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We look at a scroll of PG$(6, q)$ that uses a projectivity to rule a conic and a twisted cubic. We show this scroll is a ruled quintic surface $\mathcal{V}_2^5$, and study its geometric properties. The motivation in studying this scroll lies in its relationship with an $\mathbb{F}_q$-subplane of PG$(2, q^3)$ via the Bruck–Bose representation.

1. Introduction

In this article we consider a scroll of PG$(6, q)$ that rules a conic and a twisted cubic according to a projectivity. The motivation in studying this scroll lies in its relationship with an $\mathbb{F}_q$-subplane of PG$(2, q^3)$ via the Bruck–Bose representation as described in Section 3. In PG$(6, q)$, let $\mathcal{C}$ be a nondegenerate conic in a plane α; $\mathcal{C}$ is called the *conic directrix*. Let $\mathcal{N}_3$ be a twisted cubic in a 3-space Π_3 with $\alpha \cap \Pi_3 = \varnothing$; $\mathcal{N}_3$ is called the *twisted cubic directrix*. Let ϕ be a projectivity from the points of $\mathcal{C}$ to the points of $\mathcal{N}_3$. By this we mean that if we write the points of $\mathcal{C}$ and $\mathcal{N}_3$ using a nonhomogeneous parameter, so $\mathcal{C} = \{C_\theta = (1, \theta, \theta^2) \mid \theta \in \mathbb{F}_q \cup \{\infty\}\}$ and $\mathcal{N}_3 = \{N_\epsilon = (1, \epsilon, \epsilon^2, \epsilon^3) \mid \epsilon \in \mathbb{F}_q \cup \{\infty\}\}$, then $\phi \in \mathrm{PGL}(2, q)$ is a projectivity mapping $(1, \theta)$ to $(1, \epsilon)$. Let $\mathcal{V}$ be the set of points of PG$(6, q)$ lying on the $q+1$ lines joining each point of $\mathcal{C}$ to the corresponding point (under ϕ) of $\mathcal{N}_3$. These $q+1$ lines are called the *generators* of $\mathcal{V}$. As the two subspaces α and Π_3 are disjoint, $\mathcal{V}$ is not contained in a 5-space. We note that this construction generalises the ruled cubic surface $\mathcal{V}_2^3$ in PG$(4, q)$, a variety that has been well studied; see [Vincenti 1983].

We work with normal rational curves in PG$(6, q)$. Suppose that $\mathcal{N}$ is a normal rational curve that generates an i-dimensional space. Then we call $\mathcal{N}$ an *i-dim nrc*, and often use the notation $\mathcal{N}_i$. See [Hirschfeld and Thas 1991] for details on normal rational curves. As we will be looking at 5-dim nrcs contained in $\mathcal{V}$, we assume $q \geq 6$ throughout.

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This article studies the geometric structure of $\mathcal{V}$. In Section 2, we show that $\mathcal{V}$ is a variety $\mathcal{V}_2^5$ of order 5 and dimension 2, and that all such scrolls are projectively equivalent. Further, we show that $\mathcal{V}$ contains exactly $q+1$ lines and one nondegenerate conic. In Section 3, we describe the Bruck–Bose representation of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(6, q)$, and discuss how $\mathcal{V}$ corresponds to an $\mathbb{F}_q$-subplane of $\mathrm{PG}(2, q^3)$. We use the Bruck–Bose setting to show that $\mathcal{V}$ contains exactly q^2 twisted cubics, and that each can act as a directrix of $\mathcal{V}$. In Section 4, we count the number of 4- and 5-dim nrcs contained in $\mathcal{V}$. Further, we determine how 5-spaces meet $\mathcal{V}$, and count the number of 5-spaces of each intersection type. The main result is Theorem 4.8. In Section 5, we determine how 5-spaces meet $\mathcal{V}$ in relation to the regular 2-spread in the Bruck–Bose setting.

2. Simple properties of $\mathcal{V}$

Theorem 2.1. *Let $\mathcal{V}$ be a scroll of* $\mathrm{PG}(6, q)$ *that rules a conic and a twisted cubic according to a projectivity. Then $\mathcal{V}$ is a variety of dimension* 2 *and order* 5, *denoted $\mathcal{V}_2^5$ and called a ruled quintic surface. Further*, *any two ruled quintic surfaces are projectively equivalent.*

Proof. Let $\mathcal{V}$ be a scroll of $\mathrm{PG}(6, q)$ with conic directrix $\mathcal{C}$ in a plane α, twisted cubic directrix $\mathcal{N}_3$ in a 3-space Π_3, and ruled by a projectivity as described in Section 1. The group of collineations of $\mathrm{PG}(6, q)$ is transitive on planes, and transitive on 3-spaces. Further, all nondegenerate conics in a projective plane are projectively equivalent, and all twisted cubics in a 3-space are projectively equivalent. Hence, without loss of generality, we can coordinatise $\mathcal{V}$ as follows.

Let α be the plane which is the intersection of the four hyperplanes $x_0 = 0$, $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$. Let $\mathcal{C}$ be the nondegenerate conic in α with points $C_\theta = (0, 0, 0, 0, 1, \theta, \theta^2)$ for $\theta \in \mathbb{F}_q \cup \{\infty\}$. Note that the points of $\mathcal{C}$ are the exact intersection of α with the quadric of equation $x_5^2 = x_4x_6$. Let Π_3 be the 3-space which is the intersection of the three hyperplanes $x_4 = 0$, $x_5 = 0$, and $x_6 = 0$. Let $\mathcal{N}_3$ be the twisted cubic in Π_3 with points $N_\theta = (1, \theta, \theta^2, \theta^3, 0, 0, 0)$ for $\theta \in \mathbb{F}_q \cup \{\infty\}$. Note that the points of $\mathcal{N}_3$ are the exact intersection of Π_3 with the three quadrics with equations $x_1^2 = x_0x_2$, $x_2^2 = x_1x_3$, and $x_0x_3 = x_1x_2$. A projectivity in $\mathrm{PGL}(2, q)$ is uniquely determined by the image of three points, so without loss of generality, let $\mathcal{V}$ have generator lines $\ell_\theta = \{V_{\theta,t} = N_\theta + tC_\theta,\ t \in \mathbb{F}_q \cup \{\infty\}\}$ for $\theta \in \mathbb{F}_q \cup \{\infty\}$. That is, $V_{\theta,t} = (1, \theta, \theta^2, \theta^3, t, t\theta, t\theta^2)$. Equivalently, $\mathcal{V}$ consists of the points

$$V_{x,y,z} = (x^3,\ x^2y, xy^2,\ y^3,\ zx^2,\ zxy,\ zy^2)$$

for $x, y \in \mathbb{F}_q$ not both 0 and $z \in \mathbb{F}_q \cup \{\infty\}$. It is straightforward to verify that the pointset of $\mathcal{V}$ is the exact intersection of the following ten quadrics:

$$x_0x_5 = x_1x_4, \quad x_0x_6 = x_1x_5 = x_2x_4, \quad x_1x_6 = x_2x_5 = x_3x_4, \quad x_2x_6 = x_3x_5,$$
$$x_1^2 = x_0x_2, \quad x_2^2 = x_1x_3, \quad x_5^2 = x_4x_6, \quad x_0x_3 = x_1x_2.$$

Hence the points of $\mathcal{V}$ form a variety.

We follow [Semple and Roth 1949] to calculate the dimension and order of $\mathcal{V}$. The following map defines an algebraic one-to-one correspondence between the plane π of PG(3, q) with points $(x, y, z, 0)$, $x, y, z \in \mathbb{F}_q$ not all 0, and the points of $\mathcal{V}$:

$$\sigma : \pi \to \mathcal{V}, \quad (x, y, z, 0) \mapsto (x^3, x^2y, xy^2, y^3, x^2z, xyz, y^2z).$$

Thus $\mathcal{V}$ is an absolutely irreducible variety of dimension 2 and so we are justified in calling it a surface. Now consider a generic 4-space of PG(6, q) with equation given by the two hyperplanes $\Sigma_1 : a_0x_0+\cdots+a_6x_6=0$ and $\Sigma_2 : b_0x_0+\cdots+b_6x_6=0$ for $a_i, b_i \in \mathbb{F}_q$. The point $V_{x,y,z} = (x^3, x^2y, xy^2, y^3, x^2z, xyz, y^2z)$ lies on Σ_1 if $a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + a_4x^2z + a_5xyz + a_6y^2z = 0$. This corresponds to a cubic $\mathcal{K}$ in the plane π. Moreover, $\mathcal{K}$ contains the point $P = (0, 0, 1, 0)$, and P is a double point of $\mathcal{K}$. Similarly the set of points $V_{x,y,z} \in \Sigma_2$ corresponds to a cubic in π with a double point $(0, 0, 1, 0)$. Two cubics in a plane meet generically in nine points. As $(0, 0, 1, 0)$ lies in the kernel of σ, in PG(6, q) the 4-space $\Sigma_1 \cap \Sigma_2$ meets $\mathcal{V}$ in five points, and so $\mathcal{V}$ has order 5. □

Theorem 2.2. *Let $\mathcal{V}_2^5$ be a ruled quintic surface in* PG(6, q).

(1) *No two generators of $\mathcal{V}_2^5$ lie in a plane.*

(2) *No three generators of $\mathcal{V}_2^5$ lie in a* 4*-space.*

(3) *No four generators of $\mathcal{V}_2^5$ lie in a* 5*-space.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface of PG(6, q) with conic directrix $\mathcal{C}$ in a plane α, and twisted cubic directrix $\mathcal{N}_3$ lying in a 3-space Π_3. Suppose two generator lines ℓ_0, ℓ_1 of $\mathcal{V}_2^5$ lie in a plane. Let m be the line in α joining the distinct points $\ell_0 \cap \alpha$, $\ell_1 \cap \alpha$. Let m' be the line in Π_3 joining the distinct points $\ell_0 \cap \Pi_3$, $\ell_1 \cap \Pi_3$. The lines m, m' lie in the plane $\langle \ell_0, \ell_1 \rangle$ and so meet in a point, contradicting disjointness of α and Π_3. Hence the generator lines of $\mathcal{V}_2^5$ are pairwise skew.

For (2), suppose a 4-space Π_4 contains three distinct generators of $\mathcal{V}_2^5$. As distinct generators meet $\mathcal{C}$ in distinct points, Π_4 contains three distinct points of $\mathcal{C}$, and so contains the plane α. Further, distinct generators meet $\mathcal{N}_3$ in distinct points, hence Π_4 contains three points of $\mathcal{N}_3$, and so $\Pi_4 \cap \Pi_3$ has dimension at least 2. Hence $\langle \Pi_4, \Pi_3 \rangle$ has dimension at most $4 + 3 - 2 = 5$. However, $\mathcal{V}_2^5 \subseteq \langle \Pi_4, \Pi_3 \rangle$, a contradiction as $\mathcal{V}_2^5$ is not contained in a 5-space.

For (3), suppose a 5-space Π_5 contains four distinct generators of $\mathcal{V}_2^5$. Distinct generators meet Π_3 in distinct points of $\mathcal{N}_3$, so Π_5 contains four points of $\mathcal{N}_3$ which do not lie in a plane. Hence Π_5 contains Π_3. Similarly Π_5 contains α, and so Π_5 contains $\mathcal{V}_2^5$, a contradiction as $\mathcal{V}_2^5$ is not contained in a 5-space. □

Corollary 2.3. *No two generators of $\mathcal{V}_2^5$ lie in a 3-space containing α.*

Proof. Suppose a 3-space Π_3 contained α and two generators of $\mathcal{V}_2^5$. Let P be a point of $\mathcal{V}_2^5$ not in Π_3 and ℓ the generator of $\mathcal{V}_2^5$ through P. Then $\Pi_4 = \langle \Pi_3, P\rangle$ contains two distinct points of ℓ, namely P and $\ell \cap \mathcal{C}$, and so Π_4 contains ℓ. That is, Π_4 is a 4-space containing three generators, contradicting Theorem 2.2. □

We now show that the only lines on $\mathcal{V}_2^5$ are the generators, and the only nondegenerate conic on $\mathcal{V}_2^5$ is the conic directrix. We show later in Theorem 3.2 that there are exactly q^2 twisted cubics on $\mathcal{V}_2^5$, and that each is a directrix.

Theorem 2.4. *Let $\mathcal{V}_2^5$ be a ruled quintic surface in* $\mathrm{PG}(6, q)$. *A line of* $\mathrm{PG}(6, q)$ *meets $\mathcal{V}_2^5$ in $0, 1, 2$, or $q+1$ points. Further, $\mathcal{V}_2^5$ contains exactly $q+1$ lines, namely the generator lines.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface of $\mathrm{PG}(6, q)$ with conic directrix $\mathcal{C}$ lying in a plane α, and twisted cubic directrix $\mathcal{N}_3$ lying in the 3-space Π_3. Let m be a line of $\mathrm{PG}(6, q)$ that is not a generator of $\mathcal{V}_2^5$, and suppose m meets $\mathcal{V}_2^5$ in three points P, Q, R. As m is not a generator of $\mathcal{V}_2^5$, the points P, Q, R lie on distinct generator lines denoted ℓ_P, ℓ_Q, ℓ_R, respectively. As $\mathcal{C}$ is a nondegenerate conic, m is not a line of α and so at most one of the points P, Q, R lie in $\mathcal{C}$. Suppose firstly that $P, Q, R \notin \mathcal{C}$. Then $\langle \alpha, m\rangle$ is a 3- or 4-space that contains the three generators ℓ_P, ℓ_Q, ℓ_R, contradicting Theorem 2.2. Now suppose $P \in \mathcal{C}$ and $Q, R \notin \mathcal{C}$. Then $\Sigma_3 = \langle \alpha, m\rangle$ is a 3-space which contains the two generator lines ℓ_Q, ℓ_R. So $\Sigma_3 \cap \Pi_3$ contains the distinct points $\ell_R \cap \mathcal{N}_3$, $\ell_Q \cap \mathcal{N}_3$, and so has dimension at least 1. Hence $\langle \Sigma_3, \Pi_3\rangle$ has dimension at most $3+3-1=5$, a contradiction as $\mathcal{V}_2^5 \subset \langle \Sigma_3, \Pi_3\rangle$, but $\mathcal{V}_2^5$ is not contained in a 5-space. Hence a line of $\mathrm{PG}(6, q)$ is either a generator line of $\mathcal{V}_2^5$, or meets $\mathcal{V}_2^5$ in 0, 1, or 2 points. □

Theorem 2.5. *The ruled quintic surface $\mathcal{V}_2^5$ contains exactly one nondegenerate conic.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface with conic directrix $\mathcal{C}$ in a plane α. Suppose $\mathcal{V}_2^5$ contains another nondegenerate conic $\mathcal{C}'$ in a plane $\alpha' \neq \alpha$. If $\mathcal{C}'$ contains two points on a generator ℓ of $\mathcal{V}_2^5$, then $\alpha' \cap \mathcal{V}_2^5$ contains $\mathcal{C}'$ and ℓ. However, by the proof of Theorem 2.1, $\mathcal{V}_2^5$ is the intersection of quadrics, and the configuration $\mathcal{C}' \cup \ell$ is not contained in any planar quadric. Hence $\mathcal{C}'$ contains exactly one point on each generator of $\mathcal{V}_2^5$.

We consider the three cases where $\alpha \cap \alpha'$ is either empty, a point, or a line. Suppose $\alpha \cap \alpha' = \varnothing$. Then $\langle \alpha, \alpha'\rangle$ is a 5-space that contains $\mathcal{C}$ and $\mathcal{C}'$, and so contains two distinct points on each generator of $\mathcal{V}_2^5$. Hence $\langle \alpha, \alpha'\rangle$ contains each generator of $\mathcal{V}_2^5$ and so contains $\mathcal{V}_2^5$, a contradiction as $\mathcal{V}_2^5$ is not contained in a 5-space. Suppose $\alpha \cap \alpha'$ is a point P. Then $\langle \alpha, \alpha' \rangle$ is a 4-space that contains at least q generators of $\mathcal{V}_2^5$, contradicting Theorem 2.2 as $q \geq 6$. Finally, suppose $\alpha \cap \alpha'$ is a line. Then $\langle \alpha, \alpha' \rangle$ is a 3-space that contains at least $q-1$ generators, contradicting Theorem 2.2 as $q \geq 6$. So $\mathcal{V}_2^5$ contains exactly one nondegenerate conic. □

We aim to classify how 5-spaces meet $\mathcal{V}_2^5$, so we begin with a simple description.

Remark 2.6. Let Π_5 be a 5-space. Then $\Pi_5 \cap \mathcal{V}_2^5$ contains a set of $q+1$ points, one on each generator.

Lemma 2.7. *A* 5*-space meets* $\mathcal{V}_2^5$ *in either* (a) *a* 5*-dim nrc*, (b) *a* 4*-dim nrc and* 0 *or* 1 *generators*, (c) *a* 3*-dim nrc and* 0, 1, *or* 2 *generators*, *or* (d) *the conic directrix and* 0, 1, 2, *or* 3 *generators.*

Proof. Using properties of varieties (see, for example, [Semple and Roth 1949]) we have $\mathcal{V}_2^5 \cap \mathcal{V}_5^1 = \mathcal{V}_1^5$, that is, the variety $\mathcal{V}_2^5$ meets a 5-space $\mathcal{V}_5^1$ in a curve of degree 5. Denote this curve of PG(6, q) by $\mathcal{K}$. The degree of $\mathcal{K}$ can be partitioned as

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

By Theorem 2.4, the only lines on $\mathcal{V}_2^5$ are the generators. By Theorem 2.2, $\mathcal{K}$ does not contain more than 3 generators. By Remark 2.6, $\mathcal{K}$ contains at least one point on each generator. Hence $\mathcal{K}$ is not empty, and is not the union of 1, 2, or 3 generators, so the partition $1+1+1+1+1$ for the degree of $\mathcal{K}$ does not occur.

Suppose that the degree of $\mathcal{K}$ is partitioned as either (a) $2+2+1$ or (b) $2+1+1+1$. By Remark 2.6, $\mathcal{K}$ contains a point on each generator, so $\mathcal{K}$ contains an irreducible conic. By Theorem 2.5, this conic is the conic directrix $\mathcal{C}$ of $\mathcal{V}_2^5$, and case (a) does not occur. Hence $\mathcal{K}$ consists of $\mathcal{C}$ and 0, 1, 2, or 3 generators of $\mathcal{V}_2^5$.

Suppose that the degree of $\mathcal{K}$ is partitioned as $3+1+1$. So $\mathcal{K}$ consists of at most 2 generators, and an irreducible cubic $\mathcal{K}'$. By Remark 2.6, $\mathcal{K}$ contains a point on each generator, so $\mathcal{K}'$ contains a point on at least $q-1$ generators. If $\mathcal{K}'$ generates a 3-space, then it is a 3-dim nrc of PG(6, q). If not, $\mathcal{K}'$ is an irreducible cubic contained in a plane Π_2. By the proof of Theorem 2.1, $\mathcal{K}'$ is contained in a quadric, so $\mathcal{K}'$ is not an irreducible planar cubic. Thus $\mathcal{K}'$ is a 3-dim nrc of PG(6, q). Hence $\mathcal{K}$ consists of a 3-dim nrc and 0, 1, or 2 generators of $\mathcal{V}_2^5$.

Suppose that the degree of $\mathcal{K}$ is partitioned as $2+3$. By Remark 2.6, $\mathcal{K}$ contains a point on each generator. As argued above, $\mathcal{K}$ does not contain an irreducible planar cubic. Suppose $\mathcal{K}$ contained both an irreducible conic $\mathcal{C}$ and a twisted cubic $\mathcal{N}_3$. Then there is at least one generator ℓ that meets $\mathcal{C}$ and $\mathcal{N}_3$ in distinct points. In this case ℓ lies in the 5-space and so lies in $\mathcal{K}$, a contradiction. So $\mathcal{K}$ is not the union of an irreducible conic and a twisted cubic.

Suppose that the degree of $\mathcal{K}$ is partitioned as $4+1$. So $\mathcal{K}$ consists of at most 1 generator, and an irreducible quartic $\mathcal{K}'$. By Remark 2.6, $\mathcal{K}$ contains a point on each generator, so $\mathcal{K}'$ contains a point on at least q generators. If $\mathcal{K}'$ generates a 4-space, then it is a 4-dim nrc of $\mathrm{PG}(6, q)$. If not, $\mathcal{K}'$ is an irreducible quartic contained in a 3-space Π_3. Let ℓ, m be two generators not in $\mathcal{K}$. Then by Remark 2.6 they meet $\mathcal{K}'$. So $\langle \Pi_3, \ell, m\rangle$ has dimension at most 5, and meets $\mathcal{V}_2^5$ in an irreducible quartic and 2 lines, which is a curve of degree 6, a contradiction. Thus $\mathcal{K}'$ is a 4-dim nrc of $\mathrm{PG}(6, q)$. That is, $\mathcal{K}$ consists of a 4-dim nrc and 0 or 1 generators of $\mathcal{V}_2^5$.

Suppose the curve $\mathcal{K}$ is irreducible. By Remark 2.6, $\mathcal{K}$ contains a point on each generator. So either $\mathcal{K}$ is a 5-dim nrc of $\mathrm{PG}(6, q)$, or $\mathcal{K}$ lies in a 4-space. Suppose $\mathcal{K}$ lies in a 4-space Π_4, and let ℓ be a generator. Then $\langle \Pi_4, \ell\rangle$ has dimension at most 5 and meets $\mathcal{V}_2^5$ in a curve of degree 6, a contradiction. So $\mathcal{K}$ is a 5-dim nrc of $\mathrm{PG}(6, q)$. □

Corollary 2.8. *Let Π_r be an r-space for $r = 3, 4, 5$ that contains an r-dim nrc of $\mathcal{V}_2^5$. Then Π_r contains 0 generators of $\mathcal{V}_2^5$.*

Proof. First suppose $r = 3$. By Lemma 2.7, a 5-space containing a twisted cubic $\mathcal{N}_3$ of $\mathcal{V}_2^5$ contains at most two generators of $\mathcal{V}_2^5$. Hence a 4-space containing $\mathcal{N}_3$ contains at most one generator of $\mathcal{V}_2^5$. Hence the 3-space Π_3 containing $\mathcal{N}_3$ contains no generator of $\mathcal{V}_2^5$.

If $r = 4$, by Lemma 2.7, a 5-space containing a 4-dim nrc $\mathcal{N}_4$ of $\mathcal{V}_2^5$ contains at most one generator of $\mathcal{V}_2^5$. Hence the 4-space Π_4 containing $\mathcal{N}_4$ contains no generators of $\mathcal{V}_2^5$. If $r = 5$, then by Lemma 2.7, Π_5 contains 0 generators of $\mathcal{V}_2^5$. □

Theorem 2.9. *Let $\mathcal{N}_r$ be an r-dim nrc lying on $\mathcal{V}_2^5$ for $r = 3, 4, 5$. Then $\mathcal{N}_r$ contains exactly one point on each generator of $\mathcal{V}_2^5$.*

Proof. Let $\mathcal{N}_r$ be an r-dim nrc lying on $\mathcal{V}_2^5$ for $r = 3, 4, 5$, and denote the r-space containing $\mathcal{N}_r$ by Π_r. If Π_r contained 2 points of a generator of $\mathcal{V}_2^5$, then it contains the whole generator, so by Corollary 2.8, the $q+1$ points of $\mathcal{N}_r$ consist of one on each generator of $\mathcal{V}_2^5$. □

3. $\mathcal{V}_2^5$ and $\mathbb{F}_q$-subplanes of $\mathrm{PG}(2, q^3)$

To study $\mathcal{V}_2^5$ in more detail, we use the linear representation of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(6, q)$ developed independently by André [1954] and Bruck and Bose [1964; 1966]. Let $\mathcal{S}$ be a regular 2-spread of $\mathrm{PG}(6, q)$ in a 5-space Σ_∞. Let $\mathcal{I}$ be the incidence structure with the points of $\mathrm{PG}(6, q) \setminus \Sigma_\infty$ as *points*, the 3-spaces of $\mathrm{PG}(6, q)$ that contain a plane of $\mathcal{S}$ and are not in Σ_∞ as *lines*, and inclusion as *incidence*. Then $\mathcal{I}$ is isomorphic to $\mathrm{AG}(2, q^3)$. We can uniquely complete $\mathcal{I}$ to $\mathrm{PG}(2, q^3)$, the points on ℓ_∞ correspond to the planes of $\mathcal{S}$. We call this the *Bruck–Bose representation* of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(6, q)$; see [Barwick and Jackson 2012] for a detailed discussion on this representation. Of particular interest is the relationship between the ruled quintic surface of $\mathrm{PG}(6, q)$ and the $\mathbb{F}_q$-subplanes of $\mathrm{PG}(2, q^3)$.

To describe this relationship, we need to use the cubic extension of PG(6, q) to PG(6, q^3). The regular 2-spread $\mathcal{S}$ has a unique set of three conjugate *transversal* lines in this cubic extension, denoted g, g^q, g^{q^2}, which meet each extended plane of $\mathcal{S}$; for more details on regular spreads and transversals, see [Hirschfeld and Thas 1991, Section 25.6]. An r-space Π_r of PG(6, q) lies in a unique r-space of PG(6, q^3), denoted $\Pi_r^\star$. An nrc $\mathcal{N}$ of PG(6, q) lies in a unique nrc of PG(6, q^3), denoted $\mathcal{N}^\star$. Let $\mathcal{V}_2^5$ be a ruled quintic surface with conic directrix $\mathcal{C}$, twisted cubic directrix $\mathcal{N}_3$, and associated projectivity ϕ. Then we can extend $\mathcal{V}_2^5$ to a unique ruled quintic surface $\mathcal{V}_2^{5\star}$ of PG(6, q^3) with conic directrix $\mathcal{C}^\star$, twisted cubic directrix $\mathcal{N}_3^\star$, and the same associated projectivity, that is, extend ϕ from acting on PG(1, q) to acting on PG(1, q^3). We need the following characterisations.

Result 3.1 [Barwick and Jackson 2012; 2014]. *Let $\mathcal{S}$ be a regular 2-spread in a 5-space Σ_∞ in* PG(6, q) *and consider the Bruck–Bose plane* PG(2, q^3).

(1) *An $\mathbb{F}_q$-subline of* PG(2, q^3) *that meets ℓ_∞ in a point corresponds in* PG(6, q) *to a line not in Σ_∞.*

(2) *An $\mathbb{F}_q$-subline of* PG(2, q^3) *that is disjoint from ℓ_∞ corresponds in* PG(6, q) *to a twisted cubic $\mathcal{N}_3$ lying in a 3-space about a plane of $\mathcal{S}$ such that the extension $\mathcal{N}_3^\star$ to* PG(6, q^3) *meets each transversal of $\mathcal{S}$ in a point.*

(3) *An $\mathbb{F}_q$-subplane of* PG(2, q^3) *tangent to ℓ_∞ at the point T corresponds in* PG(6, q) *to a ruled quintic surface $\mathcal{V}_2^5$ with conic directrix in the spread plane corresponding to T such that in the cubic extension* PG(6, q^3), *the transversals g, g^q, g^{q^2} of $\mathcal{S}$ are generators of $\mathcal{V}_2^{5\star}$.*

Moreover, the converse of each is true.

We use this characterisation to show that $\mathcal{V}_2^5$ contains exactly q^2 twisted cubics.

Theorem 3.2. *The ruled quintic surface $\mathcal{V}_2^5$ contains exactly q^2 twisted cubics, and each is a directrix of $\mathcal{V}_2^5$.*

Proof. By Theorem 2.1, all ruled quintic surfaces are projectively equivalent. So without loss of generality, we can position a ruled quintic surface so that it corresponds to an $\mathbb{F}_q$-subplane of PG(2, q^3), which we denote by $\mathcal{B}$. That is, by Result 3.1, $\mathcal{S}$ is a regular 2-spread in a hyperplane Σ_∞, $\mathcal{V}_2^5 \cap \Sigma_\infty$ is the conic directrix $\mathcal{C}$ of $\mathcal{V}_2^5$, $\mathcal{C}$ lies in a plane of $\mathcal{S}$, and in the cubic extension PG(6, q^3), the transversals g, g^q, g^{q^2} of $\mathcal{S}$ are generators of $\mathcal{V}_2^{5\star}$.

Let $\mathcal{N}_3$ be a twisted cubic contained in $\mathcal{V}_2^5$, and denote the 3-space containing $\mathcal{N}_3$ by Π_3. As $\mathcal{V}_2^5 \cap \Sigma_\infty = \mathcal{C}$, Π_3 meets Σ_∞ in a plane; we show this is a plane of $\mathcal{S}$. In PG(6, q^3), $\mathcal{V}_2^{5\star}$ is a ruled quintic surface that contains the twisted cubic $\mathcal{N}_3^\star$. Moreover, the transversals g, g^q, g^{q^2} of $\mathcal{S}$ are generators of $\mathcal{V}_2^{5\star}$. So by Theorem 2.9, $\mathcal{N}_3^\star$ contains one point on each of g, g^q, and g^{q^2}. Hence the 3-space $\Pi_3^\star$ contains an extended plane of $\mathcal{S}$, and so Π_3 meets Σ_∞ in a plane of $\mathcal{S}$. Hence

$\Pi_3 \cap \alpha = \varnothing$. Further, by Theorem 2.9, $\mathcal{N}_3$ contains one point on each generator of $\mathcal{V}_2^5$, and thus $\mathcal{N}_3$ is a directrix of $\mathcal{V}_2^5$.

By Result 3.1, $\mathcal{N}_3$ corresponds in $\mathrm{PG}(2, q^3)$ to an $\mathbb{F}_q$-subline of $\mathcal{B}$ disjoint from ℓ_∞. Conversely, every $\mathbb{F}_q$-subline of $\mathcal{B}$ disjoint from ℓ_∞ corresponds to a twisted cubic on $\mathcal{V}_2^5$. Thus the twisted cubics in $\mathcal{V}_2^5$ are in one-to-one correspondence with the $\mathbb{F}_q$-sublines of $\mathcal{B}$ that are disjoint from ℓ_∞. As there are q^2 such $\mathbb{F}_q$-sublines, there are q^2 twisted cubics on $\mathcal{V}_2^5$. □

Suppose we position $\mathcal{V}_2^5$ so that it corresponds via the Bruck–Bose representation to a tangent $\mathbb{F}_q$-subplane $\mathcal{B}$ of $\mathrm{PG}(2, q^3)$. So we have a regular 2-spread $\mathcal{S}$ in a hyperplane Σ_∞, and the conic directrix of $\mathcal{V}_2^5$ lies in a plane $\alpha \in \mathcal{S}$. We define the *splash* of $\mathcal{B}$ to be the set of q^2+1 points on ℓ_∞ that lie on an extended line of $\mathcal{B}$. The *splash* of $\mathcal{V}_2^5$ is defined to be the corresponding set of q^2+1 planes of $\mathcal{S}$. We denote the splash of $\mathcal{V}_2^5$ by $\mathbb{S}$. Note that α is a plane of $\mathbb{S}$. We show that the remaining q^2 planes of $\mathbb{S}$ are related to the q^2 twisted cubics of $\mathcal{V}_2^5$.

Corollary 3.3. *Let $\mathcal{S}$ be a regular 2-spread in a hyperplane Σ_∞ of $\mathrm{PG}(6, q)$. Without loss of generality, we can position $\mathcal{V}_2^5$ so that it corresponds via the Bruck–Bose representation to a tangent $\mathbb{F}_q$-subplane of $\mathrm{PG}(2, q^3)$. Then the conic directrix of $\mathcal{V}_2^5$ lies in a plane $\alpha \in \mathcal{S}$, the q^2 3-spaces containing a twisted cubic of $\mathcal{V}_2^5$ meet Σ_∞ in distinct planes of $\mathcal{S}$, and these planes together with α form the splash $\mathbb{S}$ of $\mathcal{V}_2^5$.*

Proof. By Theorem 2.1, all ruled quintic surfaces are projectively equivalent, so without loss of generality, let $\mathcal{V}_2^5$ be positioned so that it corresponds to an $\mathbb{F}_q$-subplane $\mathcal{B}$ of $\mathrm{PG}(2, q^3)$ which is tangent to ℓ_∞. Let b be an $\mathbb{F}_q$-subline of $\mathcal{B}$ disjoint from ℓ_∞, so the extension of b meets ℓ_∞ in a point R which lies in the splash of $\mathcal{B}$. By Result 3.1, b corresponds in $\mathrm{PG}(6, q)$ to a twisted cubic of $\mathcal{V}_2^5$ which lies in a 3-space that meets Σ_∞ in the plane of $\mathbb{S}$ corresponding to the point R. □

Using this Bruck–Bose setting, we describe the 3-spaces of $\mathrm{PG}(6, q)$ that contain a plane of the regular 2-spread $\mathcal{S}$.

Corollary 3.4. *Position $\mathcal{V}_2^5$ as in Corollary 3.3, so $\mathcal{S}$ is a regular 2-spread in the hyperplane Σ_∞, and the conic directrix of $\mathcal{V}_2^5$ lies in a plane α contained in the splash $\mathbb{S} \subset \mathcal{S}$ of $\mathcal{V}_2^5$.*

1. *Let $\beta \in \mathbb{S} \setminus \alpha$. Then there exists a unique 3-space containing β that meets $\mathcal{V}_2^5$ in a twisted cubic. The remaining 3-spaces containing β (and not in Σ_∞) meet $\mathcal{V}_2^5$ in 0 or 1 point.*
2. *Let $\gamma \in \mathcal{S} \setminus \mathbb{S}$. Then each 3-space containing γ and not in Σ_∞ meets $\mathcal{V}_2^5$ in 0 or 1 point.*

Proof. By Corollary 3.3, we can position $\mathcal{V}_2^5$ so that it corresponds to an $\mathbb{F}_q$-subplane $\mathcal{B}$ of $\mathrm{PG}(2, q^3)$ which is tangent to ℓ_∞. The 3-spaces that contain a plane of $\mathcal{S}$ (and do not lie in Σ_∞) correspond to lines of PG(2, q^3). Each point on ℓ_∞ not in $\mathcal{B}$ but in the splash of $\mathcal{B}$ lies on a unique line that meets $\mathcal{B}$ in an $\mathbb{F}_q$-subline. By Result 3.1, this corresponds to a twisted cubic in $\mathcal{V}_2^5$. The remaining lines meet $\mathcal{B}$ in 0 or 1 point, so the remaining 3-spaces meet $\mathcal{V}_2^5$ in 0 or 1 point. □

As $\mathcal{V}_2^5$ corresponds to an $\mathbb{F}_q$-subplane, we have the following result.

Theorem 3.5. *Let $\mathcal{V}_2^5$ be a ruled quintic surface in* PG(6, q).

(1) *Two twisted cubics on $\mathcal{V}_2^5$ meet in a unique point.*

(2) *Let P, Q be points lying on different generators of $\mathcal{V}_2^5$, and not in the conic directrix. Then P, Q lie on a unique twisted cubic of $\mathcal{V}_2^5$.*

Proof. Without loss of generality, let $\mathcal{V}_2^5$ be positioned as described in Corollary 3.3. So the conic directrix lies in a plane α contained in a regular 2-spread $\mathcal{S}$ in Σ_∞, and $\mathcal{V}_2^5$ corresponds to an $\mathbb{F}_q$-subplane $\mathcal{B}$ of PG(2, q^3) tangent to ℓ_∞. Let $\mathcal{N}_1, \mathcal{N}_2$ be two twisted cubics contained in $\mathcal{V}_2^5$. By Result 3.1, they correspond in PG(2, q^3) to two $\mathbb{F}_q$-sublines of $\mathcal{B}$ not containing $\mathcal{B} \cap \ell_\infty$, and so meet in a unique affine point P. This corresponds to a unique point $P \in \mathcal{V}_2^5 \setminus \alpha$ lying in both $\mathcal{N}_1$ and $\mathcal{N}_2$, proving (1).

For (2), let P, Q be points lying on distinct generators of $\mathcal{V}_2^5$, $P, Q \notin \mathcal{C}$. If the line PQ met α, then $\langle \alpha, P, Q\rangle$ is a 3-space that contains α and the generators of $\mathcal{V}_2^5$ containing P and Q, contradicting Corollary 2.3. Hence the line PQ is skew to α. In PG(2, q^3), P, Q correspond to two affine points in the tangent $\mathbb{F}_q$-subplane $\mathcal{B}$, so they lie on a unique $\mathbb{F}_q$-subline b of $\mathcal{B}$. By Result 3.1, the generators of $\mathcal{V}_2^5$ correspond to the $\mathbb{F}_q$-sublines of $\mathcal{B}$ through the point $\mathcal{B} \cap \ell_\infty$. As PQ is skew to α, we have $b \cap \ell_\infty = \varnothing$. Hence, by Result 3.1, in PG(6, q) the points P, Q lie on a unique twisted cubic of $\mathcal{V}_2^5$. □

4. Intersection types for 5-spaces meeting $\mathcal{V}_2^5$

In this section we determine how 5-spaces meet $\mathcal{V}_2^5$ and count the different intersection types. A series of lemmas is used to prove the main result which is stated in Theorem 4.8.

Lemma 4.1. *Let $\mathcal{V}_2^5$ be a ruled quintic surface of* PG(6, q) *with conic directrix $\mathcal{C}$. Of the $q^3 + q^2 + q + 1$ 5-spaces of* PG(6, q) *containing $\mathcal{C}$, r_i of them meet $\mathcal{V}_2^5$ in precisely $\mathcal{C}$ and i generators, where*

$$r_3 = \frac{q^3 - q}{6}, \quad r_2 = q^2 + q, \quad r_1 = \frac{q^3}{2} + \frac{q}{2} + 1, \quad r_0 = \frac{q^3 - q}{3}.$$

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface of PG(6, q) with conic directrix $\mathcal{C}$ lying in a plane α. By Lemma 2.7, a 5-space containing $\mathcal{C}$ contains at most three generator

lines of $\mathcal{V}_2^5$. By Theorem 2.2, three generators of $\mathcal{V}_2^5$ lie in a unique 5-space. Hence there are
$$r_3 = \binom{q+1}{3}$$
5-spaces that contain three generators of $\mathcal{V}_2^5$. Such a 5-space contains three points of $\mathcal{C}$, and so contains $\mathcal{C}$ and α.

Denote the generator lines of $\mathcal{V}_2^5$ by $\ell_0, \dots, \ell_q$ and consider two generators, ℓ_0, ℓ_1 say. By Corollary 2.3, $\Sigma_4 = \langle \alpha, \ell_0, \ell_1 \rangle$ is a 4-space. By Theorem 2.2, $\langle \Sigma_4, \ell_i \rangle$ for $i = 2, \dots, q$ are distinct 5-spaces. That is, $q-1$ of the 5-spaces about Σ_4 contain 3 generators, and hence the remaining two contain ℓ_0, ℓ_1 and no further generator of $\mathcal{V}_2^5$. Hence, by Lemma 2.7, $q-1$ of the 5-spaces about Σ_4 meet $\mathcal{V}_2^5$ in exactly $\mathcal{C}$ and 3 generators; and the remaining two 5-spaces about Σ_4 meet $\mathcal{V}_2^5$ in exactly $\mathcal{C}$ and two generators. There are $\binom{q+1}{2}$ choices for Σ_4, and hence the number of 5-spaces that meet $\mathcal{V}_2^5$ in precisely $\mathcal{C}$ and two generators is
$$r_2 = 2 \times \binom{q+1}{2} = (q+1)q.$$
Next, let r_1 be the number of 5-spaces that meet $\mathcal{V}_2^5$ in precisely $\mathcal{C}$ and one generator. We count in two ways ordered pairs (ℓ, Π_5) where ℓ is a generator of $\mathcal{V}_2^5$, and Π_5 is a 5-space that contains ℓ and α, giving
$$(q+1)(q^2+q+1) = 3r_3 + 2r_2 + r_1.$$
Hence $r_1 = q^3/2 + q/2 + 1$. Finally, the number of 5-spaces containing $\mathcal{C}$ and zero generators is $r_0 = (q^3+q^2+q+1) - r_3 - r_2 - r_1 = (q^3-q)/3$, as required. □

Lemma 4.2. *Let $\mathcal{V}_2^5$ be a ruled quintic surface of* PG(6, q) *and let $\mathcal{N}_3$ be a twisted cubic directrix of $\mathcal{V}_2^5$.*

(1) *Of the q^2+q+1 5-spaces of* PG(6, q) *containing $\mathcal{N}_3$, s_i of them meet $\mathcal{V}_2^5$ in precisely $\mathcal{N}_3$ and i generators, where*
$$s_2 = \frac{q^2+q}{2}, \quad s_1 = q+1, \quad s_0 = \frac{q^2-q}{2}.$$

(2) *The total number of 5-spaces that meet $\mathcal{V}_2^5$ in a twisted cubic and i generators is $q^2 s_i$, for $i = 0, 1, 2$.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface of PG(6, q) with a twisted cubic directrix $\mathcal{N}_3$ lying in the 3-space Π_3. By Lemma 2.7, a 5-space containing $\mathcal{N}_3$ contains at most two generators of $\mathcal{V}_2^5$, so the number of 5-spaces that contain Π_3 and exactly two generator lines is $s_2 = \binom{q+1}{2}$. Let ℓ be a generator of $\mathcal{V}_2^5$ and consider the 4-space $\Pi_4 = \langle \Pi_3, \ell \rangle$. For each generator $m \neq \ell$, $\langle \Pi_4, m \rangle$ is a 5-space about Π_4 that meets $\mathcal{V}_2^5$ in $\mathcal{N}_3$, ℓ, and m, and in no further point by Lemma 2.7. This accounts for

q of the 5-spaces containing Π_4. Hence the remaining 5-space containing Π_4 meets $\mathcal{V}_2^5$ in exactly $\mathcal{N}_3$ and ℓ. That is, exactly one of the 5-spaces about $\Pi_4 = \langle \Pi_3, \ell \rangle$ meets $\mathcal{V}_2^5$ in precisely $\mathcal{N}_3$ and ℓ. There are $q+1$ choices for the generator ℓ, and hence $s_1 = q+1$. Finally $s_0 = (q^2+q+1) - s_2 - s_1 = (q^2-q)/2$, as required.

For (2), by Theorem 3.2, $\mathcal{V}_2^5$ contains q^2 twisted cubics, so the total number of 5-spaces meeting $\mathcal{V}_2^5$ in a twisted cubic and i generators is $q^2 s_i$, $i = 0, 1, 2$. □

The next result looks at properties of 4-dim nrcs contained in $\mathcal{V}_2^5$. In particular, we show that there are no 5-spaces that meet $\mathcal{V}_2^5$ in a 4-dim nrc and 0 generator lines.

Lemma 4.3. *Let $\mathcal{V}_2^5$ be a ruled quintic surface of* PG(6, q) *with conic directrix $\mathcal{C}$ in the plane α, and let $\mathcal{N}_4$ be a 4-dim nrc contained in $\mathcal{V}_2^5$.*

(1) *The $q+1$ 5-spaces containing $\mathcal{N}_4$ each contain a distinct generator line of $\mathcal{V}_2^5$.*

(2) *The 4-space containing $\mathcal{N}_4$ meets α in a point P, and either $P = \mathcal{C} \cap \mathcal{N}_4$ or q is even and P is the nucleus of $\mathcal{C}$.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface in PG(6, q) with conic directrix $\mathcal{C}$ lying in a plane α. Let $\mathcal{N}_4$ be a 4-dim nrc contained in $\mathcal{V}_2^5$, so $\mathcal{N}_4$ lies in a 4-space, which we denote Π_4. By Corollary 2.8, Π_4 does not contain a generator of $\mathcal{V}_2^5$. By Lemma 2.7, a 5-space containing $\mathcal{N}_4$ can contain at most one generator of $\mathcal{V}_2^5$. Hence each of the $q+1$ 5-spaces containing $\mathcal{N}_4$ contains a distinct generator. In particular, if we label the points of $\mathcal{C}$ by $Q_0, \ldots, Q_q$, and the generator through Q_i by ℓ_{Q_i}, then the $q+1$ 5-spaces containing $\mathcal{N}_4$ are $\Sigma_i = \langle \Pi_4, \ell_{Q_i} \rangle$, for $i = 0, \ldots, q$.

If Π_4 met the plane α in a line, then $\langle \Pi_4, \alpha \rangle$ is a 5-space whose intersection with $\mathcal{V}_2^5$ contains $\mathcal{N}_4$ and $\mathcal{C}$, contradicting Lemma 2.7. Hence Π_4 meets α in a point P. There are three possibilities for the point $P = \Pi_4 \cap \alpha$, namely $P \in \mathcal{C}$, q even and P the nucleus of $\mathcal{C}$, or q even, $P \notin \mathcal{C}$, and P not the nucleus of $\mathcal{C}$.

Case 1. Suppose $P \in \mathcal{C}$. For $i = 0, \ldots, q$, the 5-space $\Sigma_i = \langle \Pi_4, \ell_{Q_i} \rangle$ meets α in a line m_i. Label $\mathcal{C}$ so that $P = Q_0$, so the line m_0 is the tangent to $\mathcal{C}$ at P, and m_i for $i = 1, \ldots, q$, is the secant line PQ_i. We now show that $P = Q_0$ is a point of $\mathcal{N}_4$. Let $i \in \{1, \ldots, q\}$. Then by Lemma 2.7, Σ_i meets $\mathcal{V}_2^5$ in precisely $\mathcal{N}_4 \cup \ell_{Q_i}$, and $\Sigma_i \cap \mathcal{V}_2^5 \cap \alpha$ is the two points P, Q_i. As $P \notin \ell_{Q_i}$ we have $P \in \mathcal{N}_4$. That is, $P = \mathcal{C} \cap \mathcal{N}_4$.

Case 2. Suppose q is even and $P = \Pi_4 \cap \alpha$ is the nucleus of $\mathcal{C}$. For $i = 0, \ldots, q$, the 5-space $\Sigma_i = \langle \Pi_4, \ell_{Q_i} \rangle$ meets α in the tangent to $\mathcal{C}$ through Q_i. In this case, $\mathcal{C} \cap \mathcal{N}_4 = \varnothing$.

Case 3. Suppose $P = \Pi_4 \cap \alpha$ is not in $\mathcal{C}$, and P is not the nucleus of $\mathcal{C}$. Now P lies on some secant $m = QR$ of $\mathcal{C}$, for some points $Q, R \in \mathcal{C}$. The intersection of the 5-space $\langle \Pi_4, m \rangle$ with $\mathcal{V}_2^5$ contains $\mathcal{N}_4$ and two points R, Q of $\mathcal{C}$. As R, Q lie on distinct generators and are not in $\mathcal{N}_4$, this contradicts Lemma 2.7. Hence this case cannot occur. □

We can now describe how an nrc of $\mathcal{V}_2^5$ meets the conic directrix, and note that Theorem 5.1 shows that each possibility in (3) below can occur.

Corollary 4.4. *Let $\mathcal{V}_2^5$ be a ruled quintic surface of* PG(6, q) *with conic directrix $\mathcal{C}$.*

(1) *A twisted cubic $\mathcal{N}_3 \subseteq \mathcal{V}_2^5$ contains* 0 *points of $\mathcal{C}$.*

(2) *A* 4*-dim nrc $\mathcal{N}_4 \subseteq \mathcal{V}_2^5$ contains either* 1 *point of $\mathcal{C}$, or* 0 *points of $\mathcal{C}$, in which case q is even and the* 4*-space containing $\mathcal{N}_4$ contains the nucleus of $\mathcal{C}$.*

(3) *A* 5*-dim nrc $\mathcal{N}_5 \subseteq \mathcal{V}_2^5$ contains* 0, 1, *or* 2 *points of $\mathcal{C}$.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface of PG$(6, q)$ with conic directrix $\mathcal{C}$ in a plane α. Let $\mathcal{N}_3$ be a twisted cubic of $\mathcal{V}_2^5$, so by Theorem 3.2, $\mathcal{N}_3$ is a directrix of $\mathcal{V}_2^5$, and so is disjoint from α, proving (1). Next let $\mathcal{N}_4$ be a 4-dim nrc on $\mathcal{V}_2^5$, and let Π_4 be the 4-space containing $\mathcal{N}_4$. By Lemma 4.3, $\Pi_4 \cap \alpha$ is a point P, and either $P = \mathcal{C} \cap \mathcal{N}_4$, or q is even and P is the nucleus of $\mathcal{C}$. Thus, $P \notin \mathcal{V}_2^5$ and so $P \notin \mathcal{N}_4$, proving (2). Let Π_5 be a 5-space containing a 5-dim nrc of $\mathcal{V}_2^5$. By Lemma 2.7, Π_5 cannot contain α. Hence Π_5 meets α in a line, and so contains at most two points of $\mathcal{C}$, proving (3). □

We now use the Bruck–Bose setting to count the 4-dim nrcs contained in $\mathcal{V}_2^5$.

Lemma 4.5. *Let $\mathcal{S}$ be a regular* 2*-spread in a* 5*-space Σ_∞ in* PG(6, q). *Position $\mathcal{V}_2^5$ as in Corollary 3.3, so $\mathcal{V}_2^5$ has splash $\mathbb{S} \subset \mathcal{S}$. Then a* 4*- or* 5*-space about a plane $\beta \in \mathbb{S}$ cannot contain a* 4*-dim nrc of $\mathcal{V}_2^5$.*

Proof. Position $\mathcal{V}_2^5$ as described in Corollary 3.3, so $\mathcal{S}$ is a regular 2-spread in a 5-space Σ_∞, the conic directrix of $\mathcal{V}_2^5$ lies in a plane $\alpha \in \mathcal{S}$, and $\mathbb{S} \subset \mathcal{S}$ denotes the splash of $\mathcal{V}_2^5$. By Lemma 2.7, a 4-space containing α cannot contain a 4-dim nrc of $\mathcal{V}_2^5$. Let $\beta \in \mathbb{S} \setminus \alpha$. Then by Corollary 3.4, β lies in exactly one 3-space that contains a twisted cubic of $\mathcal{V}_2^5$. Denote these by Π_3 and $\mathcal{N}_3$, respectively. By Theorem 3.2, $\mathcal{N}_3$ is a directrix of $\mathcal{V}_2^5$, and so Π_3 is disjoint from α. So if ℓ_P is a generator of $\mathcal{V}_2^5$, then $\Pi_4 = \langle \Pi_3, \ell_P \rangle$ is a 4-space and $\Pi_4 \cap \alpha$ is the point $P = \ell_P \cap \mathcal{C}$. Let ℓ be a line of α through P and let $\Pi_5 = \langle \Pi_3, \ell \rangle$. If ℓ is tangent to $\mathcal{C}$, then $\Pi_5 \cap \mathcal{V}_2^5$ is exactly $\mathcal{N}_3 \cup \ell_P$. If ℓ is a secant of $\mathcal{C}$, so $\ell \cap \mathcal{C} = \{P, Q\}$, then $\Pi_5 \cap \mathcal{V}_2^5$ consists of $\mathcal{N}_3$, ℓ_P, and the generator ℓ_Q through Q. Varying ℓ_P and ℓ, we get all the 5-spaces that contain β and contain 1 or 2 generators of $\mathcal{V}_2^5$. That is, each 5-space containing β and 1 or 2 generators of $\mathcal{V}_2^5$ also contains $\mathcal{N}_3$. The remaining 5-spaces about β hence contain 0 generators of $\mathcal{V}_2^5$ and meet α in an exterior line of $\mathcal{C}$. Hence, by Lemma 4.3, none of the 5-spaces about β contain a 4-dim nrc of $\mathcal{V}_2^5$. □

Lemma 4.6. (1) *The number of* 4*-dim nrcs contained in $\mathcal{V}_2^5$ is $q^4 - q^2$.*

(2) *The number of* 5*-spaces that meet $\mathcal{V}_2^5$ in a* 4*-dim nrc and one generator is $q^5 + q^4 - q^3 - q^2$.*

Proof. Without loss of generality, position $\mathcal{V}_2^5$ as described in Corollary 3.3. That is, let $\mathcal{S}$ be a regular 2-spread in a 5-space Σ_∞, let the conic directrix of $\mathcal{V}_2^5$ lie in a plane $\alpha \in \mathcal{S}$, and let $\mathbb{S} \subset \mathcal{S}$ be the splash of $\mathcal{V}_2^5$. Straightforward counting shows that a 5-space distinct from Σ_∞ contains a unique spread plane. If this plane is in the splash $\mathbb{S}$, then by Lemma 4.5, the 5-space does not contain a 4-dim nrc of $\mathcal{V}_2^5$. So a 5-space containing a 4-dim nrc of $\mathcal{V}_2^5$ contains a unique plane of $\mathcal{S} \setminus \mathbb{S}$. Consider a plane $\gamma \in \mathcal{S} \setminus \mathbb{S}$. Let $P \in \mathcal{C}$, let ℓ_P be the generator of $\mathcal{V}_2^5$ through P, and consider the 4-space $\Pi_4 = \langle \gamma, \ell_P \rangle$. Suppose first that Π_4 contains two generators of $\mathcal{V}_2^5$. Then there is a 5-space Π_5 containing γ and two generators. By Lemma 2.7, Π_5 contains either $\mathcal{C}$ or a twisted cubic of $\mathcal{V}_2^5$. A 5-space distinct from Σ_∞ cannot contain two planes of $\mathcal{S}$, so Π_5 does not contain $\mathcal{C}$. Moreover, by Corollary 3.3, Π_5 does not contain a twisted cubic of $\mathcal{V}_2^5$. Hence Π_4 contains exactly one generator of $\mathcal{V}_2^5$. If every generator of $\mathcal{V}_2^5$ contained at least one point of Π_4, then the intersection of Π_4 with $\mathcal{V}_2^5$ contains at least ℓ_P and q further points, one on each generator. By Lemma 2.7 and Corollary 2.8, the only possibility is that $\Pi_4 \cap \mathcal{V}_2^5$ contains a twisted cubic, which is not possible by Corollary 3.3. Hence there is at least one generator which is disjoint from Π_4; denote this ℓ_Q. Label the points of ℓ_Q by $X_0, \ldots, X_q$. Then the $q+1$ 5-spaces containing Π_4 are $\Sigma_i = \langle \gamma, \ell_P, X_i \rangle$. For each $i = 0, \ldots, q$, the intersection of Σ_i with $\mathcal{V}_2^5$ contains the generator ℓ_P and the point X_i. By Corollary 3.3, Σ_i does not contain a twisted cubic of $\mathcal{V}_2^5$. Hence, by Lemma 2.7, $\Sigma_i \cap \mathcal{V}_2^5$ is ℓ_P and a 4-dim nrc.

That is, there are $(q+1)^2$ 5-spaces containing γ and one generator of $\mathcal{V}_2^5$. Each contains a 4-dim nrc of $\mathcal{V}_2^5$. Further, if Π_5 is a 5-space containing γ and zero generators of $\mathcal{V}_2^5$, then by Lemma 4.3, Π_5 does not contain a 4-dim nrc of $\mathcal{V}_2^5$. Hence, as there are $q^3 - q^2$ choices for γ, there are

$$(q+1)^2 \times (q^3 - q^2) = q^5 + q^4 - q^3 - q^2$$

5-spaces that meet $\mathcal{V}_2^5$ in one generator and a 4-dim nrc. By Lemma 4.3, every 4-dim nrc in $\mathcal{V}_2^5$ lies in $q+1$ such 5-spaces. Hence the number of 4-dim nrcs contained in $\mathcal{V}_2^5$ is $(q^5 + q^4 - q^3 - q^2)/(q+1)$ as required. □

We now count the number of 5-dim nrcs contained in $\mathcal{V}_2^5$.

Lemma 4.7. *The number of 5-spaces meeting $\mathcal{V}_2^5$ in a 5-dim nrc is $q^6 - q^4$.*

Proof. We show that the number of 5-spaces meeting $\mathcal{V}_2^5$ in a 5-dim nrc is $q^6 - q^4$ by counting in two ways the number x of incident pairs (A, Π_5) where A is a point of $\mathcal{V}_2^5$ and Π_5 is a 5-space containing A. The number of ways to choose a point A of $\mathcal{V}_2^5$ is $(q+1)^2$. The point A lies in $q^5 + q^4 + q^3 + q^2 + q + 1$ 5-spaces. So

$$x = (q+1)^2 \times (q^5+q^4+q^3+q^2+q+1) = q^7+3q^6+4q^5+4q^4+4q^3+4q^2+3q+1.$$

Alternatively, we count the 5-spaces first; there are several possibilities for Π_5. By Lemma 2.7, $\Pi_5 \cap \mathcal{V}_2^5$ is either empty, or contains an r-dim nrc for some $r \in \{2, \ldots, 5\}$. Let n_r be the number of pairs (A, Π_5) with $A \in \mathcal{V}_2^5 \cap \Pi_5$ and Π_5 containing an r-dim nrc of $\mathcal{V}_2^5$. Note that

$$x = n_2 + n_3 + n_4 + n_5. \tag{1}$$

We now calculate n_2, n_3, and n_4, and then use (1) to determine the number of 5-spaces meeting $\mathcal{V}_2^5$ in a 5-dim nrc.

For n_2, consider a 5-space Π_5 that contains the conic directrix $\mathcal{C}$, so by Lemma 4.1, Π_5 contains 0, 1, 2, or 3 generators of $\mathcal{V}_2^5$, and the number of 5-spaces meeting $\mathcal{V}_2^5$ in exactly the conic directrix and i generators is r_i. In this case the number of ways to pick a point of $\Pi_5 \cap \mathcal{V}_2^5$ is $iq + q + 1$. Hence the total number of pairs (A, Π_5) with Π_5 containing the conic directrix is

$$n_2 = \sum_{i=0}^{3} r_i (iq + q + 1) = 2q^4 + 4q^3 + 4q^2 + 3q + 1.$$

For n_3, consider a 5-space Π_5 that contains a twisted cubic. Then by Lemma 4.2, Π_5 contains 0, 1, or 2 generators of $\mathcal{V}_2^5$, and the number of 5-spaces meeting $\mathcal{V}_2^5$ in a given twisted cubic and i generators is s_i. In this case the number of ways to pick A in $\mathcal{V}_2^5 \cap \Pi_5$ is $iq + q + 1$. Hence the number of pairs (A, Π_5) with Π_5 containing a twisted cubic of $\mathcal{V}_2^5$ is

$$n_3 = q^2 \sum_{i=0}^{2} s_i (iq + q + 1) = 2q^5 + 4q^4 + 3q^3 + q^2.$$

For n_4, consider a 5-space Π_5 that contains a 4-dim nrc of $\mathcal{V}_2^5$. By Lemma 4.3, Π_5 contains 1 generator of $\mathcal{V}_2^5$. By Lemma 4.6, the number of 5-spaces meeting $\mathcal{V}_2^5$ in exactly a 4-dim nrc and one generator is $q^5 + q^4 - q^3 - q^2$. The number of ways to pick A in $\mathcal{V}_2^5 \cap \Pi_5$ is $2q + 1$. So

$$n_4 = (q^5 + q^4 - q^3 - q^2) \times (2q + 1) = 2q^6 + 3q^5 - q^4 - 3q^3 - q^2.$$

Finally, denote the number of 5-spaces containing a 5-dim nrc of $\mathcal{V}_2^5$ by y. Then the number of pairs (A, Π_5) with Π_5 containing a 5-dim nrc of $\mathcal{V}_2^5$ is

$$n_5 = y \times (q + 1).$$

Substituting the calculated values for x, n_2, n_3, n_4, n_5 into (1) and rearranging gives $y = q^6 - q^4$ as required. □

Summarising the preceding lemmas gives the following theorem describing $\mathcal{V}_2^5$.

Theorem 4.8. *Let $\mathcal{V}_2^5$ be the ruled quintic surface in* PG(6, q), $q \geq 6$.

(1) *$\mathcal{V}_2^5$ contains exactly*

$q+1$	*lines,*
1	*nondegenerate conic,*
q^2	*twisted cubics,*
q^4-q^2	4*-dim nrcs,*
q^6-q^4	5*-dim nrcs.*

(2) *A* 5*-space meets $\mathcal{V}_2^5$ in one of the following configurations*:

number of 5*-spaces*	*meeting $\mathcal{V}_2^5$ in the configuration*
q^6-q^4	5*-dim nrc,*
$q^5+q^4-q^3-q^2$	4*-dim nrc and* 1 *generator,*
$(q^4-q^3)/2$	*twisted cubic,*
q^3+q^2	*twisted cubic and* 1 *generator,*
$(q^4+q^3)/2$	*twisted cubic and* 2 *generators,*
$(q^3-q)/3$	*conic,*
$q^3/2+q/2+1$	*conic and* 1 *generator,*
q^2+q	*conic and* 2 *generators,*
$(q^3-q)/6$	*conic and* 3 *generators.*

5. The Bruck–Bose spread and 5-spaces

Let $\mathcal{S}$ be a regular 2-spread in a 5-space Σ_∞ in PG(6, q), and position $\mathcal{V}_2^5$ so that it corresponds to a tangent $\mathbb{F}_q$-subplane of PG(2, q^3). So $\mathcal{V}_2^5$ has splash $\mathbb{S} \subset \mathcal{S}$, the conic directrix $\mathcal{C}$ lies in a plane $\alpha \in \mathbb{S}$, and each of the q^2 3-spaces containing a twisted cubic directrix of $\mathcal{V}_2^5$ meets Σ_∞ in a distinct plane of $\mathbb{S} \setminus \alpha$. In Corollary 3.4, we looked at how 3-spaces containing a plane of $\mathcal{S}$ meet $\mathcal{V}_2^5$. In Lemma 4.5, we looked at how 4-spaces containing a plane of $\mathcal{S}$ meet $\mathcal{V}_2^5$. Next we look at how 5-spaces containing a plane of $\mathcal{S}$ meet $\mathcal{V}_2^5$. Note that straightforward counting shows that a 5-space distinct from Σ_∞ contains a unique plane π of $\mathcal{S}$, and meets every other plane of $\mathcal{S}$ in a line. If $\pi = \alpha$, then Lemma 4.1 describes the possible intersections with $\mathcal{V}_2^5$. The next theorem describes the possible intersections with $\mathcal{V}_2^5$ for the remaining cases $\pi \in \mathbb{S} \setminus \alpha$ and $\pi \in \mathcal{S} \setminus \mathbb{S}$.

Theorem 5.1. *Position $\mathcal{V}_2^5$ as in Corollary 3.3, so $\mathcal{S}$ is a regular* 2*-spread in a hyperplane Σ_∞, the conic directrix $\mathcal{C}$ lies in a plane $\alpha \in \mathcal{S}$, and $\mathcal{V}_2^5$ has splash $\mathbb{S} \subset \mathcal{S}$. Let ℓ be a line of α with $|\ell \cap \mathcal{C}| = i$ and let $\pi \in \mathcal{S}$, $\pi \neq \alpha$. Then the q* 5*-spaces containing π, ℓ and distinct from Σ_∞ meet $\mathcal{V}_2^5$ as follows.*

(1) *If $\pi \in \mathbb{S} \setminus \alpha$, then $q-1$ meet $\mathcal{V}_2^5$ in a* 5*-dim nrc, and* 1 *meets $\mathcal{V}_2^5$ in a twisted cubic and i generators.*

(2) *If* $\pi \in \mathcal{S} \setminus \mathbb{S}$, *then* $q - i$ *meet* $\mathcal{V}_2^5$ *in a* 5*-dim nrc, and* i *meet* $\mathcal{V}_2^5$ *in a* 4*-dim nrc and* 1 *generator.*

Proof. By [Barwick and Jackson 2012], the group of collineations of $\mathrm{PG}(6, q)$ fixing $\mathcal{S}$ and $\mathcal{V}_2^5$ is transitive on the planes of $\mathbb{S} \setminus \alpha$ and on the planes of $\mathcal{S} \setminus \mathbb{S}$. As this group fixes the conic directrix $\mathcal{C}$, it is transitive on the lines of α tangent to $\mathcal{C}$, the lines of α secant to $\mathcal{C}$, and the lines of α exterior to $\mathcal{C}$. So without loss of generality let ℓ_0 be a line of α exterior to $\mathcal{C}$, let ℓ_1 be a line of α tangent to $\mathcal{C}$, let ℓ_2 be a line of α secant to $\mathcal{C}$, let β be a plane in $\mathbb{S} \setminus \alpha$, and let γ be a plane of $\mathcal{S} \setminus \mathbb{S}$. For $i = 0, 1, 2$, label the 4-spaces $\Sigma_{4,i} = \langle \beta, \ell_i \rangle$ and $\Pi_{4,i} = \langle \gamma, \ell_i \rangle$. By Corollary 3.4, as $\beta \in \mathbb{S} \setminus \alpha$, there is a unique twisted cubic of $\mathcal{V}_2^5$ that lies in a 3-space about β. Denote this 3-space by Π_3. Hence for $i = 0, 1, 2$, there is a unique 5-space containing $\Sigma_{4,i}$ whose intersection with $\mathcal{V}_2^5$ contains a twisted cubic, namely the 5-space $\langle \Pi_3, \ell_i \rangle$.

First consider the line ℓ_0 which is exterior to $\mathcal{C}$. A 5-space meeting α in ℓ_0 contains 0 points of $\mathcal{C}$, and so contains 0 generators of $\mathcal{V}_2^5$. The 4-space $\Sigma_{4,0} = \langle \beta, \ell_0 \rangle$ lies in q 5-spaces distinct from Σ_∞, each containing 0 generators of $\mathcal{V}_2^5$. Exactly one of these 5-spaces, namely $\langle \Pi_3, \ell_0 \rangle$, contains a twisted cubic of $\mathcal{V}_2^5$. The remaining $q - 1$ 5-spaces about $\Sigma_{4,0}$ contain 0 generators, and do not contain a conic or twisted cubic of $\mathcal{V}_2^5$, so by Theorem 4.8, they meet $\mathcal{V}_2^5$ in a 5-dim nrc, proving (1) for $i = 0$. For (2), let $\Pi_5 \neq \Sigma_\infty$ be any 5-space containing $\Pi_{4,0} = \langle \gamma, \ell_0 \rangle$. As $\gamma \notin \mathbb{S}$, by Corollary 3.3, Π_5 cannot contain a twisted cubic of $\mathcal{V}_2^5$. As Π_5 contains 0 generator lines of $\mathcal{V}_2^5$ and does not contain a conic or twisted cubic of $\mathcal{V}_2^5$, by Theorem 4.8, Π_5 meets $\mathcal{V}_2^5$ in a 5-dim nrc. That is, the q 5-spaces (distinct from Σ_∞) containing $\Pi_{4,0}$ meet $\mathcal{V}_2^5$ in a 5-dim nrc, proving (2) for $i = 0$.

Next consider the line ℓ_1 which is tangent to $\mathcal{C}$. Let $P = \ell_1 \cap \mathcal{C}$ and denote the generator of $\mathcal{V}_2^5$ through P by ℓ_P. A 5-space meeting α in a tangent line contains 1 point of $\mathcal{C}$, and so contains at most one generator of $\mathcal{V}_2^5$. So exactly one 5-space contains $\Sigma_{4,1}$ and a generator, namely the 5-space $\langle \Sigma_{4,1}, \ell_P \rangle$. Consider the 5-space $\langle \Pi_3, \ell_1 \rangle$. It contains P and a twisted cubic of $\mathcal{V}_2^5$, which by Corollary 4.4 is disjoint from α. Hence $\langle \Pi_3, \ell_1 \rangle$ contains the generator ℓ_P. That is, $\langle \Pi_3, \ell_1 \rangle$ contains β, ℓ_1, ℓ_P and so $\langle \Pi_3, \ell_1 \rangle = \langle \Sigma_{4,1}, \ell_P \rangle$. That is, the intersection of $\langle \Sigma_{4,1}, \ell_P \rangle$ with $\mathcal{V}_2^5$ is a twisted cubic and one generator. Let $\Pi_5 \neq \Sigma_\infty$ be one of the remaining $q - 1$ 5-spaces (distinct from Σ_∞) that contains $\Sigma_{4,1}$, so Π_5 contains 0 generators of $\mathcal{V}_2^5$ and does not contain a conic or twisted cubic of $\mathcal{V}_2^5$. So by Theorem 4.8, Π_5 meets $\mathcal{V}_2^5$ in a 5-dim nrc, proving (1) for $i = 1$. For (2), we consider $\Pi_{4,1} = \langle \gamma, \ell_1 \rangle$. By Corollary 3.3, as $\gamma \notin \mathbb{S}$, no 5-space containing $\Pi_{4,1}$ contains a twisted cubic of $\mathcal{V}_2^5$. The 5-space $\langle \Pi_{4,1}, \ell_P \rangle$ contains one generator of $\mathcal{V}_2^5$, so by Theorem 4.8, it meets $\mathcal{V}_2^5$ in exactly a 4-dim nrc and the generator ℓ_P. Let $\Pi_5 \neq \Sigma_\infty$ be one of the remaining $q - 1$ 5-spaces containing $\Pi_{4,1}$. Then Π_5 contains 0 generators of $\mathcal{V}_2^5$. So by Theorem 4.8, Π_5 meets $\mathcal{V}_2^5$ in a 5-dim nrc, proving (2) for $i = 1$.

Finally, consider the line ℓ_2 which is secant to $\mathcal{C}$. Let $\mathcal{C} \cap \ell_2 = \{P, Q\}$ and let ℓ_P, ℓ_Q be the generators of $\mathcal{V}_2^5$ through P, Q, respectively. The intersection of the 5-space $\langle \Pi_3, \ell_2 \rangle$ and $\mathcal{V}_2^5$ contains a twisted cubic, and P and Q. By Corollary 4.4, this twisted cubic is disjoint from α, so $\langle \Pi_3, \ell_2 \rangle$ contains the two generators ℓ_P, ℓ_Q. Thus $\langle \Pi_3, \ell_2 \rangle = \langle \Sigma_{4,2}, \ell_P \rangle = \langle \Sigma_{4,2}, \ell_Q \rangle = \langle \Sigma_{4,2}, \ell_P, \ell_Q \rangle$. The remaining $q - 1$ 5-spaces (distinct from Σ_∞) about $\Sigma_{4,2}$ contain 0 generators and two points of $\mathcal{C}$. By Lemma 4.3 they cannot contain a 4-dim nrc of $\mathcal{V}_2^5$. So by Theorem 4.8, they meet $\mathcal{V}_2^5$ in a 5-dim nrc, proving (1) for $i = 2$. For (2), let $\Pi_5 \neq \Sigma_\infty$ be a 5-space containing $\Pi_{4,2} = \langle \gamma, \ell_2 \rangle$. By Corollary 3.3, Π_5 does not contain a twisted cubic of $\mathcal{V}_2^5$, as $\gamma \notin \mathbb{S}$. So by Theorem 4.8, Π_5 contains at most one generator of $\mathcal{V}_2^5$. Hence $\langle \Pi_{4,2}, \ell_P \rangle, \langle \Pi_{4,2}, \ell_Q \rangle$ are distinct 5-spaces about $\Pi_{4,2}$, and by Theorem 4.8, they each meet $\mathcal{V}_2^5$ in a 4-dim nrc and one generator. Let $\Sigma_5 \neq \Sigma_\infty$ be one of the remaining $q - 2$ 5-spaces about $\Pi_{4,2}$. Then Σ_5 contains 0 generators of $\mathcal{V}_2^5$, and so by Theorem 4.8, meets $\mathcal{V}_2^5$ in a 5-dim nrc, proving (2) for $i = 2$. □

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