemmas 5.3 and 5.4 of [18], $M \in \mathcal{S}_L^*$.

(d) $\Rightarrow$ (a): Let N' be a line such that L, M, N' form a self-polar triangle. Suppose $z_1 \notin N'$. Let $x_1, x_2 \in L$ be distinct points. For $i = 1, 2$, let J_i be the line passing through x_i and z_1. Then J_i meets M by (d). Since (a) implies (c), J_1 meets N' at a point, say w. Since (a) implies (b), we have $L \|_w M$. Thus, the four lines J_1, J_2, M, $w.x_2$ form an O'Nan configuration, which is a contradiction. ($w.x_2$ denotes the line through w and x_2.) Hence $z_1 \in N'$. Similarly, $z_2 \in N'$. So $N = N'$. $\square$

With Theorem 2.2, we characterize $\mathcal{S}_M^*$ by z-parallelism.

Lemma 2.4. *Let L, M be distinct lines of $\mathcal{U}$. Suppose $M \in \mathcal{S}_L^*$. Then*

$$\mathcal{S}_M^* = \{J \mid J \text{ is a line of } \mathcal{U} \text{ such that there is a point } y \in M \text{ such that } L\|_y J\}.$$

Proof. Let J be a line. Suppose there is a point $y \in M$ such that $L\|_y J$. Since $L \in \mathcal{S}_M^*$ by condition (p) and $L\|_y J$, we have $\overline{J} = \overline{L} \in \mathcal{F}(M, \infty_y)$ in $\mathcal{I}(y)$. Thus, $J \in \mathcal{S}_M^*$ by the construction of $\mathcal{S}_M^*$.

To prove the reverse inclusion, it suffices to show that there are $|\mathcal{S}_M^*| = n^2 - n$ J's such that $L\|_z J$ for some $z \in M$. Let N be the line such that L, M, N form a self-polar triangle. For each point z on M, there are exactly n lines z-parallel to L. By Theorem 2.2, L and N are two of these n lines. Apart from L and N, no line is z-parallel to L for distinct z's on M by Theorem 2.2. Hence, there are $2 + (n+1)(n-2) = n^2 - n$ lines z-parallel to L for some z on M, as desired. $\square$

3 Partition of $\mathcal{U}$ into a self-polar triangle and triply ruled sets

In this section, we prove that $\mathcal{U}$ can be partitioned into a self-polar triangle and $n - 2$ triply ruled sets of $(n+1)^2$ points (Theorem 3.6). This result will be used at the end of Section 5, where we relate the partition to a pencil of quadrics in a projective space.

In Dover [14, Theorem 3.2] (see also Section 4 of Baker et al. [2]), it is proved that any classical unital admits such a partition by considering coordinates and its automorphism group. The argument in this section will yield a synthetic proof for Dover's result once we prove that $\mathcal{U}$ is classical in Section 6.

We describe how lines of $\mathcal{S}_L$ correspond to circles tangent to $\overline{L}$ in $\mathcal{I}(y)$, whenever y is not a point on L.

Lemma 3.1. *Let L be a line of $\mathcal{U}$, and y be a point not on L. Let $M \in \mathcal{S}_L$ be the line of $\mathcal{U}$ through y. Let N be the line of $\mathcal{U}$ such that L, M, N form a self-polar triangle. Then the following statements hold:*

(a) *For every $J \in \mathcal{S}_L \setminus \{L, M, N\}$, $\overline{J}$ is tangent to $\overline{L}$ in $\mathcal{I}(y)$.*
(b) *Every circle of type $\mathcal{C}^y$ tangent to $\overline{L}$ in $\mathcal{I}(y)$ is $\overline{J'}$ for a unique line J' in $\mathcal{S}_L \setminus \{L, M, N\}$.*

Proof. (a) Let $J \in \mathcal{S}_L \setminus \{L, M, N\}$. It suffices to show that there is a unique line of $\mathcal{U}$ through y meeting both J and L. By Lemma 2.4, since $L \in \mathcal{S}^*_M$ and $J \in \mathcal{S}^*_L$, we have $M\|_x J$ for some point $x \in L$. Let K be the line of $\mathcal{U}$ through x and y. Then K meets both J and L.

Suppose there is a line $K' \neq K$ through y meeting both J and L. Let x' be the intersection of K' and L. By Theorem 2.2, K meets N, and K' meets N. By Theorem 2.2, $M\|_x N$. Since $M\|_x N$ and $M\|_x J$, we have $J\|_x N$ and so the line $x.(J \cap K')$ meets N. ($J \cap K'$ denotes the intersection point of J and K'.) Then the four lines K, K', N, $x.(J \cap K')$ form an O'Nan configuration. This contradicts (I).

(b) $|\mathcal{S}_L \setminus \{L, M, N\}| = (n^2 - n + 1) - 3 = (n+1)(n-2)$. This is also the number of circles of type $\mathcal{C}^y$ tangent to $\overline{L}$ in $\mathcal{I}(y)$. Since we have (a), to show (b), it suffices to show $\overline{J_1} \neq \overline{J_2}$ for any distinct $J_1, J_2 \in \mathcal{S}_L \setminus \{L, M, N\}$. Suppose not. Then $J_1\|_y J_2$. By (a), there is a line K'' through y meeting J_1, J_2, L. Let x'' be the intersection of K'' and L. Since $J_1, J_2 \in \mathcal{S}^*_L$ and $\mathcal{S}_L$ is a special spread, we have $J_1\|_{x''} J_2$ by the definition of special spread. Let K''' be the line through y and a point of J_2 not on K''. Since $J_1\|_y J_2$ and $J_1\|_{x''} J_2$, the four lines K'', K''', J_1, $x''.(J_2 \cap K''')$ form an O'Nan configuration. This is a contradiction. □

Using Lemma 3.1 and the characterization of flocks in terms of bundles in even order inversive planes by Dembowski and Hughes [13] (see also (6.2.11), (6.2.12) with footnote on p.267, and (6.2.13), of [12]), we deduce Lemma 3.2. Lemma 3.2 describes when lines of $\mathcal{S}^*_L$ meet two intersecting lines which do not belong to $\mathcal{S}^*_L$ and which miss L.

Lemma 3.2. *Let L be a line of $\mathcal{U}$. Let M, N be distinct lines not in $\mathcal{S}_L$. Suppose M and N intersect at a point y of $\mathcal{U}$. If $\overline{L}$ belongs to $\mathcal{F}(M, N)$ in $\mathcal{I}(y)$, then the following statements hold:*

(1) *Let L_1 be the line in $\mathcal{S}^*_L$ through y. In $\mathcal{I}(y)$, L_1 is a point on the circle C determined by M, N and ∞_y.*
(2) *In $\mathcal{U}$, every line in $\mathcal{S}^*_L$ meeting M meets N as well.*
(3) *In $\mathcal{U}$, there is a unique point x on L such that any line from x meeting L_1 misses all other lines in $\mathcal{S}^*_L$ that meet both M and N. Furthermore, in $\mathcal{I}(y)$, the unital line $x.y$ through x and y is a point on C.*

Proof. By Dembowski and Hughes [13], in $\mathcal{I}(y)$, the flock $\mathcal{F}(M, N)$ is the set of circles tangent to every circle in the bundle $\mathcal{B}(M, N)$, i.e. the set of circles in $\mathcal{I}(y)$ through the points M and N. Suppose $\overline{L}$ is in $\mathcal{F}(M, N)$ in $\mathcal{I}(y)$. Then $\overline{L}$ is tangent to every circle in $\mathcal{B}(M, N)$. By Lemma 3.1, $\mathcal{B}(M, N) =$

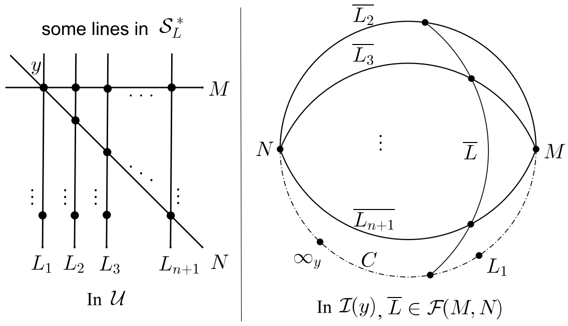

Figure 1: illustration of Lemma 3.2

$\{\overline{L_2}, \overline{L_3}, \ldots, \overline{L_{n+1}}, C\}$ for some distinct lines $L_2, L_3, \ldots, L_{n+1} \in \mathcal{S}_L^*$. Since $\mathcal{S}_L^*$ is a spread, L_1 does not meet $L_2, L_3, \ldots, L_{n+1}$. Hence in $\mathcal{I}(y)$, L_1 is not on $\overline{L_2}, \overline{L_3}, \ldots, \overline{L_{n+1}}$, but on the remaining circle C of $\mathcal{B}(M,N)$. This proves (1). The lines in $\mathcal{S}_L^*$ meeting M are $L_1, L_2, \ldots, L_{n+1}$, and they meet N. This proves (2).

Let K be the unital line on y such that K is the intersection point of C and $\overline{L}$ in $\mathcal{I}(y)$. Since K is on $\overline{L}$ in $\mathcal{I}(y)$, K meets L at a point, say x, in $\mathcal{U}$. Since K is not on $\overline{L_2}, \overline{L_3}, \ldots, \overline{L_{n+1}}$ in $\mathcal{I}(y)$, K does not meet $L_2, L_3, \ldots, L_{n+1}$ in $\mathcal{U}$. Since $\mathcal{S}_L^*$ is a special spread and K is a line from x meeting $L_1 \in \mathcal{S}_L^*$ but not $L_2, L_3, \ldots, L_{n+1} \in \mathcal{S}_L^*$, x is a point on L satisfying the property in (3). Uniqueness of x follows from that fact that in $\mathcal{I}(y)$, K is the unique point on $\overline{L}$ not on $\overline{L_2}, \overline{L_3}, \ldots, \overline{L_{n+1}}$. This proves (3). □

We introduce the notion of *triply ruled set* for a general unital: in a unital of order m, a set of $(m+1)^2$ points is *triply ruled* if there are three partitions of the $(m+1)^2$ points by lines. The following lemma suggests a method to find a triply ruled set.

Lemma 3.3. *Let L and M_1 be disjoint lines of $\mathcal{U}$ with $M_1 \notin \mathcal{S}_L$. Let $L_1, L_2, \ldots, L_{n+1}$ be the lines of $\mathcal{S}_L^*$ meeting M_1. Then there are lines $M_2, M_3, \ldots, M_{n+1}$ and $N_1, N_2, \ldots, N_{n+1}$ such that $\{M_1, M_2, \ldots, M_{n+1}\}$ and $\{N_1, N_2, \ldots, N_{n+1}\}$ are partitions of the set of points covered by the disjoint lines $L_1, L_2, \ldots, L_{n+1}$.*

Proof. Let y be a point on M_1. Since M_1 and L are disjoint, the point M_1 is not incident with $\overline{L}$ in $\mathcal{I}(y)$. Since $M_1 \notin \mathcal{S}_L^*$, condition (p) implies $L \notin \mathcal{S}_{M_1}^*$ and hence $\overline{L} \notin \mathcal{F}(M_1, \infty_y)$. Thus, there is a unital line N_1 through y such that $\overline{L} \in \mathcal{F}(M_1, N_1)$. By Lemma 3.2, $L_1, L_2, \ldots, L_{n+1}$ meet N_1. For $i = 2, 3, \ldots, n+1$, let y_i be the intersection point of N_1 and L_i. By a similar argument, for each $i = 2, 3, \ldots, n+1$, there is a unital line M_i through y_i such that $\overline{L} \in \mathcal{F}(M_i, N_1)$, and there is a unital line N_i through y_i such that $\overline{L} \in \mathcal{F}(M_i, N_i)$. By Lemma 3.2, $L_1, L_2, \ldots, L_{n+1}$ meet M_i, N_i, for all $i = 2, 3, \ldots, n+1$.

We are going to show that $M_1, M_2, \ldots, M_{n+1}$ are mutually disjoint. Suppose M_{i_1} and M_{i_2} intersect at a point z for some distinct $i_1, i_2 \in \{1, 2, \ldots, n+1\}$. Then z is on L_{i_3} for some $i_3 \in \{1, 2, \ldots, n+1\} \setminus \{i_1, i_2\}$. Let $i_4 \in \{1, 2, \ldots, n+1\} \setminus \{i_1, i_2, i_3\}$. Then the four lines $M_{i_1}, M_{i_2}, N_1, L_{i_4}$ form an O'Nan configuration. A contradiction. Hence $\{M_1, M_2, \ldots, M_{n+1}\}$ is a partition of the set of points covered by $L_1, L_2, \ldots, L_{n+1}$. By a similar argument, the same conclusion can be drawn for N_i's. □

Note that there cannot be a forth partition of the set of points covered by L_1, L_2, …, L_{n+1}; otherwise, there would be an O'Nan configuration constituted by lines of different partitions. This suggests the following notion of parallelism:

> Let M, N be lines missing L and not in $\mathcal{S}_L$. We say that M and N are *L-parallel*, denoted by $M \|_L N$, if they are identical or they are non-intersecting and meet the same lines in $\mathcal{S}^*_L$. We say that M and N are *L-non-parallel* if they intersect and meet the same lines in $\mathcal{S}^*_L$.

Lemma 3.4. *Let L be a line of $\mathcal{U}$. Then $\|_L$ defines an equivalence relation in the set of lines missing L and not in $\mathcal{S}_L$. Each class has $n+1$ lines. There are $n(n-1)(n-2)$ equivalence classes under the equivalence relation $\|_L$.*

Proof. It is clear that $\|_L$ is reflexive and symmetric. It is transitive because of Lemma 3.3 and the fact that two points determine a line. Since non-equal L-parallel line are non-intersecting, there are at most $n+1$ lines in an equivalence class. By Lemma 3.3, this upper bound is achieved. The number of lines missing L is $n(n-1)(n^2-n-1)$. Among these lines, $n(n-1)$ are in $\mathcal{S}^*_L$. The result follows from simple counting. □

By Lemma 3.2, for any distinct lines M' and N' through y, N' is L-non-parallel to M' if $\overline{L} \in \mathcal{F}(M', N')$ in $\mathcal{I}(y)$. Furthermore, the set of lines which are L-non-parallel to M' is an L-parallel class. In the context of Lemma 3.3, it is natural to ask whether $M_1, M_2, \ldots, M_{n+1}$ are in $\mathcal{S}^*_M$ for some line M. The answer is yes:

Lemma 3.5. *Refer to the set-up in Lemma 3.3. Let M be the line in $\mathcal{S}^*_L$ such that $M_1 \in \mathcal{S}^*_M$. Then for each $i = 2, 3, \ldots, n+1$, M_i belongs to $\mathcal{S}^*_M$.*

Proof. By Lemma 2.4, since $M \in \mathcal{S}^*_L$ and $M_1 \in \mathcal{S}^*_M$, there is a point $z_1 \in M$ such that $L \|_{z_1} M_1$. Let $z_2, z_3, \ldots, z_{n+1}$ be the points on M other than z_1. Let $M_{z_1} = M_1$. Let N be a line such that L, M, N form a self-polar triangle. For

$i = 2, 3, \ldots, n+1$, we claim that *there is a line* $M_{z_i} \in \mathcal{S}_M^*$ *distinct from* L *and* N*, such that* M_{z_i} *is* z_i*-parallel to* L*, and* M_{z_i} *meets* $L_1, L_2, \ldots, L_{n+1}$. If the claim is true, then $\{M_{z_i} | z_i \in M\}$ is the L-parallel class containing M_{z_1}. Indeed, if $M_{z_i} = M_{z_j}$ for some $i \neq j$, then M_{z_i}, L and N would be three lines both z_i-parallel and z_j-parallel, giving an O'Nan configuration. Hence $M_{z_i} \neq M_{z_j}$ for distinct i, j. Furthermore, for $i = 2, 3, \ldots, n+1$, since L is z_i-parallel to M_{z_i}, M_{z_i} is in $\mathcal{S}_M^*$ by Lemma 2.4. Thus M_{z_i}'s are non-intersecting. Hence $\{M_{z_i} | z_i \in M\}$ is the L-parallel class containing M_{z_1}. By uniqueness of such a class, $\{M_1, M_2, \ldots, M_{n+1}\} = \{M_{z_i} | z_i \in M\}$. The result follows.

We now prove the claim. Let $k \in \{2, 3, \ldots, n+1\}$. For $i = 1, \ldots, n+1$, let $K_i = (M_1 \cap L_i).z_1$. Since $L \|_{z_1} M_1$, we can label the points on L as $x_1, x_2, \ldots, x_{n+1}$ such that $x_i \in K_i$. Since $\mathcal{S}_L$ is a special spread, and K_i passes through $x_i \in L$ and meets both $M \in \mathcal{S}_L^*$ and $L_i \in \mathcal{S}_L^*$, we have $M \|_{x_i} L_i$. For $i = 1, 2, \ldots, n+1$, let $K_i' = z_k.x_i$. Since $M \|_{x_i} L_i$ and K_i' meets M, we conclude that K_i' meets L_i, say at w_i. If $w_1, w_2, \ldots, w_{n+1}$ are collinear, then take M_{z_k} to be the line that they are on and we find M_{z_k} (Figure 2).

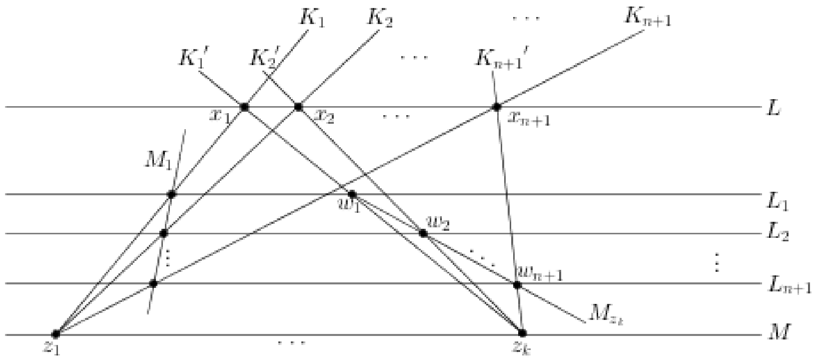

Figure 2: M_{z_k} is L-parallel to M_1

It remains the case when some of $w_1, w_2, \ldots, w_{n+1}$ are not collinear. For $i = 1, 2, \ldots, n+1$, let J_i be the line on w_i that is z_k-parallel to L. At least two of $J_1, J_2, \ldots, J_{n+1}$ are the same line. For, otherwise, there would be $n+1$ lines z_k-parallel to L. Without loss of generality, assume $J_1 = J_2$. We are going to prove that J_1 meets $L_1, L_2, \ldots, L_{n+1}$ by contradiction. Suppose J_1 misses some of $L_3, L_4, \ldots, L_{n+1}$. By Lemma 3.3, there are lines M_2 and N_2 on w_1 respectively L-parallel to M_1 and L-non-parallel to M_1. Thus, in $\mathcal{I}(w_1)$, $\mathcal{B}(M_2, N_2) = \{\overline{L_2}, \overline{L_3}, \ldots, \overline{L_{n+1}}, C\}$ for some circle C of type $\mathcal{C}_{w_1}$. Since L_1 are disjoint from $L_2, L_3, \ldots, L_{n+1}$ in $\mathcal{U}$, L_1 is a point on C in $\mathcal{I}(w_1)$. Since $J_1 = J_2$ meets L_2 in $\mathcal{U}$, J_1 is a point on $\overline{L_2}$ in $\mathcal{I}(w_1)$. Since $J_1 \neq M_2$ and $J_1 \neq N_2$ by hypothesis, and $J_1 \in \overline{L_2}$, we conclude $J_1 \notin C$. On the other hand, since $J_1\|_{z_k}L$, $\mathcal{B}(K'_1, J_1) = \{\overline{K'_2}, \overline{K'_3}, \ldots, \overline{K'_{n+1}}, D\}$ for some circle D of type $\mathcal{C}_{w_1}$. Note that $L_1 \in D$. Indeed, if L_1 is not on D, then L_1 meets K'_i in $\mathcal{U}$ for some $i \neq 1$. By Theorem 2.2, K'_i meets N. Since K'_i meets both $N \in \mathcal{S}^*_L$ and $L_1 \in \mathcal{S}^*_L$, we have $N\|_{x_i}L_1$ because $\mathcal{S}_L$ is a special spread. Thus the four lines $K'_1, K'_i, L_1, x_i.(K_1 \cap N)$ form an O'Nan configuration. Hence $L_1 \in D$. Since $J_1 \in D$ but $J_1 \notin C$, we have $C \neq D$. Since ∞_{w_1} and L_1 are points on both C and D, $K'_1 \notin C$. Then K'_1 meets L_j in $\mathcal{U}$ for some $j \neq 1$. By Theorem 2.2, K'_i meets N. Since K'_1 meets both $N \in \mathcal{S}^*_L$ and $L_j \in \mathcal{S}^*_L$, we have $N\|_{x_1}L_j$ and so the four lines K'_1, K'_j, L_j, $x_1.(K'_j \cap N)$ form an O'Nan configuration. A contradiction. Thus J_1 meets $L_1, L_2, \ldots, L_{n+1}$. Take M_{z_k} to be J_1 and we prove our claim. □

Theorem 3.6. *Let L, M, N be a self-polar triangle with respect to $\mathcal{U}$. Then $\mathcal{S}_L \setminus \{L, M, N\}$, $\mathcal{S}_M \setminus \{L, M, N\}$, $\mathcal{S}_N \setminus \{L, M, N\}$ can be respectively partitioned into $n-2$ subsets*

$$\begin{aligned}
&\{L^1_1, L^1_2, \ldots, L^1_{n+1}\}, \ldots, \{L^{n-2}_1, L^{n-2}_2, \ldots, L^{n-2}_{n+1}\};\\
&\{M^1_1, M^1_2, \ldots, M^1_{n+1}\}, \ldots, \{M^{n-2}_1, M^{n-2}_2, \ldots, M^{n-2}_{n+1}\};\\
&\{N^1_1, N^1_2, \ldots, N^1_{n+1}\}, \ldots, \{N^{n-2}_1, N^{n-2}_2, \ldots, N^{n-2}_{n+1}\};
\end{aligned}$$

each of cardinality $n+1$, such that for $i = 1, 2, \ldots, n-2$, the sets of points incident respectively on the lines of $\{L^i_1, L^i_2, \ldots, L^i_{n+1}\}$, $\{M^i_1, M^i_2, \ldots, M^i_{n+1}\}$ and $\{N^i_1, N^i_2, \ldots, N^i_{n+1}\}$ are the same.

Proof. Take a line $M^1_1 \in \mathcal{S}_M \setminus \{L, M, N\}$. Note that $M^1_1 \notin \mathcal{S}^*_L$ because $\mathcal{S}^*_L \cap \mathcal{S}^*_M = \{N\}$. Let $L^1_1, L^1_2, \ldots, L^1_{n+1}$ be the lines of $\mathcal{S}^*_L$ meeting M^1_1. By Lemmas 3.3 and 3.5, there is an L-parallel class $\{M^1_1, M^1_2, \ldots, M^1_{n+1}\} \subset \mathcal{S}^*_M$ and its L-non-parallel class $\{N^1_1, N^1_2, \ldots, N^1_{n+1}\}$ both partitioning the set of points covered by $L^1_1, L^1_2, \ldots, L^1_{n+1}$. Since $\{N^1_1, N^1_2, \ldots, N^1_{n+1}\}$ is an L-parallel class and an M-parallel class, it is a subset of $\mathcal{S}^*_{N'}$ where $N' \in \mathcal{S}^*_L \cap \mathcal{S}^*_M$ by applying Lemma 3.5 twice. Hence $N' = N$. Repeat the process $n-3$ times by taking a line $M^i_1 \in \mathcal{S}_M \setminus (\{L, M, N\} \cup \{M^k_l | k = 1, 2, \ldots, i-1, l = 1, 2, \ldots, n+1\})$. This finishes the proof. □

Remark 3.7. We may interpret Theorem 3.6 as follows. If $\mathcal{U}$ is embedded in a projective plane π as a polar unital via the construction of [18, Theorem 1.1], then $\mathcal{S}^*_L$, $\mathcal{S}^*_M$, $\mathcal{S}^*_N$ are respectively the set of lines through the pole of L, M and N. Thus, Theorem 3.6 says that in π, the set of points of $\mathcal{U}$ is partitioned into a self-polar triangle, and $n-2$ subsets of $(n+1)^2$ points triply ruled by lines through the vertices of the triangle.

Remark 3.8. In the setting in Theorem 3.6, for any disjoint index sets I_L, I_M, I_N such that $I_L \cup I_M \cup I_N = \{1, 2, \ldots, n+1\}$, the set

$$\{L, M, N\} \cup \{L^i_j \mid i \in I_L, j = 1, 2, \ldots, n+1\} \\ \cup \{M^i_j \mid i \in I_M, j = 1, 2, \ldots, n+1\} \cup \{N^i_j \mid i \in I_N, j = 1, 2, \ldots, n+1\}$$

is a spread of $\mathcal{U}$. These spreads are the subregular spreads studied by Dover [14].

4 From a special spread of a unital to a regular spread of $\mathrm{PG}(3, n)$

From now on, we fix a line L, and let $x_1, x_2, \ldots, x_{n+1}$ be the points on L.

Following Wilbrink [25], we are going to construct a generalized quadrangle $GQ(L)$, which is isomorphic to $Q(4, n)$ [25]. We will then embed $Q(4, n)$ into $\mathrm{PG}(4, n)$ and choose a 3-dimensional projective space Σ in $\mathrm{PG}(4, n)$. It turns out that the special spread $\mathcal{S}_L$ of $\mathcal{U}$ introduced in Section 2 defines a regular spread $\mathscr{S}$ of Σ (Theorem 4.4). Using the Bruck-Bose construction [6, 7], we will construct a projective plane in Section 6 using this regular spread $\mathscr{S}$, such that $\mathcal{U}$ is embedded in a way into this projective plane as a classical unital.

To construct $GQ(L)$, we recall the definition of the sets $\mathcal{A}_{ij}, 1 \le i, j \le n+1$ of $\mathcal{U}$ ([25], see also [18]). Considering $\mathcal{I}(x_1)$, denote the circles in the bundle $\mathcal{B}(L, \infty_{x_1})$ by $\{L, \infty_{x_1}\} \cup \mathcal{A}_{1j}$, where $j = 1, 2, \ldots, n+1$. We have defined $\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{1,n+1}$. Next, for each $j \in \{1, 2, \ldots, n+1\}$, consider the pencil $\langle L, \{L, \infty_{x_1}\} \cup \mathcal{A}_{1j}\rangle$ in $\mathcal{I}(x_1)$, i.e. the maximal set of mutually tangent circles through L with a member the circle $\{L, \infty_{x_1}\} \cup \mathcal{A}_{1j}$. For $k = 1, 2, \ldots, n-1$, denote by C_{jk} the remaining circles in the pencil. For $i \in \{2, 3, \ldots, n+1\}$, consider the $n-1$ lines on x_i which correspond respectively to these $n-1$ circles C_{jk}'s. Denote this set of lines by $\mathcal{A}_{ij}$. We have defined $\mathcal{A}_{2j}, \mathcal{A}_{3j}, \ldots, \mathcal{A}_{n+1,j}$, for $j = 1, 2, \ldots, n+1$. The definition of $\mathcal{A}_{ij}$ is independent of the choice of the point $x_1 \in L$.

The points of $GQ(L)$ are the points of $\mathcal{U}$ not on L, and the sets $\mathcal{A}_{ij}, i, j = 1, 2, \ldots, n+1$. The lines of $GQ(L)$ are the lines of $\mathcal{U}$ meeting L, and $2(n+1)$ new lines $A_1, A_2, \ldots, A_{n+1}, B_1 B_2, \ldots, B_{n+1}$. The incidence of $GQ(L)$ is as follows. $\mathcal{A}_{ij}$ is incident with A_k if and only if $i = k$; $\mathcal{A}_{ij}$ is incident with B_k if and only if $j = k$; for any line K of $\mathcal{U}$ meeting L, $\mathcal{A}_{ij}$ is incident with K if and only if $K \in \mathcal{A}_{ij}$; a point y of $\mathcal{U}$ is never incident with A_i or B_j for $i, j = 1, 2, \ldots, n+1$; incidence between a point and a line of $\mathcal{U}$ is the natural incidence.

Consider a parabolic quadric $\mathcal{P}$ in $\mathrm{PG}(4, n)$. The points and lines of $\mathcal{P}$ form a generalized quadrangle $Q(4, n)$ [22]. By [25], $GQ(L)$ is isomorphic under some

GQ isomorphism

$$\varphi : GQ(L) \longrightarrow Q(4,n) \tag{1}$$

to $Q(4,n)$.

Consider the 3-dimensional subspace Σ of PG$(4,n)$ determined by the skew lines $\varphi(A_1)$ and $\varphi(A_2)$. Then $\Sigma \cap \mathcal{P} = \{\varphi(\mathcal{A}_{ij}) \mid i,j = 1,2\ldots n+1\}$, and is a hyperbolic quadric $\mathcal{H}$ with regulus

$$\mathcal{R}_0 = \{\varphi(A_i) \mid i = 1,2,\ldots,n+1\} \tag{2}$$

and opposite regulus $\{\varphi(B_i) \mid i = 1,2,\ldots,n+1\}$. $\mathcal{H}$ defines a polarity

$$\alpha : \Sigma \longrightarrow \Sigma \tag{3}$$

of Σ.

The tangent spaces of $\mathcal{P}$ are concurrent at a point *nucleus* $\mathbf{N}$ of $\mathcal{P}$. Let

$$\mu : \mathcal{P} \longrightarrow \Sigma \tag{4}$$

be the function defined as follows: for any point $\mathbf{V}$ of $\mathcal{P}$, $\mu(\mathbf{V})$ is the intersection point of Σ and the line joining $\mathbf{N}$ and $\mathbf{V}$. Since $\mathcal{H} \cap \Sigma = \mathcal{H}$, μ is identity on $\mathcal{H}$. Since $|\mathcal{P}| = n^3 + n^2 + n + 1 = |\Sigma|$, μ is a bijection. Hence, we have a 1-1 correspondence between points of $GQ(L)$ and that of Σ via the composition function $\mu\varphi$. Furthermore, some quadratic cones in $\mathcal{P}$ are mapped to planes of Σ because n is even:

Lemma 4.1. *Let $\mathbf{V} \in \mathcal{P} \setminus \Sigma$. Let $\mathcal{Q}$ be the quadratic cone formed by the intersection of $\mathcal{P}$ and the tangent space of $\mathcal{P}$ at $\mathbf{V}$. Then μ maps $\mathcal{Q}$ onto the plane $\alpha(\mu(\mathbf{V}))$ in Σ. Furthermore, the plane $\alpha(\mu(\mathbf{V}))$ meets $\mathcal{H}$ in an irreducible conic with nucleus $\mu(\mathbf{V})$.*

Proof. Since $\mathbf{V} \notin \Sigma$, every generator of $\mathcal{Q}$ meets Σ in a unique point (of $\mathcal{H}$). Hence, for any generator l of $\mathcal{Q}$, $\mu(l)$ is tangent to $\mathcal{H}$ and passes through $\mu(\mathbf{V})$. By a property of hyperbolic quadric of even order, $\{\mu(l) \mid l \text{ is a generator of } \mathcal{Q}\}$ is on the plane $\alpha(\mu(\mathbf{V}))$. Since $|\mathcal{Q}| = n^2 + n + 1$, the image set $\mu(\mathcal{Q})$ is a plane. Furthermore, $\mu(\mathbf{V})$ is the nucleus of the irreducible conic formed by the intersection of $\mu(\mathcal{Q})$ and $\mathcal{H}$. $\square$

Using Lemma 4.1 and the GQ isomorphism φ, we prove that every line of $\mathcal{S}_L^*$ is mapped to a line of Σ under $\mu\varphi$:

Lemma 4.2. *Let M be a line of $\mathcal{U}$ in $\mathcal{S}_L^*$. Then $\mu(\varphi(M))$ is a line in Σ.*

Proof. By Theorem 2.2, $L\|_{z_i}M$ for $i = 1, 2, \ldots, n+1$. Let $\mathcal{Q}_i$ be the quadratic cone formed by the intersection of $\mathcal{P}$ and the tangent space of $\mathcal{P}$ at $\varphi(z_i)$. Hence $\varphi(M) \subset \mathcal{Q}_i$. By Lemma 4.1, $\mu(\varphi(M)) \subset \bigcap_{i=1}^{n+1} \alpha(\mu(\varphi(z_i)))$ and $\alpha(\mu(\varphi(z_i)))$'s are planes in Σ. Since $|\varphi(M)| = n+1$ and $\mu(\varphi(M))$ is in the intersection of $n+1$ planes, $\mu(\varphi(M))$ is a line. □

Since $\mathcal{S}_L$ is a spread of $\mathcal{U}$ and μ is a bijection, the set $\{\mu(\varphi(L')) \mid L' \in \mathcal{S}_L^*\}$ consists of disjoint lines. Let

$$\mathscr{S} = \mathcal{R}_0 \cup \{\mu(\varphi(L')) \mid L' \in \mathcal{S}_L^*\}. \tag{5}$$

Then $\mathscr{S}$ is a spread. We claim that $\mathscr{S}$ is regular (Theorem 4.4). The justification of this claim requires the notion of tube [10]:

When q is even, a *tube* in PG$(3,q)$ is a pair $\mathcal{T} = \{l, \mathcal{B}\}$, where $\{l\} \cup \mathcal{B}$ is a collection of mutually disjoint lines of PG$(3,q)$ such that for each plane Π of PG$(3,q)$ containing l, the intersection of Π with the lines of $\mathcal{B}$ is a hyperoval. For any mutually skew lines l_1, l_2, l_3 in Σ, denote by $\mathcal{R}(l_1, l_2, l_3)$ the unique regulus determined by them. According to Cameron and Knarr [10], if $\{l, \{l_0, l_1, \ldots, l_{q+1}\}\}$ is a tube, then the union $\bigcup_{i=1}^{n+1} \mathcal{R}(l, l_0, l_i)$ is a regular spread in PG$(3,q)$.

Lemma 4.3. *Let M, N be lines of $\mathcal{U}$. Suppose L, M, N form a self-polar triangle. Then the pair $\mathcal{T} = \{\mu(\varphi(M)), \{\mu(\varphi(N))\} \cup \mathcal{R}_0\}$ is a tube in Σ.*

Proof. Let $z_1, z_2, \ldots, z_{n+1}$ be the points of N. By Lemma 4.1, the $n+1$ planes in Σ containing $\mu(\varphi(M))$ are $\alpha(\mu(\varphi(z_i)))$, $i = 1, 2, \ldots, n+1$. Since $\mathcal{R}_0$ is a regulus of $\mathcal{H}$, meets the points covered by $\mathcal{R}_0$ in an irreducible conic with nucleus $\mu(\varphi(z_i))$ by Lemma 4.1. The result follows. □

Theorem 4.4. *$\mathscr{S}$ is a regular spread in Σ.*

Proof. Let M, N be lines such that L, M, N form a self-polar triangle. For $i = 1, 2, \ldots, n+1$, let $\mathcal{R}_i = \mathcal{R}(\mu(\varphi(M)), \mu(\varphi(N)), \mu(\varphi(A_i)))$. By [10] mentioned above, the union $\bigcup_{i=1}^{n+1} \mathcal{R}_i$ is a regular spread. We are done if we show $\mathscr{S} = \bigcup_{i=1}^{n+1} \mathcal{R}_i$. Thus it suffices to show $\mathscr{S} \subset \bigcup_{i=1}^{n+1} \mathcal{R}_i$.

Let $L_1 \in \mathcal{S}_L \setminus \{L, M, N\}$. Since $L \in \mathcal{S}_M^*$ and $L_1 \in \mathcal{S}_L^*$, there is a point $x_i \in L$ such that $M\|_{x_i}L_1$ by Lemma 2.4. Let $K_1, K_2, \ldots, K_{n+1}$ be the lines of $\mathcal{U}$ on x_i meeting M (and hence meeting L_1). Then they meet N by Theorem 2.2. Hence, $\mu(\varphi(K_1))$, $\mu(\varphi(K_2))$, $\ldots$, $\mu(\varphi(K_{n+1}))$ are lines in Σ meeting $\mu(\varphi(M))$, $\mu(\varphi(N))$ and $\mu(\varphi(A_i))$. Furthermore, these $n+1$ lines are disjoint. Indeed, the points $K_1, K_2, \ldots, K_{n+1}$ are incident with $\overline{M} \in \mathcal{F}(L, \infty_{x_i})$ in $\mathcal{I}(x_i)$, where $\overline{M}$ is tangent to each circle in the bundle $\mathcal{B}(L, \infty_{x_i})$ [13], and so we may assume $K_j \in \mathcal{A}_{ij}$ for $j = 1, 2, \ldots n+1$. Thus, $\{\mu(\varphi(K_j)) \mid j = 1, 2, \ldots, n+1\}$ is the opposite regulus of $\mathcal{R}_i$. Since $\mu(\varphi(L_1))$ meets every line in opposite regulus of $\mathcal{R}_i$, it is in $\mathcal{R}_i$. □

5 From a partition of $\mathcal{S}_L^*$ to a pencil of quadrics in PG$(3, n)$

We use the notations in Section 4 and continue to prove that $\mathcal{U}$ is classical. The key result in this section is Theorem 5.5. It says that the image of the partition of $\mathcal{S}_L$ in Theorem 3.6 under $\mu\varphi$ corresponds to a pencil of quadrics of two lines and $n-1$ hyperbolic quadrics of the projective space Σ, where μ, φ and Σ are defined Section 4. To prove Theorem 5.5, we have to describe every regulus of the spread $\mathscr{S}$ defined in (5), in terms of the geometry of $\mathcal{U}$. A regulus of $\mathscr{S}$ has two, one, or no common lines with $\mathcal{R}_0$. We consider these cases separately.

We first consider the reguli of $\mathscr{S}$ with exactly one common line with $\mathcal{R}_0$. They can be described using x-parallelism introduced in Section 2, where the x's are the points of the line L.

Lemma 5.1. *Let $i \in \{1, 2, \ldots, n+1\}$ and $L_1, L_2, \ldots, L_n \in \mathcal{S}_L^*$. If the set $\mathcal{R} = \{\mu(\varphi(A_i)), \mu(\varphi(L_1)), \mu(\varphi(L_2)), \ldots, \mu(\varphi(L_n))\}$ forms a regulus in $\mathscr{S}$, then $L_j \|_{x_i} L_k$ for any $j, k \in \{1, 2, \ldots, n\}$.*

Proof. Let l be a line in the opposite regulus of $\mathcal{R}$. By the definition of μ and the construction of $GQ(L)$, there is a line K of $\mathcal{U}$ on x_i such that $l = \mu(\varphi(K))$. Since l meets $\mu(\varphi(L_1)), \mu(\varphi(L_2)), \ldots, \mu(\varphi(L_n))$, the line K meets $L_1, L_2, \ldots, L_n \in \mathcal{S}_L^*$ in $\mathcal{U}$. Since $\mathcal{S}_L$ is a special spread, $L_1, L_2, \ldots, L_n$ are x_i-parallel. □

Lemma 5.3 describes the reguli of $\mathscr{S}$ with exactly two common lines with $\mathcal{R}_0$. To prove Lemma 5.3, we need Lemma 5.2.

Lemma 5.2. *Let $i_1, i_2, j_1, j_2 \in \{1, 2, \ldots, n+1\}$ with $i_1 \neq i_2$ and $j_1 \neq j_2$. Let $K_1 \in \mathcal{A}_{i_1 j_1}$ and $K_2 \in \mathcal{A}_{i_2 j_2}$ be lines of $\mathcal{U}$ meeting at a point y of $\mathcal{U}$. Let M be a line in $\mathcal{S}_L^*$ not through y. Then $\overline{M}$ is in the flock $\mathcal{F}(K_1, K_2)$ in $\mathcal{I}(y)$ if and only if there is a point $z \in M$ such that $z.x_k \in \mathcal{A}_{i_k j_k}$ for $k = 1, 2$.*

Proof. Without loss of generality, assume $i_1 = j_1 = 1$ and $i_2 = j_2 = 2$. Suppose $\overline{M} \in \mathcal{F}(K_1, K_2)$ in $\mathcal{I}(y)$. For $k = 1, 2$, let N_k be a line through x_k such that $\overline{M} \in \mathcal{F}(K_k, N_k)$ in $\mathcal{I}(x_k)$. Applying Lemma 3.2(2) respectively to $\overline{M} \in \mathcal{F}(K_1, K_2)$ in $\mathcal{I}(y)$, $\overline{M} \in \mathcal{F}(K_1, N_1)$ in $\mathcal{I}(x_1)$ and $\overline{M} \in \mathcal{F}(K_2, N_2)$ in $\mathcal{I}(x_2)$, any line of $\mathcal{S}_M^*$ meeting K_1 meets K_2,N_1 and N_2. Since $M \in \mathcal{S}_L^*$, we have $L \in \mathcal{S}_M^*$ by condition (p). Hence, L is the line in $\mathcal{S}_M^*$ that meet K_1, K_2, N_1, N_2.

By Lemma 3.2(3), there is a point z_k on M such that any line from z_k meeting L miss all other lines in $\mathcal{S}_M^*$ that meet K_1, K_2, N_1, N_2. By uniqueness in Lemma 3.2(3), $z_1 = z_2$. By Lemma 3.2(1) and (3), in $\mathcal{I}(x_k)$, the points $z_k.x_k$, K_k, L, ∞_z are concircular. By definition of $\mathcal{A}_{kk}$, we have $z_k.x_k \in \mathcal{A}_{kk}$ for $k = 1, 2$.

Conversely, note that there are exactly $n-2$ circles in $\mathcal{F}(K_1, K_2)$ in $\mathcal{I}(y)$ is of type $\mathcal{C}^y$, and each of these circles gives one $z \neq y$ such that $z.x_1 \in \mathcal{A}_{11}$ and $z.x_2 \in \mathcal{A}_{22}$. To prove the converse, it suffices to show there are exactly $n-2$ such z's. By definition of $\mathcal{A}_{kk}$, in $\mathcal{I}(x_1)$, lines of $\mathcal{A}_{22}$ correspond to circles of a pencil with carrier L, and lines of $\mathcal{A}_{11}$ correspond to points on a circle through L not in that pencil. Hence, each line of $\mathcal{A}_{11}$ meets exactly one line of $\mathcal{A}_{22}$ in $\mathcal{U}$. Since $|\mathcal{A}_{11}| = n-1$ and there is only one line of $\mathcal{A}_{11}$ passing through y, there are exactly $n-2$ such z's. □

Lemma 5.3. *Let $i_1, i_2 \in \{1, 2, \dots, n+1\}$ with $i_1 \neq i_2$. Let $L_1, L_2, \dots, L_{n-1} \in \mathcal{S}_L^*$. Suppose $\mathcal{R}$ is a regulus of $\mathscr{S}$ containing $\mu(\varphi(A_{i_1}))$, $\mu(\varphi(A_{i_2}))$, $\mu(\varphi(L_1))$, $\mu(\varphi(L_2))$, $\dots$, $\mu(\varphi(L_{n-1}))$. Then for any point $z_1 \in L_1$, if K_1 is the line passing through z_1 and x_{i_1}, and if K_2 is the line passing through z_1 and x_{i_2}, then in $\mathcal{I}(z_1)$, $\mathcal{F}(K_1, K_2) = \{\overline{L_2}, \overline{L_3}, \dots, \overline{L_{n-1}}, C\}$ for some circle C through ∞_{z_1}.*

Proof. Let z be a point on L_1. Let l be the line in the opposite regulus of $\mathcal{R}$ through $\mu(\varphi(z))$. Let $z' \in L_2$ be the unital point such that $\mu(\varphi(z')) \in l$. Consider α defined in (3). Then $\alpha(l)$ meets $\mathcal{H}$ at $\mu(\varphi(\mathcal{A}_{i_1 j_1}))$ and $\mu(\varphi(\mathcal{A}_{i_1 j_2}))$ for some j_1, j_2 with $j_1 \neq j_2$, and $\alpha(l)$ lies on the plane $\alpha(\mu(\varphi(z')))$. Let $\mathcal{Q}$ be the quadratic cone formed by the intersection of $\mathcal{P}$ and the tangent space of $\mathcal{P}$ at $\varphi(z')$. By Lemma 4.1, $\mu^{-1}(\alpha(l)) \in \mathcal{Q}$ and so $\varphi(\mathcal{A}_{i_1 j_1}), \varphi(\mathcal{A}_{i_1 j_2}) \in \mathcal{Q}$. Hence z' lies on some unital lines $K_1 \in \mathcal{A}_{i_1 j_1}$ and $K_2 \in \mathcal{A}_{i_1 j_2}$. By Lemma 5.2, $\overline{L_2} \in \mathcal{F}(K_1, K_2)$ in $\mathcal{I}(z)$. Similarly, $\overline{L_3}, \dots, \overline{L_{n-1}} \in \mathcal{F}(K_1, K_2)$. The result follows by Lemma 3.1. □

Lemma 5.4 gives a characterization of reguli of $\mathscr{S}$ with no common line with $\mathcal{R}_0$, by considering inversive planes whose blocks are defined by the reguli of $\mathscr{S}$. For each line J of $\mathcal{U}$ that misses L and not in $\mathcal{S}_L$, let $C(J)$ be the set of images of the $n+1$ lines of $\mathcal{S}_L^*$ meeting J under $\mu\varphi$.

Lemma 5.4. *A set of $n+1$ lines of $\mathscr{S} \setminus \mathcal{R}_0$ is a regulus if and only if it is $C(J)$ for some line J of $\mathcal{U}$ missing L and not in $\mathcal{S}_L$.*

Proof. Consider the incidence structure

$$\mathcal{I}_1 = (\mathscr{S}, \mathcal{C}) \tag{6}$$

where $\mathcal{C}$ is the set of the reguli of $\mathscr{S}$. By Theorem 4.5(iv) of Bruck [5], $\mathcal{I}_1$ is the Miquelian inversive plane of order n. Note that $\mathcal{R}_0$ is a circle of $\mathcal{I}_1$. We denote by $\mathcal{C}_0$ the set of those circles in $\mathcal{I}_1$ disjoint from $\mathcal{R}_0$; by $\mathcal{C}_1$ the set of those circles in $\mathcal{I}_1$ tangent to $\mathcal{R}_0$; by $\mathcal{C}_2$ the set of those circles in $\mathcal{I}_1$ secant to $\mathcal{R}_0$.

Let $\mathcal{C}_0^* = \{C(J) \mid J$ is a line of $\mathcal{U}$ missing L and not in $\mathcal{S}_L\}$. Now consider the incidence structure

$$\mathcal{I}_2 = (\mathscr{S}, (\mathcal{C} \setminus \mathcal{C}_0) \cup \mathcal{C}_0^*). \tag{7}$$

By [19, Theorem 2], provided that $\mathcal{I}_2$ is an inversive plane of order n, we will have $\mathcal{I}_1 = \mathcal{I}_2$ and thus $\mathcal{C}_0 = \mathcal{C}_0^*$. Hence, to prove Lemma 5.4, *it suffices to prove that $\mathcal{I}_2$ is a 3–$(n^2+1, n+1, 1)$ design.*

Since $\mathscr{S}$ has n^2+1 lines, $\mathcal{I}_2$ has n^2+1 points.

A block in $\mathcal{C}_0^*$ has exactly $n+1$ points because every line which is not in $\mathcal{S}_L$ and which misses L meets exactly $n+1$ lines of $\mathcal{S}_L^*$. Other blocks of $\mathcal{I}_2$ has exactly $n+1$ points because every regulus of $\mathscr{S}$ consists of $n+1$ lines.

We have $|\mathcal{C}_0^*| = n(n-1)(n-2)/2$ because there are $n(n-1)(n-2)$ L-parallel classes by Lemma 3.4, and any class and its L-non-parallel class define a same block of $\mathcal{I}_2$. Since $\mathcal{I}_1$ is an inversive plane, $|\mathcal{C} \setminus \mathcal{C}_0| = n^2(n+3)/2$. Thus $\mathcal{I}_2$ has $n(n^2+1)$ blocks.

It remains to show that any two distinct blocks of $\mathcal{I}_2$ have at most two common points. Since $\mathcal{I}_2$ only differs from the inversive plane $\mathcal{I}_1$ in blocks missing $\mathcal{R}_0$, we only need to consider the case when one of the two blocks belongs to $\mathcal{C}_0^*$. Let $C(J)$ be a block in $\mathcal{C}_0^*$. Let $\mu(\varphi(L_k)), k = 1, 2, 3$ be distinct points of $C(J)$, where $L_1, L_2, L_3 \in \mathcal{S}_L^*$.

Suppose $\mu(\varphi(L_k)), k = 1, 2, 3$, are on a block in $\mathcal{C}_1$. By Lemma 5.1, L_1, L_2, L_3 are x-parallel for some x on L. Since J meets L_1, L_2, L_3 but does not pass though x, there is an O'Nan configuration, contradicting (I).

Suppose $\mu(\varphi(L_k)), k = 1, 2, 3$, are on a block in $\mathcal{C}_2$ which contains $\mu(\varphi(A_i))$ and $\mu(\varphi(A_j))$. Let z be a point intersection of J and L_1. Let K_i, K_j be the lines through z that pass through x_i and x_j respectively. By Lemma 5.3, $\overline{L_2}, \overline{L_3}$ are distinct circles in $\mathcal{F}(K_i, K_j)$ in $\mathcal{I}(z)$. Since circles in a flock are disjoint, $\overline{L_2}$ and $\overline{L_3}$ are disjoint. This contradicts that J is a line through z meeting L_2 and L_3.

Suppose $\mu(\varphi(L_k)), k = 1, 2, 3$, are on a block $C(J_2)$ in $\mathcal{C}_0^*$. Suppose J and J' are not L-parallel or L-non-parallel. We claim that $L_2 \|_y L_3$ for any point $y \in L_1$. If the claim is true, then L_1, L_2, L_3 is a self-polar triangle by Theorem 2.2 and so $\mathcal{S}_{L_2}^* \cap \mathcal{S}_{L_3}^* = \{L_1\}$. Since $L \in \mathcal{S}_{L_2}^* \cap \mathcal{S}_{L_3}^*$ by condition (p) and $L \neq L_1$, a contradiction arises. It follows that J and J' are L-parallel or L-non-parallel, and $C(J) = C(J')$.

To see that $L_2 \|_y L_3$ for any $y \in L_1$. Note that, if y is a point on L_1, then the four lines on y which are respectively L-parallel to J_1, L-non-parallel to J_1,

L-parallel to J_2 and L-non-parallel to J_2, meet both L_2 and L_3. Thus by Wilbrink [25, Lemma 1], $L_2 \|_y L_3$.

Hence, a block in $\mathcal{C}_0^*$ has at most two common points with any other block. Thus, $\mathcal{I}_2$ is an inversive plane of order n. $\square$

To prove the main theorem in this section, we need Lemma 5.4 and that fact that when q is even, if a regular spread of PG$(3, q)$ is partitioned into two lines and $q-1$ reguli, then the two lines and the hyperbolic quadrics containing the reguli lie in a pencil of quadrics (Hirschfeld [16, Lemma 17.1.5, Corollary of Theorem 17.1.6]).

Theorem 5.5. *Let M, N be lines in $\mathcal{S}_L^*$. Suppose L, M, N form a self-polar triangle with respect to $\mathcal{U}$. Consider the lines $L_1^1, L_2^1, \ldots, L_{n+1}^1, \ldots, L_1^{n-2}, \ldots, L_{n+1}^{n-2}$, $M_1^1, M_2^1, \ldots, M_{n+1}^1$ as obtained in the construction described in Theorem 3.6. For $i = 1, 2, \ldots, n-1$, let $\mathcal{H}_i$ be the image set of the points on $L_1^i, L_2^i, \ldots, L_{n+1}^i$ under $\mu\varphi$. Consider $\mathcal{H}$ is defined in Section 4. Then $\{\mathcal{H}_i \mid i = 1, 2, \ldots, n-2\} \cup \{\mathcal{H}\} \cup \{\mu(\varphi(M)), \mu(\varphi(N))\}$ is a pencil of quadrics in Σ of two lines and $n-1$ hyperbolic quadrics.*

Proof. By Lemma 5.4, for $i = 1, 2, \ldots, n-2$, the block $C(M_1^i) = \{\mu(\varphi(L_1^i)), \mu(\varphi(L_2^i)), \ldots, \mu(\varphi(L_{n+1}^i))\}$ is a regulus. Thus $\mathcal{H}_i$ is a hyperbolic quadric. By the definition of $\mathscr{S}$, $\mathscr{S} = \left(\bigcup_{i=1}^{n-2} C(M_1^i)\right) \cup \mathcal{R}_0 \cup \{\mu(\varphi(M)), \mu(\varphi(N))\}$. Since $\mathscr{S}$ is regular by Theorem 4.4, the result follows by [16] mentioned above. $\square$

6 Completion of the proof that $\mathcal{U}$ is classical

In this section, we use the notations in Section 4 and complete the proof that $\mathcal{U}$ is classical. Recall from Section 1 that we wish to show that $\mathcal{U}$ is isomorphic to the hyperbolic Buekenhout unital $\mathcal{U}'$ in PG$(2, n^2) \cong \overline{\pi(\mathscr{S})}$ defined by $\mathcal{P}$ in Section 4 under the Bruck-Bose representation [6, 7]. To this end, we define an isomorphism φ' from φ, where φ is the isomorphism between $GQ(L)$ and $Q(4, n)$ defined in (4).

Lemma 6.1. *Let J be a line of $\mathcal{U}$ missing L and not in $\mathcal{S}_L$. Then $\varphi(J)$ lies on a plane which contains a line of $\mathscr{S}$.*

Proof. We locate $\varphi(J)$ as a subset of an intersection of two quadratic cones, as follows. Let M be the line in $\mathcal{S}_L^*$ such that $J \in \mathcal{S}_M^*$. By Lemma 2.4, there is a point $z \in M$ such that $L \|_z J$. Let $\mathcal{Q}_0$ be the quadratic cone formed by the intersection of $\mathcal{P}$ and the tangent space of $\mathcal{P}$ at $\varphi(z)$. By the definition of $GQ(L)$,

$$\varphi(J) \subset \mathcal{Q}_0. \tag{8}$$

By Lemma 4.1, $\mu(\mathcal{Q}_0) \cap \mathcal{H}$ is a base of $\mathcal{Q}_0$ and its nucleus is $\mu(\varphi(z))$. On the other hand, let $\mathcal{Q}_1$ be the cone with vertex $\mathbf{N}$ and a base $\mu(\varphi(J))$. By the definition of μ, we have

$$\varphi(J) \subset \mathcal{Q}_1. \tag{9}$$

We are going to show $\mathcal{Q}_1$ is a quadratic cone by studying $\mu(\varphi(J))$. By Lemma 4.1 and (8), $\mu(\varphi(J))$ lies on the plane $\mu(\mathcal{Q}_0)$. Let $\mathcal{H}_1 = \{\mu(\varphi(y)) \mid y$ is a point of $\mathcal{U}$ on a line of $\mathcal{S}_L^*$ meeting $J\}$. By Lemma 5.4, $\mathcal{H}_1$ is a hyperbolic quadric in Σ. Note that $\mu(\varphi(J)) \subset \mathcal{H}_1$. To see that the plane $\mu(\mathcal{Q}_0)$ is secant to $\mathcal{H}_1$, consider the line N of $\mathcal{U}$ such that L, M, N form a self-polar triangle with respect to $\mathcal{U}$. By Theorem 2.2, $L\|_z N$. Thus, $\mathcal{Q}_0$ contains $\varphi(N)$, and the plane $\mu(\mathcal{Q}_0)$ contains $\mu(\varphi(N))$. Since $L\|_z J\|_z N$ and $N \in \mathcal{S}_L^*$, we conclude that N is disjoint from any line of $\mathcal{S}_L^*$ that meets J. Thus, $\mu(\varphi(N))$ is external to $\mathcal{H}_1$. Thus, $\mu(\varphi(J)) = \mu(\mathcal{Q}_0) \cap \mathcal{H}_1$ and the base $\mu(\varphi(J))$ of $\mathcal{Q}_1$ is an irreducible conic (Figure 3).

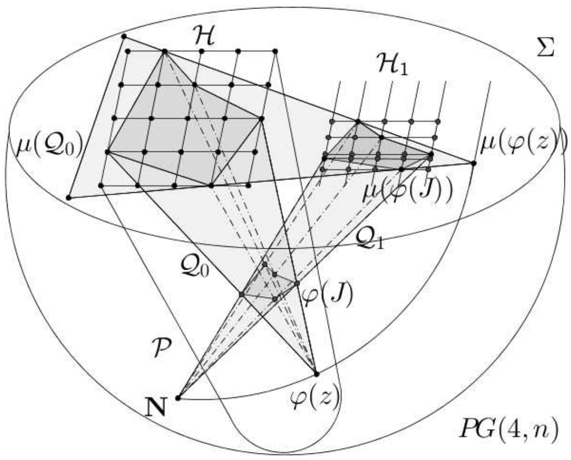

Figure 3: $\varphi(J) \subset \mathcal{Q}_0 \cap \mathcal{Q}_1$

We then study $\mathcal{Q}_0 \cap \mathcal{Q}_1$. Let Π be the plane determined by three distinct points in $\mathcal{Q}_0 \cap \mathcal{Q}_1$. Then $\mathcal{Q}_0 \cap \Pi$ and $\mathcal{Q}_1 \cap \Pi$ are irreducible conics. Since L is z-parallel to J, the nucleus of $\mu(\varphi(J))$ is $\mu(\varphi(z))$. Since the nuclei of $\mathcal{Q}_0 \cap \mu(\mathcal{Q}_0)$ and $\mathcal{Q}_1 \cap \mu(\mathcal{Q}_0)$ are both $\mu(\varphi(z))$, the conics $\mathcal{Q}_0 \cap \Pi$ and $\mathcal{Q}_1 \cap \Pi$ have a same nucleus, namely the intersection of Π and the line through $\mathbf{N}$, $\varphi(z)$ and $\mu(\varphi(z))$. Since $\mathcal{Q}_0 \cap \Pi$ and $\mathcal{Q}_1 \cap \Pi$ are irreducible conics with the same nucleus and containing three distinct common points, $\mathcal{Q}_0 \cap \Pi = \mathcal{Q}_1 \cap \Pi$ by Lemma 2.1 of Luyckx [20]. Thus, $\varphi(J) \subset \Pi$.

By Theorem 5.5, $\mathcal{H}_1, \mathcal{H}, \mu(\varphi(M)), \mu(\varphi(N))$ are quadrics in a same pencil. Thus, $\mathcal{H}_1$ and $\mathcal{H}$ meet $\mu(\varphi(N))$ in the same conjugate pair of points with respect to $\mathbb{F}_{q^2}$ [8, Theorem 5.1]. Hence the bases of $\mathcal{Q}_0$ and $\mathcal{Q}_1$ pass through the conjugate pair of points in PG$(4, q^2)$, and so do $\mathcal{Q}_0 \cap \mathcal{Q}_1$ and Π. Hence, the plane Π, where $\varphi(J)$ lies on, contains the line $\mu(\varphi(N)) \in \mathscr{S}$. $\square$

We prove that a unital $\mathcal{U}$ is classical by studying the hyperbolic Buekenhout unital $\mathcal{U}'$ in $\overline{\pi(\mathscr{S})}$ defined by $\mathcal{P}$, where $\overline{\pi(\mathscr{S})}$ is the projective plane constructed by the Bruck-Bose construction [6, 7] (see André [1] for an alternative treatment).

Theorem 6.2. *$\mathcal{U}$ is classical.*

Proof. Consider the incidence structure $\pi(\mathscr{S})$ whose points are the affine points of PG$(4, q)$ and whose lines are the affine planes of PG$(4, q)$ each containing a line of $\mathscr{S}$. The incidence of $\pi(\mathscr{S})$ is the incidence of PG$(4, q)$. By Bruck and Bose [6], $\pi(\mathscr{S})$ is an affine translation plane of order q^2. Complete $\pi(\mathscr{S})$ to a projective plane $\overline{\pi(\mathscr{S})}$ by adding a line at infinity, L_∞. Since $\mathscr{S}$ is regular, $\overline{\pi(\mathscr{S})}$ is Desarguesian [7]. By Buekenhout [9], $\mathcal{P}$ corresponds to a hyperbolic Buekenhout unital $\mathcal{U}'$ in $\overline{\pi(\mathscr{S})}$. Let a_i be the points on L_∞ corresponding to the $\varphi(A_i)$'s in $\mathcal{R}_0$. Then the point set of $\mathcal{U}'$ is $(\mathcal{P} \setminus \Sigma) \cup \{a_1, a_2, \ldots, a_{q+1}\}$. Since $\mathscr{S}$ is regular by Theorem 4.4, $\mathcal{U}'$ is classical by Barwick [3].

Let $\varphi' : \mathcal{U} \longrightarrow \mathcal{U}'$ be a function defined by $\varphi'(x_i) = a_i$ for $i = 1, 2, \ldots, n+1$; $\varphi'(y) = \varphi(y)$ for any point y of $\mathcal{U}$ not on L; $\varphi'(J) = \{\varphi(y) \mid y \in J\}$ for any line J of $\mathcal{U}$ missing L; $\varphi'(K) = \{\varphi(y) \mid y \in K \setminus L\} \cup \{a_i\}$ for any line K of $\mathcal{U}$ meeting L at some point x_i; $\varphi'(L) = \{a_1, a_2, \ldots, a_{q+1}\}$.

Note that φ' is a well-defined function. Indeed, for any line $J_1 \notin \mathcal{S}_L^*$ of $\mathcal{U}$ missing L, its image $\varphi'(J_1)$ is on a secant plane on a line of $\mathscr{S}$ by Lemma 6.1, and hence is a line of $\mathcal{U}'$. As for a line $J_2 \in \mathcal{S}_L^*$ of $\mathcal{U}$ missing L, its image $\varphi'(J_2)$ is on the secant plane of $\mathcal{P}$ determined by the point $\mathbf{N}$ and the line $\mu(\varphi(J_2)) \in \mathscr{S}$, and hence $\varphi'(J_2)$ is a line of $\mathcal{U}'$. As for a line K meeting L at some x_i, its image $\varphi'(K)$ consists of a_i and the n affine points of PG$(4, n)$ on a line of $\mathcal{P}$ meeting $\varphi(A_i)$, and hence is a line of $\mathcal{U}'$. Since φ is an isomorphism, φ' preserves incidence. Clearly, φ' is injective. Thus, φ' is a design isomorphism and $\mathcal{U}$ is classical. $\square$

Remark 6.3. By [18, Theorem 1.1], since $\mathcal{U}$ satisfies (p), $\mathcal{U}$ can be embedded in a projective plane π as a polar unital. The author does not know whether π is Desarguesian or not.

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Alice M. W. Hui

Statistics Program, BNU-HKBU United International College, Zhuhai, China, and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

e-mail: huimanwa@gmail.com, alicemwhui@uic.edu.hk