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Unitary subspaces of unitary Grassmannians

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Abstract

The purpose of this article is to characterize those subspaces of a unitary Grassmannian which are isomorphic to a unitary Grassmannian.

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1 Introduction and basic concepts

We assume the reader is familiar with the concepts of a *partial linear rank two incidence geometry* $\Gamma = (\mathcal{P}, \mathcal{L})$ (also called a point-line geometry) and the Lie incidence geometries. For the former we refer to [3] and for the latter see the paper [4].

The *collinearity graph* of Γ is the graph $(\mathcal{P}, \Delta)$ where Δ consists of all pairs of points belonging to a common line. For a point $x \in \mathcal{P}$ we will denote by $\Delta(x)$ the collection of all points collinear with x. For points $x, y \in \mathcal{P}$ and a positive integer t a *path of length* t from x to y is a sequence $x_0 = x, x_1, \ldots, x_t = y$ such that $\{x_i, x_{i+1}\} \in \Delta$ for each $i = 0, 1, \ldots, t-1$. The *distance* from x to y, denoted by $d(x, y)$, is defined to be the length of a shortest path from x to y if some path exists and otherwise is $+\infty$.

By a *subspace* of Γ we mean a subset S of $\mathcal{P}$ such that if $l \in \mathcal{L}$ and $l \cap S$ contains at least two points, then $l \subset S$. $(\mathcal{P}, \mathcal{L})$ is said to be a *Gamma space* if, for every $x \in \mathcal{P}, \{x\} \cup \Delta(x)$ is a subspace. A subspace S is *singular* provided each pair of points in S is collinear, that is, S is a clique in the collinearity graph of Γ. For a Lie incidence geometry with respect to a "good node" every singular subspace, together with the lines it contains, is isomorphic to a projective space, see [4]. Clearly the intersection of subspaces is a subspace and, consequently, it is natural to define the subspace generated by a subset X of $\mathcal{P}, \langle X \rangle_\Gamma$, to be the intersection of all subspaces of Γ which contain X. Note that if $(\mathcal{P}, \mathcal{L})$ is a Gamma space and X is a clique then $\langle X \rangle_\Gamma$ will be a singular subspace.

A *polar space* is an incidence geometry $(\mathcal{P}, \mathcal{L})$ which satisfies: (i) For any point x and line l either x is collinear with every point of l or a unique point of l; and (ii) For each point x there exists a point y such that x and y are non-collinear. A polar space in which lines are maximal singular subspaces is a *generalized quadrangle*.

1.1 Ordinary Grassmannians

Let $\mathbb{F}$ be a field and W be a vector space of dimension m over $\mathbb{F}$. For $1 \leq i \leq m-1$, let $L_i(W)$ be the collection of all i-dimensional subspaces of W. Now fix $j, 2 \leq j \leq m-2$ and set $\mathcal{P} = L_j(W)$.

For pairs (C, A) of incident subspaces of W with $\dim(A) = a$, $\dim(C) = c$, let $S(C, A)$ consist of all the j-subspaces B of W such that $A \subset B \subset C$.

Finally, let $\mathcal{L}$ consist of all the sets $S(C, A)$ where $\dim A = j-1, \dim C = j+1$ and $A \subset C$. The rank two incidence geometry $(\mathcal{P}, \mathcal{L})$ is the incidence geometry of j-Grassmannian of W, denoted by $\mathcal{G}_j(W)$. We also use the notation $\mathcal{G}_{m,j}(\mathbb{F})$ for the isomorphism type of this geometry and sometimes $A_{m-1,j}(\mathbb{F})$.

We note that the incidence geometry $\mathcal{G}_{4,2}(\mathbb{F})$ is a polar space which is isomorphic to the incidence geometry of singular one-spaces and totally singular two-spaces on a hyperbolic orthogonal space in a vector space of dimension six, $D_{3,1}(\mathbb{F}) \cong Q^+(6, \mathbb{F})$.

1.2 The unitary Grassmannians

Let $\mathbb{E} \subset \mathbb{F}$ be a Galois extension of fields of degree two and let σ be the generator of the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{E})$. We will often denote the image of an element $a \in \mathbb{F}$ under σ by $\overline{a}$. Let V be a space of dimension n over the field $\mathbb{F}$ and f be a non-degenerate σ-Hermitian form.

For $X \subset V$ let $X^\perp = \{\mathbf{v} \in V : f(\mathbf{x}, \mathbf{v}) = \mathbf{0}, \forall \mathbf{x} \in X\}$. Recall that a subspace U of V is *totally isotropic* if $U \subset U^\perp$. The Witt index of (V, f) is the dimension of a maximal totally isotropic subspace of V. This is an invariant of f. Because (V, f) is non-degenerate the dimension of a totally isotropic subspace is at most $\lfloor \frac{n}{2} \rfloor$. We will say that (V, f) has maximal Witt index if there are totally isotropic subspaces of dimension $\lfloor \frac{n}{2} \rfloor$. Hereafter we assume (V, f) is non-degenerate of dimension n with Witt index equal to $n' = \lfloor \frac{n}{2} \rfloor$.

For $1 \leq k \leq n' = \lfloor \frac{n}{2} \rfloor$, let $\mathcal{I}_k(V)$ consist of all totally isotropic k-dimensional subspaces of V. More generally, if W is a subspace of V then we will denote by

$\mathcal{I}_k(W)$ the set of all elements of $\mathcal{I}_k(V)$ which are contained in W. We will set $P = \mathcal{I}_1(V)$, the collection of all one-dimensional subspaces of V and $L = \mathcal{I}_2(V)$, the collection of totally isotropic two-spaces (projective lines). The incidence geometry (P, L) is the unitary polar space of rank n' over the field $\mathbb{F}$, which we will denote by ${}^2A_{n-1,1}(\mathbb{F})$.

Now fix l with $2 \leq l \leq n' - 1$ and set $\mathcal{P} = \mathcal{I}_l$. For a pair of subspaces $C \subset D \subset C^\perp$ (so C is totally isotropic) where $\dim(C) = c < l < d = \dim(D)$ let $T_l(D, C)$ consist of all the l-dimensional totally isotropic subspaces U such that $C \subset U \subset D$. When $c = l - 1, d = l + 1$ we set $\lambda(D, C) = T_l(D, C)$ and $\mathcal{L} = \{\lambda(D, C) : C \subset D \subset C^\perp, \dim C = l - 1, \dim D = l + 1\}$. In this way we obtain a rank 2 incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ which we refer to as the unitary l-Grassmannian of V. We denote the isomorphism type of this geometry by ${}^2A_{n-1,l}(\mathbb{F})$. Note that two totally isotropic l-subspaces, when viewed as points of Γ, are on a line if and only if they span a totally isotropic $(l + 1)$-dimensional subspace. We remark that the automorphism group of the geometry $(\mathcal{P}, \mathcal{L})$ is isomorphic to $PU_n(\mathbb{F})$.

When the subspace C has dimension $l - k$ and D is totally isotropic and has dimension $l + m - k$ then $T_l(D, C)$ is an ordinary Grassmannian isomorphic to $\mathcal{G}_{m,k}(\mathbb{F})$. Subspaces arising this way are said to be *parabolic* since their stabilzers in $\mathrm{Aut}(\Gamma) \cong U_n(\mathbb{F})$ are parabolic subgroups. In [1] we classified subspaces of ${}^2A_{n-1,l}(\mathbb{F})$ which are isomorphic to $\mathcal{G}_{m,k}(\mathbb{F})$ and proved that they are all parabolic.

Assume $\dim(V) = n = 2n'$. By a hyperbolic basis of the unitary space (V, f) we will mean a vector space basis $(\mathbf{x}_1, \dots, \mathbf{x}_{n'}, \mathbf{y}_1, \dots, \mathbf{y}_{n'})$ such that each vector $\mathbf{x}_i, \mathbf{y}_i$ is isotropic, $f(\mathbf{x}_i, \mathbf{y}_i) = 1$ and $f(\mathbf{x}_i, \mathbf{x}_j) = f(\mathbf{x}_i, \mathbf{y}_j) = f(\mathbf{y}_i, \mathbf{y}_j) = 0$ for all $i \neq j$. The existence of a hyperbolic basis can be shown by an easy induction on the Witt index m of f.

If $\dim(V) = n = 2n' + 1$ then by a "near" hyperbolic basis of the unitary space (V, f) we will mean a vector space basis $(\mathbf{x}_1, \dots, \mathbf{x}_{n'}, \mathbf{y}_1, \dots, \mathbf{y}_{n'}, \mathbf{z})$ such that each vector $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}$ is isotropic, $f(\mathbf{x}_i, \mathbf{y}_i) = 1, f(\mathbf{x}_i, \mathbf{x}_j) = f(\mathbf{x}_i, \mathbf{y}_j) = f(\mathbf{y}_i, \mathbf{y}_j) = 0$, and $f(\mathbf{x}_i, \mathbf{z}) = f(\mathbf{y}_i, \mathbf{z}) = 0$ for all $i < n'$. The existence of a near hyperbolic basis can also be shown by an easy induction on the Witt index n' of f.

1.3 Subspaces of unitary Grassmannians

We continue with the notation from section 1.2 so that (V, f) is a non-degenerate unitary space of dimension n and $1 \leq l \leq n' - 1$. Assume $\Gamma = (\mathcal{P}, \mathcal{L})$ is isomorphic to ${}^2A_{n-1,l}(\mathbb{F})$ with $\mathcal{P} = \mathcal{I}_l(V)$. Suppose X is a subset of $\mathcal{P}$. We will denote by $\Sigma(X)$ the vector subspace of V that is spanned by all $\mathbf{x} \in X$ and by $I(X)$ the intersection of all $\mathbf{x} \in X$.

Let $A \in \mathcal{I}_{l-k}$ with $k \geq 1$ and B a subspace of $A^{\perp}$ with $A \subset B$ and such that the quotient space B/A is non-degenerate of dimension q. In this situation the collection of isotropic subspaces $T_l(B, A)$ is a subspace of the incidence geometry $(\mathcal{P}, \mathcal{L})$ and is isomorphic to a unitary Grassmannian space ${}^2A_{q-1,k}(\mathbb{F})$. In our main result we show that these “natural” examples are the only subspaces of $(\mathcal{P}, \mathcal{L})$ which are isomorphic to a unitary Grassmannian.

Main theorem. *Let S be a subspace of ${}^2A_{n-1,l}(\mathbb{F})$ isomorphic to ${}^2A_{n'-1,l'}(\mathbb{F})$. Then there exists a totally isotropic subspace A of dimension $l - l'$ and a subspace B with $A \subset B \subset A^{\perp}$ with B/A non-degenerate of dimension n' and such that $S = T_l(B, A)$.*

It may be possible that a result like this can be obtained more generally by relaxing the condition that the unitary space have maximal Witt index but we have chosen not to do so because of the many technical obstacles that would have to be overcome. In any case, the result is applicable whenever the field is finite.

The proof will be very much in the spirit of [5] where we proved a similar result for symplectic Grassmanians. Before proceeding to the proof we introduce some notation:

Notation 1.1. Since we will generate all kinds of subspaces, of the unitary space V, of the geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, etc., we need to distinguish between these. When X is some collection of subspaces or vectors from V we will denote the subspace of V spanned by X by $\langle X \rangle_{\mathbb{F}}$. When X is a subset of $\mathcal{P}$ we will denote the subspace of $\Gamma = (\mathcal{P}, \mathcal{L})$ generated by X by $\langle X \rangle_{\Gamma}$.

For a point $p \in \mathcal{P}$ we will denote by $\Delta^{\Gamma}(p)$ the collection of all points of $\mathcal{P}$ which are collinear with p in $(\mathcal{P}, \mathcal{L})$ (including p).

2 Properties of unitary Grassmannians

In this short section we review some properties of unitary Grassmannians. We omit the proofs of most because these propositions are either well known or easy to prove.

Lemma 2.1. *Let (V, f) be a non-degenerate unitary space of dimension n and maximal Witt index m. Then the following hold:*

(i) *The unitary Grassmannian space ${}^2A_l(V)$, which is isomorphic to ${}^2A_{n-1,l}(\mathbb{F})$, has two classes of maximal singular subspaces with representatives $T_l(B, 0)$ where B is a totally isotropic subspace of $V, \dim B = l + 1$, and $T_l(C, A)$*

where A and C are totally isotropic subspaces of V with $A \subset B$ and where $\dim(A) = l-1, \dim(C) = m$. In the former case $T_l(B,0) \cong \mathbb{PG}_l(\mathbb{F})$ and in the latter $T_l(C,A) \cong \mathbb{PG}_{m-l}(\mathbb{F})$. We refer to the first class as type one maximal singular subspaces and the latter as type two singular subspaces.

(ii) *If M_1 and M_2 are maximal singular subspaces of different types then either $M_1 \cap M_2$ is empty, a point, or a line.*

(iii) *If M_1 and M_2 are distinct maximal singular subspaces of the same type then $M_1 \cap M_2$ is either empty or a point.*

Definition 2.2. A **symp** of $(\mathcal{P}, \mathcal{L})$ is a maximal geodesically closed subspace which is isomorphic to a polar space.

Lemma 2.3. *There are two classes of symps in $(\mathcal{P}, \mathcal{L})$. One class has representative $T_l(E,D)$ where $D \subset E$ are totally isotropic subspaces, $\dim D = l-2, \dim E = l+2$. In this case $T_l(E,D) \cong D_{3,1}(\mathbb{F})$ the polar space of a non-degenerate six dimensional orthogonal space with maximal Witt index. The second class has representative $T_l(C^\perp, C)$ where C is a totally isotropic subspace, $\dim C = l-1$. In this case $T_l(C^\perp, C)$ is isomorphic to the polar space of a non-degenerate unitary space of dimension $n-2(l-1)$.*

Definition 2.4. We refer to a member of the first class of symps in Lemma 2.3 as a **type one symp** and and a member of the second class as a **type two symp**.

Lemma 2.5. *There are three classes of points at distance two in $\Gamma = (\mathcal{P}, \mathcal{L})$:*

(i) *The pairs $\{x,y\}$ which satisfy $\dim(x \cap y) = l-2$ and $x \perp y$. Such a pair $\{x,y\}$ lies in a unique symp which is $T_l(x+y, x\cap y)$. Note this only occurs if the Witt index of the unitary space is greater than or equal to four.*

(ii) *The pairs $\{x,y\}$ that satisfy $\dim(x \cap y) = l-1$ and $(x+y)/(x\cap y)$ is a non-degenerate two-space. This pair belongs to a unique symp which is $T_l((x\cap y)^\perp, x \cap y)$.*

(iii) *The pairs $\{x,y\}$ which satisfy $\dim(x \cap y) = l-2$ and $\dim([x+y]\cap[x+y]^\perp) = l$. There is a unique point (of the geometry $(\mathcal{P}, \mathcal{L})$) collinear with both x and y, namely $[x+y] \cap [x+y]^\perp$.*

Definition 2.6. The first class of pairs in Lemma 2.5 will be referred to as **type one symp pairs**, the second as **type two symp pairs** and the third type as **special pairs**. For a point x we will denote by $\Delta_{(2,i)}(x)$ all the points y such that the pair x, y is a type i symp pair and by $\Delta_{(2,s)}(x)$ the points y such that x, y is a special pair.

Lemma 2.7. *Let S be a type two symp of the incidence geometry $(\mathcal{P}, \mathcal{L}) \cong {}^2A_{n-1,l}(\mathbb{F})$ and $x \in \mathcal{P} \setminus S$. Then $\Delta^\Gamma(x) \cap S$ is either empty or a line.*

Lemma 2.8. *Let $(\mathcal{P}, \mathcal{L}) = {}^2A_l(V) \cong {}^2A_{n-1,l}(\mathbb{F})$ and let $p \neq q \in \mathcal{I}_1(W)$.*

(i) *Assume $p \perp q$ and let $x \in T_l(p^\perp, p)$. Then one of the following occurs:*

(α) *$q \subset x$ and $x \in T_l(q^\perp, q)$;*

(β) *q is not contained in $x, x \subset q^\perp$ and $\Delta^\Gamma(x) \cap T_l(q^\perp, q) = T_l(x+q, q)$ is a singular subspace isomorphic to $\mathbb{PG}_{l-1}(\mathbb{F})$; or*

(γ) *x is not contained in $q^\perp$ and $\langle x \cap q^\perp, q\rangle$ is the unique point in $\Delta_{(2,2)}(x) \cap T_l(q^\perp, q)$.*

(ii) *Assume p and q are non-orthogonal. Then $T_l(p^\perp, p) \cap T_l(q^\perp, q) = \emptyset$. If $x \in T_l(p^\perp, p)$ then $y = \langle x \cap q^\perp, q\rangle_V$ is the unique point in $T_l(q^\perp, q) \cap \Delta_{(2,2)}(x)$.*

2.1 Properties of the geometry ${}^2A_{5,2}(\mathbb{F})$

The particular geometry ${}^2A_{5,2}(\mathbb{F})$ plays a prominent role in our proof and we use several properties of this geometry which we will make explicit here for later reference. Throughout this subsection we will let W be a non-degenerate six dimensional unitary space over $\mathbb{F}$ and $(\mathcal{P}, \mathcal{L})$ will be the geometry ${}^2A_2(W) \cong {}^2A_{5,2}(\mathbb{F})$.

Lemma 2.9. *The maximal singular subspaces of ${}^2A_2(W)$ are projective planes. If M_1, M_2 are two such subspaces then $M_1 \cap M_2$ is either empty or a point.*

Proof. Suppose x and y are collinear points of ${}^2A_2(W)$. Then $x \cap y \in \mathcal{I}_1(W)$ and $T_2([x \cap y]^\perp, [x \cap y])$ is a generalized quadrangle and therefore its lines are maximal singular subspaces. Therefore, if z is collinear with both x and y but does not lie on the line $T_2(x+y, x \cap y)$ then z must lie in the totally isotropic three space $x+y$ and $\langle x, y, z\rangle_\Gamma = T_2(x+y, 0)$ is a projective plane (dual to $T_1(x+y, 0)$.) We have therefore shown that the maximal singular subspaces of ${}^2A_2(W)$ are all of the form $T_2(U, 0)$ for U a totally isotropic subspace of W of dimension three.

Now let $M_i = T_2(U_i, 0), i = 1, 2$ where U_i are distinct maximal totally isotropic subspaces of W. Then $\dim(U_1 \cap U_2) \leq 2$. If $\dim(U_1 \cap U_2) = 2$ then $U_1 \cap U_2$ is the unique point in $M_1 \cap M_2$. Otherwise $M_1 \cap M_2 = \emptyset$. □

Lemma 2.10. *Let $M_i = T_2(U_i, 0), i = 1, 2$ where U_i are maximal isotropic subspaces of W. Assume $M_1 \cap M_2 = \{x\}$. For a point $y \in M_1, y \neq x$ we have the following:*

(i) $\Delta^\Gamma(y) \cap M_2 = \{x\}$.

(ii) $[\Delta_{(2,2)}(y) \cap M_2] \cup \{x\}$ *is the line $T_2(M_2, x \cap y)$.*

Proof. Note that $U_1 \cap U_2 = \{x\}$. Let $y \in M_1$. Then $y \cap x = p$ is a projective point of W. Since U_2 is a maximal totally singular subspace and y is not contained in U_2 it follows that $y^\perp \cap U_2 = x$ and so $\{x\} = \Delta^\Gamma(y) \cap M_2$. On the other hand, suppose $z \in M_2$ and $z \cap x = p$. Then $y \cap z = p$ and the pair y, z is a type two symp pair. Thus, every point of the line $T_2(U_2, p)$, apart from x belongs to $\Delta_{(2,2)}(y)$. Moreover, if $w \in M_2$ and $w \cap x = q \neq p$ then $\{y, w\}$ is a special pair. Thus, we have shown (i) and (ii). □

Lemma 2.11. *Let M_1 and M_2 be maximal singular subspaces of ${}^2A_2(W)$ such that $M_1 \cap M_2 = \emptyset$. Then one of the following occurs:*

(i) *There are lines $m_i \subset M_i, i = 1, 2$, satisfying the following: For each point $x \in m_1, \Delta^\Gamma(x) \cap M_2 \in m_2$ is a point and $m_2 \subset \Delta^\Gamma(x) \cup \Delta_{(2,2)}(x)$. In particular, for every $x \in m_1, y \in m_2, \dim(x \cap y) = 1$. Moreover, if $x_1 \in m_1, x_2 \in M_2 \setminus m_2$ then $\{x_1, x_2\}$ is a special pair, whereas if $x_i \notin m_i, i = 1, 2$ then $d(x_1, x_2) = 3$.*

(ii) *For each point $x \in M_1, M_2 \cap \Delta^\Gamma(x) = \emptyset$. For each $x \in M_1, \Delta_{(2,s)}(x) \cap M_2$ is a line and if $y \in M_2, y \notin \Delta_{(2,s)}(x) \cap M_2$ then $d(x, y) = 3$.*

Proof. (i) Let $M_i = T_2(U_i, 0), i = 1, 2$. Then we have either $U_1 \cap U_2 = \{\mathbf{0}_W\}$ or $U_1 \cap U_2 = \{p\}$ where p is an isotropic point of W. Assume first that $U_1 \cap U_2 = \{p\}$. We show that (i) holds. Set $m_i = T_2(U_i, p), i = 1, 2$, lines of M_1, M_2 respectively. Suppose $x \in m_1$. Let $y = U_2 \cap x^\perp$. Then $y \in m_2$ and it is the unique point of M_2 collinear with x. For any other point $y' \in m_2, \dim(x \cap y') = 1$ and therefore $\{x, y'\}$ is a type two symp pair. On the other hand, if $z \in M_2 \setminus m_2$ then $x \cap z = \{\mathbf{0}_W\}$. However, $p \subset x^{\perp_W} \cap z$ and therefore $\{x, z\}$ is a special pair. On the other hand, suppose $x_i \in M_2, i = 1, 2$ and p is not contained in $x_1 \cap x_2$. Then $x_1 \cap x_2 = \{\mathbf{0}\}$ and $x_1 \cap x_2^\perp = \{\mathbf{0}\}$ and $d(x_1, x_2) = 3$.

(ii) Now assume that $U_1 \cap U_2 = \{\mathbf{0}_W\}$. Then for each $x \in M_1$ and $y \in M_2$ we have $x \cap y = 0$ and $\{x, y\}$ cannot be collinear or a type two symp pair and so either $\{x, y\}$ is special pair or $d(x, y) = 3$. However, for $x \in M_1, x^\perp \cap U_2 = p$ is a projective point of W and all the points of the line $T_2(U_2, p)$ are in $\Delta_{(2,s)}(x)$. This proves (ii). □

Notation 2.12. If M_1, M_2 are maximal singular subspaces of ${}^2A_2(W)$ we will write $M_1 \sim M_2$ if $M_1 \cap M_2$ is a point and $M_1 * M_2$ if M_1, M_2 are as in Lemma 2.11 part (i).

Lemma 2.13. *Let $\mathcal{M}$ be the collection of all maximal singular subspaces of ${}^2A_2(W)$. Then*

(i) *The graph $(\mathcal{M}, \sim)$ is connected.*

(ii) *The graph* $(\mathcal{M}, *)$ *is connected.*

Proof. (i) The graph $(\mathcal{M}, \sim)$ is the collinearity graph of the dual polar space ${}^2A_{5,3}(\mathbb{F}) = DU(6, \mathbb{F})$ which is known to be connected.

(ii) In light of (i) it suffices to prove that if $M_1 \sim M_2$ then there exists a $*$ path from M_1 to M_2. Suppose $M_i = T_2(U_i, 0), i = 1, 2$ where $U_1 \cap U_2 \in \mathcal{I}_2(W)$. Let $(\mathbf{v}_1, \mathbf{v}_2)$ be a basis for $U_1 \cap U_2$. Extend this to a basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ for U_1 and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_3)$ for U_2. Now $\mathbf{v}_3$ and $\mathbf{w}_3$ are non-orthogonal. Then $(\mathbf{v}_3 + \mathbf{w}_3)^{\perp}$ is a non-degenerate four dimensional subspace of W which contains $\mathbf{v}_1$ and $\mathbf{v}_2$. Extend this to a base $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2)$ where $\mathbf{v}_i \perp \mathbf{w}_j$ for $i \neq j$ and $\mathbf{w}_1 \perp \mathbf{w}_2$. Now set $M_3 = \langle v_1, w_2, v_3 + w_3 \rangle_{\mathbb{F}}$. Then $M_1 * M_3 * M_2$. □

3 Proof of the main theorem

In this section we prove our main theorem. Let (V, f) be a non-degenerate unitary space of dimension n over $\mathbb{F}$ and (W, g) a non-degenerate unitary space of dimension m over $\mathbb{F}$. When necessary, we will distinguish orthogonality in V by writing $\perp_V$ and in W by $\perp_W$. Before proceeding to the proof we introduce some notation: When A, B are subspaces of V and l is an positive integer we will denote by $T_{(V,l)}(B, A)$ the collection of l-dimensional totally isotropic subspaces of V which satisfy $A \subset C \subset B$ and in a similar fashion we define $T_{(W,k)}(E, D)$.

Fix an $l, 1 \leq l \leq n-1$ and let $\Gamma = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P} = \mathcal{I}_l(V)$ and $\mathcal{L}$ consists of all sets $\lambda(B, A) = T_{(V,l)}(B, A)$ where $A \subset B \subset B^{\perp_V}$ are subspaces of $V, \dim A = l-1$ and $\dim B = l+1$.

Now fix $k, 1 \leq k \leq m-1$ and set $\mathcal{P}' = \mathcal{I}_k(W)$ and set $\mathcal{L}'$ equal to the collection of all set $\lambda(B', A') = T_{(W,k)}(B', A')$ where $A' \subset B' \subset (B')^{\perp_W}$ are subspaces of $W, \dim A' = k-1$ and $\dim B' = k+1$ so that $\Gamma' = (\mathcal{P}', \mathcal{L}') \cong {}^2A_{m-1,k}(\mathbb{F})$. Now assume that S is a subspace of $\Gamma, S = (\mathcal{P}_S, \mathcal{L}_S) \cong (\mathcal{P}', \mathcal{L}')$. Let $\sigma : \Gamma' \to S$ be an isomorphism. For a totally isotropic subspace $U \in \mathcal{I}_t(W), 1 \leq t \leq m$, we will denote by S_U the image under σ of $T_{(W,k)}(U^{\perp_W}, U)$.

Notation 3.1. For a subset X of $\mathcal{P}$ we will denote by $\Sigma(X)$ the subspace of V spanned by all $U \in X$.

We will show that the conclusions of our main theorem hold in a sequence of lemmas. Our proof is by induction on $N = n + l + m + k$.

Lemma 3.2. *Let* $x, y \in \mathcal{P}$ *be collinear and* z *on the line* xy*. Then* $x \cap y \subset z \subset x + y$*.*

Proof. This is an immediate consequence of the definition of collinearity in ${}^2A_{n-1,l}$ and of a line. □

Lemma 3.3. *Let S be a subspace of $\Gamma = {}^2A_l(V) \cong {}^2A_{n-1,l}(\mathbb{F})$ and X a generating set of S, that is, a subset X of S such that $\langle X\rangle_\Gamma = S$. Then $\Sigma(S) = \Sigma(X)$.*

Proof. We define a sequence of sets $P_j(X) \subset \mathcal{P}, j \geq 0$ inductively as follows: $P_0(X) = X$ and

$$P_{j+1}(X) = P_j(X) \cup \bigcup_{\{\lambda\in\mathcal{L}:|\lambda\cap P_j(X)|\geq 2\}} \lambda$$

and set $P(X) = \cup_{j\geq 0}P_j(X)$. Note that $P_{j+1}(X) \supset P_j(X)$. We claim that $P(X)$ is a subspace of Γ. For suppose that λ is a line and $x \neq y \in \lambda \cap P(X)$. Then there are natural numbers s, t such that $x \in P_s(X), y \in P_t(X)$. If $t' = \max\{s, t\}$ then $x, y \in P_{t'}(X)$ and then $\lambda \subset P_{t'+1}(X)$. This proves that $P(X)$ is a subspace.

Since $X \subset P(X)$ and X generates S we can conclude that $S \subset P(X)$. On the other hand, a simple induction implies that $P_j(X) \subset S$ for each $j \geq 0$, whence $P(X) \subset S$ and consequently, $P(X) = S$.

We next claim that $\Sigma(P_j(X)) \subset \Sigma(X)$ for all $j \geq 0$. The proof is by induction on j. Since $P_0(X) = X$ the base case is clear.

Now assume that $\Sigma(P_j(X)) \subset \Sigma(X)$ and let $z \in P_{j+1}(X) \setminus P_j(X)$. Then there is a line λ containing z with $|\lambda \cap P_j(X)| \geq 2$. Let $x \neq y \in \lambda \cap P_n(X)$. By the inductive hypothesis, $x, y \subset \Sigma(X)$ and then by Lemma 3.2 it follows that $z \subset \Sigma(X)$.

Since $P_j(X) \subset P_{j+1}(X)$ it follows that $\Sigma(P_j(X)) \subset \Sigma(P_{j+1}(X))$ and consequently that $\cup_{j\geq 0}\Sigma(P_j(X))$ is a subspace of V and equal to $\Sigma(\cup_{j\geq 0}P_j(X))$. We can then conclude that

$$\begin{aligned}\Sigma(X) &\supseteq \cup_{j\geq 0}\,\Sigma(P_j(X))\\ &= \Sigma(\cup_{j\geq 0}\,P_j(X))\\ &= \Sigma(S).\end{aligned}$$

□

Before getting to the next result we need a lemma on the generation of unitary polar spaces of which have maximal Witt index.

Lemma 3.4. *Let $\Pi = (P, L)$ be the polar space of isotropic points and totally isotropic lines of a non-degenerate unitary space (V, f) of dimension n and maximal Witt index m. Then the following occurs:*

(i) *Assume $n = 2m$ and the Witt index of (V, f) is m. Then any subgraph of the collinearity graph of (P, L) with isomorphism type $K_{2,2,\ldots,2}$ (m 2's) generates P.*

(ii) *Assume* $n = 2m+1$ *and the Witt index of* (V, f) *is* m. *Let* $p_i, q_i, 1 \le i \le m$ *and* r *be isotropic points such that*

(1) $p_i \perp p_j, p_i \perp q_j, q_i \perp q_j$ *for* $i \ne j, 1 \le i, j \le m$ *whereas* $p_i \not\perp q_i$ *for* $1 \le i \le m$;

(2) $p_i \perp r_j \perp q_i$ *for* $1 \le i \le m-1, 1 \le j \le 3$, $p_m \not\perp r \not\perp q_m$; *and*

(3) $p_m^\perp \cap r^\perp \ne q_m^\perp \cap r^\perp$.

Then $\{p_i, q_i | 1 \le i \le m\} \cup \{r\}$ *generates* P.

Proof. (i) This is proved by Blok and Cooperstein in [2].

(ii) Set $X = \{p_i, q_i | 1 \le i \le m\}$ and $Y = X \cup \{r\}$. Also set $U = \langle X \rangle_{\mathbb{F}}, H = \langle X \rangle_\Pi$, and $S = \langle Y \rangle_\Pi$. By (i) $H = \mathcal{I}_1(U)$, the point set of ${}^2A_{2m-1,1}(U)$. Also, since U is a linear hyperplane of V, H is a geometric hyperplane of P. Note, by assumption (3) that $r \notin H$. Let $x \in P, x \perp r$. We claim that $x \in S$. If $x \in H$ then $x \in S$ since $H \subset S$. So assume that $x \notin H$. Let λ be the line of Π containing x and r and let y be the point of λ in H. Since $r, y \in S$ it follows that $\lambda \subset S$, whence $x \in S$. In a similar fashion, if $x, y \in P \setminus H$ and $x \perp y \perp r$ then $x \in S$. We now claim that $S = P$. Suppose $x \in P \setminus H$. Let $z_1, z_2 \in H, z_1 \perp r \perp z_2, z_1 \not\perp z_2$. Let $\lambda_i, i = 1, 2$ be the line on z_i and r. Of course, we can assume that $x \not\perp r_3$. Suppose $x \not\perp z_1$. Let y be the point on λ_1 such that $x \perp y$. Then $r_3 \perp y \perp x$ whence $x \in S$ by the above. We get a similar conclusion if $x \not\perp z_2$. So we may now assume that $z_1 \perp x \perp z_2$. Let λ_3 be the line on x and z_2 and choose a point y on $\lambda_1, y \ne r, z_1$ and let y' be the point on λ_3 with $y \perp y'$. Observe that $y' \ne z_2$ since $z_1 \not\perp z_2$. Now $r \perp y \perp y'$ and therefore $y' \in S$. Then y' and $z_2 \in S$ from which we can conclude that $\lambda_3 \subset S$. Thus, $x \in S$. □

Lemma 3.5. *If* $k = 1$, *that is,* S *is isomorphic to* ${}^2A_{m-1,1}(\mathbb{F})$ *with* $m \ge 4$, *then there exists a totally isotropic subspace* D *of dimension* $l-1$, *and a subspace* E *contained in* $D^\perp$ *and containing* D *such that* E/D *non-degenerate of dimension* n' *and* $S = T_{(V,l)}(E, D)$.

Proof. The subspace S is a polar space and therefore contained in one of the two types of symps because any polar space is the convex hull of any two of its points at distance two. Suppose S is contained in a type two symp, $T_{(V,l)}(D^\perp, D)$, where D is totally isotropic, $\dim D = l-1$. Suppose $n' = 2s$ is even. Let $p_i, q_i, 1 \le i \le s$, be points of S such that the pairs $\{p_i, p_j\}, \{p_i, q_j\}, \{q_i, q_j\}$ are collinear for $i \ne j$ and $\{p_i, q_i\}$ are not collinear for $1 \le i \le s$. By (i) of Lemma 3.4, $\langle p_i, q_i | 1 \le i \le s \rangle_\Gamma = S$. Since $p_i, q_i \in T_{(V,l)}(D^\perp, D)$ and p_i, q_i are not collinear we must have $p_i^\perp \cap q_i = D, p_i \perp_V p_j, p_i \perp_V q_j$, and $q_i \perp_V q_j$ for $i \ne j$. Then the space $E = \sum_{i=1}^s (p_i + q_i)$ has dimension $2s + (l-1)$ and E/D

is non-degenerate. Since $\{p_i, q_i | 1 \leq i \leq s\}$ generates the subspace S it follows from Lemma 3.3 that $\Sigma(S) = E$. So in this case the conclusion of the theorem holds.

Suppose $m = 2s + 1$. Let $p_i, q_i, 1 \leq i \leq s$ and r be points of S such that the pairs $\{p_i, p_j\}, \{p_i, q_j\}, \{q_i, q_j\}, i \neq j$, are collinear for $1 \leq i \leq s$, that $\{p_i, r\}$ and $\{q_i, r\}$ are collinear for $1 \leq s - 1$ and all other pairs are non-collinear. Further, assume that r is not in $\langle p_i, q_i | 1 \leq i \leq s\rangle_\Gamma$. Then by (ii) of Lemma 3.4, S is generated by $\{p_i, q_i | 1 \leq s\} \cup \{r\}$. Note that D is a subset of p_i, q_i for every i and r since all these points are in $T_{(V,l)}(D^\perp, D)$. Since p_i, q_i are not collinear it follows that $p_i^\perp \cap q_i = D$ and $p_i/D, q_i/D$ are two non-orthogonal isotropic points of $D^\perp/D$ as are p_m/D and r/D as well as q_m/D and r/D. It then follows that the dimension of $r/D + \sum_{i=1}^{s}(p_i/D + q_i/D)$ is $2s + 1 = n'$. Consequently, if $E = r + \sum_{i=1}^{s}(p_i + q_i)$ then $\dim(E) = (l - 1) + 2s + 1 = (l - 1) + n'$. Since $\{p_i, q_i | 1 \leq i \leq s\} \cup \{r\}$ generates S it follows that $E = \Sigma(S)$. Thus, in this case the result holds.

We now show that S cannot be contained in a type one symp. Suppose to the contrary that S is contained in $T_{(V,l)}(B, A)$ with $A \subset B \subset A^{\perp_V}$, subspaces of V with $\dim(A) = l - 2$ and $\dim(B) = l + 2$. Let p_1, p_2, q_1, q_2 be points in S such that all pairs are collinear except $\{p_1, q_1\}$ and $\{p_2, q_2\}$. Then $\langle p_1, p_2, q_1, q_2\rangle_\Gamma$ is a quadrangle of S isomorphic to ${}^2A_{3,1}(\mathbb{F})$. However, for four such points in $T_{(V,l)}(B, A)$, $\langle p_1, p_2, q_1, q_2\rangle_\Gamma$ is a grid and we have a contradiction. $\square$

We will next be treating the case that $m \geq 6$, S is isomorphic to ${}^2A_{n'-1,2}(\mathbb{F})$ and is a subspace of ${}^2A_l(V)$ which is isomorphic to ${}^2A_{n-1,l}(\mathbb{F})$. Let p be a point of $\mathcal{I}_1(W)$ and denote by S_p those elements of S which are the image of point x of the geometry ${}^2A_2(W)$ such that $p \subset x$. Then S_p is isomorphic to ${}^2A_{m-3,1}(\mathbb{F})$. By Lemma 3.5 there is a totally isotropic subspace A_p of dimension $l - 1$ and a subspace B_p contained in $A_p^{\perp_V}$ and containing A_p such that B_p/A_p non-degenerate of dimension $n' - 2$ and $S_p = T_{(V,l)}(B_p, A_p)$.

Lemma 3.6. *For $p \neq q \in \mathcal{I}_1(W), A_p \neq A_q$.*

Proof. Suppose to the contrary that $A_p = A_q$ for some pair $p \neq q \in \mathcal{I}_1(W)$. Set $U = A_p = A_q$. Then S_p, S_q are both subspaces of $T_{(V,l)}(U^{\perp_V}, U)$ which is a type two symp. By Lemma 2.7 for any point $y \in S_q \setminus S_p, \Delta^\Gamma(y) \cap S_p$ is either empty or a singular subspace. In particular, S_p is not contained in $\Delta^\Gamma(y)$.

Choose a $y \in S_q \setminus S_p$ and let x be a point in S_p which is not collinear with y. Let w, z be points of S_p which are non-collinear but are both collinear with x. Since S_p, y are contained in the symp $T_{(V,l)}(U^{\perp_V}, U)$, y is collinear with a point $w' \neq x$ on the line xw and a point $z' \neq x$ on the line xz. However, the points w'

and z' are non-collinear and this contradicts the fact that $S_p \cap \Delta^\Gamma(y)$ is empty or a singular subspace. Thus, $A_p \neq A_q$ for $p \neq q \in \mathcal{I}_1(W)$. $\square$

We shall now deal with the case $k = l = 2$.

Lemma 3.7. *Let $n \geq m \geq 6$. Assume $m = l = 2$. Then there is a non-degenerate m-dimensional subspace B of V such that $S = \mathcal{I}_2(B) = T_{(V,2)}(B, 0)$.*

Proof. Let (W, g) be a non-degenerate unitary space of dimension m and maximal Witt index $m' = \lfloor \frac{m}{2} \rfloor$ and let $\sigma : {}^2A_{m-1,2} \to S$ be a isomorphism. For $p \in \mathcal{I}_1(W)$ let $S_p = \sigma(T_{(W,2)}(p^\perp, p))$ which is isomorphic to ${}^2A_{m-3,1}(\mathbb{F})$, a symp of S. By Lemma 3.5 there is a point A_p of V and a subspace $B_p \subset A_p^{\perp_V}$ such that B_p/A_p non-degenerate of dimension $m-2$ such that $S_p = T_{(V,2)}(B_p, A_p)$. We have seen for $p \neq q \in \mathcal{I}_1(W)$ that $A_p \neq A_q$. Thus the map $p \to A_p$ of points of $\mathcal{I}_1(W)$ to $\mathcal{I}_1(V)$ is injective.

Next note that if $p \perp_W q$ then $T_{(W,2)}(p^{\perp_W}, p) \cap T_{(W,2)}(q^{\perp_W}, q) = \{\langle p, q\rangle_W\}$. If $x = \sigma(\langle p, q\rangle_W)$ then A_p and A_q must be contained in x. Then they are distinct hyperplanes of x and consequently, $x = \langle A_p, A_q\rangle_V$. In particular, $A_p + A_q = \langle A_p, A_q\rangle_V$ is totally isotropic.

Next suppose $r \neq p$ is a point of $\mathcal{I}_1(\langle p, q\rangle_W)$. Then $\langle p, q\rangle_W = \langle p, r\rangle_W$ from which it follows that $A_p + A_r = A_p + A_q$ which implies in turn that $A_r \in T_{(V,2)}(A_p + A_q, 0)$; since, for $l = 2$, $A_p \cap A_q = 0$.

Finally, suppose that $p, q \in \mathcal{I}_1(W), p$ and q are non-orthogonal. We claim that A_p and A_q are non orthogonal. Suppose to the contrary that $A_p \perp A_q$. Let $r \in \mathcal{I}_1(W)$ with $p \perp r \perp q$ so that $\langle p, r\rangle_W, \langle q, r\rangle_W$ are two points of ${}^2A_2(W)$ which are non-collinear. Then $\sigma(\langle p, r\rangle_W) = A_p + A_r$ and $\sigma(\langle q, r\rangle_W) = A_q + A_r$ are not collinear. However, since $A_p + A_q + A_r \in \mathcal{I}_3(V)$ and $(A_p + A_r) \cap (A_q + A_r) = A_r \neq 0$, $A_p + A_r$ it follows that $A_p + A_r$ and $A_q + A_r$ are collinear points of ${}^2A_{n-1}(V)$, a contradiction.

Assume that $A \in \mathbb{PG}(A_p + A_q)$. We claim that there exists an $r \in \mathbb{PG}(p + q)$ such that $A_r = A$. Towards that end, let s_1, s_2 be non-collinear points of W with $p \perp s_i \perp q$ for $i = 1, 2$. The totally isotropic lines $p + s_i$ and $q + s_i$ meet at s_i and their sum is $p + q + s_i$, which is totally isotropic. Therefore $p + s_i$ and $q + s_i$ are collinear in ${}^2A_2(W)$. Now set $x_i = \sigma(p + s_i), y_i = \sigma(q + s_i), i = 1, 2$. Then $x_i \in T_{(V,2)}(A_p^\perp, A_p)$ and $y_i \in T_{(V,2)}(A_q^\perp, A_q)$ are collinear. It follows that there is a unique point z_i on the line $T_{(V,2)}(x_i + y_i, A_{s_i})$ contained in $T_{(V,2)}(A^\perp, A)$. Since S is a subspace, $z_i \in S$. Since σ is an isomorphism of ${}^2A_2(W)$ onto S there are points $u_i \in {}^2A_2(W)$ such that $\sigma(u_i) = z_i, i = 1, 2$. In fact, u_i belongs to the line $T_{(W,2)}(p + q + s_i, s_i)$. Also, since z_1, z_2 are contained in the type two symp $T_{(V,2)}(A^\perp, A)$ it also follows that $u_1 \cap u_2$ is a point $r \in \mathcal{I}_1(W)$ which belongs to

$(p+q+s_1) \cap (p+q+s_2) = p+q$. It now follows that $S_r \subset T_{(V,2)}(A^\perp, A)$ and consequently that $A_r = A$.

We can now conclude that the injective map $p \to A_p$ defines an isomorphism of the polar space ${}^2A_1(W)$, which is isomorphic to ${}^2A_{m-1,1}(\mathbb{F})$ into ${}^2A_1(V)$. It follows that the image of this map is a non-degenerate m-dimensional subspace B of V. We claim that $S = T_{(V,2)}(B, 0)$.

If $x = \sigma(\langle p, q\rangle_W)$ then $x = A_p + A_q \subset B$ and consequenlty, $S \subset \mathcal{I}_2(B)$. Since S is isomorphic to $T_{(V,2)}(B, 0)$ it follows that $S = T_{(V,2)}(B, 0)$ as claimed. $\square$

Lemma 3.8. *Assume that S is isomorphic to ${}^2A_{5,2}(\mathbb{F})$ and Γ is isomorphic to ${}^2A_{n-1,l}(\mathbb{F})$ with $l > 2$. Then there is a totally isotropic subspace $A, \dim A = l-2$, a subspace B containing A and contained in $A^{\perp_V}$ such that B/A is a six-dimensional non-degenerate space and $S = T_{(V,l)}(B, A)$.*

Proof. Let $U \in \mathcal{I}_3(W)$. We set $M(U) = \sigma(T_{(W,2)}(U, 0))$ which is a singular plane of S. There are two possibilities for M: (i) $M = T_{(V,l)}(D, C)$ with $C \subset D$ totally singular subspaces, $\dim(C) = l-1, \dim(D) = l+2$; or (ii) $M = T_{(V,l)}(D, C)$ with $C \subset D$ totally singular subspaces, $\dim(C) = l-2, \dim(D) = l+1$. We want to show that the first case cannot occur. Toward that end we first show that it is not possible for two different types of planes to occur in S.

By Lemma 2.13 the graph on $\mathcal{I}_3(W)$ given by $U_1 * U_2$ if $U_1 \cap U_2 \in \mathcal{I}_1(W)$ is connected. Consequently, it suffices to show for any such pair that $M(U_1)$ and $M(U_2)$ have the same type. So, let $U_1, U_2 \in \mathcal{I}_3(W)$ with $U_1 \cap U_2 \in \mathcal{I}_1(W)$ and set $M_i = M(U_i), i = 1, 2$ and suppose $M_i = T_{(V,l)}(D_i, C_i)$ where $\dim(C_1) = l-1, \dim(C_2) = l-2, \dim(D_1) = l+2, \dim(D_2) = l+1$.

By Lemma 2.11 there are lines $m_i \subset M_i, i = 1, 2$ such that if $x \in m_1$ then $M_2 \cap \Delta^\Gamma(x) = m_2 \cap \Delta^\Gamma(x)$ is a point, x', and for $y \in m_2 \setminus \{x'\}$ the pair x, y is a type two symp pair. Let $m_1 = T_{(V,l)}(E_1, C_1)$ where E_1 is contained in D_1 and $\dim(E_1) = l+1$ and $m_2 = T_{(V,l)}(D_2, E_2)$ where E_2 is contained in D_2 and $\dim(E_2) = l-1$. We claim that there is no $x \in m_1$ with $x \subset D_2$ and no $y \in m_2$ such that $y \subset D_1$. Suppose to the contrary that $x \in m_1$ and $x \subset D_2$. Then x is a hyperplane of D_2. In particular, $D_2 \subset x^\perp$. Since for all $y \in m_2, \dim(x \cap y) = l-1$ it is then the case that $m_2 \subset \Delta^\Gamma(x)$, a contradiction. We get a similar contradiction if there is a $y \in m_2$ such that $y \subset D_1$.

We next claim that for $x \in m_1, y \in m_2$ the intersection $x \cap y$ is independent of x and y. Assume to the contrary that there are $x \in m_1, y_1, y_2 \in m_2$ such that $x \cap y_1 \neq x \cap y_2$. Since $x \cap y_1$ and $x \cap y_2$ are hyperplanes of x we then get that $x = x \cap y_1 + x \cap y_2 \subset D_1 \cap D_2$, contradicting the above. Wet get a similar contradiction if there are $x_1, x_2 \in m_1, y \in m_2$ such that $x_1 \cap y \neq x_2 \cap y$.

Let $x \in m_1, y \in m_2$. Since $x \cap y$ is independent of the choice of $x, x \cap y$ is contained in $I(m_1) = C_1$. Likewise, $x \cap y$ is contained in $I(m_2) = E_2$. However, $\dim(x \cap y) = l - 1 = \dim(C_1) = \dim(E_2)$ and therefore $C_1 = E_2$.

Now assume that $y \in m_2 = T_{(V,l)}(D_2, E_2) = T_{(V,l)})(D_2, C_1)$ and $x \in M_1 = T_{(V,l)}(D_1, C_1)$. Then $C_1 = x \cap y$ and therefore $\{x, y\}$ is either a collinear or a type two symp pair. However, this contradicts part (i) of Lemma 2.11. Thus, only one type of plane can occur. We show that, in fact, type (i) planes do not occur.

Suppose to the contrary that all the planes of S are type of (i). Let $U_1, U_2 \in \mathcal{I}_3(W)$ with $U_1 \cap U_2 \in \mathcal{I}_2(W)$ and set $M_i = M(U_i) = T_{(V,l)}(D_i, C_i)$ where D_i, C_i are isotropic subspaces with $C_i \subset D_i, \dim(C_i) = l - 1$ and $\dim(D_i) = l + 2$. Set $x = \sigma(U_1 \cap U_2)$. Since $M_1 \cap M_2 = \{x\}$ either $C_1 + C_2 = x$ or $C_1 = C_2$ and $D_1 \cap D_2 = x$. Suppose $C_1 + C_2 = x$. Let $y \in M_1, y \neq x$ so that C_2 is not contained in y. By pulling back to ${}^2A_2(W)$ and using the isomorphism σ we can conclude that there is a line $\lambda_y \subset M_2$ containing x such that if $y' \in \lambda_y \setminus \{x\}$ then y, y' is a symp pair and therefore $\dim(y \cap y') = l - 1$. Now the line λ_y must be of the form $T_{(V,l)}(D, C_2)$ for some subspace D of $D_2, \dim(D) = l + 1$. But then $I(\lambda_y) = C_2$. Since $\dim(y \cap z) = l - 1$ for all $z \in \lambda_y$ and $\cap_{z \in \lambda_y} z = C_2$ is not contained in y it follows that there are $z_1, z_2 \in \lambda_y$ such that $y \cap z_1 \neq y \cap z_2$. Then $y = y \cap z_1 + y \cap z_2 \subset D_1 \cap D_2$. Since y is arbitrary and $y \neq x$ it follows that $D_1 = \Sigma(M_1) \subset D_2$ and therefore $D_1 = D_2$. But then for each point $y \in M_1 \setminus \{x\}, M_2 \cap \Delta^\Gamma(y)$ is a line, a contradiction. Thus, $C_1 = C_2$ in this case as well. However, since the graph on $\mathcal{I}_3(W)$ given by $U_1 \sim U_2$ if and only if $U_1 \cap U_2 \in \mathcal{I}_2(W)$ is connected, it must be the case that for any $U_1, U_2 \in \mathcal{I}_3(W)$ if $M_i = M(U_i) = T_{(V,l)}(D_i, C_i)$ then $C_1 = C_2 = C$. But then it follows that $S \subset T_{(V,l)}(C^\perp, C)$ a symp, which is a contradiction. Thus, every singular plane of S is of type (ii).

Now let $U_1, U_2 \in \mathcal{I}_3(W)$ with $U = U_1 \cap U_2 \in \mathcal{I}_2(W)$ and set $M_i = M(U_i) = T_{(V,l)}(D_i, C_i), i = 1, 2$ and $x = \sigma(U) \in M_1 \cap M_2$. Then $x \subset D_1 \cap D_2$ and $C_1 + C_2 \subset x$. If $D_1 \cap D_2 \neq x$ and $C_1 + C_2 \neq x$ then $T_{(V,l)}(D_1 \cap D_2, C_1 + C_2)$ is contained in $M_1 \cap M_2$ has points in addition to x, a contradiction. We claim that $C_1 = C_2$. Suppose to the contrary that $C_1 \neq C_2$. As in the previous paragraph, for $y \in M_1$ we will denote by λ_y a line in M_2 containing x such that for $x \neq y' \in \lambda_y$ the pair y, y' is a symp pair and therefore $\dim(y \cap y') = l - 1$. And, as shown above, $I(\lambda_y) = C_2$.

We have $I(M_1) = C_1 \neq C_2 = I(M_2)$. Since $\cap_{z \in \lambda_y}(y \cap z) \subset C_2$ has dimension $l-2$ and $\dim(y \cap z) = l-1$ for $z \in \lambda_y$ there must be $z_1, z_2 \in \lambda_y$ with $y \cap z_1 \neq y \cap z_2$. Then $y \cap z_1, y \cap z_2$ are distinct hyperplanes of y and $y = (y \cap z_1) + (y \cap z_2) \subset D_1 \cap D_2$. Since y is arbitrary, $D_1 = \Sigma(M_1) \subset D_2$ and therefore $D_1 = D_2$. But any two hyperplanes of $D_1 = D_2$ are then collinear, whence every point of M_1 with every point of M_2, a contradiction. Thus, $C_1 = C_2$.

As argued previously, this implies there is a fixed $l-2$ dimensional subspace C such that $M(U) = T_{(V,l)}(D, C)$ for all $U \in \mathcal{I}_3(W)$. But then S is contained in $T_{(V,l)}(C^\perp, C)$ which is isomorphic to ${}^2A_{n-l+1,2}(\mathbb{F})$ and we are done by Lemma 3.7. $\square$

Lemma 3.9. *Assume that S is isomorphic to ${}^2A_{m-1,2}(\mathbb{F})$ and Γ is isomorphic to ${}^2A_{n-1,l}(\mathbb{F})$ with $l > 2$. Then there is a totally isotropic subspace $A, \dim(A) = l-2$, and a subspace B which contains A and is contained in $A^{\perp_V}$ and such that B/A is an m-dimensional non-degenerate space and $S = T_{(V,l)}(B, A)$.*

Proof. For a point $p \in \mathcal{I}_1(W)$ we let $S_p = \sigma(T_{(W,2)}(p^\perp, p))$ which is isomorphic to ${}^2A_{m-3,1}(\mathbb{F})$. By Lemma 3.5, $S_p = T_{(V,l)}(B_p, A_p)$ where A_p is a totally isotropic space of dimension $l-1, B_p$ contains A_p and is a subset of $A_p^{\perp_V}$ and B_p/A_p is non-degenerate of dimension $m-2$. From Lemma 3.6 the map $p \to A_p$ is injective. Now suppose q_1, q_2 are two isotropic points of W such that $q_1 \perp_W p \perp_W q_2$. We claim that $A_p \cap A_{q_1} = A_p \cap A_{q_2}$.

Let W' be the non-degenerate six dimensional subspace of W which contains $p + q_1 + q_2$ and let $S' = \sigma(T_{(W,2)}(W', 0))$ which is isomorphic to ${}^2A_{5,2}(\mathbb{F})$. By Lemma 3.8 it follows that $S' = T_{(V,l)}(D, A)$ where A is a totally isotropic subspace, $\dim(A) = l-2, D$ is a subspace containing A and contained in $A^{\perp_V}$, and D/A is non-degenerate of dimension six. For a point $y \in \mathcal{I}_1(W')$ set $S'_y = S_y \cap S'$. Then S'_y is isomorphic to ${}^2A_{3,1}(\mathbb{F})$ and $S'_y = T_{(V,l)}(B_y \cap D, A_y)$. Now for all $y \in S', A_y \supset A$. On the other hand, if $y, z \in S'$ with $A_y \neq A_z$ then $A_y \cap A_z = A$. In particular, $A_p \cap A_{q_1} = A = A_p \cap A_{q_2}$. Now the graph whose vertices consists of those of pairs $\{p, q\}$ in $\mathcal{I}_1(W)$ with $p \perp_W q$ given by $\alpha \sim \beta$ if and only if $|\alpha \cap \beta| = 1$ is connected. From this it follows that $I(S) = A$ and $S \subset T_{(V,l)}(A^\perp, A)$. Applying Lemma 3.7 completes the result. $\square$

We next take up the case where S is a subspace of ${}^2A_l(V)$ which is isomorphic to ${}^2A_{m-1,l}(\mathbb{F})$. We will make use of our inductive hypothesis: if S' is isomorphic to ${}^2A_{m^*,l^*}(\mathbb{F})$ is a subspace of ${}^2A_l(V)$ with $m^* + l^* < m + l$ then the conclusion of our theorem holds: there is a totally isotropic subspace A of dimension $l - l^*$ and a subspace B with $A \subset B \subset A^{\perp_V}$ such that B/A is non-degenerate of dimension m^* with $S' = T_{(V,l)}(B, A)$.

Before proceeding to the proof we obtain a lemma about "large" subspaces of unitary polar spaces which will be used in the succeeding result.

Lemma 3.10. *Let (V, f) be a non-degenerate unitary space of dimension n and Witt index $n' = \lfloor \frac{n}{2} \rfloor > 2$ and let $1 < l \leq n'$. Let X be a proper subspace of W and assume for every element of $x \in \mathcal{I}_l(W)$ that $x \subset X$ or $x \cap X$ is a hyperplane of x. Then X is a hyperplane of W.*

Proof. We claim that for every $z \in \mathcal{I}_2(V)$ that $z \cap X \neq \{\mathbf{0}\}$ from which it will follow that $\mathcal{I}_1(X)$ is a geometric hyperplane of the polar space $(\mathcal{I}_1(V), \mathcal{I}_2(V))$ and then X is a linear hyperplane of V. If $l = 2$ then there is nothing to prove. Suppose $2 < l$ and $z \in \mathcal{I}_2(V)$. Let $x \in \mathcal{I}_l(V)$ with $z \subset x$. If $x \subset X$ then $z \subset X$ so we may assume that $x \not\subset X$ so that $x \cap X$ is a hyperplane of x. Then we have either $z \subset x \cap X \subset X$ or $z \cap [x \cap X]$ is a point. Since $z \cap [x \cap X] \subset z \cap X$ it follows that $z \cap X \neq \{\mathbf{0}\}$. $\square$

Lemma 3.11. *Assume $l \geq 3, m \geq 2(l+1)$ and S is a subspace of ${}^2A_l(V)$ is isomorphic to ${}^2A_{m-1,l}(\mathbb{F})$. Then there is non-degenerate subspace B of dimension m such that $S = T_{(V,l)}(B, 0)$.*

Proof. The proof of this closely follows the proof of Lemma 3.7 but differs in enough of its details to warrant its inclusion.

As previously defined, for a point $p \in \mathcal{I}_1(W)$ we let $S_p = \sigma(T_{(W,l)}(p^\perp, p))$ which is isomorphic to ${}^2A_{m-3,l-1}(\mathbb{F})$. By our inductive hypothesis there is an isotropic point A_p and a subspace B_p satisfying $A_p \subset B_p \subset A_p^{\perp_V}$ with B_p/A_p non-degenerate of dimension $m-2$ and $S_p = T_{(V,l)}(B_p, A_p)$. We first show that the map $p \to A_p$ from $\mathcal{I}_1(W)$ to $\mathcal{I}_1(V)$ is injective.

Suppose first that $p \neq q \in \mathcal{I}_1(W)$ are orthogonal and $A_p = A_q$. Set $A = A_p = A_q$. Note that $S_p \cap S_q = \sigma(T_{(W,k)}(\langle p, q\rangle^{\perp_W}, \langle p, q\rangle)) = T_{(V,l)}(B_p \cap B_q, A)$ is isomorphic to ${}^2A_{m-5,l-2}(\mathbb{F})$. Consequently, the dimension of $[B_p \cap B_q]/A$ is $m-4$.

By Lemma 2.8 if $x \in S_p$ then either $x \in S_q, \Delta^\Gamma(x) \cap S_q$ is a singular subspace isomorphic to $\mathbb{PG}_{l-1}(\mathbb{F})$, or there is a unique $y \in \Delta_{(2,2)}(x) \cap S_q$.

In the first case $x \subset B_q$. In the second case, if $y \in S_q \cap \Delta^\Gamma(x)$, then $x \cap y$ is a hyperplane of x and therefore we can conclude that $B_q \cap x$ contains a hyperplane of x. Finally, in the third case, if $y \in \Delta_{(2,2)}(x) \cap S_q$ then $x \cap y$ is a hyperplane of x and again $B_q \cap x$ contains a hyperplane of x.

It therefore follows that either $x/A \subset (B_p \cap B_q)/A$ or the intersection of x/A and $(B_p \cap B_q)/A$ is a hyperplane of x/A for every $x \in T_{(V,l)}(B_p, A)$. Since $l > 2$, Lemma 3.10 applies and $(B_p \cap B_q)/A$ is a hyperplane of B_p/A. In particular, $\dim([B_p \cap B_q]/A) = m - 3$, a contradiction.

Now assume that p and q are non-orthogonal points of W and that $A_p = A_q = A$. Note that $S_p \cap S_q = \emptyset$. Let $x \in S_p = T_{(V,l)}(B_p, A_p) = T_{(V,l)}(B_p, A)$. Then it cannot be the case that $x \subset B_q$ because otherwise we would have $x \in T_{(V,l)}(B_q, A) = S_q$ contradicting $S_p \cap S_q = \emptyset$. On the other hand, there is a unique point $y \in \Delta_{(2,2)}(x) \cap S_q$. Then $x \cap y$ is a hyperplane of x contained in $B_q \cap x$. It therefore follows that for every $x \in T_{(V,l)}(B_p, A)$, either x/A is contained in $(B_p \cap B_q)/A$ or else $(B_p \cap B_q)/A$ meets x/A in a hyperplane.

By Lemma 3.10 it follows that $(B_p \cap B_q)/A$ is a hyperplane of B_p/A. Since $l < m = \lfloor \frac{n}{2} \rfloor$, the index of (V, f), it must be the case that $B_p \cap B_q$ contains an element of $T_{(V,l)}(B_p, A)$ contradicting $S_p \cap S_q = \emptyset$. Thus, the map from $\mathcal{I}_1(W)$ to $\mathcal{I}_1(V), p \to A_p$ is injective.

When $p \neq q \in \mathcal{I}_1(W)$ and $p \perp_W q$ then $S_p \cap S_q \neq \emptyset$ from which it follows that $A_p \perp_V A_q$. On the other hand, suppose $p, q \in \mathcal{I}_1(W)$ and are non-orthogonal. We claim that A_p and A_q are non-orthogonal. Suppose to the contrary that $A_p \perp_V A_q$. We will get a contradiction.

We first show that either $A_p \subset B_q$ or $A_q \subset B_p$. Let $U \in \mathcal{I}_{l-1}(W)$ be contained in $p^{\perp_W} \cap q^{\perp_W}$ and set $X = \langle U, p \rangle_W, Y = \langle U, q \rangle_W$. Then $X, Y \in {}^2A_l(W)$ and belong to the type two symp $T_{(W,l)}(U^{\perp}, U)$. Set $x = \sigma(X), y = \sigma(Y)$. Then $(x, y) \in \Delta_{(2,2)}$ and so $x \cap y \in \mathcal{I}_{l-1}(V)$. Suppose neither A_p nor A_q is contained in $x \cap y$. Then $x + y = \langle x \cap y, A_p, A_q \rangle_{\mathbb{F}}$ is totally isotropic which means that x and y are collinear, a contradiction. This proves our assertion. Without loss of generality we can assume that $A_p \subset B_q$.

By Lemma 2.8, for each $x' \in S_p$ there is a unique point $y' \in S_q$ with $(x', y') \in \Delta_{(2,2)}$. Then $x' \cap y' \subset B_q$ is a hyperplane. By Lemma 3.10, $(B_p \cap B_q)/A_p$ is a hyperplane of B_p/A_p and consequently, $B_p \cap B_q$ is a hyperplane of B_p. It then follows that $T_{(V,l)}(B_p \cap B_q, A_p) \neq \emptyset$. Let $x' \in T_{(V,l)}(B_p \cap B_q, A_p)$. If $A_q \subset x'$ then $x' \in S_p \cap S_q$, a contradiction. However, it now follows that $\langle x', A_q \rangle_V \subset B_q$ and that $T_{(V,l)}(\langle x', A_q \rangle_V, A_q) \subset \Delta^{\Gamma}(x') \cap S_q$, a contradiction. Thus, if $p, q \in \mathcal{I}_1(W)$ are non-orthogonal then the points A_p and A_q in V are non-orthogonal.

We next show that if $X \in \mathcal{I}_2(W)$ then $\{A_p : p \in \mathbb{PG}(X)\}$ is contained in a totally singular line of V. Let $p \neq q \in \mathcal{I}_1(W)$ with $p \perp_W q$ and let $r \in \mathbb{PG}(\langle p, q \rangle_W), r \neq p$. Then $S_p \cap S_q = S_p \cap S_r$. Therefore, $T_{(V,l)}(B_p \cap B_q, A_p + A_q) = T_{(V,l)}(B_p \cap B_r, A_p + A_r)$. In particular, $A_r \in \mathbb{PG}(A_p + A_q)$.

Finally, we prove that if $p \neq q \in \mathcal{I}_1(W)$ with $p \perp q$ then the collection $\{A_r : r \in \mathbb{PG}(\langle p, q \rangle_W\} = \mathbb{PG}(\langle A_p, A_q \rangle_V)$. Let $U \in \mathcal{I}_{l+1}(W)$ with $\langle p, q \rangle_W \subset U$. Then $T_{(W,l)}(U, 0)$ is a type one maximal singular subspace of ${}^2A_l(W)$ and isomorphic to $\mathbb{PG}_l(\mathbb{F})$. Set $X = \sigma(T_{(W,l)}(U, 0))$, a singular subspace of ${}^2A_l(V)$ (here we are making use of the assumption that S is a subspace, not just a subgeometry, of ${}^2A_l(V)$). Note that $X \cap S_p$ is a type one maximal singular subspace of $T_{(V,l)}(B_p, A_p)$ and consequently must be of the form $T_{(V,l)}(U', A_p)$ for $U' \in \mathcal{I}_{l+1}(V)$. It follows that $X = T_{(V,l)}(U', 0)$. Now suppose that $A \in \mathbb{PG}(A_p + A_q)$. Then $X_A = X \cap T_{(V,l)}(A^{\perp}, A)$ is a hyperplane of X and so, $\sigma^{-1}(X_A)$ is a hyperplane of $T_{(W,l)}(U, 0)$ and therefore there must be a point $r \in \mathbb{PG}(\langle p, q \rangle_W)$ such that $\sigma^{-1}(X_A) = T_{(W,k)}(U, r)$. Then $X_A \subset S_r = T_{(V,l)}(B_r, A_r)$. Note that $I(X_A) = A$ and consequently, $A = A_r$ completing the assertion.

We can now say that the map $p \to A_p$ of $\mathcal{I}_1(W)$ into $\mathcal{I}_1(V)$ is a full embedding of the polar space $(\mathcal{I}_1(W), \mathcal{I}_2(W))$ into the polar space $(\mathcal{I}_1(V), \mathcal{I}_2(V))$. This implies that $B = \{A_p : p \in \mathcal{I}_1(W)\}$ is a non-degenerate m-dimensional space of V. This completes the lemma. $\square$

We now complete our main result. We can assume that S is isomorphic to ${}^2A_{n'-1,l'}(\mathbb{F})$ is a subspace of ${}^2A_l(V)$, which is isomorphic to ${}^2A_{n-1,l}(\mathbb{F})$, with $l' < l$. We will show that there is a totally isotropic subspace A of dimension $l - l'$ such that $S \subset T_{(V,l)}(A^{\perp_V}, A)$ and then the result will follow from Lemma 3.11.

Let U be a non-degenerate subspace of W of dimension $2(l'+1)$ and Witt index $l'+1$, Y a maximal totally singular subspace of U and X a subspace of Y of dimension $l'-2$. Set $M = M(Y) = \sigma(T_{(W,l')}(Y,0))$ a singular subspace of S isomorphic to $\mathbb{PG}(l', \mathbb{F})$. Also, set $S(U) = \sigma(T_{(W,l')}(U,0))$ which is isomorphic to ${}^2A_{l'+1,l'}(\mathbb{F})$ and $S' = \sigma(T_{(W,l')}(U \cap X^{\perp_W}, X))$ which is isomorphic to ${}^2A_{5,2}(\mathbb{F})$ and $M' = S' \cap M$. We have seen in Lemma 3.8 that $M' = T_{(V,l)}(D, C)$ for totally isotropic subspaces $C \subset D$ with $\dim(C) = l-2, \dim(D) = l+1$. $T_{(V,l)}(D,0)$ is the unique maximal singular subspace of Γ containing M'. Since $M' \subset M$ it follows that $M \subset T_{(V,l)}(D,0)$ and consequently, $M = T_{(V,l)}(D_Y, A_Y)$ where $\dim(A_Y) = l - l'$ and $\dim(D_Y) = l+1$. Note that since $D_Y \subset D$ and $\dim(D_Y) = \dim(D)$ it follows that $D_Y = D$.

We next claim that if Y_1, Y_2 are totally isotropic subspaces of W of dimension $l'+1$ which satisfy $\dim(Y_1 \cap Y_2) = l'$ and Y_1, Y_2 are not orthogonal then $A_{Y_1} = A_{Y_2}$. For convenience set $D_{Y_i} = D_i, A_{Y_i} = A_i, i = 1, 2$. The singular subspaces $M_i = T_{(V,l)}(D_i, A_i)$ have a common point x. Moreover, there is a one-to-one correspondence between the lines on x in M_1 and the lines on x in M_2 such that if λ is a line on x in M_1 and λ' is the corresponding line in M_2 then for $x \neq y \in \lambda$, $x \neq z \in \lambda'$ it follows that y, z is a type two symp pair and $y \cap z$ has dimension $l-1$.

Fix y in $M_1, y \neq x$ and let λ_y be the line on x and y. Suppose there are $z_1, z_2 \in \lambda'_y$ such that $y \cap z_1 \neq y \cap z_2$ then $y \cap z_1$ and $y \cap z_2$ are distinct hyperplanes of y and then $y = y \cap z_1 + y \cap z_2 \subset D_1 \cap D_2$. Then y is a hyperplane of D_2 in which case $M_2 \subset \Delta^\Gamma(y)$, a contradiction. Thus, $y \cap z_1 = y \cap z_2$ for any points $z_1, z_2 \subset \lambda'_y$. By reversing the argument we can conclude that $\cap_{w \in \lambda_y} w = \cap_{z \in \lambda'_y} z$ has dimension $l-1$. From this it follows that $A_1 = I(M_1) = \cap_{y \in M_1} y = \cap_{z \in M_2} z = I(M_2) = A_2$ as claimed.

Finally, since the graph on $\mathcal{I}_{l'+1}(W)$ given by $Y_1 \sim Y_2$ if and only if $\dim(Y_1 \cap Y_2) = l'$ and Y_1 and Y_2 non-orthogonal is connected, it follows for any two $Y_1, Y_2 \in \mathcal{I}_{l'+1}(W)$ that $A_{Y_1} = A_{Y_2}$. Let $A = A_Y$ for some totally isotropic subspace of dimension $l'+1$ in W. Since every point x of S belongs to a singular subspace $M(Y)$ for some totally isotropic subspace Y of W of dimension $l'+1$, it follows that $x \in T_{(V,l)}(A^{\perp_V}, A)$ and the proof of the main result is complete. $\square$

References

[1] **Rieuwert J.Blok** and **Bruce N. Cooperstein**, Projective subgrassmannians of polar Grassmannians, *Bull. Belg. Math. Soc. Simon Stevin* **17** (2010), no. 4, 675–691.

[2] ______, The generating rank of unitary and symplectic Grassmannians, *J. Combin. Theory Ser. A* **119** (2012), 675–691.

[3] **Francis Buekenhout**, The foundations of incidence geometry, **in** *Handbook of Incidence Geometry, Buildings and Foundations* (ed. F. Buekenhout), Chapter 9, Elsevier, North-Holland, 1995, 63–106.

[4] **Bruce N. Cooperstein**, Some geometries associated with parabolic representations of groups of Lie type, *Canad. J. Math.* **28**, no. 5 (1976), 1021–1031.

[5] ______, Symplectic subspaces of symplectic Grassmannians, *European J. Combin.* **28**, no. 5 (2007), 1442–1454

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