Hyperplanes of symplectic dual polar spaces: a survey

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Abstract

This paper surveys the most important results about hyperplanes of symplectic dual polar spaces. These results concern constructions of such hyperplanes, classification results and characterization results. Also the problem which hyperplanes arise from full projective embeddings will be considered here.

1 Introduction

Let V be a vector space of even dimension $2n \geq 4$ over a field $\mathbb{F}$ that is equipped with a nondegenerate alternating bilinear form f. For every subspace U of V, we denote by $U^{\perp}$ the subspace of V consisting of all vectors $\bar{u} \in V$ for which $f(\bar{u}, \bar{u}') = 0$, $\forall \bar{u}' \in U$. We denote by ζ the symplectic polarity of $\mathrm{PG}(V)$ associated with f. Then $\mathrm{PG}(U)^{\zeta} = \mathrm{PG}(U^{\perp})$ for every subspace U of V. A subspace U of V is called *totally isotropic* (*with respect to* f) if $U \subseteq U^{\perp}$. A subspace α of $\mathrm{PG}(V)$ is called *totally isotropic* (*with respect to* ζ) if $\alpha \subseteq \alpha^{\zeta}$. We denote by $\mathcal{P}$ the set of all points of $\mathrm{PG}(V)$ and by Σ the set of all totally isotropic subspaces of $\mathrm{PG}(V)$.

The structure $\Pi = (\mathcal{P}, \Sigma)$ is a *polar space* of *rank* n in the sense of Tits [62], which means that $(\mathcal{P}, \Sigma)$ satisfies the following four properties:

(P1) Any subspace of Π (i.e. any element of Σ) together with the subspaces contained in it define a projective space of dimension at most $n - 1$.

(P2) The intersection of two subspaces of Π is again a subspace of Π.

(P3) If α is a maximal subspace of Π (i.e. a subspace of projective dimension $n-1$), and p is a point of Π (i.e. an element of $\mathcal{P}$) not contained in α, then there is a unique maximal subspace β of Π that contains p and intersects α in a subspace of dimension $n-2$. Moreover, the intersection $\alpha \cap \beta$ consists precisely of those points of α that are contained in a 1-dimensional subspace together with p.

(P4) There exist two disjoint maximal subspaces.

We will denote Π also by $W(2n-1, \mathbb{F})$ and call it a *symplectic polar space*. If $\mathbb{F}$ is a finite field with q elements, then we denote $W(2n-1, \mathbb{F})$ also by $W(2n-1, q)$.

With the polar space Π, there is associated a so-called *dual polar space* Δ of *rank* n (see Cameron [7]). This is the point-line geometry $(\mathcal{M}, \mathcal{L}, \mathrm{I})$, where

- $\mathcal{M}$ is the set of maximal subspaces of Π;

- $\mathcal{L}$ is the set of next-to-maximal subspaces of Π, i.e. the subspaces of Π of projective dimension $n-2$;

- The incidence relation $\mathrm{I} \subseteq \mathcal{M} \times \mathcal{L}$ corresponds to reverse containment, i.e. if $p \in \mathcal{M}$ and $L \in \mathcal{L}$, then $(p, L) \in \mathrm{I}$ if and only if $L \subseteq p$.

We denote Δ also by $DW(2n - 1, \mathbb{F})$ and call it a *symplectic dual polar space*. If $\mathbb{F}$ is a finite field with q elements, then we denote $DW(2n - 1, \mathbb{F})$ also by $DW(2n - 1, q)$. Distances between points or nonempty sets of points of $\Delta = DW(2n - 1, \mathbb{F})$ will always be measured in the collinearity graph of Δ. The maximal distance between two points of $DW(2n - 1, \mathbb{F})$ is equal to n.

A set H of points of $\Delta = DW(2n-1, \mathbb{F})$ is called a *hyperplane* if $H \neq \mathcal{M}$ and if every line of Δ intersects H in either a singleton or the whole line.

A full projective embedding of $\Delta = DW(2n-1, \mathbb{F})$ is an injective map e from $\mathcal{M}$ to the point set of a projective space $\mathrm{PG}(W)$ satisfying the following:

(1) the image of e generates the whole projective space $\mathrm{PG}(W)$;

(2) e maps lines of Δ to full lines of $\mathrm{PG}(W)$.

We denote a full projective embedding e of Δ into $\mathrm{PG}(W)$ also by $e : \Delta \to \mathrm{PG}(W)$.

Suppose $e : \Delta \to \mathrm{PG}(W)$ is a full projective embedding of the dual polar space $\Delta = DW(2n - 1, \mathbb{F})$ into $\mathrm{PG}(W)$ and α is a hyperplane of $\mathrm{PG}(W)$. Then the set $H_\alpha := e^{-1}(e(\mathcal{M}) \cap \alpha)$ is a hyperplane of Δ. Any hyperplane of Δ which can be obtained in this way is said to *arise from the embedding e*. A hyperplane of Δ is called *classical* if it arises from some full projective embedding of Δ.

Suppose now that $e : \Delta \to \mathrm{PG}(W)$ is a full projective embedding of $\Delta = DW(2n-1, \mathbb{F})$ and α is a subspace of $\mathrm{PG}(W)$ disjoint from the image of e. Denote by $\mathrm{PG}(W)_\alpha$ the quotient projective space whose points are the subspaces of $\mathrm{PG}(W)$ that contain α as a hyperplane. For every point x of Δ, let $e_\alpha(x)$ be the point $\langle e(x), \alpha \rangle$ of $\mathrm{PG}(W)_\alpha$. Then e_α defines a full projective embedding of Δ into $\mathrm{PG}(W)_\alpha$. We call e_α a *quotient* of e.

Suppose $e_1 : \Delta \to \mathrm{PG}(W_1)$ and $e_2 : \Delta \to \mathrm{PG}(W_2)$ are two full projective embeddings of the dual polar space $\Delta = DW(2n-1, \mathbb{F})$. Then e_1 and e_2 are said to be *isomorphic* if there exists an isomorphism $\phi : \mathrm{PG}(W_1) \to \mathrm{PG}(W_2)$ such that $e_2 = \phi \circ e_1$. We write $e_1 \geq e_2$ if e_2 is isomorphic to a quotient of e_1.

Suppose $\mathcal{E}$ is a set of (mutually nonisomorphic) full projective embeddings of $\Delta = DW(2n-1, \mathbb{F})$ such that every full projective embedding of Δ is isomorphic to precisely one element of $\mathcal{E}$. Then $(\mathcal{E}, \leq)$ is a poset.

Let $\bigwedge^n V$ denote the n-th exterior power of V. If $p = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$ is an n-dimensional subspace of V, then $e_{gr}(p) := \langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \rangle$ is a point of $\mathrm{PG}(\bigwedge^n V)$. The subspace $\mathrm{PG}(W)$ of $\mathrm{PG}(\bigwedge^n V)$ generated by all points $e_{gr}(p)$, where p is an n-dimensional totally isotropic subspace of V, has projective dimension $\binom{2n}{n} - \binom{2n}{n-2} - 1$, see e.g. Bourbaki [4, 13.3], Burau [6, 82.7], De Bruyn [18, Theorem 1.1] and Premet & Suprunenko [54, p. 1337]. Moreover, the map $p \mapsto e_{gr}(p)$ defines a full projective embedding of $\Delta = DW(2n-1, \mathbb{F})$ into $\mathrm{PG}(W)$, see Cooperstein [10, Proposition 5.1]. This embedding is called the *Grassmann embedding* of Δ.

A full projective embedding $\widetilde{e} : \Delta \to \mathrm{PG}(\widetilde{W})$ of $\Delta = DW(2n-1, \mathbb{F})$ is called *absolutely universal* if $\widetilde{e} \geq e$ for any full projective embedding e of Δ. By results of Tits [62, 8.6] and Kasikova & Shult [38, 4.6], we know that a dual polar space admits an absolutely universal full projective embedding if it has at least one full projective embedding. In particular, we thus know that the dual polar space $\Delta = DW(2n-1, \mathbb{F})$ has an absolutely universal embedding. If $|\mathbb{F}| \geq 3$, then by results of Cooperstein [10, Theorem B] and De Bruyn & Pasini [32, Corollary 1.2], we know that the Grassmann embedding of $DW(2n-1, \mathbb{F})$ is absolutely universal. This is false if $|\mathbb{F}| = 2$ and $n \geq 3$. Li [39] and Blokhuis & Brouwer [3] independently showed that the absolutely universal embedding of $DW(2n-1, 2)$, $n \geq 2$, has vector dimension $\frac{(2^n+1)(2^{n-1}+1)}{3}$.

If e_1 and e_2 are two full projective embeddings of $\Delta = DW(2n-1, \mathbb{F})$ such that $e_1 \geq e_2$, then any hyperplane of Δ arising from e_2 also arises from e_1. Hence, the classical hyperplanes of $\Delta = DW(2n-1, \mathbb{F})$ are precisely the hyperplanes of Δ arising from the absolutely universal embedding of Δ.

Suppose α is a subspace of $\Pi = W(2n-1, \mathbb{F})$ of dimension $n-1-k$ where $k \in \{0, 1, \ldots, n\}$, and denote by F_α the set of all maximal subspaces of Π containing α. Then F_α, regarded as a set of points of $\Delta = DW(2n-1, \mathbb{F})$, satisfies the

following properties:

(i) If a line of Δ has at least two of its points in F_α, then all its points are contained in F_α.

(ii) If x and y are two points of F_α, then any point of Δ on a shortest path between x and y also belongs to F_α.

So, F_α is a convex subspace. It can be shown that every nonempty convex subspace of Δ arises in this way. We denote by $\widetilde{F_\alpha}$ the point-line geometry induced on F_α by the lines of Δ that have all their points in F_α. If δ denotes the maximal distance between two points of F_α, then $\widetilde{F_\alpha}$ is a point if $\delta = 0$, a line if $\delta = 1$ and a symplectic dual polar space isomorphic to $DW(2\delta - 1, \mathbb{F})$ if $\delta \geq 2$. The convex subspace F_α is called a *quad* if $\delta = 2$ and a *hex* if $\delta = 3$. If F is a quad, then $\widetilde{F}$ is a generalized quadrangle, that is, a point-line geometry that satisfies the following properties:

(GQ1) every two distinct points are incident with at most one line;

(QQ2) every point is incident with at least two lines;

(GQ3) for every non-incident point-line pair (x, L), there exists a unique point on L collinear with x.

If H is a hyperplane and F a nonempty convex subspace of $DW(2n - 1, \mathbb{F})$, then either $F \subseteq H$ or $H \cap F$ is a hyperplane of $\widetilde{F}$.

Let $Q(2n, \mathbb{F})$ denote the quadric of $\mathrm{PG}(2n, \mathbb{F})$ defined by the equation $X_0^2 + X_1 X_2 + \cdots + X_{2n-1} X_{2n} = 0$. Here, $(X_0, X_1, \ldots, X_{2n})$ are the coordinates of a generic point of $\mathrm{PG}(2n, \mathbb{F})$ with respect to some fixed reference system. The points and subspaces of $\mathrm{PG}(2n, \mathbb{F})$ contained in $Q(2n, \mathbb{F})$ define a polar space which we will also denote by $Q(2n, \mathbb{F})$. With $Q(2n, \mathbb{F})$, there is also associated a dual polar space which we denote by $DQ(2n, \mathbb{F})$. If $\mathbb{F}$ is a finite field with q elements, then we denote $(D)Q(2n, \mathbb{F})$ also by $(D)Q(2n, q)$.

The (dual) polar spaces $(D)Q(2n, \mathbb{F})$ and $(D)W(2n - 1, \mathbb{F})$ are isomorphic if and only if $\mathbb{F}$ is a perfect field of characteristic 2, i.e. a field of characteristic 2 in which each element is a square. In the finite case, we thus know that the (dual) polar spaces $(D)Q(2n, q)$ and $(D)W(2n - 1, q)$ are isomorphic if and only if q is an even prime power.

This paper intends to give a survey of the most important results regarding hyperplanes of symplectic dual polar spaces. We provide answers to the problems mentioned in the following list:

(1) Find families of hyperplanes of symplectic dual polar spaces.

(2) Characterize families of hyperplanes by means of their intersections with convex subspaces.

(3) For particular values of n and $\mathbb{F}$, classify all hyperplanes of $DW(2n-1, \mathbb{F})$.

(4) For particular values of n and $\mathbb{F}$, classify all classical hyperplanes of $DW(2n-1, \mathbb{F})$.

(5) For which n and $\mathbb{F}$ are all hyperplanes of $DW(2n-1, \mathbb{F})$ classical?

(6) If not all hyperplanes of $DW(2n-1, \mathbb{F})$ are classical, then find necessary and sufficient conditions for a hyperplane of $DW(2n-1, \mathbb{F})$ to be classical.

2   Some families of hyperplanes

2.1   Hyperplanes constructed from other hyperplanes

Before describing some families of hyperplanes of symplectic dual polar spaces, we give two constructions that allow to construct hyperplanes from other hyperplanes.

Construction 1. Suppose $\mathbb{F}$ is the field with two elements. Then every line of $DW(2n-1, \mathbb{F}) = DW(2n-1, 2)$ is incident with precisely three points. If X_1 and X_2 are two sets of points of $DW(2n-1, 2)$, then we denote by $X_1 * X_2$ the complement of the symmetric difference $(X_1 \setminus X_2) \cup (X_2 \setminus X_1)$ of X_1 and X_2. If H_1 and H_2 are two distinct hyperplanes of $DW(2n-1, 2)$, then $H_1 * H_2$ is again a hyperplane of $DW(2n-1, 2)$.

Construction 2. Suppose F is a convex subspace of diameter δ of the dual polar space $\Delta = DW(2n-1, \mathbb{F})$. Suppose G is a hyperplane of $\widetilde{F}$. The maximal distance from a point of Δ to F is equal to $n - \delta$. Every point x of Δ at distance $k \in \{0, \ldots, n - \delta\}$ from F lies at distance k from a unique point $\pi_F(x)$ of F. Denote now by H the set of points of Δ consisting of all points at distance at most $n - \delta - 1$ from F together with all points x of Δ at distance $n - \delta$ from F for which $\pi_F(x) \in G$. By De Bruyn and Vandecasteele [34, Proposition 1], we then know that H is a hyperplane of Δ. We will call H *the extension* of G. If $F = \Delta$, then we say that H is a *trivial extension* of G.

2.2   Singular hyperplanes

Recall that the maximal distance between two points of $\Delta = DW(2n-1, \mathbb{F})$ is equal to n.

For every point x of Δ, we denote by H_x the set of points of Δ at non-maximal distance from x, i.e. the set of points of Δ at distance at most $n-1$ from x. Then H_x is a hyperplane of Δ, which is called the *singular hyperplane of Δ with deepest point x*.

Suppose F is a convex subspace of diameter δ of Δ and G is the singular hyperplane of $\widetilde{F}$ with deepest point x. Then the extension of G is the singular hyperplane of $DW(2n-1, \mathbb{F})$ with deepest point x.

2.3 Ovoids

An *ovoid* of a point-line geometry is a set of points meeting each line in a singleton. For infinite fields $\mathbb{F}$, the dual polar space $\Delta = DW(2n-1, \mathbb{F})$ always has ovoids due to the possibility to construct such hyperplanes by means of transfinite induction. The situation is quite different in the finite case.

Let us first discuss the case $n = 2$. Then $\Delta = DW(3, q) \cong Q(4, q)$. The generalized quadrangle $Q(4, q)$ always has classical ovoids. All of these are elliptic quadrics $Q^-(3, q) \subset Q(4, q)$. There are many values of q for which the generalized quadrangle $Q(4, q)$ has nonclassical ovoids:

- $q = p^h$ with p an odd prime and $h \in \mathbb{N} \setminus \{0, 1\}$ ([37]);

- $q = 2^{2h+1}$ with $h \in \mathbb{N} \setminus \{0\}$ ([61]);

- $q = 3^{2h+1}$ with $h \in \mathbb{N} \setminus \{0\}$ ([37]);

- $q = 3^h$ with $h \in \mathbb{N} \setminus \{0, 1, 2\}$ ([59]);

- $q = 3^5$ ([48]).

For certain values of q, it is known that all ovoids of $Q(4, q)$ are classical:

Theorem 2.1.   (i) *Every ovoid of $Q(4, 4)$ is classical (*[2, 44]*).*

(ii) *Every ovoid of $Q(4, 16)$ is classical (*[41, 42]*).*

(iii) *Every ovoid of $Q(4, q)$, q prime, is classical (*[1, page 137, Corollary 1]*).*

The following theorem, due to Thomas [60, Theorem 3.2] and Cooperstein & Pasini [12], deals with the case $n = 3$.

Theorem 2.2 ([12, 60]). *The dual polar space $DW(5, q)$ does not have ovoids.*

The following is an immediate consequence of Theorem 2.2.

Corollary 2.3. *For every $n \geq 3$, the dual polar space $DW(2n-1, q)$ does not have ovoids.*

Proof. Let F be a convex subspace of diameter 3 of $\Delta = DW(2n-1,q)$ and suppose O is an ovoid of Δ. Then $F \cap O$ is an ovoid of $\widetilde{F} \cong DW(5,q)$. By Theorem 2.2, this is impossible. $\qquad\square$

2.4 Semi-singular hyperplanes

Suppose x is a point of $\Delta = DW(2n-1,\mathbb{F})$ and X is a set of points at distance n from x such that every line at distance $n-1$ from x has a unique point in common with X. Denote by H the union of X and the set of points at distance at most $n-2$ from x. Then H is a hyperplane of Δ, which is called a *semi-singular hyperplane (with deep point x)*. If $n = 2$, then semi-singular hyperplanes and ovoids are the same objects.

For infinite fields $\mathbb{F}$, the dual polar space $\Delta = DW(2n-1,\mathbb{F})$ always has semi-singular hyperplanes due to the possibility to construct such hyperplanes by means of transfinite induction. The situation is quite different in the finite case.

For the finite dual polar space $DW(5,q)$, the following was proved in [25, Corollary 18 and Theorem 19].

Theorem 2.4. *If q is prime or an even prime power, then $DW(5,q)$ does not have semi-singular hyperplanes.*

The proof of Theorem 2.4 for the case where q is prime relies on the above-mentioned result of Ball, Govaerts and Storme [1, Corollary 1, page 137] stating that every ovoid of $Q(4,q)$, q prime, is classical, i.e. an elliptic quadric $Q^-(3,q) \subset Q(4,q)$.

Suppose now that $2 \leq n_1 \leq n_2$. Suppose H is a semi-singular hyperplane of $DW(2n_2-1,q)$ with deep point x. Let F be a convex subspace of diameter n_1 of $DW(2n_2-1,q)$ containing a point at maximal distance n from x. Then $\widetilde{F} \cong DW(2n_1-1,q)$ and $F \cap H$ is a semi-singular hyperplane of $\widetilde{F}$ with deep point $\pi_F(x)$. In view of Theorem 2.4, this implies the following.

Corollary 2.5. *If $n \geq 3$ and q is prime or an even prime power, then the symplectic dual polar space $DW(2n-1,q)$ does not have semi-singular hyperplanes.*

2.5 Full subgrids of $DW(3,\mathbb{F}) \cong Q(4,\mathbb{F})$

Consider the ambient projective space $\mathrm{PG}(4,\mathbb{F})$ of $Q(4,\mathbb{F})$. If L_1 and L_2 are two disjoint lines of $Q(4,\mathbb{F})$, then the 3-space of $\mathrm{PG}(4,\mathbb{F})$ generated by L_1 and L_2 intersects $Q(4,\mathbb{F})$ in a full subgrid. This full subgrid is a hyperplane of $Q(4,\mathbb{F}) \cong DW(3,\mathbb{F})$.

2.6   The hyperplanes of hyperbolic type

Consider the quadric $Q(2n, 2)$ in the projective space $\mathrm{PG}(2n, 2)$, with $n \geq 2$, and a hyperplane Π intersecting $Q(2n, 2)$ in a nonsingular hyperbolic quadric $Q^+(2n - 1, 2)$. The subspaces of $Q(2n, 2)$ of maximal dimension $n - 1$ that are not contained in Π then define a hyperplane of the dual polar space $DQ(2n, 2)$ associated with $Q(2n, 2)$. Any such hyperplane of $DQ(2n, 2)$ is called a *hyperplane of hyperbolic type*. As $DW(2n - 1, 2) \cong DQ(2n, 2)$, hyperplanes of hyperbolic type also live in the symplectic dual polar space $DW(2n - 1, 2)$. The hyperplanes of hyperbolic type of the generalized quadrangle $DW(3, 2)$ are precisely the full subgrids of $DW(3, 2)$. Hyperplanes of hyperbolic type were introduced in Section 1.2 of Pasini and Shpectorov [46].

2.7   Hyperplanes of subspace type

Suppose again that $\Delta = DW(2n - 1, \mathbb{F})$ is the dual polar space associated with the polar space $W(2n - 1, \mathbb{F})$ and that the subspaces of $W(2n - 1, \mathbb{F})$ are the subspaces of $\mathrm{PG}(V)$ that are totally isotropic with respect to ζ.

Suppose α is an arbitrary $(n-1)$-dimensional subspace of $\mathrm{PG}(V)$, and denote by H_α the set of all totally isotropic $(n - 1)$-dimensional subspaces of $\mathrm{PG}(V)$ meeting α. Then H_α is a hyperplane. Any hyperplane of $DW(2n - 1, \mathbb{F})$ which can be obtained in this way is called a *hyperplane of subspace type*.

In the special case that α itself is also totally isotropic, we know that H_α is a singular hyperplane, namely the singular hyperplane with deepest point α. So, singular hyperplanes are examples of hyperplanes of subspace type. The hyperplanes of subspace type of $DW(3, \mathbb{F}) \cong Q(4, \mathbb{F})$ are precisely the singular hyperplanes or the full subgrids. The extension of any hyperplane of subspace type is again a hyperplane of subspace type.

Hyperplanes of subspace type were introduced in [20] and proofs of the above-mentioned facts can also be found in this paper (see Propositions 2.7, 2.8 and 2.10).

2.8   The hexagonal hyperplanes

A *generalized hexagon* is a point-line geometry $\mathcal{S}$ that satisfies the following properties:

- Every two distinct points of $\mathcal{S}$ are incident with at most one line.

- $\mathcal{S}$ has no subgeometries that are ordinary k-gons with $k \in \{3, 4, 5\}$.

- Two distinct objects of $\mathcal{S}$ are contained in a subgeometry that is an ordinary 6-gon.

With an object of $\mathcal{S}$, we mean a point or a line.

By Shult [57] (finite case) and Pralle [51] (general case), the dual polar space $DQ(6, \mathbb{F})$ has hyperplanes H with the property that the points and lines contained in H define a generalized hexagon $\widetilde{H}$ (being isomorphic to the so-called split Cayley hexagon $H(\mathbb{F})$). These hyperplanes are therefore called *hexagonal hyperplanes*.

If $\mathbb{F}$ is a perfect field of characteristic 2, then the polar spaces $DW(5, \mathbb{F})$ and $DQ(6, \mathbb{F})$ are isomorphic and therefore $DW(5, \mathbb{F})$ also has hexagonal hyperplanes.

2.9 SDPS-hyperplanes

Suppose Δ is a dual polar space of even rank $2n \geq 2$. A nonempty set X of points of Δ is called an *SDPS-set* if it satisfies the following properties:

- No two distinct points of X are collinear.

- If a quad Q contains two distinct points of X, then it intersects X in an ovoid of $\widetilde{Q}$.

- The partial linear space Δ' whose points are the elements of X and whose lines are the quad intersections of size at least two (natural incidence) is a dual polar space of rank n.

- If x_1 and x_2 are two points of X, then the distance between x_1 and x_2 in the geometry Δ is twice the distance between these points in the geometry Δ'.

- Every line of Δ meeting X is contained in a (necessarily unique) quad which intersects X in at least two points.

An SDPS-set of a dual polar space of rank 0 consists of the unique point of that geometry. An SDPS-set of a generalized quadrangle is just an ovoid of that geometry. The word SDPS is an abbreviation of *Sub Dual Polar Space* and refers to the fact that Δ' can be regarded as a sub dual polar space of Δ. SDPS-sets were introduced in De Bruyn and Vandecasteele [34]. In [34] (finite case) and [13, Section 5.6.7] (general case), it was shown that if X is an SDPS-set of a thick dual polar space Δ of rank $2n$, then the maximal distance from a point of Δ to X is equal to n. Moreover, the set of points of Δ at distance at most $n - 1$ from X is a hyperplane. Any hyperplane of a dual polar space that can

be obtained in this way is called an *SDPS-hyperplane.* An SDPS-hyperplane of a dual polar space of rank 2 is an ovoid.

The following theorem was proved in [19].

Theorem 2.6 ([19])**.** *For every $m \in \mathbb{N} \setminus \{0, 1\}$, the dual polar space $DW(4m - 1, q)$ has up to isomorphism a unique SDPS-set.*

The proof of Theorem 2.6 given in [19] invokes a classification result of Pralle and Shpectorov [53, Theorem 2], whose proof itself relied on the classification of the flag-transitive ovoid complements of $Q(4, q)$ obtained in Pasini and Shpectorov [45, Proposition 4.1].

We will now describe the unique SDPS-set of $DW(4m - 1, q)$, $m \geq 2$. Consider the finite field $\mathbb{F}_{q^2}$ with q^2 elements and let $\mathbb{F}_q$ denote the subfield of order q of $\mathbb{F}_q$. Let δ denote an arbitrary element of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Then $\mathbb{F}_{q^2} = \{x_1 + x_2\delta \,|\, x_1, x_2 \in \mathbb{F}_q\}$. Define $\tau : \mathbb{F}_{q^2} \to \mathbb{F}_q; x_1 + x_2\delta \mapsto x_1$. Consider the following bijection ϕ between the vector spaces $\mathbb{F}_q^{4m}$ and $\mathbb{F}_{q^2}^{2m}$:

$$\phi(x_1, x_2, \ldots, x_{4m}) = (x_1 + \delta x_2, \ldots, x_{4m-1} + \delta x_{4m}).$$

Let $\langle \cdot, \cdot \rangle$ be a nondegenerate alternating bilinear form on the vector space $\mathbb{F}_{q^2}^{2m}$. Then $\tau(\langle \phi(\cdot), \phi(\cdot) \rangle)$ is a nondegenerate alternating bilinear form on $\mathbb{F}_q^{4m}$. If α is a totally isotropic m-dimensional subspace of $\mathbb{F}_{q^2}^{2m}$, then $\phi^{-1}(\alpha)$ is a $2m$-dimensional totally isotropic subspace of $\mathbb{F}_q^{4m}$. The set of all $2m$-dimensional totally isotropic subspaces of $\mathbb{F}_q^{4m}$ which can be obtained in this way is an SDPS-set of $DW(4m - 1, q)$.

3   On classical hyperplanes of $DW(2n - 1, \mathbb{F})$

Ronan [56, Corollary 2, page 180] proved that if a point-line geometry with three points per line has a full projective embedding, then all its hyperplanes are classical. In particular, this means that all hyperplanes of the dual polar space $DW(2n - 1, 2)$ are classical. Sometimes, the fact that certain hyperplanes are classical also implies that other hyperplanes are classical as well. In this context, we wish to mention the following result which was proved in [24, Theorem 1.2(1)].

Theorem 3.1 ([24])**.** *Let F be a convex subspace of diameter $\delta \geq 2$ of the dual polar space $\Delta = DW(2n - 1, \mathbb{F})$. Let G be a hyperplane of $\widetilde{F}$ and let H denote the hyperplane of Δ that arises by extending G. Then H is classical if and only if G is classical.*

As the generalized quadrangle $DW(3, \mathbb{F}) \cong Q(4, \mathbb{F})$ admits nonclassical ovoids for every infinite field $\mathbb{F}$ and certain finite fields $\mathbb{F}$, Theorem 3.1 thus implies that the extensions of all these nonclassical ovoids are nonclassical hyperplanes as well.

Since not all hyperplanes of the dual polar space $DW(2n-1, \mathbb{F})$ are necessarily classical, one can ask for necessary and sufficient conditions for hyperplanes of $DW(2n - 1, \mathbb{F})$ to be classical. Such necessary and sufficient conditions were obtained in [23, Main Theorem].

Theorem 3.2 ([23])**.** *Suppose H is a hyperplane of the dual polar space $DW(2n-1, \mathbb{F})$. Then the following are equivalent:*

(a) *H is a classical hyperplane;*

(b) *for every quad Q intersecting H in an ovoid of $\widetilde{Q}$, the intersection $H \cap Q$ is a classical ovoid of $\widetilde{Q}$.*

Theorem 3.2 would also be a consequence of Cardinali, De Bruyn and Pasini [9, Corollary 1.5] and the results on simple connectedness of hyperplane complements on dual polar spaces obtained in Cardinali, De Bruyn & Pasini [9] and McInroy & Shpectorov [40].

For certain values of q, it is known that all ovoids of the generalized quadrangle $DW(3, q) \cong Q(4, q)$ are classical, see Theorem 2.1. Theorems 2.1 and 3.2 thus imply the following.

Corollary 3.3.   (i) *Every hyperplane of $DW(2n - 1, p)$, p prime, is classical.*

 (ii) *If $q \in \{4, 16\}$, then every hyperplane of $DW(2n - 1, q)$ is classical.*

4   Some classification results

4.1   Hyperplanes of $DW(3, \mathbb{F}) \cong Q(4, \mathbb{F})$

The following result is well known and easy to prove.

Theorem 4.1. *Every hyperplane of $DW(3, \mathbb{F})$ is either a singular hyperplane, an ovoid or a full subgrid.*

Proof. Let H be a hyperplane of $DW(3, \mathbb{F})$ and let $\mathcal{L}$ denote the set of lines of $DW(3, \mathbb{F})$ contained in H. If $\mathcal{L} \neq \emptyset$, then Property (GQ3) and the fact that H is a subspace imply that every point of H is contained in some element of $\mathcal{L}$.

If $\mathcal{L} = \emptyset$, then any line of $DW(3, \mathbb{F})$ intersects H in a singleton, showing that H is an ovoid.

Suppose $\mathcal{L} \neq \emptyset$ and that any two distinct lines of $\mathcal{L}$ meet. Then there exists a point p that belongs to every line of $\mathcal{L}$, implying that $H \subseteq p^{\perp}$. Since every line not containing p contains a point of H, we necessarily have $H = p^{\perp}$.

Suppose that $\mathcal{L}$ contains two disjoint lines L_1 and L_2. Let $\mathcal{L}'$ denote the set of lines meeting L_1 and L_2. Then $G := \bigcup_{L \in \mathcal{L}'} L$ is a full subgrid of $DW(3, \mathbb{F})$ contained in H. Since there are no subspaces between G and the entire point set of $DW(3, \mathbb{F})$, we thus see that H must coincide with the full subgrid G. $\square$

4.2   Uniform hyperplanes of $DW(2n-1, \mathbb{F})$

The following is an immediate consequence of Theorem 4.1.

Corollary 4.2. *If H is a hyperplane of $\Delta = DW(2n-1, \mathbb{F})$ and Q is a quad of Δ, then either $Q \subseteq H$ or $Q \cap H$ is a singular hyperplane of $\widetilde{Q}$, $Q \cap H$ is an ovoid of $\widetilde{Q}$ or $Q \cap H$ is a full subgrid of $\widetilde{Q}$.*

If Q is a quad and H is a hyperplane of $DW(2n-1, \mathbb{F})$, then Q is called *deep, singular, ovoidal* or *subquadrangular* (*with respect to H*) depending on whether $Q \cap H$ is either Q, a singular hyperplane of $\widetilde{Q}$, an ovoid of $\widetilde{Q}$ or a full subgrid of $\widetilde{Q}$.

Suppose H is a hyperplane of $\Delta = DW(2n-1, \mathbb{F})$, $n \geq 2$. Then H is called locally singular (locally ovoidal, respectively locally subquadrangular) if every non-deep quad is singular (ovoidal, respectively subquadrangular) with respect to H. The hyperplane H is called *uniform* if it is either locally singular, locally subquadrangular or locally ovoidal.

It can easily be seen that the locally ovoidal hyperplanes of $\Delta = DW(2n-1, \mathbb{F})$ are precisely the ovoids and we refer to Section 2.3 for the known classification results about these hyperplanes. Regarding locally subquadrangular hyperplanes of Δ, a complete classification was obtained by Pasini & Shpectorov [46, Theorems 1.3 and 3.1] (finite case) and De Bruyn [22, Proposition 2.1] (infinite case).

Theorem 4.3 ([46, 22]).   (i) *The locally subquadrangular hyperplanes of the dual polar space $DW(2n-1, 2)$ are precisely the hyperplanes of hyperbolic type.*

 (ii) *If $\mathbb{F}$ is a field containing at least three elements, then $DW(2n-1, \mathbb{F})$ does not have locally subquadrangular hyperplanes.*

Regarding locally singular hyperplanes, the following classification results do exist. The first one is due to Cardinali, De Bruyn and Pasini [8, Theorem 3.5].

Theorem 4.4 ([8]). *Suppose the field $\mathbb{F}$ is not a perfect field of characteristic 2. Then the locally singular hyperplanes of $DW(2n-1,\mathbb{F})$ are precisely the singular hyperplanes of $DW(2n-1,\mathbb{F})$.*

The following result is due to Shult [57] (finite case) and Pralle [51] (general case).

Theorem 4.5 ([57, 51]). *Suppose $\mathbb{F}$ is a perfect field of characteristic 2. Then the locally singular hyperplanes of $DW(5,\mathbb{F})$ are precisely the singular and hexagonal hyperplanes of $DW(5,\mathbb{F})$.*

The dual polar space $DQ(2n,\mathbb{F})$ has a full projective embedding in a projective space of dimension $2^n - 1$ over $\mathbb{F}$, which is called the *spin-embedding* of $DQ(2n,\mathbb{F})$. A description of this spin-embedding can be found in Buekenhout and Cameron [5, Section 7]. The following result is due to Shult & Thas [58] and De Bruyn [15, Proposition 1.2 and Theorem 1.3].

Theorem 4.6 ([58, 15]). *Suppose $\mathbb{F}$ is a perfect field of characteristic 2. Then the locally singular hyperplanes of $DW(2n-1,\mathbb{F}) \cong DQ(2n,\mathbb{F})$ are precisely the hyperplanes of $DW(2n-1,\mathbb{F})$ arising from its spin-embedding.*

Remark 4.7. As a consequence of Pralle [50, Theorem 1], we know that if H is a non-uniform hyperplane of $DW(2n-1,\mathbb{F})$, $n \geq 3$, then there is a quad which is singular with respect to H. So, the only hyperplanes of $DW(2n-1,\mathbb{F})$, $n \geq 3$, not having singular quads are the locally subquadrangular hyperplanes of $DW(2n-1,\mathbb{F}) = DW(2n-1,2)$.

4.3   Hyperplanes of $DW(5,2)$

The hyperplanes of $DW(5,2)$ have been classified by Pralle [52] (using a computer) and De Bruyn [17, Section 9] (without a computer). They showed that there are up to isomorphism 12 such hyperplanes.

Theorem 4.8 ([52, 17]). *Each hyperplane of $DW(5,2)$ is one of the following:*

(1) *a singular hyperplane;*

(2) *the extension of a full subgrid of a quad;*

(3) *the extension of an ovoid of a quad;*

(4) *a hexagonal hyperplane;*

(5) *a hyperplane of hyperbolic type;*

(6) *a hyperplane of the form $H * H_x$, where H is the extension of a full subgrid of a quad Q and H_x is a singular hyperplane whose deepest point x is contained in $H \setminus Q$;*

(7) *a hyperplane of the form $H * H_x$, where H is the extension of a full subgrid of a quad Q and H_x is a singular hyperplane whose deepest point x is not contained in H;*

(8) *a hyperplane of the form $H * H_x$, where H is a hyperplane of hyperbolic type and H_x is a singular hyperplane whose deepest point x is contained in H;*

(9) *a hyperplane of the form $H * H_x$, where H is a hyperplane of hyperbolic type and H_x is a singular hyperplane whose deepest point x is not contained in H;*

(10) *a hyperplane of the form $H_1 * H_2$, where H_1 is a hyperplane of hyperbolic type and H_2 is a hexagonal hyperplane;*

(11) *a hyperplane of the form $H_1 * H_2$, where H_1 is a hyperplane of hyperbolic type and H_2 is the extension of a full subgrid of a quad Q such that $Q \subseteq H_1$;*

(12) *a hyperplane of the form $H_1 * H_2$, where H_1 is a hyperplane of hyperbolic type and H_2 is the extension of an ovoid of a quad Q such that $Q \subseteq H_1$.*

4.4   Classical hyperplanes of $DW(5, \mathbb{F})$

All hyperplanes of $DW(5, 2)$ are classical, and in Section 4.3 we already provided a complete classification of these hyperplanes. We may therefore assume here that $|\mathbb{F}| \geq 3$. If $|\mathbb{F}| \geq 3$, then the classical hyperplanes of $DW(5, \mathbb{F})$ are precisely the hyperplanes arising from the Grassmann embedding. The classification of all hyperplanes of $DW(5, \mathbb{F})$ arising from the Grassmann embedding was achieved in a series of papers. A complete classification for the finite dual polar spaces $DW(5, q)$ was achieved in De Bruyn [17] for q even and in Cooperstein & De Bruyn [11] for q odd. A complete classification for the dual polar spaces $DW(5, \mathbb{F})$, where $\mathbb{F}$ is a perfect field of characteristic 2, was achieved in De Bruyn [21].

In joint work with M. Kwiatkowski [31], the author has obtained an "almost complete" classification of all hyperplanes of $DW(5, \mathbb{F})$ arising from the Grassmann embedding. This classification is complete if $\mathbb{F}$ is a field of characteristic distinct from 2, and partial for fields of characteristic 2. The statement of the classification in [31] takes a few pages and will therefore be omitted here. We only mention that the classification of the hyperplanes invokes multilinear algebra (use of exterior powers of vector spaces) and relies on the classification of the quasi-$Sp(V, f)$-equivalence classes of trivectors which the authors obtained in [30]. This classification itself relies on the lengthy classification of the $Sp(V, f)$-equivalence classes of trivectors[1] which the authors obtained in

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[1]Classification results for $Sp(V, f)$-equivalence classes of trivectors were also obtained by Igusa [36, Proposition 7, p. 1026] and Popov [49, Section 3] in the case the underlying field $\mathbb{F}$ is algebraically closed and of characteristic distinct from 2.

a series of four papers (see [29] for a survey). In the proof of the classification of the $Sp(V, f)$-equivalence classes itself, Revoy's classification [55] of the $GL(V)$-equivalence classes of trivectors was invoked.

4.5   Hyperplanes of $DW(5, \mathbb{F})$ containing a quad

For infinite fields $\mathbb{F}$, it seems not possible to obtain a complete classification of all hyperplanes of $DW(5, \mathbb{F})$ due to the possibility to construct such hyperplanes by means of transfinite induction. However, a classification might be possible under additional restrictions. The following result was proved in [26].

Theorem 4.9 ([26]). *Suppose H is a hyperplane of the dual polar space $DW(5, \mathbb{F})$ containing a quad Q. Then H is either classical or the extension of a nonclassical ovoid of a quad.*

4.6   Nonclassical hyperplanes of $DW(2n - 1, \mathbb{F})$

Nonclassical hyperplanes of $DW(5, q)$ were studied in [25]. The following is the main result of that paper (see [25, Theorem 1]):

Theorem 4.10 ([25]). *Every nonclassical hyperplane of $DW(5, q)$ is either a semi-singular hyperplane or the extension of a nonclassical ovoid of a quad of $DW(5, q)$.*

Among other things, nonclassical hyperplanes of $DW(2n-1, q)$ were studied in the paper [27]. By relying on Theorem 4.10, the following result was proved there ([27, Theorem 1.1(1)]).

Theorem 4.11 ([27]). *Suppose H is a nonclassical hyperplane of $DW(2n-1, q)$, $n \geq 3$. Then one of the following cases occurs:*

(1) *H is the extension of a nonclassical ovoid of a quad;*

(2) *there exists a hex F such that $F \cap H$ is a semi-singular hyperplane of $\widetilde{F}$.*

Currently, no example of a semi-singular hyperplane of $DW(5, q)$ is known. Combining Theorem 4.11 with Theorem 2.4, we find:

Corollary 4.12. *Let q be a prime power for which the dual polar space $DW(5, q)$ has no semi-singular hyperplanes. Then every hyperplane of $DW(2n-1, q)$, $n \geq 2$, is either classical or the extension of a nonclassical ovoid of a quad. In particular, this holds if q is even.*

There exists a complete classification of all ovoids of the generalized quadrangle $Q(4,q) \cong DW(3,q)$ if $q \in \{2,4,8,16,32\}$ (see [2, 44] for $q \in \{2,4\}$, [35, 47] for $q = 8$, [41, 42] for $q = 16$ and [43] for $q = 32$). If $q \in \{2,4,16\}$, then every ovoid of $Q(4,q)$ is classical, but for these values of q we already mentioned that every hyperplane of $DW(2n-1,q)$, $n \geq 2$, is classical: see Corollary 3.3. If $q \in \{8,32\}$, then every ovoid of $Q(4,q)$ is either a classical ovoid or a so-called Tits ovoid. So, Corollary 4.12 implies the following:

Corollary 4.13. *Let $q \in \{8,32\}$ and $n \geq 2$. Then every nonclassical hyperplane of $DW(2n-1,q)$ is the extension of a Tits ovoid of a quad of $DW(2n-1,q)$.*

5 Some characterization results

We characterized above already some classes of hyperplanes of $\Delta = DW(2n-1,\mathbb{F})$. For instance, in Theorem 4.3 we saw that the hyperplanes of hyperbolic type of $DW(2n-1,2)$ are precisely the locally subquadrangular hyperplanes of $DW(2n-1,2)$ and in Theorem 4.4, we saw that the singular hyperplanes are precisely the locally singular hyperplanes in case $\mathbb{F}$ is not a perfect field of characteristic 2. This characterization of singular hyperplanes no longer is valid for perfect fields of characteristic 2. However, it can be adapted in the following way; see Lemma 3.4 of Cardinali, De Bruyn and Pasini [8].

Theorem 5.1 ([8]). *The following are equivalent for a hyperplane H of $\Delta = DW(2n-1,\mathbb{F})$:*

(a) *H is a singular hyperplane;*

(b) *for every hex F of Δ not contained in H, the intersection $H \cap F$ is a singular hyperplane of $\widetilde{F}$.*

Hexagonal hyperplanes of $DW(5,\mathbb{F})$ with $\mathbb{F}$ a perfect field of characteristic 2 can be characterized as follows.

Theorem 5.2. *Let $\mathbb{F}$ be a perfect field of characteristic 2 and suppose H is a hyperplane of $DW(2n-1,\mathbb{F})$ having the property that every quad is singular with respect to H. Then $n = 3$ and H is a hexagonal hyperplane.*

Proof. We note that a hexagonal hyperplane of $DQ(6,\mathbb{F})$ does not have deep points, that is, no points x for which $\{x\} \cup x^\perp$ is contained in the hyperplane. We will use this fact later in the proof.

By Theorem 4.5, we know that the result is valid if $n = 3$.

Suppose therefore that $n \geq 4$. Since the theorem is true for $n = 3$, we know that for every hex F the intersection $H \cap F$ is a hexagonal hyperplane of $\widetilde{F}$.

Let x be a point of H. The lines and quads through x define an $(n-1)$-dimensional projective space $\mathcal{P}_x$. Denote by Λ_x the set of lines through x contained in H, and regard Λ_x as a set of points of $\mathcal{P}_x$. If L_1 and L_2 are two distinct elements of Λ_x, then the unique quad Q through L_1 and L_2 is singular with respect to H, implying that every line through x contained in Q belongs to Λ_x. So, Λ_x is a subspace of $\mathcal{P}_x$. Every quad through x is singular with respect to H and hence contains an element of Λ_x, showing that Λ_x is either a hyperplane or the whole set of points of $\mathcal{P}_x$. Since $n \geq 4$, this implies that there exists a hex F through x such that every line through x contained in F belongs to Λ_x. This implies that the hexagonal hyperplane $F \cap H$ of $\widetilde{F}$ contains a deep point, which is impossible. $\qquad\square$

In De Bruyn and Pralle [33, Proposition 4.2], the following result was proved.

Theorem 5.3 ([33]). *Suppose H is a hyperplane of $DW(5,q)$ without ovoidal quads. Then H is one of the following:*

(1) *a singular hyperplane;*

(2) *the extension of a full subgrid of a quad;*

(3) *(only if q is even) a hexagonal hyperplane;*

(4) *(only if $q = 2$) a hyperplane of hyperbolic type of $DW(5,q) = DW(5,2)$;*

(5) *(only if $q = 2$) a hyperplane of the form $H_1 * H_2$ where H_1 is a hyperplane of hyperbolic type of $DW(5,q)$ and H_2 is the extension of a full subgrid of a quad Q of $DW(5,2)$ such that $Q \subseteq H_1$.*

In the following two results, we characterize the hyperplanes of $DW(2n - 1, \mathbb{F})$ of subspace type. The first result was proved in [16, Main Theorem] and relies on Theorem 5.3. The second result was proved in [28, Theorem 1.2].

Theorem 5.4 ([16]). *Let H be a hyperplane of $DW(2n-1, q)$, $n \geq 2$ and $q \neq 2$, without ovoidal quads. Then the following hold:*

(i) *If q is odd, then H is a hyperplane of subspace-type.*

(ii) *If $q \geq 4$ is even, then H is either a hyperplane of subspace-type or arises from the spin-embedding of $DW(2n - 1, q) \cong DQ(2n, q)$.*

Theorem 5.5 ([28]). *The following are equivalent for a hyperplane H of $DW(2n-1, \mathbb{F})$, $n \geq 3$:*

(a) *H is a hyperplane of subspace-type;*

(b) *for every hex F of $DW(2n-1, \mathbb{F})$, the intersection $F \cap H$ is either F, a singular hyperplane of $\widetilde{F}$ or the extension of a full subgrid of a quad of $\widetilde{F}$.*

In the following theorem, we characterize the SDPS-hyperplanes of the dual polar spaces $DW(2n - 1, \mathbb{F})$ (and their extensions). This characterization result is taken from [14, Main Theorem].

Theorem 5.6 ([14]). *The following are equivalent for a hyperplane H of $\Delta = DW(2n - 1, \mathbb{F})$, $n \geq 3$:*

(a) *H is the possibly trivial extension of an SDPS-hyperplane of a convex subspace of even diameter of Δ;*

(b) *for every hex F of Δ, the intersection $H \cap F$ is either F, a singular hyperplane of $\widetilde{F}$ or the extension of an ovoid of a quad of $\widetilde{F}$.*

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