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A characterization of Clifford parallelism by automorphisms

Rainer Löwen

Betten and Riesinger have shown that Clifford parallelism on real projective space is the only topological parallelism that is left invariant by a group of dimension at least 5. We improve the bound to 4. Examples of different parallelisms admitting a group of dimension ≤ 3 are known, so 3 is the "critical dimension".

Consider \mathbb{R}^4 as the quaternion skew field \mathbb{H} . Then the orthogonal group SO(4, \mathbb{R}) may be described as the product of two commuting copies $\tilde{\Lambda}$, $\tilde{\Phi}$ of the unitary group U(2, \mathbb{C}), consisting of the maps $q \mapsto aq$ and $q \mapsto qb$, respectively, where a, b are quaternions of norm one and multiplication is quaternion multiplication. The intersection of the two factors is of order two, containing the map -id. Thus, passing to projective space, we get PSO(4, \mathbb{R}) = $\Lambda \times \Phi$, a direct product of two copies of SO(3, \mathbb{R}). The left and right Clifford parallelisms are defined as the equivalence relations on the line space of PG(3, \mathbb{R}) formed by the orbits of Λ and Φ , respectively.

The two Clifford parallelisms are equivalent under quaternion conjugation $q \rightarrow \bar{q}$; this is immediate from their definition in view of the fact that conjugation does not change the norm and is an antiautomorphism, i.e., that $\bar{pq} = \bar{q} \bar{p}$. Note that both Λ and Φ are transitive on the point set of projective space. Since they centralize one another, each acts transitively on the parallelism defined by the other, and the group PSO(4, \mathbb{R}) leaves both parallelisms invariant (we say that it consists of *automorphisms* of these parallelisms). For more information on Clifford parallels, see [Berger 1987; Klingenberg 1984; Betten and Riesinger 2012]. For generalizations to other dimensions, compare also [Tyrrell and Semple 1971].

The notion of a *topological parallelism* on real projective 3-space $PG(3, \mathbb{R})$ generalizes this example. A *spread* is a set C of lines such that every point is incident with exactly one of them, and a topological parallelism may be defined

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as a compact set Π of compact spreads such that every line belongs to exactly one of them; see, e.g., [Betten and Riesinger 2014b] for details. Many examples of different topological parallelisms have been constructed in a series of papers by Betten and Riesinger, see, e.g., [Betten and Riesinger 2009].

The group $\Sigma = \operatorname{Aut} \Pi$ of automorphisms of a topological parallelism is a closed subgroup of the Lie group PGL(4, \mathbb{R}), hence it is a Lie group, as well. In particular, the identity component Σ^1 is an open subgroup of Σ and has the same (manifold) dimension as Σ . We know that Σ^1 is compact [Betten and Löwen 2017], and hence (conjugate to) a subgroup of PSO(4, \mathbb{R}) \cong SO(3, \mathbb{R}) \times SO(3, \mathbb{R}). The group SO(3, \mathbb{R}) does not have any 2-dimensional closed subgroups, because its Lie algebra is \mathbb{R}^3 with the vector product \times and $x \times y$ is always orthogonal to both x and y. Moreover, the 1-dimensional closed subgroups of SO(3, \mathbb{R}) form a single conjugacy class. It follows easily that there are no closed 5-dimensional subgroups of SO(3, \mathbb{R}) \times SO(3, \mathbb{R}) and all 4-dimensional ones are isomorphic to SO(3, \mathbb{R}) \times SO(2, \mathbb{R}).

We see that in the case of the Clifford parallelism, Σ^1 is the 6-dimensional group PSO(4, \mathbb{R}) that we used to define the parallelism. Betten and Riesinger [2014b] proved that no other topological parallelism has a group of dimension dim $\Sigma \geq 5$. Examples of parallelisms with 1-, 2- or 3-dimensional automorphism groups are known; see [Betten and Riesinger 2014a; 2009; 2011]. Here we consider parallelisms with a 4-dimensional group.

Theorem 1. Let Σ be the automorphism group of a topological parallelism Π on PG(3, \mathbb{R}). If dim $\Sigma \ge 4$, then Π is equivalent to the Clifford parallelism.

Proof. Recall that a topological parallelism Π is homeomorphic to the real projective plane in the Hausdorff topology on the space of compact sets of lines, and that every equivalence class is a compact spread and homeomorphic to the 2-sphere; compare [Betten and Riesinger 2014b].

The remarks preceding the theorem show that a group Σ of dimension at least 4 contains a 4-dimensional connected closed subgroup Δ , and it will suffice for our proof to use this group. Further, up to equivalence, we may assume that $\Delta = \Lambda \cdot \Gamma$, where $\Gamma \leq \Phi$ is the subgroup defined by restricting the factor *b* to be a complex number (here we use the notation of the introduction). Since Λ does not have any one-dimensional coset spaces, we know that Λ acts on Π either transitively or trivially. If it acts trivially, then the classes of Π are the Λ -orbits of lines, and we have the Clifford parallelism. Observe here that every Λ -orbit is contained in a single class, and both the orbit and the class are 2-spheres.

In what follows, assume therefore that Λ acts transitively on Π . There is only one possibility for this action, namely, the standard transitive action of SO(3, \mathbb{R}) on the real projective plane. Every 2-dimensional subgroup of Δ contains Γ . Hence,

there is no effective action of Δ on the projective plane Π , and the kernel can only be Γ since the only other proper normal subgroup is Λ , which is transitive. If $C \in \Pi$ is any equivalence class, then the stabilizer Λ_C is a product of a 1-torus and a group of order two. Hence Δ_C contains a 2-torus *T*. There is only one conjugacy class of 2-tori in Δ , represented by the group

$$T_0 = \{ \langle q \rangle \mapsto \langle aqb \rangle \mid a, b \in \mathbb{C}, |a| = |b| = 1 \}.$$

Here, $\langle q \rangle$ denotes the 1-dimensional real vector space spanned by q. We may assume that $T = T_0$. Write quaternions as pairs of complex numbers with multiplication $(x, y)(u, v) = (xu - \bar{v}y, vx + y\bar{u})$; see 11.1 of [Salzmann et al. 1995]. Then complex numbers become pairs (a, 0), and the elements of T are now given by

$$\langle (z, w) \rangle \mapsto \langle (azb, awb) \rangle.$$

The kernel of ineffectivity of *T* on the 2-sphere *C* must be a 1-torus Ξ , and the elements of the kernel other than the identity cannot have eigenvalue 1 — otherwise they would be axial collineations of the translation plane defined by the spread *C* and would act nontrivially on *C*. There are only two subgroups of the 2-torus satisfying these conditions, given by b = 1 and by a = 1, respectively. In other words, the kernel Ξ is a subgroup either of Λ or of Φ . In both cases, *C* consists of the fixed lines of Ξ . If $\Xi \leq \Phi$, then Λ permutes these lines, contrary to the transitivity of Λ on Π . If $\Xi \leq \Lambda$, then Φ permutes the fixed lines, which means that *C* is a Φ -orbit. Now Λ is transitive both on Π and on the set of Φ -orbits, hence Π equals the Clifford parallelism formed by the Φ -orbits.

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