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# The exterior splash in PG(6, $q$): transversals

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Let $\pi$ be an order-$q$-subplane of $\mathrm{PG}(2, q^3)$ that is exterior to $\ell_\infty$. Then the exterior splash of $\pi$ is the set of $q^2+q+1$ points on $\ell_\infty$ that lie on an extended line of $\pi$. Exterior splashes are projectively equivalent to scattered linear sets of rank 3, covers of the circle geometry $CG(3, q)$, and hyper-reguli in $\mathrm{PG}(5, q)$. We use the Bruck–Bose representation in $\mathrm{PG}(6, q)$ to investigate the structure of $\pi$, and the interaction between $\pi$ and its exterior splash. We show that the point set of $\mathrm{PG}(6, q)$ corresponding to $\pi$ is the intersection of nine quadrics, and that there is a unique tangent plane at each point, namely the intersection of the tangent spaces of the nine quadrics. In $\mathrm{PG}(6, q)$, an exterior splash $\mathbb{S}$ has two sets of cover planes (which are hyper-reguli) and we show that each set has three unique transversal lines in the cubic extension $\mathrm{PG}(6, q^3)$. These transversal lines are used to characterise the carriers and the sublines of $\mathbb{S}$.

## 1. Introduction

In [Barwick and Jackson 2012; 2014], we studied order-$q$-subplanes of $\mathrm{PG}(2, q^3)$ and determined their representation in the Bruck–Bose representation in $\mathrm{PG}(6, q)$. A full characterisation in $\mathrm{PG}(6, q)$ was given for order-$q$-subplanes that are secant or tangent to $\ell_\infty$ in $\mathrm{PG}(2, q^3)$. In [Rottey et al. 2015], this was generalised to study subplanes of $\mathrm{PG}(2, q^n)$ in $\mathrm{PG}(2n, q)$. The cases when the subplane is secant or tangent to $\ell_\infty$ yield nice geometric characterisations. However, the case of an order-$q$-subplane $\pi$ of $\mathrm{PG}(2, q^3)$ that is exterior to $\ell_\infty$ yields a complex structure denoted $[\pi]$ in $\mathrm{PG}(6, q)$. Our main motivation in this article is to investigate the geometric properties of the structure $[\pi]$. The splash of $\pi$ gives crucial information about the geometrical properties of $[\pi]$, and so we also study the interplay in $\mathrm{PG}(6, q)$ between $[\pi]$ and its splash.

The splash of a subplane $\pi$ of $\mathrm{PG}(2, q^n)$ is defined to be the set of points on $\ell_\infty$ that lie on an extended line of $\pi$. In [Barwick and Jackson 2015] it was shown that

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the splash of a tangent order-q-subplane of $\mathrm{PG}(2, q^3)$ is a linear set. In [Lavrauw and Zanella 2015] the notion of splash was generalised from subplanes to subgeometries, and to general field extensions. It was shown that a splash is a linear set, and conversely, a linear set is a splash.

In this article we let $\pi$ be a subplane of $\mathrm{PG}(2, q^3)$ of order $q$ that is exterior to $\ell_\infty$. The lines of $\pi$ meet $\ell_\infty$ in a set $\mathbb{S}$ of size $q^2+q+1$, which we call the *exterior splash* of $\pi$. Properties of the exterior splash of $\mathrm{PG}(2, q^3)$ were studied in [Barwick and Jackson 2016]. The sets of points in an exterior splash has arisen in many different situations, namely scattered $\mathbb{F}_q$-linear sets of rank 3, covers of the circle geometry $\mathrm{CG}(3, q)$, hyper-reguli in $\mathrm{PG}(5, q)$, and Sherk surfaces of size $q^2+q+1$. Scattered linear sets are surveyed in [Lavrauw 2016]. An important result is that all scattered $\mathbb{F}_q$-linear sets of rank 3 are projectively equivalent [Lavrauw and Zanella 2015].

This article proceeds as follows. In Section 2 we introduce the notation we use for the Bruck–Bose representation of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(6, q)$, as well as presenting some other preliminary results.

We next introduce coordinates; as all scattered $\mathbb{F}_q$-linear sets of rank 3 are projectively equivalent, we will work with an exterior splash equivalent to the set of points

$$\{(x, x^q) : x \in \mathrm{GF}(q^3)\setminus\{0\}\}.$$

In Section 3 we coordinatise an order-$q$-subplane $\mathcal{B}$ in $\mathrm{PG}(2, q^3)$ that is exterior to $\ell_\infty$, with this exterior splash. This order-$q$-subplane will be used in many of the proofs in this article.

In Section 4, we study the structure of an order-$q$-subplane in $\mathrm{PG}(6, q)$. We show that it contains $q^2 + q + 1$ twisted cubics and is the intersection of nine quadrics. Further, we show that there is a unique tangent plane at each point, which is the intersection of the tangent spaces of these nine quadrics.

We next study the exterior splash $\mathbb{S}$ of $\ell_\infty$ in the Bruck–Bose representation in $\mathrm{PG}(5, q)$. By results of Bruck [1973], $\mathbb{S}$ has two switching sets denoted $\mathbb{X}$, $\mathbb{Y}$, which we call covers of $\mathbb{S}$. The three sets $\mathbb{S}$, $\mathbb{X}$, $\mathbb{Y}$ are called hyper-reguli in [Ostrom 1993]. In Section 5, we look at the exterior splash

$$\{(x, x^q) : x \in \mathrm{GF}(q^3)\setminus\{0\}\},$$

and working in $\mathrm{PG}(6, q)$, find coordinates for the two covers $\mathbb{X}$, $\mathbb{Y}$. In Section 6, we show that each of the sets $\mathbb{S}$, $\mathbb{X}$, $\mathbb{Y}$ has a unique triple of conjugate transversal lines in the cubic extension $\mathrm{PG}(5, q^3)$. Theorem 6.5 characterises the carriers of an exterior splash as the only planes of the regular spread that meet all nine transversal lines. Theorem 6.6 shows that the nine transversal lines are common to the set of $q - 1$ disjoint splashes of $\ell_\infty$ that have common carriers. We interpret this result in terms of replacing hyper-reguli to create André planes. In Section 7 we use the transversal lines to characterise the order-$q$-sublines of an exterior splash in terms of how the corresponding 2-reguli meet the cover planes.

## 2. The Bruck–Bose representation

**2A.** ***The Bruck–Bose representation of* PG$(2, q^3)$ *in* PG$(6, q)$.** We introduce the notation we will use for the Bruck–Bose representation of PG$(2, q^3)$ in PG$(6, q)$. We work with the finite field $\mathbb{F}_q$ of order $q$. A 2-*spread* of PG$(5, q)$ is a set of $q^3 + 1$ planes that partition PG$(5, q)$. A 2-*regulus* of PG$(5, q)$ is a set of $q + 1$ mutually disjoint planes $\pi_1, \ldots, \pi_{q+1}$ with the property that if a line meets three of the planes, then it meets all $q + 1$ of them. A 2-regulus $\mathcal{R}$ has a set of $q^2 + q + 1$ mutually disjoint *ruling lines* that meet every plane of $\mathcal{R}$. A 2-regulus is uniquely determined by three mutually disjoint planes, or four (ruling) lines (mutually disjoint and lying in general position). A 2-spread $\mathcal{S}$ is *regular* if for any three planes in $\mathcal{S}$, the 2-regulus containing them is contained in $\mathcal{S}$. See [Hirschfeld and Thas 1991] for more information on 2-spreads.

The following construction of a regular 2-spread of PG$(5, q)$ will be needed. Embed PG$(5, q)$ in PG$(5, q^3)$ and let $g$ be a line of PG$(5, q^3)$ disjoint from PG$(5, q)$. Let $g^q$, $g^{q^2}$ be the conjugate lines of $g$; both of these are disjoint from PG$(5, q)$. Let $P_i$ be a point on $g$; then the plane $\langle P_i, P_i^q, P_i^{q^2} \rangle$ meets PG$(5, q)$ in a plane. As $P_i$ ranges over all the points of $g$, we get $q^3 + 1$ planes of PG$(5, q)$ that partition PG$(5, q)$. These planes form a regular 2-spread $\mathcal{S}$ of PG$(5, q)$. The lines $g$, $g^q$, $g^{q^2}$ are called the (conjugate skew) *transversal lines* of the 2-spread $\mathcal{S}$. Conversely, given a regular 2-spread in PG$(5, q)$, there is a unique set of three (conjugate skew) transversal lines in PG$(5, q^3)$ that generate $\mathcal{S}$ in this way.

We will use the linear representation of a finite translation plane $\mathcal{P}$ of dimension at most three over its kernel, due independently to André [1954] and Bruck and Bose [1964; 1966]. Let $\Sigma_\infty$ be a hyperplane of PG$(6, q)$ and let $\mathcal{S}$ be a 2-spread of $\Sigma_\infty$. We use the phrase *a subspace of* PG$(6, q)\backslash\Sigma_\infty$ to mean a subspace of PG$(6, q)$ that is not contained in $\Sigma_\infty$. Consider the following incidence structure: the *points* of $\mathcal{A}(\mathcal{S})$ are the points of PG$(6, q)\backslash\Sigma_\infty$; the *lines* of $\mathcal{A}(\mathcal{S})$ are the 3-spaces of PG$(6, q)\backslash\Sigma_\infty$ that contain an element of $\mathcal{S}$; and *incidence* in $\mathcal{A}(\mathcal{S})$ is induced by incidence in PG$(6, q)\backslash\Sigma_\infty$. Then the incidence structure $\mathcal{A}(\mathcal{S})$ is an affine plane of order $q^3$. We can complete $\mathcal{A}(\mathcal{S})$ to a projective plane $\mathcal{P}(\mathcal{S})$; the points on the line at infinity $\ell_\infty$ have a natural correspondence to the elements of the 2-spread $\mathcal{S}$. The projective plane $\mathcal{P}(\mathcal{S})$ is the Desarguesian plane PG$(2, q^3)$ if and only if $\mathcal{S}$ is a regular 2-spread of $\Sigma_\infty \cong$ PG$(5, q)$ (see [Bruck 1969]). For the remainder of this article, we use $\mathcal{S}$ to denote a regular 2-spread of $\Sigma_\infty \cong$ PG$(5, q)$.

We use the following notation. If $T$ is a point of $\ell_\infty$ in PG$(2, q^3)$, we use $[T]$ to refer to the plane of $\mathcal{S}$ corresponding to $T$. More generally, if $X$ is a set of points of PG$(2, q^3)$, then we let $[X]$ denote the corresponding set in PG$(6, q)$. If $P$ is an affine point of PG$(2, q^3)$, we generally simplify the notation and also use $P$ to refer to the corresponding affine point in PG$(6, q)$, although in some cases, to avoid confusion, we use $[P]$.

When $\mathcal{S}$ is a regular 2-spread, we can relate the coordinates of $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2, q^3)$ and $\mathrm{PG}(6, q)$ as follows. Let $\tau$ be a primitive element in $\mathbb{F}_{q^3}$ with primitive polynomial $x^3 - t_2x^2 - t_1x - t_0$. Every element $\alpha \in \mathbb{F}_{q^3}$ can be uniquely written as $\alpha = a_0 + a_1\tau + a_2\tau^2$ with $a_0, a_1, a_2 \in \mathbb{F}_q$. Points in $\mathrm{PG}(2, q^3)$ have homogeneous coordinates $(x, y, z)$ with $x, y, z \in \mathbb{F}_{q^3}$, not all zero. Let the line at infinity $\ell_\infty$ have equation $z = 0$; so the affine points of $\mathrm{PG}(2, q^3)$ have coordinates $(x, y, 1)$. Points in $\mathrm{PG}(6, q)$ have homogeneous coordinates $(x_0, x_1, x_2, y_0, y_1, y_2, z)$ with $x_0, x_1, x_2, y_0, y_1, y_2, z \in \mathbb{F}_q$. Let $\Sigma_\infty$ have equation $z = 0$. Let $P = (\alpha, \beta, 1)$ be a point of $\mathrm{PG}(2, q^3)$. We can write $\alpha = a_0 + a_1\tau + a_2\tau^2$ and $\beta = b_0 + b_1\tau + b_2\tau^2$ with $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{F}_q$. We want to map the element $\alpha$ of $\mathbb{F}_{q^3}$ to the vector $(a_0, a_1, a_2)$, and we use the following notation to do this:

$$[\alpha] = (a_0, a_1, a_2).$$

This gives some notation for the Bruck–Bose map, denoted $\epsilon$, from an affine point $P = (\alpha, \beta, 1) \in \mathrm{PG}(2, q^3)\setminus\ell_\infty$ to the corresponding affine point $[P] \in \mathrm{PG}(6, q)\setminus\Sigma_\infty$, namely

$$\epsilon(\alpha, \beta, 1) = [(\alpha, \beta, 1)] = ([\alpha], [\beta], 1) = (a_0, a_1, a_2, b_0, b_1, b_2, 1).$$

More generally, if $z \in \mathbb{F}_q$, then $\epsilon(\alpha, \beta, z) = ([\alpha], [\beta], z) = (a_0, a_1, a_2, b_0, b_1, b_2, z)$.

Consider the case when $z = 0$, that is, a point on $\ell_\infty$ in $\mathrm{PG}(2, q^3)$ has coordinates $L = (\alpha, \beta, 0)$ for some $\alpha, \beta \in \mathbb{F}_{q^3}$. In $\mathrm{PG}(6, q)$, the point $\epsilon(\alpha, \beta, 0) = ([\alpha], [\beta], 0)$ is one point in the spread element $[L]$ corresponding to $L$. Moreover, the spread element $[L]$ consists of all the points $\{([\alpha x], [\beta x], 0) : x \in \mathbb{F}'_{q^3}\}$. Hence the regular 2-spread $\mathcal{S}$ consists of the planes $\{[kx], [x], 0] : x \in \mathbb{F}'_{q^3}\}$ for $k \in \mathbb{F}_{q^3} \cup \{\infty\}$.

With this coordinatisation for the Bruck–Bose map, we can calculate the coordinates of the transversal lines of the regular 2-spread $\mathcal{S}$.

**Lemma 2.1** [Barwick and Jackson 2012]. *Let* $p_0 = t_1 + t_2\tau - \tau^2 = -\tau^q\tau^{q^2}$, $p_1 = t_2 - \tau = \tau^q + \tau^{q^2}$, $p_2 = -1$, *and* $A = (p_0, p_1, p_2)$. *Then in the cubic extension* $\mathrm{PG}(6, q^3)$, *one transversal line of the regular* 2*-spread* $\mathcal{S}$ *contains the two points* $A_1 = (p_0, p_1, p_2, 0, 0, 0, 0) = (A, [0], 0)$ *and* $A_2 = (0, 0, 0, p_0, p_1, p_2, 0) = ([0], A, 0)$.

**2B.** ***Some useful homographies.*** In order to simplify the notation in some of the following coordinate-based proofs, we define some homographies which will be useful. We can represent an element $x = x_0 + x_1\tau + x_2\tau^2 \in \mathbb{F}_{q^3}$ as a point $[x] = (x_0, x_1, x_2)$ in $\mathrm{PG}(2, q)$. For $k \in \mathbb{F}'_{q^3}$, consider the homography $\zeta_k$ in $\mathrm{PGL}(3, q)$ with matrix $M_k$ that maps $[x]$ to $[kx]$. Let $k \in \mathbb{F}'_{q^3}$ and write $k = k_0 + k_1\tau + k_2\tau^2$, then $M_k = k_0M_1 + k_1M_\tau + k_2M_{\tau^2}$, and hence

$$M_kA = kA \quad \text{and} \quad M_kA^{q^2} = k^{q^2}A^{q^2}, \tag{1}$$

where $A=(p_0,\,p_1,\,p_2)^t$ is defined in Lemma 2.1. We use $\zeta_k$ to define the homography $\theta_k$ of PG(5, $q$), $k\in\mathbb{F}_{q^3}$:

$$\theta_k\colon ([x],[y])\to([kx],[y])=(M_k[x],[y]).$$

From the matrix $M_\tau$, we construct three more homographies of PG(2, $q$) with matrices $U_0, U_1, U_2$ that help with the notation in the proof of Theorem 7.4. For $i=0,1,2$, (with $p_i$ as in Lemma 2.1), let

$$U_i=(p_0I+p_1M_\tau+p_2M_\tau^2)^{q^i}=\begin{pmatrix} p_0^{q^i} & \tau^{q^i}p_0^{q^i} & \tau^{2q^i}p_0^{q^i}\\ p_1^{q^i} & \tau^{q^i}p_1^{q^i} & \tau^{2q^i}p_1^{q^i}\\ p_2^{q^i} & \tau^{q^i}p_2^{q^i} & \tau^{2q^i}p_2^{q^i}\end{pmatrix}.$$

Then

$$U_i\begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix}=(a_0+a_1\tau^{q^i}+a_2\tau^{2q^i})\begin{pmatrix}p_0^{q^i}\\p_1^{q^i}\\p_2^{q^i}\end{pmatrix},\qquad a_0,a_1,a_2\in\mathbb{F}_{q^3}.$$

Note that if $a_0,a_1,a_2\in\mathbb{F}_q$, and $\alpha=a_0+a_1\tau+a_2\tau^2$, then $[\alpha]=(a_0,\,a_1,\,a_2)^t$, and we write the matrix equation as $U_i[\alpha]=\alpha^{q^i}A^{q^i}$.

**2C. *Sublines in the Bruck–Bose representation.*** An *order-$q$-subplane* of PG(2, $q^3$) is a subplane of PG(2, $q^3$) of order $q$. Equivalently, it is an image of PG(2, $q$) under PGL(3, $q^3$). An *order-$q$-subline* of PG(2, $q^3$) is a line of an order-$q$-subplane of PG(2, $q^3$). An *order-$q$-subline* of PG(1, $q^3$) is defined to be one of the images of PG(1, $q$) $=\{(a,1):a\in\mathbb{F}_q\}\cup\{(1,0)\}$ under PGL(2, $q^3$).

In [Barwick and Jackson 2012; 2014], we determine the representation of order-$q$-subplanes and order-$q$-sublines of PG(2, $q^3$) in the Bruck–Bose representation in PG(6, $q$), and we quote the results for order-$q$-sublines which are needed in this article. We first introduce some terminology to simplify the statements. Recall that $\mathcal{S}$ is a regular 2-spread in the hyperplane at infinity $\Sigma_\infty$ in PG(6, $q$).

**Definition 2.2.** (i) An *$\mathcal{S}$-special conic* is a nondegenerate conic $\mathcal{C}$ contained in a plane of $\mathcal{S}$, such that the extension of $\mathcal{C}$ to PG(6, $q^3$) meets the transversals of $\mathcal{S}$.

(ii) An *$\mathcal{S}$-special twisted cubic* is a twisted cubic $\mathcal{N}$ in a 3-space of PG(6, $q$)$\backslash\Sigma_\infty$ about a plane of $\mathcal{S}$, such that the extension of $\mathcal{N}$ to PG(6, $q^3$) meets the transversals of $\mathcal{S}$.

**Theorem 2.3** [Barwick and Jackson 2012]. *Let $b$ be an order-$q$-subline of* PG(2, $q^3$).

(i) *If $b\subset\ell_\infty$, then in* PG(6, $q$), *$b$ corresponds to a* 2*-regulus of $\mathcal{S}$. Conversely every* 2*-regulus of $\mathcal{S}$ corresponds to an order-$q$-subline of $\ell_\infty$.*

(ii) *If $b$ meets $\ell_\infty$ in a point, then $b$ corresponds to a line of* $\mathrm{PG}(6,q)\backslash\Sigma_\infty$. *Conversely every line of* $\mathrm{PG}(6,q)\backslash\Sigma_\infty$ *corresponds to an order-$q$-subline of* $\mathrm{PG}(2,q^3)$ *tangent to $\ell_\infty$.*

(iii) *If $b$ is disjoint from $\ell_\infty$, then in* $\mathrm{PG}(6,q)$, *$b$ corresponds to an $\mathcal{S}$-special twisted cubic. Further, a twisted cubic $\mathcal{N}$ of* $\mathrm{PG}(6,q)$ *corresponds to an order-$q$-subline of* $\mathrm{PG}(2,q^3)$ *if and only if $\mathcal{N}$ is $\mathcal{S}$-special.*

In [Barwick and Jackson 2012], we also determine the representation of secant and tangent order-$q$-subplanes of $\mathrm{PG}(2,q^3)$ in $\mathrm{PG}(6,q)$. The representation of an exterior order-$q$-subplane in $\mathrm{PG}(6,q)$ is more complex to describe. One of the motivations of this work is to investigate this representation in more detail. Some aspects of the representation are discussed in more detail in Section 4.

**2D. *Properties of exterior splashes.*** We need some group theoretic results about order-$q$-subplanes and exterior splashes; the first appears in [Barwick and Jackson 2016].

**Theorem 2.4.** *Let $G = \mathrm{PGL}(3,q^3)$ be the collineation group acting on* $\mathrm{PG}(2,q^3)$. *The subgroup $G_\ell$ fixing a line $\ell$ is transitive on the order-$q$-subplanes that are exterior to $\ell$, and is transitive on the exterior splashes of $\ell$.*

This theorem can be proved by generalising the arguments in [Barwick and Jackson 2015]. In particular, it involves looking at two important subgroups of $G$. The first subgroup fixes an order-$q$-subplane, and the following property will be very useful.

**Theorem 2.5.** *The group $K = \mathrm{PGL}(3,q^3)_\pi$ acting on* $\mathrm{PG}(2,q^3)$ *and fixing an order-$q$-subplane $\pi$ is transitive on the points of $\pi$.*

The second important subgroup is $I = G_{\pi,\ell}$ which fixes an order-$q$-subplane $\pi$, and a line $\ell$ exterior to $\pi$. By [Barwick and Jackson 2016], $I$ fixes exactly three lines: $\ell$, and its conjugates $m, n$ with respect to $\pi$; and $I$ fixes exactly three points: $E_1 = \ell\cap m,\ E_2 = \ell\cap n,\ E_3 = m\cap n$, which are conjugate with respect to $\pi$. Further $I$ identifies two fixed points $E_1 = \ell\cap m,\ E_2 = \ell\cap n$ on $\ell$ which are called the *carriers* of the exterior splash $\mathbb{S}$ of $\pi$. This is consistent with the definition of carriers of a circle geometry $\mathrm{CG}(3,q)$; see [Barwick and Jackson 2016]. In [Lunardon et al. 2014], scattered linear sets of pseudoregulus type are considered, and they use the term "transversal points". The fixed points and fixed lines of $I$ are used to define an important class of conics in an order-$q$-subplane $\pi$ with respect to an exterior line $\ell$. A conic of $\pi$ whose extension to $\mathrm{PG}(2,q^3)$ contains the three points $E_1, E_2, E_3$ is called a *$(\pi,\ell)$-carrier conic* of $\pi$. A dual conic of $\pi$ whose extension to $\mathrm{PG}(2,q^3)$ contains the three lines $\ell, m, n$ is called a *$(\pi,\ell)$-carrier-dual conic*. Note that carrier-conics/dual conics were called special-conics/dual conics in [Barwick and Jackson 2016]; we change the name here so that the term "special" is reserved for objects in PG(6, $q$).

## 3. Coordinatising an exterior order-$q$-subplane

Recall from Theorem 2.4 that the group of homographies of PG(2, $q^3$) is transitive on pairs $(\pi, \ell)$ where $\pi$ is an order-$q$-subplane exterior to the line $\ell$. So if we want to use coordinates to prove a result about exterior order-$q$-subplanes, we can without loss of generality prove it for a particular exterior order-$q$-subplane. In this section we calculate the coordinates for an exterior order-$q$-subplane $\mathcal{B}$ of PG(2, $q^3$) whose exterior splash has a simple form. Set

$$K = \begin{pmatrix} -\tau & 1 & 0 \\ -\tau^q & 1 & 0 \\ \tau\tau^q & -\tau-\tau^q & 1 \end{pmatrix}, \qquad K' = \begin{pmatrix} -1 & 1 & 0 \\ -\tau^q & \tau & 0 \\ -\tau^{2q} & \tau^2 & \tau-\tau^q \end{pmatrix}. \tag{2}$$

Let $\sigma$ be the homography of PG(2, $q^3$) with matrix $K$. Note that as $KK'$ is a $\mathbb{F}_{q^3}$-multiple of the identity matrix, it follows that $K'$ is a matrix for the inverse homography $\sigma^{-1}$. Thus, if we write the points $X$ of PG(2, $q^3$) as column vectors, and the lines $\ell$ of PG(2, $q^3$) as row vectors, then $\sigma(X) = KX$ and $\sigma(\ell) = \ell K'$. The order-$q$-subplane $\pi_0 = \text{PG}(2, q)$ is secant to $\ell_\infty$. We show that the subplane $\sigma(\pi_0)$ is exterior to $\ell_\infty$ and has the desired simple form as exterior splash.

**Theorem 3.1.** *In* PG(2, $q^3$), *let* $\pi_0 = \text{PG}(2, q)$, *let* $\sigma$ *be the homography with matrix* $K$ *given in* (2), *and let* $\mathcal{B} = \sigma(\pi_0)$. *Then* $\mathcal{B}$ *is an order-q-subplane exterior to* $\ell_\infty$ *with exterior splash* $\mathbb{S} = \{(k, 1, 0) : k \in \mathbb{F}_{q^3}, k^{q^2+q+1} = 1\}$ *and carriers* $E_1 = (1, 0, 0)$ *and* $E_2 = (0, 1, 0)$.

*Proof.* Note that $\sigma$ maps $\pi_0 = \text{PG}(2, q)$ to $\mathcal{B}$ and the line $\ell = [-\tau\tau^q, \tau+\tau^q, -1]$ to $\ell_\infty = [0, 0, 1]$. By [Barwick and Jackson 2016], $\pi_0$ is exterior to $\ell$ and has carriers $E = (1, \tau, \tau^2)$ and $E^q = (1, \tau^q, \tau^{2q})$ on $\ell$. Hence $\mathcal{B}$ is exterior to $\ell_\infty$ and has carriers $\sigma(E) = (0, 1, 0)$ and $\sigma(E^q) = (1, 0, 0)$ on $\ell_\infty$. By considering the action of $\sigma$ on the lines $[l, m, n]$ ($l, m, n \in \mathbb{F}_q$, not all zero) of $\pi_0$, we calculate the lines of $\mathcal{B}$ are $\ell_{l,m,n} = [-l - \tau^q m - \tau^{2q} n,\ l + \tau m + \tau^2 n,\ n(\tau - \tau^q)]$, with $l, m, n \in \mathbb{F}_q$, not all zero. The exterior splash of $\mathcal{B}$ consists of the points $Q_{l,m,n} = \ell_{l,m,n} \cap \ell_\infty = (l + \tau m + \tau^2 n,\ (l + \tau m + \tau^2 n)^q, 0)$. Writing $y = l + \tau m + \tau^2 n$, gives $Q_{l,m,n} \equiv (y, y^q, 0) \equiv (y^{1-q}, 1, 0)$ and writing $y = \tau^{-j}$ for some $j \in \{0, \dots, q^3 - 2\}$ yields $Q_{l,m,n} \equiv (\tau^{j(q-1)}, 1, 0)$. Note that if we write $j = n(q^2+q+1) + i$ where $0 \le i < q^2+q+1$, then $\tau^{j(q-1)} = \tau^{i(q-1)}$. So we may assume that $Q_{l,m,n} = (\tau^{i(q-1)}, 1, 0)$ with $0 \le i < q^2+q+1$. It is useful to observe that

$$\mathbb{S} = \{(k, 1, 0) : k \in \mathbb{F}_{q^3}, k^{q^2+q+1} = 1\} \equiv \{(\tau^{(q-1)i}, 1, 0) : 0 \le i < q^2+q+1\}$$

as the solutions to $k^{q^2+q+1} = 1$ are $\tau^{i(q-1)}$, $0 \le i < q^2+q+1$. □

## 4. The structure of the subplane in PG(6, $q$)

If $\pi$ is an exterior order-$q$-subplane of PG(2, $q^3$), then in the Bruck–Bose representation in PG(6, $q$), $\pi$ corresponds to a set of $q^2+q+1$ affine points denoted $[\pi]$. It is difficult to characterise the structure of $[\pi]$. We note that as $\pi$ contains $q^2+q+1$ order-$q$-sublines that are exterior to $\ell_\infty$, then by Theorem 2.3, $[\pi]$ contains $q^2+q+1$ $\mathcal{S}$-special twisted cubics, each lying in a 3-space through a distinct plane of the exterior splash of $\pi$. In this section we aim to determine more about the structure of $[\pi]$.

**4A.** ***The intersection of nine quadrics.*** We show that the structure $[\pi]$ of PG(6, $q$) corresponding to an exterior order-$q$-subplane $\pi$ of PG(2, $q^3$) is the intersection of nine quadrics in PG(6, $q$). This is analogous to [Barwick and Jackson 2015, Theorem 9.2] which shows that a *tangent* order-$q$-subplane of PG(2, $q^3$) corresponds to a structure in PG(6, $q$) that is the intersection of nine quadrics.

**Theorem 4.1.** *Let $\pi$ be an exterior order-q-subplane in* PG(2, $q^3$). *The corresponding set $[\pi]$ in* PG(6, $q$) *is the intersection of nine quadrics.*

*Proof.* By Theorem 2.4, we can without loss of generality prove this for the order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3. We use the homographies $\sigma, \sigma^{-1}$ with matrices $K$, $K'$ respectively, given in (2). A point $P=(x, y, 1)\in$ PG(2, $q^3$) belongs to $\mathcal{B}$ if its preimage $K'P=(-x+y,\ -\tau^q x+\tau y,\ -\tau^{2q}x+\tau^2 y+(\tau-\tau^q))$ belongs to $\pi_0=$ PG(2, $q$). Suppose firstly that $-x+y\neq 0$, then

$$K'P\equiv\left(1,\ \frac{-\tau^q x+\tau y}{-x+y},\ \frac{-\tau^{2q}x+\tau^2 y+(\tau-\tau^q)}{-x+y}\right).$$

This belongs to $\pi_0=$ PG(2, $q$) if and only if the second and third coordinates belong to $\mathbb{F}_q$, that is,

$$\left(\frac{-\tau^q x+\tau y}{-x+y}\right)^q=\frac{-\tau^q x+\tau y}{-x+y}, \tag{3}$$

$$\left(\frac{-\tau^{2q}x+\tau^2 y+(\tau-\tau^q)}{-x+y}\right)^q=\frac{-\tau^{2q}x+\tau^2 y+(\tau-\tau^q)}{-x+y}. \tag{4}$$

Writing $x=x_0+x_1\tau+x_2\tau^2$ and $y=y_0+y_1\tau+y_2\tau^2$, where $x_i, y_i\in\mathbb{F}_q$ and $i=1,2,3$, then equating powers of $1, \tau, \tau^2$, yields three quadratic equations from each condition, a total of six, each of which represents a quadric in PG(6, $q$).

Secondly, suppose $-\tau^q x+\tau y\neq 0$, then

$$K'P\equiv\left(\frac{-x+y}{-\tau^q x+\tau y},\ 1,\ \frac{-\tau^{2q}x+\tau^2 y+(\tau-\tau^q)}{-\tau^q x+\tau y}\right).$$

As before, this lies in $\pi_0$ if and only if

$$\left(\frac{-x+y}{-\tau^q x+\tau y}\right)^q=\frac{-x+y}{-\tau^q x+\tau y}, \tag{5}$$

$$\left(\frac{-\tau^{2q}x+\tau^2 y+(\tau-\tau^q)}{-\tau^q x+\tau y}\right)^q=\frac{-\tau^{2q}x+\tau^2 y+(\tau-\tau^q)}{-\tau^q x+\tau y}, \tag{6}$$

leading to a further six quadrics in PG(6, $q$). The equations (3) and (5) give the same triple of quadrics. Hence the point $P$ lies in $\mathcal{B}$ if and only if the point $[P]$ lies on a total of nine quadrics in PG(6, $q$). Finally, note that if both $-x+y=0$ and $-\tau^q x+\tau y=0$, then $x=y=0$ and the point $P$ has coordinates (0, 0, 1). This satisfies all the quadratic equations from (3), (4), (6), and so in PG(6, $q$), $[P]$ lies on each of the nine quadrics. □

**4B.** ***Tangent planes at points of an exterior subplane.*** We now consider a point $P$ lying in an exterior order-$q$-subplane $\pi$ of PG(2, $q^3$). In the Bruck–Bose representation in PG(6, $q$), $P$ corresponds to an affine point which we also denote by $P$. We show that in PG(6, $q$), there is a unique *tangent plane* $\mathcal{T}_P$ at $P$ to the structure $[\pi]$. We show that there are two equivalent ways to define this tangent plane. Recall from Theorem 2.3 that the order-$q$-sublines of $\pi$ correspond to twisted cubics in PG(6, $q$). Theorem 4.2 shows that we can define $\mathcal{T}_P$ by looking at the tangent lines at $P$ to these twisted cubics. Then Theorem 4.3 shows that we can define $\mathcal{T}_P$ by looking at the tangent space of $P$ with respect to the nine quadrics defined by $[\pi]$.

**Theorem 4.2.** *Let $\pi$ be an exterior order-$q$-subplane of* PG(2, $q^3$), *and let $P$ be a point of $\pi$. Label the lines of $\pi$ through $P$ by $\ell_0, \dots, \ell_q$. In* PG(6, $q$), *$\ell_i$ corresponds to a twisted cubic $[\ell_i]$. Let $m_i$ be the unique tangent line to $[\ell_i]$ through $P$. Then the lines $m_0, \dots, m_q$ lie in a plane $\mathcal{T}_P$, called the* tangent plane *of $[\pi]$ at $P$.*

*Proof.* By Theorems 2.4 and 2.5, we can without loss of generality prove this for the order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3, and the point $P=(0,0,1)$ of $\mathcal{B}$. First consider the order-$q$-subplane $\pi_0=\mathrm{PG}(2,q)$. The point $P=(0,0,1)$ lies in $\pi_0$, and the lines of $\pi_0$ through $P$ have coordinates $\ell'_m=[m,1,0]$, $m\in\mathbb{F}_q\cup\{\infty\}$. Points on the line $\ell'_m$ distinct from $P$ have coordinates $P'_x=(1,-m,x)$ for $x\in\mathbb{F}_q$. We map the plane $\pi_0$ to $\mathcal{B}$ using the homography $\sigma$ with matrix $K$ given in (2). As $\sigma(P)=P$, the lines of $\mathcal{B}$ through $P$ are $\ell_m=\sigma(\ell'_m)$, $m\in\mathbb{F}_q\cup\{\infty\}$. Points on the line $\ell_m$ distinct from $P$ have coordinates

$$P_x=\sigma(P'_x)=(-\tau-m,\ -\tau^q-m,\ \tau\tau^q+(\tau+\tau^q)m+x),$$

for $x\in\mathbb{F}_q$.

To convert this to a coordinate in PG(6, $q$), we need to multiply by an element of $\mathbb{F}_{q^3}$ so that the last coordinate lies in $\mathbb{F}_q$. Let $F(x)=\tau\tau^q+(\tau+\tau^q)m+x$ (the third coordinate in $P_x$). As $F(x)\in\mathbb{F}_{q^3}$, we have $F(x)^{q^2+q+1}\in\mathbb{F}_q$. So in PG(6, $q$), we have the point $P_x = ([-(\tau+m)F(x)^{q^2+q}],\ [-(\tau^q+m)F(x)^{q^2+q}],\ F(x)^{q^2+q+1})$.

By Theorem 2.3, the line $\ell_m$ of $\mathrm{PG}(2, q^3)$ corresponds to a twisted cubic $[\ell_m] = \{P_x : x \in \mathbb{F}_q\} \cup \{P\}$ of $\mathrm{PG}(6, q)$. Consider the unique tangent to $[\ell_m]$ through $P$, and let $I_m$ be the intersection of this tangent with $\Sigma_\infty$. We will show that the points $I_m,\ m \in \mathbb{F}_q \cup \{\infty\}$, form a line. To calculate the coordinates of $I_m$, we let $Q_x = PP_x \cap \Sigma_\infty$. To calculate $I_m = Q_\infty$, we use the homogeneous coordinate technique of dividing by the largest power of $x$, and then substituting $x = \infty$, that is, replacing $1/x$ by 0. We use the notation $\lim_{x\to\infty}$ to describe this technique.

$$I_m = \lim_{x\to\infty} PP_x \cap \Sigma_\infty = \lim_{x\to\infty} ([-(\tau+m)F(x)^{q^2+q}], [-(\tau^q+m)F(x)^{q^2+q}], 0)$$
$$= ([-(\tau+m)],\ -[\tau^q+m], 0).$$

Hence the points $I_m,\ m \in \mathbb{F}_q \cup \{\infty\}$, form a line $\ell = \langle([1], [1], 0), ([\tau], [\tau^q], 0)\rangle$ in $\Sigma_\infty$. Hence the tangent lines $m_0, \dots, m_q$ to the twisted cubics of $[\pi]$ through $P$ form a plane $\mathcal{T}_P = \langle \ell, P\rangle$ through $P$, as required. □

**Theorem 4.3.** *Let $\pi$ be an exterior order-$q$-subplane of* $\mathrm{PG}(2, q^3)$, *and let $P$ be a point of $\pi$. In* $\mathrm{PG}(6, q)$, *consider the intersection of the tangent spaces at $P$ of the nine quadrics corresponding to $[\pi]$. Then this intersection is equal to the tangent plane $\mathcal{T}_P$ of $[\pi]$ at $P$ as defined in Theorem 4.2.*

*Proof.* By Theorems 2.4 and 2.5, we can without loss of generality prove this for the order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3, and the point $P = (0, 0, 1)$ of $\mathcal{B}$. In $\mathrm{PG}(6, q)$, consider the nine quadrics corresponding to $[\mathcal{B}]$ which are given in equations (4), (5) and (6). We want to find the set of lines through $P$ that meet each of these nine quadrics twice at $P$. Every line $\ell$ of $\mathrm{PG}(6, q)$ through $P$ has the form $\ell = RP$ for some point $R = ([u], [v], 0) \in \Sigma_\infty,\ u, v \in \mathbb{F}_{q^3}$. So the points of $\ell$ are of the form $P_s = P + sR = ([su], [sv], 1)$ where $s \in \mathbb{F}_q$. Substituting the point $P_s$ into the quadrics of (4) gives

$$(-\tau^{2q}su+\tau^{2q}sv+(\tau-\tau^q))^q(-su+sv) = (-\tau^{2q}su+\tau^2sv+(\tau-\tau^q))(-su+sv)^q.$$

This expression is a polynomial of degree two in $s$. The line $\ell = PR$ is tangent to the three quadrics of (4) if this expression has a repeated root $s = 0$, that is, if the coefficient of $s$ is equal to zero. That is,

$$(\tau - \tau^q)^q(-u + v) = (\tau - \tau^q)(-u + v)^q,$$

and so $k = (-u + v)/(\tau - \tau^q)$ is in $\mathbb{F}_q$. Rearranging gives $v = k(\tau - \tau^q) + u$. Substituting the point $P_s$ into the quadrics of (5) gives no constraints. Substituting the point $P_s$ into the quadrics of (6) and simplifying gives that the constraint $m = (-\tau^q u + \tau v)/(\tau - \tau^q)$ lies in $\mathbb{F}_q$, and so $v = (m(\tau - \tau^q) + \tau^q u)/\tau$. Equating this with the expression for $v$ obtained from (4) gives $u = m - k\tau$, and so $v = m - k\tau^q$.

Hence the line $\ell = PR$ is tangent to all nine quadrics when $R$ has form

$$R = ([u], [v], 0) = ([m - k\tau], [m - k\tau^q], 0) = m([1], [1], 0) - k([\tau], [\tau^q], 0).$$

Thus the tangent space to $[\mathcal{B}]$ at $P$ is the plane through $P$ and the line

$$\ell = \langle([1], [1], 0), ([\tau], [\tau^q], 0)\rangle$$

of $\Sigma_\infty$. This is the same as the tangent plane $\mathcal{T}_P$ to $[\mathcal{B}]$ at $P$ calculated in the proof of Theorem 4.2. □

## 5. Coordinatising the exterior splash and its covers

Let $\mathbb{S}$ be an exterior splash of PG$(1, q^3)$. In the Bruck–Bose representation, $\mathbb{S}$ corresponds to a set of $q^2+q+1$ planes of the regular 2-spread $\mathcal{S}$ in $\Sigma_\infty \cong$ PG$(5, q)$. To simplify the notation, we use the same symbol $\mathbb{S}$ to denote both the points of the exterior splash on $\ell_\infty$, and the planes of the exterior splash contained in $\mathcal{S}$. In [Barwick and Jackson 2016], we show that an exterior splash is projectively equivalent to a cover of the circle geometry CG$(3, q)$. Hence by Bruck [1973], there are two *switching sets* $\mathbb{X}$, $\mathbb{Y}$ for $\mathbb{S}$. That is, $\mathbb{X}$ and $\mathbb{Y}$ consist of $q^2 + q + 1$ planes each, such that the planes of the three sets $\mathbb{S}$, $\mathbb{X}$ and $\mathbb{Y}$ each cover the same set of points. Further, planes from different sets meet in unique points, and planes in the same set are disjoint. The three sets $\mathbb{S}$, $\mathbb{X}$, $\mathbb{Y}$ are called *hyper-reguli* in [Culbert and Ebert 2005; Ostrom 1993]. In this article, we call the families $\mathbb{X}$ and $\mathbb{Y}$ *covers* of the exterior splash $\mathbb{S}$.

In this section we take the order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3, with exterior splash $\mathbb{S}$, and use [Ostrom 1993] to calculate the coordinates of the two covers of $\mathbb{S}$. We will characterise the two covers in terms of the subplane $\mathcal{B}$.

We call one cover of $\mathbb{S}$ the *tangent cover with respect to* $\mathcal{B}$, and denote it by $\mathbb{T}_\mathcal{B}$, or if there is only one subplane under consideration, we shorten this to $\mathbb{T}$. The nomenclature for tangent covers comes from Theorem 5.3 which shows that the tangent planes $\mathcal{T}_P$ of $[\mathcal{B}]$ meet $\Sigma_\infty$ in lines that lie in distinct planes of the cover $\mathbb{T}$.

We call the other cover of $\mathbb{S}$ the *conic cover with respect to* $\mathcal{B}$, and denote it by $\mathbb{C}_\mathcal{B}$, or $\mathbb{C}$. The nomenclature for the conic cover comes from [Barwick and Jackson 2017] which shows that the planes in the cover $\mathbb{C}$ are related to the $(\mathcal{B}, \ell_\infty)$-carrier conics of $\mathcal{B}$.

A certain type of embedding is looked at in [Lavrauw et al. 2015]. Specialising their results to PG$(5, q)$, their embedding $\mathcal{Q}_{2,q}$ is equivalent to the set $\mathbb{S} \cup \mathbb{C} \cup \mathbb{T}$. They determine the collineation group stabilising $\mathcal{Q}_{2,q}$. In particular they demonstrate: a collineation of PG$(5, q)$ that fixes $\mathcal{Q}_{2,q}$ and permutes the families $\mathbb{S}$, $\mathbb{C}$, $\mathbb{T}$; and a collineation fixing $\mathcal{Q}_{2,q}$ that permutes the planes in each family. Further, [Lavrauw et al. 2015] determines the equation of $\mathcal{Q}_{2,q}$. In Lemma 5.1 we describe the homogeneous coordinates for the planes in $\mathbb{S}, \mathbb{C}, \mathbb{T}$ in the format we will work with, and in Lemma 5.2 we calculate the matrix of a homography that fixes the planes in $\mathbb{S}$, permutes the planes of $\mathbb{T}$, and permutes the planes of $\mathbb{C}$ (this is the map $\varphi_{0,0}(\tau, \tau)$ of [Lavrauw et al. 2015]).

**Lemma 5.1.** *Let $\mathbb{S}$ be the exterior splash of the exterior order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3. Let $\mathcal{K} = \{k = \tau^{i(q-1)} : 0 \le i < q^2 + q + 1\}$. In* PG$(6, q)$, $\mathbb{S}$ *and its two covers* $\mathbb{T}, \mathbb{C}$ *have planes given by*

$$\mathbb{S} = \{[S_k] = \{([kx], [x], 0) : x \in \mathbb{F}'_{q^3}\} : k \in \mathcal{K}\},$$
$$\mathbb{T} = \{[T_k] = \{([kx], [x^q], 0) : x \in \mathbb{F}'_{q^3}\} : k \in \mathcal{K}\},$$
$$\mathbb{C} = \{[C_k] = \{([kx], [x^{q^2}], 0) : x \in \mathbb{F}'_{q^3}\} : k \in \mathcal{K}\}.$$

*Proof.* The points of $\ell_\infty$ in PG$(2, q^3)$ have coordinates $S_k = (k, 1, 0)$ for $k \in \mathbb{F}_{q^3} \cup \{\infty\}$. Hence in the Bruck–Bose representation of $\ell_\infty$ in $\Sigma_\infty \cong$ PG$(5, q)$, planes of the regular 2-spread $\mathcal{S}$ are given by $[S_k] = \{([kx], [x]) : x \in \mathbb{F}'_{q^3}\}$, for $k \in \mathbb{F}_{q^3} \cup \{\infty\}$. Consider the homography $\beta$ (of order 3) of $\Sigma_\infty \cong$ PG$(5, q)$ defined by

$$\beta : ([x], [y]) \to ([x], [y^q]). \tag{7}$$

We consider the action of $\beta$ on the planes of $[S_k]$. For each $k \in \mathbb{F}_{q^3} \cup \{\infty\}$, define the planes $[T_k], [C_k]$ by $\beta([S_k]) = [T_k]$ and $\beta([T_k]) = [C_k]$. That is, $[T_k] = \{([kx], [x^q]) : x \in \mathbb{F}'_{q^3}\}$, and $[C_k] = \{([kx], [x^{q^2}]) : x \in \mathbb{F}'_{q^3}\}$.

We now consider the exterior order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3 which by Theorem 3.1 has exterior splash $\mathbb{S} = \{S_k = (k, 1, 0) : k \in \mathcal{K}\} \subset \ell_\infty$, and carriers $S_\infty = (1, 0, 0)$, $S_0 = (0, 1, 0)$. Note that in PG$(5, q)$, the carriers of $\mathcal{B}$ lie in each of the three sets of planes, as $[S_0] = [T_0] = [C_0]$ and $[S_\infty] = [T_\infty] = [C_\infty]$. In PG$(5, q)$, we have $\mathbb{S} = \{[S_k] : k \in \mathcal{K}\}$. Let $\mathbb{T} = \{[T_k] : k \in \mathcal{K}\}$ and $\mathbb{C} = \{[C_k] : k \in \mathcal{K}\}$, then $\beta : \mathbb{S} \mapsto \mathbb{T} \mapsto \mathbb{C}$. By [Ostrom 1993], the sets $\mathbb{S}$, $\mathbb{T}$, $\mathbb{C}$ cover the same set of points. Moreover, planes in the same set are disjoint, and planes from different sets meet in one point. That is, $\mathbb{T}$ and $\mathbb{C}$ are the two covers of $\mathbb{S}$. □

The next lemma calculates the action of a useful homography of PG$(6, q)$ (this is the map $\varphi_{0,0}(\tau, \tau)$ of [Lavrauw et al. 2015]). Recall that $\tau$ is a zero of the primitive polynomial $x^3 - t_2x^2 - t_1x - t_0$.

**Lemma 5.2.** *Let $\mathbb{S}$ be the exterior splash of the exterior order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3 with covers $\mathbb{C}$ and $\mathbb{T}$ coordinatised in Lemma 5.1. Consider the homography $\Theta \in$ PGL$(7, q)$ with $7 \times 7$ matrix*

$$\begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} 0 & 0 & t_0 \\ 1 & 0 & t_1 \\ 0 & 1 & t_2 \end{pmatrix}.$$

*Then in* PG(6, $q$), $\Theta$ *fixes each plane of the regular* 2*-spread* $\mathcal{S}$, *maps the cover plane* $[C_k]\in\mathbb{C}$ *to* $[C_{\tau^{1-q}k}]\in\mathbb{C}$, *and the cover plane* $[T_k]\in\mathbb{T}$ *to* $[T_{\tau^{1-q^2}k}]\in\mathbb{T}$, $k\in\mathcal{K}$.

*Proof.* It is straightforward to show that $\Theta$ fixes the planes of the regular 2-spread $\mathcal{S}$ (so it also fixes the planes of the exterior splash $\mathbb{S}$). In fact $\langle\Theta\rangle$ acts regularly on the set of points, and on the set of lines, of each spread element. Note that $M$ is the matrix $M_\tau$ defined in Section 2B, and so $M[x]=[\tau x]$. Consider the action of $\Theta$ on a point of the cover plane $[C_k]\in\mathbb{C}$ coordinatised in Lemma 5.1. We have

$$([kx],[x^{q^2}],0)^\Theta=([\tau kx],[\tau x^{q^2}],0)\equiv([\tau^{1-q}k(\tau^q x)],[(\tau^q x)^{q^2}],0)$$

which lies in the cover plane $[C_{\tau^{1-q}k}]$ of $\mathbb{C}$. Similarly a point $([kx],[x^q],0)$ in the cover plane $[T_k]\in\mathbb{T}$ maps under $\Theta$ to the point $([\tau^{1-q^2}k(\tau^{q^2}x)],[(\tau^{q^2}x)^q],0)$ which lies in the cover plane $[T_{\tau^{1-q^2}k}]$ of $\mathbb{T}$. □

**Theorem 5.3.** *Let $P$ be a point of an exterior order-$q$-subplane $\pi$. In* PG(6, $q$), *the tangent plane $\mathcal{T}_P$ at $P$ to $[\pi]$ meets $\Sigma_\infty$ in a line that lies in a plane of the tangent cover $\mathbb{T}$ of $[\pi]$. Moreover, distinct points of $\pi$ correspond to distinct cover planes of $\mathbb{T}$.*

*Proof.* By Theorems 2.4 and 2.5, we can without loss of generality prove this result for the order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3 and the point $P=(0,0,1)\in\mathcal{B}$. In PG(6, $q$), let $\mathcal{T}_P$ be the tangent plane at $P$. The line $\ell=\mathcal{T}_P\cap\Sigma_\infty$ was calculated in the proof of Theorem 4.2 to be

$$\ell=\{a([1],[1],0)+b([\tau],[\tau^q],0):a,b\in\mathbb{F}_q\}.$$

The points of $\ell$ all lie in the plane $[T_1]=\{[x],[x^q],0)\mid x\in\mathbb{F}'_{q^3}\}$, which by Lemma 5.1 is a plane of the tangent cover $\mathbb{T}$ of $\mathcal{B}$. The collineation of Lemma 5.2 is transitive on the cover planes of $\mathbb{T}$, hence each cover plane contains a line of a distinct tangent plane. Hence there is a one-to-one correspondence between points of $\pi$ and cover planes of $\mathbb{T}$. □

## 6. Transversal lines of covers

Recall that a regular 2-spread in PG(5, $q$) has three (conjugate skew) transversals in PG(5, $q^3$) which meet each (extended) plane of $\mathcal{S}$. In this section we consider an exterior splash $\mathbb{S}\subset\mathcal{S}$, and show in Lemma 6.1 that the transversals of the 2-spread $\mathcal{S}$ are the only lines of PG(5, $q^3$) that meet every extended plane of $\mathbb{S}$. We then consider the two sets of cover planes $\mathbb{T}$ and $\mathbb{C}$. Corollary 6.2 shows that each can be uniquely extended to regular 2-spread, and we calculate the coordinates of the corresponding transversal lines in Theorem 6.3. Theorem 6.5 shows that the nine transversals of $\mathbb{S}$, $\mathbb{C}$ and $\mathbb{T}$ can be used to characterise the carriers of the exterior splash $\mathbb{S}$. Theorem 6.6, looks at the transversal lines in the situation when $\ell_\infty$ is partitioned into exterior splashes with common carriers.

**6A.** ***The exterior splash and its covers have unique transversals.*** If $\mathcal{X}$ is a set in PG(6, $q$) (such as a line, a plane, or a conic), then we denote its natural extension to PG(6, $q^3$) by $\mathcal{X}^*$. Let $\mathcal{S}$ be the regular 2-spread in $\Sigma_\infty$ of the Bruck–Bose representation in PG(6, $q$). If we extend the planes of $\mathcal{S}$ to PG(6, $q^3$), yielding $\mathcal{S}^*$, then there are exactly three transversal lines to $\mathcal{S}^*$, that is, three lines that meet every plane of $\mathcal{S}^*$. These three lines are conjugate and skew. We now consider an exterior splash $\mathbb{S} \subset \mathcal{S}$ and extend the planes of $\mathbb{S}$ to PG(6, $q^3$), yielding $\mathbb{S}^*$. We show that there are exactly three lines of PG(6, $q^3$) that meet every plane of $\mathbb{S}^*$, namely the three transversals of $\mathcal{S}$.

**Lemma 6.1.** *Let $\mathcal{S}$ be a regular* 2*-spread in* PG(5, $q$), *and let $\mathbb{S} \subset \mathcal{S}$ be an exterior splash. In the cubic extension* PG(5, $q^3$), *there are exactly three transversals to $\mathbb{S}$, namely the three transversals of $\mathcal{S}$. Hence $\mathbb{S}$ lies in a unique regular* 2*-spread, namely $\mathcal{S}$.*

*Proof.* The three conjugate transversal lines of the regular 2-spread $\mathcal{S}$, denoted $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}$, are also transversals of $\mathbb{S}$. Suppose there is a fourth transversal line $\ell$ of $\mathbb{S}$. Then the four lines $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}, \ell$ are pairwise skew. Further, these four lines are ruling lines of a unique 2-regulus $\mathfrak{R}$ of $\Sigma_\infty^* \cong$ PG(5, $q^3$), which contains the set of extended planes $\mathbb{S}^*$. Now consider two planes $[L], [M] \in \mathbb{S}$; the corresponding points $L, M$ of $\ell_\infty$ in PG(2, $q^3$) lie in two order-$q$-sublines contained in $\mathbb{S}$ by [Lavrauw and Van de Voorde 2010, Corollary 15]. Hence by Theorem 2.3, $[L], [M]$ lie in two 2-reguli $\mathcal{R}_1, \mathcal{R}_2$ which are contained in $\mathbb{S}$. Let $P$ be a point in $[L]$, then there are unique lines $m_1, m_2$ through $P$ that are ruling lines of $\mathcal{R}_1, \mathcal{R}_2$ respectively. Now $\mathcal{R}_1$, $\mathcal{R}_2$ lie in $\mathbb{S}$, and so lie in $\mathfrak{R}$, so the extended lines $m_i^*$, $i = 1, 2$, are two ruling lines of $\mathfrak{R}$ that meet in a point $P$, a contradiction. Hence the line $\ell$ cannot exist. That is, there are only three transversal lines to $\mathbb{S}$, and these are necessarily the transversals of $\mathcal{S}$. □

As $\mathbb{S}$, $\mathbb{C}$, $\mathbb{T}$ are projectively equivalent by [Lavrauw et al. 2015, Theorem 16], an analogous result holds for the two covers of $\mathbb{S}$.

**Corollary 6.2.** *In* PG(5, $q$), *let $\mathbb{S}$ be an exterior splash with covers $\mathbb{T}$ and $\mathbb{C}$. Then in the cubic extension* PG(5, $q^3$),

(i) *the cover $\mathbb{T}$ has exactly three transversal lines in* PG(5, $q^3$)\PG(5, $q$), *denoted $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$, and so $\mathbb{T}$ lies in a unique regular* 2*-spread,*

(ii) *the cover $\mathbb{C}$ has exactly three transversal lines in* PG(5, $q^3$)\PG(5, $q$), *denoted $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$, and so $\mathbb{C}$ lies in a unique regular* 2*-spread.*

Later we will need the coordinates of the point of intersection of the transversal lines with the corresponding cover planes, and we calculate these next.

**Theorem 6.3.** *Let $\mathcal{B}$ be the order-$q$-subplane coordinatised in Section 3 with exterior splash $\mathbb{S}$ and covers $\mathbb{C}$, $\mathbb{T}$. Let $p_0 = t_1 + t_2\tau - \tau^2 = -\tau^q\tau^{q^2}$, $p_1 = t_2 - \tau =$*

$\tau^q + \tau^{q^2}$, $p_2 = -1$, *and* $\eta = p_0 + p_1\tau + p_2\tau^2$. *Let* $A_1 = (p_0, p_1, p_2, 0, 0, 0, 0)$, $A_2 = (0, 0, 0, p_0, p_1, p_2, 0)$. *Then in* PG$(6, q^3)$,

(i) *one transversal line of* $\mathbb{S}$ *is* $g_{\mathbb{S}} = \langle A_1, A_2\rangle$, *and* $g_{\mathbb{S}} \cap [S_k]^* = kA_1 + A_2$,

(ii) *one transversal line of* $\mathbb{T}$ *is* $g_{\mathbb{T}} = \langle A_1, A_2^{q^2}\rangle$, *and* $g_{\mathbb{T}} \cap [T_k]^* = kA_1 + \eta^{1-q^2} A_2^{q^2}$,

(iii) *one transversal line of* $\mathbb{C}$ *is* $g_{\mathbb{C}} = \langle A_1, A_2^{q}\rangle$, *and* $g_{\mathbb{C}} \cap [C_k]^* = kA_1 + \eta^{1-q} A_2^{q}$.

*Proof.* We use the coordinatisation in PG$(5, q)$ of the exterior splash $\mathbb{S}$ of $\mathcal{B}$ and the two covers $\mathbb{T}$, $\mathbb{C}$ given in Lemma 5.1. Lemma 2.1 shows that $g_{\mathbb{S}} = \langle A_1, A_2\rangle$ is a transversal line for the regular 2-spread $\mathcal{S}$, where $A_1 = (p_0, p_1, p_2, 0, 0, 0) = (A, [0])$ and $A_2 = (0, 0, 0, p_0, p_1, p_2) = ([0], A)$. Hence $g_{\mathbb{S}} = \langle A_1, A_2\rangle$ is a transversal line for the exterior splash $\mathbb{S}$. The planes of the regular 2-spread $\mathcal{S}$ are $[S_k] = \{([kx], [x]) : x \in \mathbb{F}'_{q^3}\}$, $k \in \mathbb{F}_{q^3} \cup \{\infty\}$. We first show that the extended plane $[S_k]^*$ meets the line $g_{\mathbb{S}}$ in the point $kA_1 + A_2$. Consider the point $P = p_0([k], [1]) + p_1([k\tau], [\tau]) + p_2([k\tau^2], [\tau^2])$ of PG$(5, q^3)$, and note that $P \in [S_k]^*$. Using the matrix $M_k$ defined in Section 2B, we have

$$P = p_0(M_k[1], [1]) + p_1(M_k[\tau], [\tau]) + p_2(M_k[\tau^2], [\tau^2]) = (M_k A, A) = (kA, A)$$

by (1). Hence $P = kA_1 + A_2$ which lies in $g_{\mathbb{S}} = \langle A_1, A_2\rangle$, that is, $P$ is the intersection of $g_{\mathbb{S}}$ and $[S_k]^*$ proving part (i).

Consider the homography $\beta$ defined in (7), acting on PG$(5, q^3)$. The proof of Lemma 5.1 shows that $\beta$ maps $g_{\mathbb{S}}$ to $g_{\mathbb{T}}$, and maps $g_{\mathbb{T}}$ to $g_{\mathbb{C}}$. Each element $y \in \mathbb{F}'_{q^3}$ can be considered as a point $[y]$ in PG$(2, q)$. The collineation of PG$(2, q)$ mapping the point $[y]$ to $[y^q]$ is a homography, and can be represented using a matrix $N$ with entries in $\mathbb{F}_q$. We omit the transpose notation, and write $N[y] = [y^q]$. Hence we can write the collineation $\beta$ as $\beta([x], [y]) = ([x], N[y])$. Clearly $\beta(A_1) = A_1$, and we show that $\beta(A_2) = A_2^{q^2}$. Recall the point $A = (p_0, p_1, p_2) = p_0[1] + p_1[\tau] + p_2[\tau^2]$, so $NA = p_0[1] + p_1[\tau^q] + p_2[\tau^{2q}]$. Using the matrix $M_k$ from Section 2B, it is straightforward to write this as $NA = (p_0^{q^2} I + p_1^{q^2} M_\tau + p_2^{q^2} M_{\tau^2})^q\,[1]$. Now

$$(p_0^{q^2} I + p_1^{q^2} M_\tau + p_2^{q^2} M_{\tau^2})[1] = A^{q^2} \quad \text{and} \quad (p_0^{q^2} I + p_1^{q^2} M_\tau + p_2^{q^2} M_{\tau^2})A^{q^2} = \eta^{q^2} A^{q^2}$$

by (1). So repeated use of (1) yields $NA = \eta^{q^2(q-1)} A^{q^2} = \eta^{1-q^2} A^{q^2}$. Further, as $N$ is over $\mathbb{F}_q$, we have

$$NA = \eta^{1-q^2} A^{q^2}, \quad NA^q = \eta^{q-1} A, \quad NA^{q^2} = \eta^{q^2-q} A^q. \tag{8}$$

Hence $\beta(kA_1 + A_2) = kA_1 + \eta^{1-q^2} A_2^{q^2}$. As $\beta : g_{\mathbb{S}} \mapsto g_{\mathbb{T}}$, we have $g_{\mathbb{T}} \cap [T_k]^* = kA_1 + \eta^{1-q^2} A_2^{q^2}$ and $g_{\mathbb{T}} = \langle A_1, A_2^{q^2}\rangle$, proving part (ii). Similarly, calculating

$$\beta(kA_1 + \eta^{1-q^2} A_2^{q^2}) = kA_1 + \eta^{1-q^2+q^2-q} A_2^{q} = kA_1 + \eta^{1-q} A_2^{q},$$

and using $\beta : g_{\mathbb{T}} \mapsto g_{\mathbb{C}}$, we get $g_{\mathbb{C}} \cap [C_k]^* = kA_1 + \eta^{1-q} A_2^{q}$ and $g_{\mathbb{C}} = \langle A_1, A_2^{q}\rangle$. □

We can use the transversals of the covers $\mathbb{T}$ and $\mathbb{C}$ to generalise the notion of $\mathcal{S}$-special conics and twisted cubics in PG(6, $q$) defined in Definition 2.2. We define $\mathbb{C}$-special here, $\mathbb{T}$-special is similarly defined.

**Definition 6.4.** (i) A $\mathbb{C}$-*special conic* is a nondegenerate conic $\mathcal{C}$ contained in a plane of $\mathbb{C}$, such that the extension of $\mathcal{C}$ to PG(6, $q^3$) meets the transversals of $\mathbb{C}$.

(ii) A $\mathbb{C}$-*special twisted cubic* is a twisted cubic $\mathcal{N}$ in a 3-space of PG(6, $q$)$\backslash\Sigma_\infty$ about a plane of $\mathbb{C}$, such that the extension of $\mathcal{N}$ to PG(6, $q^3$) meets the transversals of $\mathbb{C}$.

**6B. *Characterising the carriers in* PG(6, *q*).** Letting $\mathcal{S}$ be a regular 2-spread of PG(5, $q$), and $\mathbb{S}$ be an exterior splash contained in $\mathcal{S}$, with covers $\mathbb{C}$ and $\mathbb{T}$, we can then characterise the carriers of $\mathbb{S}$ in terms of the nine transversals of $\mathbb{S}$, $\mathbb{C}$ and $\mathbb{T}$.

**Theorem 6.5.** *Let $\mathcal{S}$ be a regular* 2*-spread of* PG(5, $q$), *and let $\mathbb{S}\subset\mathcal{S}$ be an exterior splash with covers $\mathbb{C}$, $\mathbb{T}$, whose corresponding triples of transversal lines are $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}$, $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$, and $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$, respectively. Then the carriers of $\mathbb{S}$ are the only two planes of $\mathcal{S}$ whose extension to* PG(5, $q^3$) *meets all nine transversal lines.*

*Proof.* By Theorem 2.4, we can without loss of generality show this for the exterior splash $\mathbb{S}$ of the exterior order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3, with carriers $E_1=(1,0,0)$, $E_2=(0,1,0)$. In PG(6, $q$), the transversal lines $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}$ each meet the carriers $[E_1]$, $[E_2]$ of $\mathbb{S}$. We use the notation for planes $[S_k]\in\mathcal{S}$, $[T_k]\in\mathbb{T}$ and $[C_k]\in\mathbb{C}$ from Lemma 5.1. By Corollary 6.2, in the cubic extension PG(5, $q^3$), the transversal lines $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$ meet each plane $[T_k]$, $k\in\mathbb{F}_{q^3}\cup\{\infty\}$; and the transversal lines $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$ meet each plane $[C_k]$, $k\in\mathbb{F}_{q^3}\cup\{\infty\}$. The carriers of $\mathbb{S}$ satisfy $[E_2]=[S_0]=[T_0]=[C_0]$ and $[E_1]=[S_\infty]=[T_\infty]=[C_\infty]$. Hence in the cubic extension PG(5, $q^3$), all nine transversal lines meet the carriers of $\mathbb{S}$.

We now show that no other plane of the regular 2-spread $\mathcal{S}$ meets all nine transversal lines. We use the homography with matrix $M_k$ defined in Section 2B. A plane of the regular 2-spread $\mathcal{S}$ distinct from $[E_1]$, $[E_2]$ has the form $[S_k]=\{([kx],[x],0) : x\in\mathbb{F}'_{q^3}\}$, for some $k\in\mathbb{F}'_{q^3}$. This plane is spanned by the three points

$$\begin{aligned} S_{0,k} &= ([k],[1],0) = (M_k(1,0,0),(1,0,0)),\\ S_{1,k} &= ([k\tau],[\tau],0) = (M_k(0,1,0),(0,1,0))\\ S_{2,k} &= ([k\tau^2],[\tau^2],0) = (M_k(0,0,1),(0,0,1)).\end{aligned}$$

Hence the extension $[S_k]^*$ to PG(5, $q^3$) contains the points

$$S_{k,j} = c_0S_{0,j}+c_1S_{1,j}+c_2S_{2,j},$$

where $c_i \in \mathbb{F}_{q^3}$, not all zero. By Theorem 6.3, a general point $X$ on the transversal line $g_{\mathbb{T}}$ has coordinates $X = rA_1 + A_2^{q^2} = (rp_0,\ rp_1,\ rp_2,\ p_0^{q^2},\ p_1^{q^2},\ p_2^{q^2})$, for some $r \in \mathbb{F}_{q^3} \cup \{\infty\}$. Now $S_{j,k} = X$ if and only if $c_i = p_i^{q^2}$, $i = 0, 1, 2$, and $M_k(c_0, c_1, c_2) = r(p_0, p_1, p_2)$. That is, $M_k A^{q^2} = rA$. However, $M_k A^{q^2} = k^{q^2} A^{q^2}$, by (1), so there are no solutions to $c_0, c_1, c_2$. Hence the transversal line $g_{\mathbb{T}}$ does not meet any further plane of the regular 2-spread $\mathcal{S}$, and so $g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$ do not meet any further plane of $\mathcal{S}$. A similar argument shows that the lines $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$ do not meet any further plane of the regular 2-spread $\mathcal{S}$. □

**6C.** ***Transversal lines of exterior splashes with common carriers.*** As exterior splashes are equivalent to covers of the circle geometry CG(3, $q$), there are $q - 1$ disjoint exterior splashes on $\ell_\infty$ with common carriers $E_1, E_2$. We show that in PG(6, $q$), the covers of these disjoint exterior splashes have common transversals.

**Theorem 6.6.** *Let $\mathbb{S}_0, \ldots, \mathbb{S}_{q-1}$ be $q - 1$ disjoint exterior splashes on $\ell_\infty$ with common carriers $E_1, E_2$, and let exterior splash $\mathbb{S}_j$ have covers $\mathbb{C}_j$, $\mathbb{T}_j$. Then the covers $\mathbb{C}_0, \ldots, \mathbb{C}_{q-1}$ have common transversal lines $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$, and the covers $\mathbb{T}_0, \ldots, \mathbb{T}_{q-1}$ have common transversal lines $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$.*

*Proof.* By Theorem 2.4, we can without loss of generality prove this for the order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3. Let $\mathcal{K} = \{k \in \mathbb{F}'_{q^3} : k^{q^2+q+1} = 1\} = \{k = \tau^{i(q-1)} : 0 \le i < q^2 + q + 1\}$. Recall that $\mathcal{B}$ has carriers $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$, and exterior splash $\mathbb{S}_0 = \{S_{k,0} = (k, 1, 0) : k \in \mathcal{K}\}$. Let $\mathcal{K}_j = \tau^j \mathcal{K}$, for $j = 0, \ldots, q-2$, be the $q-1$ cosets of $\mathcal{K}$ in $\mathbb{F}'_{q^3}$. Let $\mathbb{S}_j = \{S_{k,j} = (k, 1, 0) : k \in \mathcal{K}_j\}$, $0 \le j \le q-2$. Consider the homography $\xi$ acting on $\ell_\infty$ that maps the point $(x, y, 0)$ to $(\tau x, y, 0)$. Then $\xi$ fixes $E_1, E_2$, maps $\mathbb{S}_j$ to $\mathbb{S}_{j+1}$ $(0 \le j \le q - 3)$, and maps $\mathbb{S}_{q-2}$ to $\mathbb{S}_0$. Hence $\mathbb{S}_0, \ldots, \mathbb{S}_{q-1}$ are the $q - 1$ disjoint exterior splashes on $\ell_\infty$ with carriers $(1, 0, 0)$ and $(0, 1, 0)$.

In $\Sigma_\infty \cong \mathrm{PG}(5, q)$, we have planes $[S_{k,j}] = \{([kx], [x]) : x \in \mathbb{F}'_{q^3}\} \in \mathbb{S}$, and define the planes $[T_{k,j}] = \{([kx], [x^q]) : x \in \mathbb{F}'_{q^3}\}$, and $[C_{k,j}] = \{([kx], [x^{q^2}]) : x \in \mathbb{F}'_{q^3}\}$, for $k \in \mathcal{K}_j$. So $\mathbb{S}_j = \{[S_{k,j}],\ k \in \mathcal{K}_j\}$, and define $\mathbb{T}_j = \{[T_{k,j}], k \in \mathcal{K}_j\}$ and $\mathbb{C}_j = \{[C_{k,j}], k \in \mathcal{K}_j\}$. Note that $\mathbb{T}_0$, $\mathbb{C}_0$ are the covers of the exterior splash $\mathbb{S}_0$ of $\mathcal{B}$. Now consider the map $\theta_\tau$ of PG(5, $q$) acting on $\Sigma_\infty$ defined in Section 2B; it maps $\mathbb{S}_j$ to $\mathbb{S}_{j+1}$, $\mathbb{T}_j$ to $\mathbb{T}_{j+1}$, and $\mathbb{C}_j$ to $\mathbb{C}_{j+1}$. Hence $\mathbb{T}_j$ and $\mathbb{C}_j$ are covers for $\mathbb{S}_j$. By Theorem 6.3, the transversal line of $\mathbb{T}_0$ is $g_{\mathbb{T}} = \langle A_1, A_2^{q^2} \rangle$. Using (1), we see that the homography $\theta_\tau$ fixes $g_{\mathbb{T}}$, and so $g_{\mathbb{T}}$ is a transversal for all $\mathbb{T}_j$. So $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$ are transversal lines of $\mathbb{T}_j$ for each $j = 0, \ldots, q - 2$. Similarly, $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$ are transversal lines of $\mathbb{C}_j$ for each $j = 0, \ldots, q - 2$. □

**Remark 6.7.** We can interpret this result using the terminology of [Culbert and Ebert 2005]. We can partition the planes of a regular 2-spread into $q - 1$ disjoint hyper-reguli with common carriers. Each hyper-regulus has two replacement hyper-reguli, which correspond to our conic and tangent covers. If we replace all $q - 1$

hyper-reguli of $\mathcal{S}$ with hyper-reguli of the *same type* (that is, all belonging to $\mathbb{C}$, or all belonging to $\mathbb{T}$), then the resulting 2-spread has transversals either $g_{\mathbb{C}}$, $g_{\mathbb{C}}^{q}$, $g_{\mathbb{C}}^{q^2}$ or $g_{\mathbb{T}}$, $g_{\mathbb{T}}^{q}$, $g_{\mathbb{T}}^{q^2}$, and so is regular. Hence the resulting André plane is Desarguesian. If we replace all the hyper-reguli of $\mathcal{S}$ with a combination of hyper-reguli from each type, then the resulting 2-spread is not regular, and so the resulting André plane is non-Desarguesian.

## 7. Sublines of an exterior splash

In this section we characterise the order-$q$-sublines of $\mathbb{S}$ with respect to the covers of $\mathbb{S}$ and their transversal lines.

**7A.** ***Background.*** Let $\pi$ be an exterior order-$q$-subplane of $\mathrm{PG}(2, q^3)$ with exterior splash $\mathbb{S}$ on $\ell_\infty$. There are $2(q^2+q+1)$ order-$q$-sublines in an exterior splash which lie in two families of size $q^2+q+1$. These families are studied in [Lavrauw and Van de Voorde 2010; Barwick and Jackson 2016].

We first describe properties of the two families given in [Lavrauw and Van de Voorde 2010]; here the two families are called regular and irregular with respect to a plane in one of the covers. That is, let $\mathbb{S}$ be an exterior splash in $\mathrm{PG}(5, q)$, and let $\alpha$ be a plane that meets each plane of $\mathbb{S}$ in a point, so $\alpha$ lies in one of the covers $\mathbb{X}$ or $\mathbb{Y}$ of $\mathbb{S}$. In $\mathrm{PG}(2, q^3)$, let $b$ be an $\mathbb{F}_q$-subline contained in $\mathbb{S}$, so by Theorem 2.3, in $\mathrm{PG}(6, q)$, $[b]$ is a 2-regulus. The subline $b$ is called regular with respect to $\alpha$ if $\alpha \cap [b]$ is a line, otherwise $b$ is irregular. Suppose $\alpha$ lies in the cover $\mathbb{X}$, and $\alpha \cap [b]$ is a line, then each plane in the cover $\mathbb{X}$ meets $[b]$ in a line, and each plane in the cover $\mathbb{Y}$ meets $[b]$ in a set of points which is not collinear. We adapt the phrases regular and irregular with respect to $\alpha$ in terms of the covers of $\mathbb{S}$. We say $b$ is both $\mathbb{X}$-*regular* and $\mathbb{Y}$-*irregular* if each plane in $\mathbb{X}$ meets $[b]$ in a line. In particular, we note that if we start with a scattered $\mathbb{F}_q$-linear set of rank 3 of $\mathrm{PG}(1, q^3)$, then an $\mathbb{F}_q$-subline $b$ contained in the linear set can be categorised as both regular and irregular (by choosing $\alpha$ in different covers).

In [Lunardon and Polverino 2004], it is shown that if $\mathbb{S}$ is an exterior splash of $\ell_\infty$ in $\mathrm{PG}(2, q^3)$, then there is an order-$q$-subplane $\beta$ and point $P$ such that $\mathbb{S}$ is the projection of $\beta$ from $P$ onto $\ell_\infty$. In [Barwick and Jackson 2016, Theorem 5.2], the projection and splash constructions are compared, and it is shown that in almost all cases, the projection and exterior splash of $\beta$ are distinct. In [Lavrauw and Van de Voorde 2010], the two families of sublines of $\mathbb{S}$ are characterised in relation to a point $P$ and subplane $\beta$ which project $\mathbb{S}$: one family arises from projecting the sublines of $\beta$, the other arises from projecting certain conics of $\beta$. The latter family are described as irregular in [Lavrauw and Van de Voorde 2010], although it is not specified which cover these sublines are irregular with respect to.

Now we describe properties of the two families given in [Barwick and Jackson 2016]. Here the two families of order-$q$-sublines of $\mathbb{S}$ are characterised with respect to geometric objects of an exterior $\pi$ with exterior splash $\mathbb{S}$. If $A$ is a point of $\pi$, then the pencil of $q+1$ lines of $\pi$ through $A$ meets $\ell_\infty$ in an order-$q$-subline of $\mathbb{S}$, called a *$\pi$-pencil-subline* of $\mathbb{S}$. Recall from Section 2D that a $(\pi, \ell_\infty)$-carrier-dual conic of $\pi$ is a dual conic that contains the three lines fixed by the subgroup $I$ fixing $\pi$ and $\ell$. If $\Gamma$ is a $(\pi, \ell_\infty)$-carrier-dual conic of $\pi$, then the lines of $\Gamma$ meet $\ell_\infty$ in an order-$q$-subline of $\mathbb{S}$, called a *$\pi$-dual-conic-subline* of $\mathbb{S}$. Note that in [Barwick and Jackson 2016, Theorem 4.4], we show that it is possible to switch the roles of the two families by considering different associated order-$q$-subplanes.

**7B.** ***A characterisation of the sublines of an exterior splash.*** We now consider the interaction in PG(6, $q$) of the two families of order-$q$-sublines of $\mathbb{S}$ with the two covers of $\mathbb{S}$. We show in Theorem 7.1 that each family meets planes from one cover in lines, and planes from the other cover in conics. Theorem 7.2 shows that the converse is true, and so we have a characterisation of the order-$q$-sublines of $\mathbb{S}$. This allows us to relate the families from [Barwick and Jackson 2016] and [Lavrauw and Van de Voorde 2010]. Theorem 7.4 shows that the conics concerned in each case are special with respect to the conic cover.

Suppose $\mathcal{R}$ is a 2-regulus in PG(5, $q$), and consider a plane $\alpha$ that meets $\mathcal{R}$ in a set of $q+1$ points. Then an easy counting argument shows that these points form either a line or a conic in $\alpha$. We abbreviate this to "$\mathcal{R}$ meets $\alpha$ in a line or a conic".

**Theorem 7.1.** *Let $\pi$ be an exterior order-$q$-subplane with exterior splash $\mathbb{S}$, conic cover $\mathbb{C}$, and tangent cover $\mathbb{T}$.*

(i) *A $\pi$-pencil-subline of $\mathbb{S}$ corresponds in* PG(6, $q$) *to a 2-regulus that meets each plane of $\mathbb{T}$ in a distinct line, and meets each plane of $\mathbb{C}$ in a conic.*

(ii) *A $\pi$-dual-conic-subline of $\mathbb{S}$ corresponds in* PG(6, $q$) *to a 2-regulus that meets each plane of $\mathbb{T}$ in a conic, and meets each plane of $\mathbb{C}$ in a distinct line.*

*Proof.* Let $P$ be a point in the exterior order-$q$-subplane $\pi$, and let $d$ be the corresponding $\pi$-pencil-subline of $\mathbb{S}$. By Theorem 2.3, in PG(6, $q$), $[d]$ is a 2-regulus contained in $\mathbb{S}$. Consider the tangent plane $\mathcal{T}_P$ to $[\pi]$ at $P$. By Theorem 4.2, the lines of $\mathcal{T}_P$ through $P$ meet $\Sigma_\infty$ in points that lie in distinct planes of the 2-regulus $[d]$. Hence $\mathcal{T}_P \cap \Sigma_\infty$ is a ruling line of the 2-regulus $[d]$. By Theorem 5.3, this ruling line $\mathcal{T}_P \cap \Sigma_\infty$ lies in a tangent cover plane. The homography $\Theta$ of Lemma 5.2 fixes the planes of $[b]$ and is transitive on the cover planes of $\mathbb{T}$. Hence each ruling line of $[b]$ meets a unique cover plane of $\mathbb{T}$.

A straightforward geometric argument shows that planes of $\mathbb{T}, \mathbb{C}$ meet a 2-regulus of $\mathbb{S}$ in a line or a conic. Hence a conic cover plane meets the 2-regulus $[d]$ in a conic. As there are $q^2+q+1$ $\pi$-pencil-sublines of $\mathbb{S}$, every line in a plane of $\mathbb{T}$ is a ruling line for some 2-regulus corresponding to a $\pi$-pencil-subline. Hence if $[d']$ is a 2-regulus of $\mathbb{S}$ corresponding to a $\pi$-dual-conic-subline, then planes of $\mathbb{T}$ meet $[d']$ in conics, and so planes of $\mathbb{C}$ meet $[d']$ in ruling lines of $[d']$. Moreover, applying the homography of Lemma 5.2 shows that each ruling line of $[d']$ lies in a unique conic cover plane. □

By Theorem 2.3, there is a one-to-one correspondence between the order-$q$-sublines of $\mathbb{S}$ in $\mathrm{PG}(2, q^3)$, and the 2-reguli contained in $\mathbb{S}$ in $\mathrm{PG}(6, q)$. Hence the converse of Theorem 7.1 is also true, and so we have a characterisation of order-$q$-sublines of $\mathbb{S}$ relating to the cover planes of the associated order-$q$-subplane.

**Theorem 7.2.** *Let $\pi$ be an exterior order-$q$-subplane with exterior splash $\mathbb{S}$, conic cover $\mathbb{C}$, and tangent cover $\mathbb{T}$.*

(i) *A 2-regulus contained in $\mathbb{S}$ that meets some plane of $\mathbb{T}$ in a line corresponds to a $\pi$-pencil-subline of $\mathbb{S}$.*

(ii) *A 2-regulus contained in $\mathbb{S}$ that meets some plane of $\mathbb{C}$ in a conic corresponds to a $\pi$-pencil-subline of $\mathbb{S}$.*

(iii) *A 2-regulus contained in $\mathbb{S}$ that meets some plane of $\mathbb{T}$ in a conic corresponds to a $\pi$-dual-conic-subline of $\mathbb{S}$.*

(iv) *A 2-regulus contained in $\mathbb{S}$ that meets some plane of $\mathbb{C}$ in a line corresponds to a $\pi$-dual-conic-subline of $\mathbb{S}$.*

This allows us to determine the relationship between the different family naming used in [Barwick and Jackson 2016] and [Lavrauw and Van de Voorde 2010].

**Corollary 7.3.** *Let $\pi$ be an exterior order-$q$-subplane with exterior splash $\mathbb{S}$, conic cover $\mathbb{C}$, and tangent cover $\mathbb{T}$.*

(i) *Let $b$ be a $\pi$-pencil-subline of $\mathbb{S}$, then $b$ is $\mathbb{T}$-regular and $\mathbb{C}$-irregular.*

(ii) *Let $d$ be a $\pi$-dual-conic-subline of $\mathbb{S}$, then $d$ is $\mathbb{C}$-regular and $\mathbb{T}$-irregular.*

In fact, we can give a stronger characterisation of the order-$q$-sublines of $\mathbb{S}$, namely that the conics of Theorem 7.1 are *special* with respect to the associated cover. In order to prove that the conics are special, we need to introduce coordinates, and the proof is calculation intensive.

**Theorem 7.4.** *Let $\pi$ be an exterior order-$q$-subplane with exterior splash $\mathbb{S}$, conic cover $\mathbb{C}$, and tangent cover $\mathbb{T}$.*

(i) *A 2-regulus of $\mathbb{S}$ corresponding to a $\pi$-pencil-subline of $\mathbb{S}$ meets each plane of $\mathbb{C}$ in a $\mathbb{C}$-special conic.*

(ii) *A 2-regulus of $\mathbb{S}$ corresponding to a $\pi$-dual-conic-subline of $\mathbb{S}$ meets each plane of $\mathbb{T}$ in a $\mathbb{T}$-special conic.*

*Proof.* By Theorem 2.4, we can without loss of generality prove this for the exterior order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3. We start with the order-$q$-subplane $\pi_0 = \mathrm{PG}(2, q)$ and the line $\ell = [-\tau\tau^q, \tau + \tau^q, -1]$ which is exterior to $\pi_0$. Note that using the notation for $p_0, p_1, p_2$ given in Theorem 6.3, we have $\ell = [p_0^{q^2}, p_1^{q^2}, p_2^{q^2}]$. A line of $\pi_0$ has coordinates $[l, m, n]$ for $l, m, n \in \mathbb{F}_q$, and meets $\ell$ in the point $W'_{l,m,n} = (-n(\tau + \tau^q) - m,\ l - n\tau\tau^q,\ m\tau\tau^q + l(\tau + \tau^q))$. We apply the homography $\sigma$ of Section 3 with matrix $K$ to map $\pi_0$ and $\ell$ to $\mathcal{B}$ and $\ell_\infty$, respectively. The point $W'_{l,m,n}$ of $\ell$ maps to the point $W_{l,m,n} = KW'_{l,m,n} = (l + m\tau + n\tau^2,\ l + m\tau^q + n\tau^{2q},\ 0)$ of $\ell_\infty$. Writing $\varepsilon = \varepsilon_{l,m,n} = l + m\tau + n\tau^2$, we have $W_\varepsilon = W_{l,m,n} = (\varepsilon, \varepsilon^q, 0) \equiv (\varepsilon^{1-q}, 1, 0)$. Using the notation from Lemma 5.1, this is the point $S_{\varepsilon^{1-q}} \in \ell_\infty$. In PG(6, $q$), $W_\varepsilon$ corresponds to the spread plane $[W_\varepsilon] = [W_{l,m,n}] = \{([\varepsilon x], [\varepsilon^q x], 0) \equiv ([\varepsilon^{1-q} x], [x], 0) : x \in \mathbb{F}'_{q^3}\} = [S_{\varepsilon^{1-q}}]$.

Fix a point $P = (a, b, c)$ of $\pi_0$, so $a, b, c, \in \mathbb{F}_q$, not all zero. Let

$$\mathcal{L} = \{(l, m, n) : l, m, n \in \mathbb{F}_q \text{ not all zero, and } la + mb + nc = 0\}.$$

The $q + 1$ lines of $\pi_0$ through $P$ have coordinates $[l, m, n] \in \mathcal{L}$. These $q + 1$ lines meet the exterior line $\ell$ of $\pi_0$ in a $\pi_0$-pencil-subline which, under the collineation $\sigma$, maps to a $\mathcal{B}$-pencil-subline $d$ of $\ell_\infty$. By Theorem 2.3, in PG(6, $q$), $d$ corresponds to the 2-regulus $[d]$ which we denote by $\mathcal{R}$, so $\mathcal{R} = [d] = \{[W_\varepsilon] = [S_{\varepsilon^{1-q}}] : \varepsilon \in \mathcal{W}\}$, where $\mathcal{W} = \{\varepsilon = \varepsilon_{l,m,n} = l + m\tau + n\tau^2 : (l, m, n) \in \mathcal{L}\}$. For each $\alpha \in \mathbb{F}'_{q^3}$, consider the set of points $t_\alpha = \{([\varepsilon\alpha], [\varepsilon^q\alpha], 0) : \varepsilon \in \mathcal{W}\}$. As $\mathcal{W}$ is closed under addition, $t_\alpha$ is a line of $\Sigma_\infty \cong \mathrm{PG}(5, q)$; further $t_\alpha$ meets every plane in $\mathcal{R}$. Hence $t_\alpha$ is a ruling line of the 2-regulus $\mathcal{R}$.

By Theorem 7.2(ii), the 2-regulus $\mathcal{R}$ meets a cover plane of the conic cover $\mathbb{C}$ in a conic $\mathcal{C}_k = [C_k] \cap \mathcal{R}$ for $k \in \mathcal{K}$. To show that the conic $\mathcal{C}_k$ is $\mathbb{C}$-special, we need to extend it to PG(5, $q^3$), and show that it meets the three transversal lines of $\mathbb{C}$. To do this, we extend the 2-regulus $\mathcal{R}$ of $\Sigma_\infty \cong \mathrm{PG}(5, q)$ to a 2-regulus $\mathcal{R}^*$ of PG(5, $q^3$), so $\mathcal{C}_k^* = [C_k]^* \cap \mathcal{R}^*$. We then use coordinates to show that one of the planes of $\mathcal{R}^*$ contains the transversal line $g_{\mathbb{C}}^{q^2}$ of $\mathbb{C}$, and then deduce that $\mathcal{C}_k^*$ meets $g_{\mathbb{C}}^{q^2}$.

To extend $\mathcal{R}$ to a 2-regulus $\mathcal{R}^*$ of PG(5, $q^3$), we find four lines in PG(5, $q^3$) that meet each extended plane of $\mathcal{R}$. As a 2-regulus is uniquely determined by four ruling lines in general position, we can use these four lines to define the 2-regulus $\mathcal{R}^*$. The transversal line $g_{\mathbb{S}}$ of the regular 2-spread $\mathcal{S}$ can be used as one of our ruling lines; for the other three ruling lines, we use the extended lines $t_1^*$, $t_\tau^*$, $t_{\tau^2}^*$, which each meet every plane of $\mathcal{R}$. So $\mathcal{R}^*$ is the 2-regulus of PG(5, $q^3$) determined by the four ruling lines $t_1^*$, $t_\tau^*$, $t_{\tau^2}^*$, $g_{\mathbb{S}}$ (which are in general position), and further $\mathcal{R}^* \cap \Sigma_\infty = \mathcal{R}$.

We now exhibit a plane $\gamma$ of $\mathcal{R}^*$ that contains the transversal line $g_{\mathbb{C}}^{q^2}$ of the conic cover $\mathbb{C}$. Extend the set $\mathcal{L}$ to

$$\mathcal{L}^* = \{(l, m, n) : l, m, n \in \mathbb{F}_{q^3} \text{ not all zero, and } la + mb + nc = 0\}.$$

We use the matrix $M_\tau$ defined in Section 2B, and write $M = M_\tau$. The ruling line $t^*_{\tau^i}$, $i = 0, 1, 2$, has points $P_{\tau^i,l,m,n}$ with $(l, m, n) \in \mathcal{L}^*$, where $P_{\tau^i,l,m,n} = l(M^i[1], M^i[1], 0) + m(M^i[\tau], M^i[\tau^q], 0) + n(M^i[\tau^2], M^i[\tau^{2q}], 0)$. Recall that the order-$q$-subline $d$ corresponds to the fixed point $P = (a, b, c) \in \pi_0$. Consider the following $(l, m, n) \in \mathcal{L}^*$:

$$l = c\tau - b\tau^2, \quad m = a\tau^2 - c, \quad n = b - a\tau. \tag{9}$$

Note that for these $l, m, n$ we have

$$l + m\tau + n\tau^2 = 0. \tag{10}$$

For $l, m, n$ as in (9), consider the plane $\gamma$ spanned by the three points $P_{1,l,m,n} \in t^*_1$, $P_{\tau,l,m,n} \in t^*_\tau$, $P_{\tau^2,l,m,n} \in t^*_{\tau^2}$. We first show that $\gamma$ is a plane of the 2-regulus $\mathcal{R}^*$ by showing that the fourth ruling line $g_{\mathbb{S}}$ of $\mathcal{R}^*$ also meets $\gamma$. By Theorem 6.3, $g_{\mathbb{S}} = \langle A_1, A_2\rangle$, and we show that $g_{\mathbb{S}}$ meets $\gamma$ by showing that the point $A_2$ lies in $\gamma$. With $l, m, n$ given by (9), consider the point $F = p_0 P_{1,l,m,n} + p_1 P_{\tau,l,m,n} + p_2 P_{\tau^2,l,m,n}$ of $\gamma$. To simplify the notation, we use the point $A = (p_0, p_1, p_2)^t$, and matrix $U_0 = p_0 I + p_1 M + p_1 M^2$ defined in Section 2B, and note that $U_0[\alpha] = \alpha A$. We have

$$\begin{aligned} F &= (lU_0[1] + mU_0[\tau] + nU_0[\tau^2],\ lU_0[1] + mU_0[\tau^q] + nU_0[\tau^{2q}],\ 0) \\ &= (lA + m\tau A + n\tau^2 A,\ lA + m\tau^q A + n\tau^{2q} A,\ 0). \end{aligned}$$

By (10), $F \equiv ([0], A, 0) = A_2$, and by Lemma 2.1, $g_{\mathbb{S}} = \langle A_1, A_2\rangle$, so $F \in g_{\mathbb{S}} \cap \gamma$. That is, the four ruling lines $t^*_1, t^*_\tau, t^*_{\tau^2}, g_{\mathbb{S}}$ of the 2-regulus $\mathcal{R}^*$ all meet the plane $\gamma$, and so $\gamma$ is a plane of $\mathcal{R}^*$.

We now show that the transversal line $g_{\mathbb{C}}^{q^2}$ of $\mathbb{C}$ lies in the plane $\gamma$ of $\mathcal{R}^*$. Let $G = p_0^{q^2} P_{1,l,m,n} + p_1^{q^2} P_{\tau,l,m,n} + p_2^{q^2} P_{\tau^2,l,m,n}$, and note that $G \in \gamma$. We use the matrix

$$U_2 = p_0^{q^2} I + p_1^{q^2} M + p_1^{q^2} M^2$$

defined in Section 2B, and note that $U_2[\alpha] = \alpha^{q^2} A^{q^2}$, so we have

$$\begin{aligned} G &= (lU_2[1] + mU_2[\tau] + n^2 U_2[\tau^2],\ lU_2[1] + mU_2[\tau^q] + nU_2[\tau^{2q}],\ 0) \\ &= (lA^{q^2} + m\tau^{q^2} A^{q^2} + n\tau^{2q^2} A^{q^2},\ lA^{q^2} + m\tau A^{q^2} + n\tau^2 A^{q^2},\ 0). \end{aligned}$$

By (10), $G \equiv (A^{q^2}, [0], 0) = A_1^{q^2}$, so $\gamma$ contains the points $G = A_1^{q^2}$ and $F = A_2$. Hence by Theorem 6.3, $\gamma$ contains the transversal line $g_{\mathbb{C}}^{q^2} = \langle A_1^{q^2}, A_2\rangle$ of $\mathbb{C}$.

We showed above that the 2-regulus $[d] = \mathcal{R}$ meets a cover plane $[C_i]$ of $\mathbb{C}$ in a conic $\mathcal{C}_i$. We want to show that $\mathcal{C}_i$ is a $\mathbb{C}$-special conic, that is, we want to show that in PG$(6, q^3)$, the extended conic $\mathcal{C}_i^* = [C_i]^* \cap \mathcal{R}^*$ contains the three points $g_{\mathbb{C}} \cap [C_i]^*$, $g_{\mathbb{C}}^q \cap [C_i]^*$, $g_{\mathbb{C}}^{q^2} \cap [C_i]^*$. We have shown that the transversal line $g_{\mathbb{C}}^{q^2}$ of $\mathbb{C}$ lies in a plane $\gamma$ of $\mathcal{R}^*$. As the extended cover plane $[C_i]^*$ meets the transversal line $g_{\mathbb{C}}^{q^2}$ in a unique point denoted $P_i$, we have

$$P_i = [C_i]^* \cap g_{\mathbb{C}}^{q^2} = [C_i]^* \cap \gamma \in [C_i]^* \cap \mathcal{R}^* = \mathcal{C}_i^*.$$

Hence $\mathcal{C}_i^*$ contains the point $g_{\mathbb{C}}^{q^2} \cap [C_i]^*$, and hence it also contains the conjugate points $g_{\mathbb{C}}^{q} \cap [C_i]^*$, $g_{\mathbb{C}} \cap [C_i]^*$. That is, the conic $\mathcal{C}_i = [C_i] \cap \mathcal{R}$ is a $\mathbb{C}$-special conic, completing the proof of part (i). As $\mathbb{C}$ and $\mathbb{T}$ are projectively equivalent by [Lavrauw et al. 2015, Theorem 16], part (ii) holds by symmetry. □

## 8. Conclusion

An investigation into the interaction between an exterior order-$q$-subplane $\pi$ of PG(2, $q^3$), and its exterior splash on $\ell_\infty$ began in [Barwick and Jackson 2016]. The main focus of that paper was to show that exterior splashes are projectively equivalent to scattered $\mathbb{F}_q$-linear sets of rank 3, covers of circle geometries, Sherk sets of size $q^2+q+1$. Further, we investigated the geometric relationship between the order-$q$-sublines of $\mathbb{S}$ and the points of $\pi$. The current article focusses on using the Bruck–Bose representation in PG(6, $q$) to continue the study of exterior splashes, in particular their interplay with order-$q$-subplanes. The notion of special conics and special twisted cubics is closely tied with this interplay.

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