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The exterior splash in PG(6, q): transversals

Susan G. Barwick and Wen-Ai Jackson

Let π be an order-q-subplane of $\mathrm{PG}(2, q^3)$ that is exterior to ℓ_∞. Then the exterior splash of π is the set of q^2+q+1 points on ℓ_∞ that lie on an extended line of π. Exterior splashes are projectively equivalent to scattered linear sets of rank 3, covers of the circle geometry $CG(3, q)$, and hyper-reguli in $\mathrm{PG}(5, q)$. We use the Bruck–Bose representation in $\mathrm{PG}(6, q)$ to investigate the structure of π, and the interaction between π and its exterior splash. We show that the point set of $\mathrm{PG}(6, q)$ corresponding to π is the intersection of nine quadrics, and that there is a unique tangent plane at each point, namely the intersection of the tangent spaces of the nine quadrics. In $\mathrm{PG}(6, q)$, an exterior splash $\mathbb{S}$ has two sets of cover planes (which are hyper-reguli) and we show that each set has three unique transversal lines in the cubic extension $\mathrm{PG}(6, q^3)$. These transversal lines are used to characterise the carriers and the sublines of $\mathbb{S}$.

1. Introduction

In [Barwick and Jackson 2012; 2014], we studied order-q-subplanes of $\mathrm{PG}(2, q^3)$ and determined their representation in the Bruck–Bose representation in $\mathrm{PG}(6, q)$. A full characterisation in $\mathrm{PG}(6, q)$ was given for order-q-subplanes that are secant or tangent to ℓ_∞ in $\mathrm{PG}(2, q^3)$. In [Rottey et al. 2015], this was generalised to study subplanes of $\mathrm{PG}(2, q^n)$ in $\mathrm{PG}(2n, q)$. The cases when the subplane is secant or tangent to ℓ_∞ yield nice geometric characterisations. However, the case of an order-q-subplane π of $\mathrm{PG}(2, q^3)$ that is exterior to ℓ_∞ yields a complex structure denoted $[\pi]$ in $\mathrm{PG}(6, q)$. Our main motivation in this article is to investigate the geometric properties of the structure $[\pi]$. The splash of π gives crucial information about the geometrical properties of $[\pi]$, and so we also study the interplay in $\mathrm{PG}(6, q)$ between $[\pi]$ and its splash.

The splash of a subplane π of $\mathrm{PG}(2, q^n)$ is defined to be the set of points on ℓ_∞ that lie on an extended line of π. In [Barwick and Jackson 2015] it was shown that

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the splash of a tangent order-q-subplane of $\mathrm{PG}(2, q^3)$ is a linear set. In [Lavrauw and Zanella 2015] the notion of splash was generalised from subplanes to subgeometries, and to general field extensions. It was shown that a splash is a linear set, and conversely, a linear set is a splash.

In this article we let π be a subplane of $\mathrm{PG}(2, q^3)$ of order q that is exterior to ℓ_∞. The lines of π meet ℓ_∞ in a set $\mathbb{S}$ of size q^2+q+1, which we call the *exterior splash* of π. Properties of the exterior splash of $\mathrm{PG}(2, q^3)$ were studied in [Barwick and Jackson 2016]. The sets of points in an exterior splash has arisen in many different situations, namely scattered $\mathbb{F}_q$-linear sets of rank 3, covers of the circle geometry $\mathrm{CG}(3, q)$, hyper-reguli in $\mathrm{PG}(5, q)$, and Sherk surfaces of size q^2+q+1. Scattered linear sets are surveyed in [Lavrauw 2016]. An important result is that all scattered $\mathbb{F}_q$-linear sets of rank 3 are projectively equivalent [Lavrauw and Zanella 2015].

This article proceeds as follows. In Section 2 we introduce the notation we use for the Bruck–Bose representation of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(6, q)$, as well as presenting some other preliminary results.

We next introduce coordinates; as all scattered $\mathbb{F}_q$-linear sets of rank 3 are projectively equivalent, we will work with an exterior splash equivalent to the set of points

$$\{(x, x^q) : x \in \mathrm{GF}(q^3)\backslash\{0\}\}.$$

In Section 3 we coordinatise an order-q-subplane $\mathcal{B}$ in $\mathrm{PG}(2, q^3)$ that is exterior to ℓ_∞, with this exterior splash. This order-q-subplane will be used in many of the proofs in this article.

In Section 4, we study the structure of an order-q-subplane in $\mathrm{PG}(6, q)$. We show that it contains $q^2 + q + 1$ twisted cubics and is the intersection of nine quadrics. Further, we show that there is a unique tangent plane at each point, which is the intersection of the tangent spaces of these nine quadrics.

We next study the exterior splash $\mathbb{S}$ of ℓ_∞ in the Bruck–Bose representation in $\mathrm{PG}(5, q)$. By results of Bruck [1973], $\mathbb{S}$ has two switching sets denoted $\mathbb{X}$, $\mathbb{Y}$, which we call covers of $\mathbb{S}$. The three sets $\mathbb{S}$, $\mathbb{X}$, $\mathbb{Y}$ are called hyper-reguli in [Ostrom 1993]. In Section 5, we look at the exterior splash

$$\{(x, x^q) : x \in \mathrm{GF}(q^3)\backslash\{0\}\},$$

and working in $\mathrm{PG}(6, q)$, find coordinates for the two covers $\mathbb{X}$, $\mathbb{Y}$. In Section 6, we show that each of the sets $\mathbb{S}$, $\mathbb{X}$, $\mathbb{Y}$ has a unique triple of conjugate transversal lines in the cubic extension $\mathrm{PG}(5, q^3)$. Theorem 6.5 characterises the carriers of an exterior splash as the only planes of the regular spread that meet all nine transversal lines. Theorem 6.6 shows that the nine transversal lines are common to the set of $q - 1$ disjoint splashes of ℓ_∞ that have common carriers. We interpret this result in terms of replacing hyper-reguli to create André planes. In Section 7 we use the transversal lines to characterise the order-q-sublines of an exterior splash in terms of how the corresponding 2-reguli meet the cover planes.

2. The Bruck–Bose representation

2A. ***The Bruck–Bose representation of*** **PG(2, q^3)** ***in*** **PG(6, q).** We introduce the notation we will use for the Bruck–Bose representation of PG(2, q^3) in PG(6, q). We work with the finite field $\mathbb{F}_q$ of order q. A 2-*spread* of PG(5, q) is a set of q^3+1 planes that partition PG(5, q). A 2-*regulus* of PG(5, q) is a set of $q+1$ mutually disjoint planes $\pi_1, \dots, \pi_{q+1}$ with the property that if a line meets three of the planes, then it meets all $q+1$ of them. A 2-regulus $\mathcal{R}$ has a set of q^2+q+1 mutually disjoint *ruling lines* that meet every plane of $\mathcal{R}$. A 2-regulus is uniquely determined by three mutually disjoint planes, or four (ruling) lines (mutually disjoint and lying in general position). A 2-spread $\mathcal{S}$ is *regular* if for any three planes in $\mathcal{S}$, the 2-regulus containing them is contained in $\mathcal{S}$. See [Hirschfeld and Thas 1991] for more information on 2-spreads.

The following construction of a regular 2-spread of PG(5, q) will be needed. Embed PG(5, q) in PG(5, q^3) and let g be a line of PG(5, q^3) disjoint from PG(5, q). Let g^q, g^{q^2} be the conjugate lines of g; both of these are disjoint from PG(5, q). Let P_i be a point on g; then the plane $\langle P_i, P_i^q, P_i^{q^2}\rangle$ meets PG(5, q) in a plane. As P_i ranges over all the points of g, we get q^3+1 planes of PG(5, q) that partition PG(5, q). These planes form a regular 2-spread $\mathcal{S}$ of PG(5, q). The lines g, g^q, g^{q^2} are called the (conjugate skew) *transversal lines* of the 2-spread $\mathcal{S}$. Conversely, given a regular 2-spread in PG(5, q), there is a unique set of three (conjugate skew) transversal lines in PG(5, q^3) that generate $\mathcal{S}$ in this way.

We will use the linear representation of a finite translation plane $\mathcal{P}$ of dimension at most three over its kernel, due independently to André [1954] and Bruck and Bose [1964; 1966]. Let Σ_∞ be a hyperplane of PG(6, q) and let $\mathcal{S}$ be a 2-spread of Σ_∞. We use the phrase *a subspace of* PG(6, q)$\backslash\Sigma_\infty$ to mean a subspace of PG(6, q) that is not contained in Σ_∞. Consider the following incidence structure: the *points* of $\mathcal{A}(\mathcal{S})$ are the points of PG(6, q)$\backslash\Sigma_\infty$; the *lines* of $\mathcal{A}(\mathcal{S})$ are the 3-spaces of PG(6, q)$\backslash\Sigma_\infty$ that contain an element of $\mathcal{S}$; and *incidence* in $\mathcal{A}(\mathcal{S})$ is induced by incidence in PG(6, q)$\backslash\Sigma_\infty$. Then the incidence structure $\mathcal{A}(\mathcal{S})$ is an affine plane of order q^3. We can complete $\mathcal{A}(\mathcal{S})$ to a projective plane $\mathcal{P}(\mathcal{S})$; the points on the line at infinity ℓ_∞ have a natural correspondence to the elements of the 2-spread $\mathcal{S}$. The projective plane $\mathcal{P}(\mathcal{S})$ is the Desarguesian plane PG(2, q^3) if and only if $\mathcal{S}$ is a regular 2-spread of $\Sigma_\infty \cong$ PG(5, q) (see [Bruck 1969]). For the remainder of this article, we use $\mathcal{S}$ to denote a regular 2-spread of $\Sigma_\infty \cong$ PG(5, q).

We use the following notation. If T is a point of ℓ_∞ in PG(2, q^3), we use $[T]$ to refer to the plane of $\mathcal{S}$ corresponding to T. More generally, if X is a set of points of PG(2, q^3), then we let $[X]$ denote the corresponding set in PG(6, q). If P is an affine point of PG(2, q^3), we generally simplify the notation and also use P to refer to the corresponding affine point in PG(6, q), although in some cases, to avoid confusion, we use $[P]$.

When $\mathcal{S}$ is a regular 2-spread, we can relate the coordinates of $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2, q^3)$ and $\mathrm{PG}(6, q)$ as follows. Let τ be a primitive element in $\mathbb{F}_{q^3}$ with primitive polynomial $x^3 - t_2x^2 - t_1x - t_0$. Every element $\alpha \in \mathbb{F}_{q^3}$ can be uniquely written as $\alpha = a_0 + a_1\tau + a_2\tau^2$ with $a_0, a_1, a_2 \in \mathbb{F}_q$. Points in $\mathrm{PG}(2, q^3)$ have homogeneous coordinates (x, y, z) with $x, y, z \in \mathbb{F}_{q^3}$, not all zero. Let the line at infinity ℓ_∞ have equation $z = 0$; so the affine points of $\mathrm{PG}(2, q^3)$ have coordinates $(x, y, 1)$. Points in $\mathrm{PG}(6, q)$ have homogeneous coordinates $(x_0, x_1, x_2, y_0, y_1, y_2, z)$ with $x_0, x_1, x_2, y_0, y_1, y_2, z \in \mathbb{F}_q$. Let Σ_∞ have equation $z = 0$. Let $P = (\alpha, \beta, 1)$ be a point of $\mathrm{PG}(2, q^3)$. We can write $\alpha = a_0 + a_1\tau + a_2\tau^2$ and $\beta = b_0 + b_1\tau + b_2\tau^2$ with $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{F}_q$. We want to map the element α of $\mathbb{F}_{q^3}$ to the vector (a_0, a_1, a_2), and we use the following notation to do this:

$$[\alpha] = (a_0, a_1, a_2).$$

This gives some notation for the Bruck–Bose map, denoted ϵ, from an affine point $P = (\alpha, \beta, 1) \in \mathrm{PG}(2, q^3)\setminus\ell_\infty$ to the corresponding affine point $[P] \in \mathrm{PG}(6, q)\setminus\Sigma_\infty$, namely

$$\epsilon(\alpha, \beta, 1) = [(\alpha, \beta, 1)] = ([\alpha], [\beta], 1) = (a_0, a_1, a_2, b_0, b_1, b_2, 1).$$

More generally, if $z \in \mathbb{F}_q$, then $\epsilon(\alpha, \beta, z) = ([\alpha], [\beta], z) = (a_0, a_1, a_2, b_0, b_1, b_2, z)$.

Consider the case when $z = 0$, that is, a point on ℓ_∞ in $\mathrm{PG}(2, q^3)$ has coordinates $L = (\alpha, \beta, 0)$ for some $\alpha, \beta \in \mathbb{F}_{q^3}$. In $\mathrm{PG}(6, q)$, the point $\epsilon(\alpha, \beta, 0) = ([\alpha], [\beta], 0)$ is one point in the spread element $[L]$ corresponding to L. Moreover, the spread element $[L]$ consists of all the points $\{([\alpha x], [\beta x], 0) : x \in \mathbb{F}'_{q^3}\}$. Hence the regular 2-spread $\mathcal{S}$ consists of the planes $\{[kx], [x], 0] : x \in \mathbb{F}'_{q^3}\}$ for $k \in \mathbb{F}_{q^3} \cup \{\infty\}$.

With this coordinatisation for the Bruck–Bose map, we can calculate the coordinates of the transversal lines of the regular 2-spread $\mathcal{S}$.

Lemma 2.1 [Barwick and Jackson 2012]. *Let* $p_0 = t_1 + t_2\tau - \tau^2 = -\tau^q\tau^{q^2}$, $p_1 = t_2 - \tau = \tau^q + \tau^{q^2}$, $p_2 = -1$, *and* $A = (p_0, p_1, p_2)$. *Then in the cubic extension* $\mathrm{PG}(6, q^3)$, *one transversal line of the regular* 2*-spread* $\mathcal{S}$ *contains the two points* $A_1 = (p_0, p_1, p_2, 0, 0, 0, 0) = (A, [0], 0)$ *and* $A_2 = (0, 0, 0, p_0, p_1, p_2, 0) = ([0], A, 0)$.

2B. ***Some useful homographies.*** In order to simplify the notation in some of the following coordinate-based proofs, we define some homographies which will be useful. We can represent an element $x = x_0 + x_1\tau + x_2\tau^2 \in \mathbb{F}_{q^3}$ as a point $[x] = (x_0, x_1, x_2)$ in $\mathrm{PG}(2, q)$. For $k \in \mathbb{F}'_{q^3}$, consider the homography ζ_k in $\mathrm{PGL}(3, q)$ with matrix M_k that maps $[x]$ to $[kx]$. Let $k \in \mathbb{F}'_{q^3}$ and write $k = k_0 + k_1\tau + k_2\tau^2$, then $M_k = k_0M_1 + k_1M_\tau + k_2M_{\tau^2}$, and hence

$$M_kA = kA \quad \text{and} \quad M_kA^{q^2} = k^{q^2}A^{q^2}, \tag{1}$$

where $A = (p_0, p_1, p_2)^t$ is defined in Lemma 2.1. We use ζ_k to define the homography θ_k of PG(5, q), $k \in \mathbb{F}_{q^3}$:

$$\theta_k\colon ([x], [y]) \to ([kx], [y]) = (M_k[x], [y]).$$

From the matrix M_τ, we construct three more homographies of PG(2, q) with matrices U_0, U_1, U_2 that help with the notation in the proof of Theorem 7.4. For $i = 0, 1, 2$, (with p_i as in Lemma 2.1), let

$$U_i = (p_0 I + p_1 M_\tau + p_2 M_\tau^2)^{q^i} = \begin{pmatrix} p_0^{q^i} & \tau^{q^i} p_0^{q^i} & \tau^{2q^i} p_0^{q^i} \\ p_1^{q^i} & \tau^{q^i} p_1^{q^i} & \tau^{2q^i} p_1^{q^i} \\ p_2^{q^i} & \tau^{q^i} p_2^{q^i} & \tau^{2q^i} p_2^{q^i} \end{pmatrix}.$$

Then

$$U_i \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = (a_0 + a_1 \tau^{q^i} + a_2 \tau^{2q^i}) \begin{pmatrix} p_0^{q^i} \\ p_1^{q^i} \\ p_2^{q^i} \end{pmatrix}, \quad a_0, a_1, a_2 \in \mathbb{F}_{q^3}.$$

Note that if $a_0, a_1, a_2 \in \mathbb{F}_q$, and $\alpha = a_0 + a_1\tau + a_2\tau^2$, then $[\alpha] = (a_0, a_1, a_2)^t$, and we write the matrix equation as $U_i[\alpha] = \alpha^{q^i} A^{q^i}$.

2C. ***Sublines in the Bruck–Bose representation.*** An *order-q-subplane* of PG(2, q^3) is a subplane of PG(2, q^3) of order q. Equivalently, it is an image of PG(2, q) under PGL(3, q^3). An *order-q-subline* of PG(2, q^3) is a line of an order-q-subplane of PG(2, q^3). An *order-q-subline* of PG(1, q^3) is defined to be one of the images of PG(1, q) = $\{(a, 1) : a \in \mathbb{F}_q\} \cup \{(1, 0)\}$ under PGL(2, q^3).

In [Barwick and Jackson 2012; 2014], we determine the representation of order-q-subplanes and order-q-sublines of PG(2, q^3) in the Bruck–Bose representation in PG(6, q), and we quote the results for order-q-sublines which are needed in this article. We first introduce some terminology to simplify the statements. Recall that $\mathcal{S}$ is a regular 2-spread in the hyperplane at infinity Σ_∞ in PG(6, q).

Definition 2.2. (i) An *$\mathcal{S}$-special conic* is a nondegenerate conic $\mathcal{C}$ contained in a plane of $\mathcal{S}$, such that the extension of $\mathcal{C}$ to PG(6, q^3) meets the transversals of $\mathcal{S}$.

(ii) An *$\mathcal{S}$-special twisted cubic* is a twisted cubic $\mathcal{N}$ in a 3-space of PG(6, q)$\setminus\Sigma_\infty$ about a plane of $\mathcal{S}$, such that the extension of $\mathcal{N}$ to PG(6, q^3) meets the transversals of $\mathcal{S}$.

Theorem 2.3 [Barwick and Jackson 2012]. *Let b be an order-q-subline of* PG(2, q^3).

(i) *If $b \subset \ell_\infty$, then in* PG(6, q), *b corresponds to a* 2*-regulus of $\mathcal{S}$. Conversely every* 2*-regulus of $\mathcal{S}$ corresponds to an order-q-subline of ℓ_∞.*

(ii) *If b meets ℓ_∞ in a point, then b corresponds to a line of* $\mathrm{PG}(6,q)\backslash\Sigma_\infty$. *Conversely every line of* $\mathrm{PG}(6,q)\backslash\Sigma_\infty$ *corresponds to an order-q-subline of* $\mathrm{PG}(2,q^3)$ *tangent to ℓ_∞.*

(iii) *If b is disjoint from ℓ_∞, then in* $\mathrm{PG}(6,q)$, *b corresponds to an $\mathcal{S}$-special twisted cubic. Further, a twisted cubic $\mathcal{N}$ of* $\mathrm{PG}(6,q)$ *corresponds to an order-q-subline of* $\mathrm{PG}(2,q^3)$ *if and only if $\mathcal{N}$ is $\mathcal{S}$-special.*

In [Barwick and Jackson 2012], we also determine the representation of secant and tangent order-q-subplanes of $\mathrm{PG}(2,q^3)$ in $\mathrm{PG}(6,q)$. The representation of an exterior order-q-subplane in $\mathrm{PG}(6,q)$ is more complex to describe. One of the motivations of this work is to investigate this representation in more detail. Some aspects of the representation are discussed in more detail in Section 4.

2D. *Properties of exterior splashes.* We need some group theoretic results about order-q-subplanes and exterior splashes; the first appears in [Barwick and Jackson 2016].

Theorem 2.4. *Let $G=\mathrm{PGL}(3,q^3)$ be the collineation group acting on $\mathrm{PG}(2,q^3)$. The subgroup G_ℓ fixing a line ℓ is transitive on the order-q-subplanes that are exterior to ℓ, and is transitive on the exterior splashes of ℓ.*

This theorem can be proved by generalising the arguments in [Barwick and Jackson 2015]. In particular, it involves looking at two important subgroups of G. The first subgroup fixes an order-q-subplane, and the following property will be very useful.

Theorem 2.5. *The group $K=\mathrm{PGL}(3,q^3)_\pi$ acting on $\mathrm{PG}(2,q^3)$ and fixing an order-q-subplane π is transitive on the points of π.*

The second important subgroup is $I=G_{\pi,\ell}$ which fixes an order-q-subplane π, and a line ℓ exterior to π. By [Barwick and Jackson 2016], I fixes exactly three lines: ℓ, and its conjugates m,n with respect to π; and I fixes exactly three points: $E_1=\ell\cap m$, $E_2=\ell\cap n$, $E_3=m\cap n$, which are conjugate with respect to π. Further I identifies two fixed points $E_1=\ell\cap m$, $E_2=\ell\cap n$ on ℓ which are called the *carriers* of the exterior splash $\mathbb{S}$ of π. This is consistent with the definition of carriers of a circle geometry $\mathrm{CG}(3,q)$; see [Barwick and Jackson 2016]. In [Lunardon et al. 2014], scattered linear sets of pseudoregulus type are considered, and they use the term "transversal points". The fixed points and fixed lines of I are used to define an important class of conics in an order-q-subplane π with respect to an exterior line ℓ. A conic of π whose extension to $\mathrm{PG}(2,q^3)$ contains the three points E_1,E_2,E_3 is called a (π,ℓ)-*carrier conic* of π. A dual conic of π whose extension to $\mathrm{PG}(2,q^3)$ contains the three lines ℓ,m,n is called a (π,ℓ)-*carrier-dual conic*. Note that carrier-conics/dual conics were called special-conics/dual conics in [Barwick and Jackson 2016]; we change the name here so that the term "special" is reserved for objects in PG(6, q).

3. Coordinatising an exterior order-q-subplane

Recall from Theorem 2.4 that the group of homographies of PG(2, q^3) is transitive on pairs (π, ℓ) where π is an order-q-subplane exterior to the line ℓ. So if we want to use coordinates to prove a result about exterior order-q-subplanes, we can without loss of generality prove it for a particular exterior order-q-subplane. In this section we calculate the coordinates for an exterior order-q-subplane $\mathcal{B}$ of PG(2, q^3) whose exterior splash has a simple form. Set

$$K = \begin{pmatrix} -\tau & 1 & 0 \\ -\tau^q & 1 & 0 \\ \tau\tau^q & -\tau-\tau^q & 1 \end{pmatrix}, \qquad K' = \begin{pmatrix} -1 & 1 & 0 \\ -\tau^q & \tau & 0 \\ -\tau^{2q} & \tau^2 & \tau-\tau^q \end{pmatrix}. \tag{2}$$

Let σ be the homography of PG(2, q^3) with matrix K. Note that as KK' is a $\mathbb{F}_{q^3}$-multiple of the identity matrix, it follows that K' is a matrix for the inverse homography σ^{-1}. Thus, if we write the points X of PG(2, q^3) as column vectors, and the lines ℓ of PG(2, q^3) as row vectors, then $\sigma(X) = KX$ and $\sigma(\ell) = \ell K'$. The order-q-subplane $\pi_0 = \mathrm{PG}(2, q)$ is secant to ℓ_∞. We show that the subplane $\sigma(\pi_0)$ is exterior to ℓ_∞ and has the desired simple form as exterior splash.

Theorem 3.1. *In* PG(2, q^3), *let* $\pi_0 = \mathrm{PG}(2, q)$, *let* σ *be the homography with matrix* K *given in* (2), *and let* $\mathcal{B} = \sigma(\pi_0)$. *Then* $\mathcal{B}$ *is an order-q-subplane exterior to* ℓ_∞ *with exterior splash* $\mathbb{S} = \{(k, 1, 0) : k \in \mathbb{F}_{q^3}, k^{q^2+q+1} = 1\}$ *and carriers* $E_1 = (1, 0, 0)$ *and* $E_2 = (0, 1, 0)$.

Proof. Note that σ maps $\pi_0 = \mathrm{PG}(2, q)$ to $\mathcal{B}$ and the line $\ell = [-\tau\tau^q, \tau+\tau^q, -1]$ to $\ell_\infty = [0, 0, 1]$. By [Barwick and Jackson 2016], π_0 is exterior to ℓ and has carriers $E = (1, \tau, \tau^2)$ and $E^q = (1, \tau^q, \tau^{2q})$ on ℓ. Hence $\mathcal{B}$ is exterior to ℓ_∞ and has carriers $\sigma(E) = (0, 1, 0)$ and $\sigma(E^q) = (1, 0, 0)$ on ℓ_∞. By considering the action of σ on the lines $[l, m, n]$ ($l, m, n \in \mathbb{F}_q$, not all zero) of π_0, we calculate the lines of $\mathcal{B}$ are $\ell_{l,m,n} = [-l - \tau^q m - \tau^{2q} n,\ l + \tau m + \tau^2 n,\ n(\tau - \tau^q)]$, with $l, m, n \in \mathbb{F}_q$, not all zero. The exterior splash of $\mathcal{B}$ consists of the points $Q_{l,m,n} = \ell_{l,m,n} \cap \ell_\infty = (l + \tau m + \tau^2 n,\ (l + \tau m + \tau^2 n)^q, 0)$. Writing $y = l + \tau m + \tau^2 n$, gives $Q_{l,m,n} \equiv (y, y^q, 0) \equiv (y^{1-q}, 1, 0)$ and writing $y = \tau^{-j}$ for some $j \in \{0, \dots, q^3 - 2\}$ yields $Q_{l,m,n} \equiv (\tau^{j(q-1)}, 1, 0)$. Note that if we write $j = n(q^2+q+1) + i$ where $0 \le i < q^2 + q + 1$, then $\tau^{j(q-1)} = \tau^{i(q-1)}$. So we may assume that $Q_{l,m,n} = (\tau^{i(q-1)}, 1, 0)$ with $0 \le i < q^2 + q + 1$. It is useful to observe that

$$\mathbb{S} = \{(k, 1, 0) : k \in \mathbb{F}_{q^3}, k^{q^2+q+1} = 1\} \equiv \{(\tau^{(q-1)i}, 1, 0) : 0 \le i < q^2 + q + 1\}$$

as the solutions to $k^{q^2+q+1} = 1$ are $\tau^{i(q-1)}$, $0 \le i < q^2 + q + 1$. $\square$

4. The structure of the subplane in PG(6, q)

If π is an exterior order-q-subplane of PG(2, q^3), then in the Bruck–Bose representation in PG(6, q), π corresponds to a set of q^2+q+1 affine points denoted $[\pi]$. It is difficult to characterise the structure of $[\pi]$. We note that as π contains q^2+q+1 order-q-sublines that are exterior to ℓ_∞, then by Theorem 2.3, $[\pi]$ contains q^2+q+1 $\mathcal{S}$-special twisted cubics, each lying in a 3-space through a distinct plane of the exterior splash of π. In this section we aim to determine more about the structure of $[\pi]$.

4A. ***The intersection of nine quadrics.*** We show that the structure $[\pi]$ of PG(6, q) corresponding to an exterior order-q-subplane π of PG(2, q^3) is the intersection of nine quadrics in PG(6, q). This is analogous to [Barwick and Jackson 2015, Theorem 9.2] which shows that a *tangent* order-q-subplane of PG(2, q^3) corresponds to a structure in PG(6, q) that is the intersection of nine quadrics.

Theorem 4.1. *Let π be an exterior order-q-subplane in* PG(2, q^3). *The corresponding set* $[\pi]$ *in* PG(6, q) *is the intersection of nine quadrics.*

Proof. By Theorem 2.4, we can without loss of generality prove this for the order-q-subplane $\mathscr{B}$ coordinatised in Section 3. We use the homographies σ, σ^{-1} with matrices K, K' respectively, given in (2). A point $P=(x, y, 1)\in$ PG(2, q^3) belongs to $\mathscr{B}$ if its preimage $K'P=(-x+y,\ -\tau^q x+\tau y,\ -\tau^{2q}x+\tau^2 y+(\tau-\tau^q))$ belongs to $\pi_0=$ PG(2, q). Suppose firstly that $-x+y\neq 0$, then

$$K'P \equiv \left(1,\ \frac{-\tau^q x+\tau y}{-x+y},\ \frac{-\tau^{2q}x+\tau^2 y+(\tau-\tau^q)}{-x+y}\right).$$

This belongs to $\pi_0=$ PG(2, q) if and only if the second and third coordinates belong to $\mathbb{F}_q$, that is,

$$\left(\frac{-\tau^q x+\tau y}{-x+y}\right)^q = \frac{-\tau^q x+\tau y}{-x+y}, \tag{3}$$

$$\left(\frac{-\tau^{2q}x+\tau^2 y+(\tau-\tau^q)}{-x+y}\right)^q = \frac{-\tau^{2q}x+\tau^2 y+(\tau-\tau^q)}{-x+y}. \tag{4}$$

Writing $x=x_0+x_1\tau+x_2\tau^2$ and $y=y_0+y_1\tau+y_2\tau^2$, where $x_i, y_i\in\mathbb{F}_q$ and $i=1,2,3$, then equating powers of $1, \tau, \tau^2$, yields three quadratic equations from each condition, a total of six, each of which represents a quadric in PG(6, q).

Secondly, suppose $-\tau^q x+\tau y\neq 0$, then

$$K'P \equiv \left(\frac{-x+y}{-\tau^q x+\tau y},\ 1,\ \frac{-\tau^{2q}x+\tau^2 y+(\tau-\tau^q)}{-\tau^q x+\tau y}\right).$$

As before, this lies in π_0 if and only if

$$\left(\frac{-x+y}{-\tau^q x+\tau y}\right)^q = \frac{-x+y}{-\tau^q x+\tau y}, \tag{5}$$

$$\left(\frac{-\tau^{2q} x+\tau^2 y+(\tau-\tau^q)}{-\tau^q x+\tau y}\right)^q = \frac{-\tau^{2q} x+\tau^2 y+(\tau-\tau^q)}{-\tau^q x+\tau y}, \tag{6}$$

leading to a further six quadrics in PG(6, q). The equations (3) and (5) give the same triple of quadrics. Hence the point P lies in $\mathcal{B}$ if and only if the point $[P]$ lies on a total of nine quadrics in PG(6, q). Finally, note that if both $-x+y=0$ and $-\tau^q x+\tau y=0$, then $x=y=0$ and the point P has coordinates (0, 0, 1). This satisfies all the quadratic equations from (3), (4), (6), and so in PG(6, q), $[P]$ lies on each of the nine quadrics. □

4B. ***Tangent planes at points of an exterior subplane.*** We now consider a point P lying in an exterior order-q-subplane π of PG(2, q^3). In the Bruck–Bose representation in PG(6, q), P corresponds to an affine point which we also denote by P. We show that in PG(6, q), there is a unique *tangent plane* $\mathcal{T}_P$ at P to the structure $[\pi]$. We show that there are two equivalent ways to define this tangent plane. Recall from Theorem 2.3 that the order-q-sublines of π correspond to twisted cubics in PG(6, q). Theorem 4.2 shows that we can define $\mathcal{T}_P$ by looking at the tangent lines at P to these twisted cubics. Then Theorem 4.3 shows that we can define $\mathcal{T}_P$ by looking at the tangent space of P with respect to the nine quadrics defined by $[\pi]$.

Theorem 4.2. *Let π be an exterior order-q-subplane of* PG(2, q^3), *and let P be a point of π. Label the lines of π through P by $\ell_0, \dots, \ell_q$. In* PG(6, q), *ℓ_i corresponds to a twisted cubic $[\ell_i]$. Let m_i be the unique tangent line to $[\ell_i]$ through P. Then the lines $m_0, \dots, m_q$ lie in a plane $\mathcal{T}_P$, called the* tangent plane *of $[\pi]$ at P.*

Proof. By Theorems 2.4 and 2.5, we can without loss of generality prove this for the order-q-subplane $\mathcal{B}$ coordinatised in Section 3, and the point $P=(0,0,1)$ of $\mathcal{B}$. First consider the order-q-subplane $\pi_0=\mathrm{PG}(2,q)$. The point $P=(0,0,1)$ lies in π_0, and the lines of π_0 through P have coordinates $\ell'_m=[m,1,0]$, $m\in\mathbb{F}_q\cup\{\infty\}$. Points on the line ℓ'_m distinct from P have coordinates $P'_x=(1,-m,x)$ for $x\in\mathbb{F}_q$. We map the plane π_0 to $\mathcal{B}$ using the homography σ with matrix K given in (2). As $\sigma(P)=P$, the lines of $\mathcal{B}$ through P are $\ell_m=\sigma(\ell'_m)$, $m\in\mathbb{F}_q\cup\{\infty\}$. Points on the line ℓ_m distinct from P have coordinates

$$P_x=\sigma(P'_x)=(-\tau-m,\ -\tau^q-m,\ \tau\tau^q+(\tau+\tau^q)m+x),$$

for $x\in\mathbb{F}_q$.

To convert this to a coordinate in PG(6, q), we need to multiply by an element of $\mathbb{F}_{q^3}$ so that the last coordinate lies in $\mathbb{F}_q$. Let $F(x)=\tau\tau^q+(\tau+\tau^q)m+x$ (the third coordinate in P_x). As $F(x)\in\mathbb{F}_{q^3}$, we have $F(x)^{q^2+q+1}\in\mathbb{F}_q$. So in PG(6, q), we have the point $P_x = ([-(\tau+m)F(x)^{q^2+q}],\ [-(\tau^q+m)F(x)^{q^2+q}],\ F(x)^{q^2+q+1})$.

By Theorem 2.3, the line ℓ_m of $\mathrm{PG}(2,q^3)$ corresponds to a twisted cubic $[\ell_m] = \{P_x : x \in \mathbb{F}_q\} \cup \{P\}$ of $\mathrm{PG}(6,q)$. Consider the unique tangent to $[\ell_m]$ through P, and let I_m be the intersection of this tangent with Σ_∞. We will show that the points I_m, $m \in \mathbb{F}_q \cup \{\infty\}$, form a line. To calculate the coordinates of I_m, we let $Q_x = PP_x \cap \Sigma_\infty$. To calculate $I_m = Q_\infty$, we use the homogeneous coordinate technique of dividing by the largest power of x, and then substituting $x = \infty$, that is, replacing $1/x$ by 0. We use the notation $\lim_{x\to\infty}$ to describe this technique.

$$\begin{aligned} I_m &= \lim_{x\to\infty} PP_x \cap \Sigma_\infty = \lim_{x\to\infty} ([-(\tau+m)F(x)^{q^2+q}], [-(\tau^q+m)F(x)^{q^2+q}], 0) \\ &= ([-(\tau+m)],\ -[\tau^q+m], 0). \end{aligned}$$

Hence the points I_m, $m \in \mathbb{F}_q \cup \{\infty\}$, form a line $\ell = \langle ([1],[1],0), ([\tau],[\tau^q],0)\rangle$ in Σ_∞. Hence the tangent lines $m_0, \ldots, m_q$ to the twisted cubics of $[\pi]$ through P form a plane $\mathcal{T}_P = \langle \ell, P\rangle$ through P, as required. □

Theorem 4.3. *Let π be an exterior order-q-subplane of* $\mathrm{PG}(2,q^3)$, *and let P be a point of π. In* $\mathrm{PG}(6,q)$, *consider the intersection of the tangent spaces at P of the nine quadrics corresponding to $[\pi]$. Then this intersection is equal to the tangent plane $\mathcal{T}_P$ of $[\pi]$ at P as defined in Theorem 4.2.*

Proof. By Theorems 2.4 and 2.5, we can without loss of generality prove this for the order-q-subplane $\mathcal{B}$ coordinatised in Section 3, and the point $P = (0,0,1)$ of $\mathcal{B}$. In $\mathrm{PG}(6,q)$, consider the nine quadrics corresponding to $[\mathcal{B}]$ which are given in equations (4), (5) and (6). We want to find the set of lines through P that meet each of these nine quadrics twice at P. Every line ℓ of $\mathrm{PG}(6,q)$ through P has the form $\ell = RP$ for some point $R = ([u],[v],0) \in \Sigma_\infty$, $u, v \in \mathbb{F}_{q^3}$. So the points of ℓ are of the form $P_s = P + sR = ([su],[sv],1)$ where $s \in \mathbb{F}_q$. Substituting the point P_s into the quadrics of (4) gives

$$(-\tau^{2q}su+\tau^{2q}sv+(\tau-\tau^q))^q(-su+sv) = (-\tau^{2q}su+\tau^2sv+(\tau-\tau^q))(-su+sv)^q.$$

This expression is a polynomial of degree two in s. The line $\ell = PR$ is tangent to the three quadrics of (4) if this expression has a repeated root $s=0$, that is, if the coefficient of s is equal to zero. That is,

$$(\tau-\tau^q)^q(-u+v) = (\tau-\tau^q)(-u+v)^q,$$

and so $k = (-u+v)/(\tau-\tau^q)$ is in $\mathbb{F}_q$. Rearranging gives $v = k(\tau-\tau^q)+u$. Substituting the point P_s into the quadrics of (5) gives no constraints. Substituting the point P_s into the quadrics of (6) and simplifying gives that the constraint $m = (-\tau^q u + \tau v)/(\tau-\tau^q)$ lies in $\mathbb{F}_q$, and so $v = (m(\tau-\tau^q)+\tau^q u)/\tau$. Equating this with the expression for v obtained from (4) gives $u = m - k\tau$, and so $v = m - k\tau^q$.

Hence the line $\ell = PR$ is tangent to all nine quadrics when R has form

$$R = ([u], [v], 0) = ([m - k\tau], [m - k\tau^q], 0) = m([1], [1], 0) - k([\tau], [\tau^q], 0).$$

Thus the tangent space to $[\mathcal{B}]$ at P is the plane through P and the line

$$\ell = \langle([1], [1], 0), ([\tau], [\tau^q], 0)\rangle$$

of Σ_∞. This is the same as the tangent plane $\mathcal{T}_P$ to $[\mathcal{B}]$ at P calculated in the proof of Theorem 4.2. □

5. Coordinatising the exterior splash and its covers

Let $\mathbb{S}$ be an exterior splash of PG$(1, q^3)$. In the Bruck–Bose representation, $\mathbb{S}$ corresponds to a set of q^2+q+1 planes of the regular 2-spread $\mathcal{S}$ in $\Sigma_\infty \cong$ PG$(5, q)$. To simplify the notation, we use the same symbol $\mathbb{S}$ to denote both the points of the exterior splash on ℓ_∞, and the planes of the exterior splash contained in $\mathcal{S}$. In [Barwick and Jackson 2016], we show that an exterior splash is projectively equivalent to a cover of the circle geometry CG$(3, q)$. Hence by Bruck [1973], there are two *switching sets* $\mathbb{X}$, $\mathbb{Y}$ for $\mathbb{S}$. That is, $\mathbb{X}$ and $\mathbb{Y}$ consist of q^2+q+1 planes each, such that the planes of the three sets $\mathbb{S}$, $\mathbb{X}$ and $\mathbb{Y}$ each cover the same set of points. Further, planes from different sets meet in unique points, and planes in the same set are disjoint. The three sets $\mathbb{S}$, $\mathbb{X}$, $\mathbb{Y}$ are called *hyper-reguli* in [Culbert and Ebert 2005; Ostrom 1993]. In this article, we call the families $\mathbb{X}$ and $\mathbb{Y}$ *covers* of the exterior splash $\mathbb{S}$.

In this section we take the order-q-subplane $\mathcal{B}$ coordinatised in Section 3, with exterior splash $\mathbb{S}$, and use [Ostrom 1993] to calculate the coordinates of the two covers of $\mathbb{S}$. We will characterise the two covers in terms of the subplane $\mathcal{B}$.

We call one cover of $\mathbb{S}$ the *tangent cover with respect to* $\mathcal{B}$, and denote it by $\mathbb{T}_\mathcal{B}$, or if there is only one subplane under consideration, we shorten this to $\mathbb{T}$. The nomenclature for tangent covers comes from Theorem 5.3 which shows that the tangent planes $\mathcal{T}_P$ of $[\mathcal{B}]$ meet Σ_∞ in lines that lie in distinct planes of the cover $\mathbb{T}$.

We call the other cover of $\mathbb{S}$ the *conic cover with respect to* $\mathcal{B}$, and denote it by $\mathbb{C}_\mathcal{B}$, or $\mathbb{C}$. The nomenclature for the conic cover comes from [Barwick and Jackson 2017] which shows that the planes in the cover $\mathbb{C}$ are related to the $(\mathcal{B}, \ell_\infty)$-carrier conics of $\mathcal{B}$.

A certain type of embedding is looked at in [Lavrauw et al. 2015]. Specialising their results to PG$(5, q)$, their embedding $\mathcal{Q}_{2,q}$ is equivalent to the set $\mathbb{S} \cup \mathbb{C} \cup \mathbb{T}$. They determine the collineation group stabilising $\mathcal{Q}_{2,q}$. In particular they demonstrate: a collineation of PG$(5, q)$ that fixes $\mathcal{Q}_{2,q}$ and permutes the families $\mathbb{S}$, $\mathbb{C}$, $\mathbb{T}$; and a collineation fixing $\mathcal{Q}_{2,q}$ that permutes the planes in each family. Further, [Lavrauw et al. 2015] determines the equation of $\mathcal{Q}_{2,q}$. In Lemma 5.1 we describe the homogeneous coordinates for the planes in $\mathbb{S}$, $\mathbb{C}$, $\mathbb{T}$ in the format we will work with, and in Lemma 5.2 we calculate the matrix of a homography that fixes the planes in $\mathbb{S}$, permutes the planes of $\mathbb{T}$, and permutes the planes of $\mathbb{C}$ (this is the map $\varphi_{0,0}(\tau,\tau)$ of [Lavrauw et al. 2015]).

Lemma 5.1. *Let $\mathbb{S}$ be the exterior splash of the exterior order-q-subplane $\mathcal{B}$ coordinatised in Section 3. Let $\mathcal{K}=\{k=\tau^{i(q-1)}:0\le i<q^2+q+1\}$. In* PG$(6,q)$, $\mathbb{S}$ *and its two covers $\mathbb{T}$, $\mathbb{C}$ have planes given by*

$$\mathbb{S}=\{[S_k]=\{([kx],[x],0):x\in\mathbb{F}'_{q^3}\}:k\in\mathcal{K}\},$$
$$\mathbb{T}=\{[T_k]=\{([kx],[x^q],0):x\in\mathbb{F}'_{q^3}\}:k\in\mathcal{K}\},$$
$$\mathbb{C}=\{[C_k]=\{([kx],[x^{q^2}],0):x\in\mathbb{F}'_{q^3}\}:k\in\mathcal{K}\}.$$

Proof. The points of ℓ_∞ in PG$(2,q^3)$ have coordinates $S_k=(k,1,0)$ for $k\in\mathbb{F}_{q^3}\cup\{\infty\}$. Hence in the Bruck–Bose representation of ℓ_∞ in $\Sigma_\infty\cong$ PG$(5,q)$, planes of the regular 2-spread $\mathcal{S}$ are given by $[S_k]=\{([kx],[x]):x\in\mathbb{F}'_{q^3}\}$, for $k\in\mathbb{F}_{q^3}\cup\{\infty\}$. Consider the homography β (of order 3) of $\Sigma_\infty\cong$ PG$(5,q)$ defined by

$$\beta:([x],[y])\to([x],[y^q]). \tag{7}$$

We consider the action of β on the planes of $[S_k]$. For each $k\in\mathbb{F}_{q^3}\cup\{\infty\}$, define the planes $[T_k]$, $[C_k]$ by $\beta([S_k])=[T_k]$ and $\beta([T_k])=[C_k]$. That is, $[T_k]=\{([kx],[x^q]):x\in\mathbb{F}'_{q^3}\}$, and $[C_k]=\{([kx],[x^{q^2}]):x\in\mathbb{F}'_{q^3}\}$.

We now consider the exterior order-q-subplane $\mathcal{B}$ coordinatised in Section 3 which by Theorem 3.1 has exterior splash $\mathbb{S}=\{S_k=(k,1,0):k\in\mathcal{K}\}\subset\ell_\infty$, and carriers $S_\infty=(1,0,0)$, $S_0=(0,1,0)$. Note that in PG$(5,q)$, the carriers of $\mathcal{B}$ lie in each of the three sets of planes, as $[S_0]=[T_0]=[C_0]$ and $[S_\infty]=[T_\infty]=[C_\infty]$. In PG$(5,q)$, we have $\mathbb{S}=\{[S_k]:k\in\mathcal{K}\}$. Let $\mathbb{T}=\{[T_k]:k\in\mathcal{K}\}$ and $\mathbb{C}=\{[C_k]:k\in\mathcal{K}\}$, then $\beta:\mathbb{S}\mapsto\mathbb{T}\mapsto\mathbb{C}$. By [Ostrom 1993], the sets $\mathbb{S}$, $\mathbb{T}$, $\mathbb{C}$ cover the same set of points. Moreover, planes in the same set are disjoint, and planes from different sets meet in one point. That is, $\mathbb{T}$ and $\mathbb{C}$ are the two covers of $\mathbb{S}$. □

The next lemma calculates the action of a useful homography of PG$(6,q)$ (this is the map $\varphi_{0,0}(\tau,\tau)$ of [Lavrauw et al. 2015]). Recall that τ is a zero of the primitive polynomial $x^3-t_2x^2-t_1x-t_0$.

Lemma 5.2. *Let $\mathbb{S}$ be the exterior splash of the exterior order-q-subplane $\mathcal{B}$ coordinatised in Section 3 with covers $\mathbb{C}$ and $\mathbb{T}$ coordinatised in Lemma 5.1. Consider the homography $\Theta\in$* PGL$(7,q)$ *with 7×7 matrix*

$$\begin{pmatrix} M & 0 & 0\\ 0 & M & 0\\ 0 & 0 & 1\end{pmatrix},\quad \text{where } M=\begin{pmatrix}0&0&t_0\\1&0&t_1\\0&1&t_2\end{pmatrix}.$$

Then in PG(6, q), Θ *fixes each plane of the regular* 2*-spread* $\mathcal{S}$, *maps the cover plane* $[C_k] \in \mathbb{C}$ *to* $[C_{\tau^{1-q}k}] \in \mathbb{C}$, *and the cover plane* $[T_k] \in \mathbb{T}$ *to* $[T_{\tau^{1-q^2}k}] \in \mathbb{T}$, $k \in \mathcal{K}$.

Proof. It is straightforward to show that Θ fixes the planes of the regular 2-spread $\mathcal{S}$ (so it also fixes the planes of the exterior splash $\mathbb{S}$). In fact $\langle\Theta\rangle$ acts regularly on the set of points, and on the set of lines, of each spread element. Note that M is the matrix M_τ defined in Section 2B, and so $M[x] = [\tau x]$. Consider the action of Θ on a point of the cover plane $[C_k] \in \mathbb{C}$ coordinatised in Lemma 5.1. We have

$$([kx], [x^{q^2}], 0)^\Theta = ([\tau kx], [\tau x^{q^2}], 0) \equiv ([\tau^{1-q} k(\tau^q x)], [(\tau^q x)^{q^2}], 0)$$

which lies in the cover plane $[C_{\tau^{1-q}k}]$ of $\mathbb{C}$. Similarly a point $([kx], [x^q], 0)$ in the cover plane $[T_k] \in \mathbb{T}$ maps under Θ to the point $([\tau^{1-q^2} k(\tau^{q^2} x)], [(\tau^{q^2} x)^q], 0)$ which lies in the cover plane $[T_{\tau^{1-q^2}k}]$ of $\mathbb{T}$. □

Theorem 5.3. *Let* P *be a point of an exterior order-*q*-subplane* π. *In* PG(6, q), *the tangent plane* $\mathcal{T}_P$ *at* P *to* $[\pi]$ *meets* Σ_∞ *in a line that lies in a plane of the tangent cover* $\mathbb{T}$ *of* $[\pi]$. *Moreover, distinct points of* π *correspond to distinct cover planes of* $\mathbb{T}$.

Proof. By Theorems 2.4 and 2.5, we can without loss of generality prove this result for the order-q-subplane $\mathcal{B}$ coordinatised in Section 3 and the point $P = (0, 0, 1) \in \mathcal{B}$. In PG(6, q), let $\mathcal{T}_P$ be the tangent plane at P. The line $\ell = \mathcal{T}_P \cap \Sigma_\infty$ was calculated in the proof of Theorem 4.2 to be

$$\ell = \{a([1], [1], 0) + b([\tau], [\tau^q], 0) : a, b \in \mathbb{F}_q\}.$$

The points of ℓ all lie in the plane $[T_1] = \{[x], [x^q], 0) \mid x \in \mathbb{F}'_{q^3}\}$, which by Lemma 5.1 is a plane of the tangent cover $\mathbb{T}$ of $\mathcal{B}$. The collineation of Lemma 5.2 is transitive on the cover planes of $\mathbb{T}$, hence each cover plane contains a line of a distinct tangent plane. Hence there is a one-to-one correspondence between points of π and cover planes of $\mathbb{T}$. □

6. Transversal lines of covers

Recall that a regular 2-spread in PG(5, q) has three (conjugate skew) transversals in PG(5, q^3) which meet each (extended) plane of $\mathcal{S}$. In this section we consider an exterior splash $\mathbb{S} \subset \mathcal{S}$, and show in Lemma 6.1 that the transversals of the 2-spread $\mathcal{S}$ are the only lines of PG(5, q^3) that meet every extended plane of $\mathbb{S}$. We then consider the two sets of cover planes $\mathbb{T}$ and $\mathbb{C}$. Corollary 6.2 shows that each can be uniquely extended to regular 2-spread, and we calculate the coordinates of the corresponding transversal lines in Theorem 6.3. Theorem 6.5 shows that the nine transversals of $\mathbb{S}$, $\mathbb{C}$ and $\mathbb{T}$ can be used to characterise the carriers of the exterior splash $\mathbb{S}$. Theorem 6.6, looks at the transversal lines in the situation when ℓ_∞ is partitioned into exterior splashes with common carriers.

6A. ***The exterior splash and its covers have unique transversals.*** If $\mathcal{X}$ is a set in PG(6, q) (such as a line, a plane, or a conic), then we denote its natural extension to PG(6, q^3) by $\mathcal{X}^*$. Let $\mathcal{S}$ be the regular 2-spread in Σ_∞ of the Bruck–Bose representation in PG(6, q). If we extend the planes of $\mathcal{S}$ to PG(6, q^3), yielding $\mathcal{S}^*$, then there are exactly three transversal lines to $\mathcal{S}^*$, that is, three lines that meet every plane of $\mathcal{S}^*$. These three lines are conjugate and skew. We now consider an exterior splash $\mathbb{S} \subset \mathcal{S}$ and extend the planes of $\mathbb{S}$ to PG(6, q^3), yielding $\mathbb{S}^*$. We show that there are exactly three lines of PG(6, q^3) that meet every plane of $\mathbb{S}^*$, namely the three transversals of $\mathcal{S}$.

Lemma 6.1. *Let $\mathcal{S}$ be a regular* 2*-spread in* PG(5, q), *and let $\mathbb{S} \subset \mathcal{S}$ be an exterior splash. In the cubic extension* PG(5, q^3), *there are exactly three transversals to $\mathbb{S}$, namely the three transversals of $\mathcal{S}$. Hence $\mathbb{S}$ lies in a unique regular* 2*-spread, namely $\mathcal{S}$.*

Proof. The three conjugate transversal lines of the regular 2-spread $\mathcal{S}$, denoted $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}$, are also transversals of $\mathbb{S}$. Suppose there is a fourth transversal line ℓ of $\mathbb{S}$. Then the four lines $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}, \ell$ are pairwise skew. Further, these four lines are ruling lines of a unique 2-regulus $\mathcal{R}$ of $\Sigma_\infty^* \cong$ PG(5, q^3), which contains the set of extended planes $\mathbb{S}^*$. Now consider two planes $[L], [M] \in \mathbb{S}$; the corresponding points L, M of ℓ_∞ in PG(2, q^3) lie in two order-q-sublines contained in $\mathbb{S}$ by [Lavrauw and Van de Voorde 2010, Corollary 15]. Hence by Theorem 2.3, $[L], [M]$ lie in two 2-reguli $\mathcal{R}_1, \mathcal{R}_2$ which are contained in $\mathbb{S}$. Let P be a point in $[L]$, then there are unique lines m_1, m_2 through P that are ruling lines of $\mathcal{R}_1, \mathcal{R}_2$ respectively. Now $\mathcal{R}_1$, $\mathcal{R}_2$ lie in $\mathbb{S}$, and so lie in $\mathcal{R}$, so the extended lines m_i^*, $i = 1, 2$, are two ruling lines of $\mathcal{R}$ that meet in a point P, a contradiction. Hence the line ℓ cannot exist. That is, there are only three transversal lines to $\mathbb{S}$, and these are necessarily the transversals of $\mathcal{S}$. □

As $\mathbb{S}$, $\mathbb{C}$, $\mathbb{T}$ are projectively equivalent by [Lavrauw et al. 2015, Theorem 16], an analogous result holds for the two covers of $\mathbb{S}$.

Corollary 6.2. *In* PG(5, q), *let $\mathbb{S}$ be an exterior splash with covers $\mathbb{T}$ and $\mathbb{C}$. Then in the cubic extension* PG(5, q^3),

(i) *the cover $\mathbb{T}$ has exactly three transversal lines in* PG(5, q^3)\PG(5, q), *denoted $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$, and so $\mathbb{T}$ lies in a unique regular* 2*-spread,*

(ii) *the cover $\mathbb{C}$ has exactly three transversal lines in* PG(5, q^3)\PG(5, q), *denoted $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$, and so $\mathbb{C}$ lies in a unique regular* 2*-spread.*

Later we will need the coordinates of the point of intersection of the transversal lines with the corresponding cover planes, and we calculate these next.

Theorem 6.3. *Let $\mathcal{B}$ be the order-q-subplane coordinatised in Section 3 with exterior splash $\mathbb{S}$ and covers $\mathbb{C}, \mathbb{T}$. Let $p_0 = t_1 + t_2\tau - \tau^2 = -\tau^q\tau^{q^2}$, $p_1 = t_2 - \tau =$*

$\tau^q + \tau^{q^2}$, $p_2 = -1$, *and* $\eta = p_0 + p_1\tau + p_2\tau^2$. *Let* $A_1 = (p_0, p_1, p_2, 0, 0, 0, 0)$, $A_2 = (0, 0, 0, p_0, p_1, p_2, 0)$. *Then in* PG(6, q^3),

(i) *one transversal line of* $\mathbb{S}$ *is* $g_{\mathbb{S}} = \langle A_1, A_2\rangle$, *and* $g_{\mathbb{S}} \cap [S_k]^* = kA_1 + A_2$,

(ii) *one transversal line of* $\mathbb{T}$ *is* $g_{\mathbb{T}} = \langle A_1, A_2^{q^2}\rangle$, *and* $g_{\mathbb{T}} \cap [T_k]^* = kA_1 + \eta^{1-q^2} A_2^{q^2}$,

(iii) *one transversal line of* $\mathbb{C}$ *is* $g_{\mathbb{C}} = \langle A_1, A_2^{q}\rangle$, *and* $g_{\mathbb{C}} \cap [C_k]^* = kA_1 + \eta^{1-q} A_2^{q}$.

Proof. We use the coordinatisation in PG(5, q) of the exterior splash $\mathbb{S}$ of $\mathcal{B}$ and the two covers $\mathbb{T}$, $\mathbb{C}$ given in Lemma 5.1. Lemma 2.1 shows that $g_{\mathbb{S}} = \langle A_1, A_2\rangle$ is a transversal line for the regular 2-spread $\mathcal{S}$, where $A_1 = (p_0, p_1, p_2, 0, 0, 0) = (A, [0])$ and $A_2 = (0, 0, 0, p_0, p_1, p_2) = ([0], A)$. Hence $g_{\mathbb{S}} = \langle A_1, A_2\rangle$ is a transversal line for the exterior splash $\mathbb{S}$. The planes of the regular 2-spread $\mathcal{S}$ are $[S_k] = \{([kx], [x]) : x \in \mathbb{F}'_{q^3}\}$, $k \in \mathbb{F}_{q^3} \cup \{\infty\}$. We first show that the extended plane $[S_k]^*$ meets the line $g_{\mathbb{S}}$ in the point $kA_1 + A_2$. Consider the point $P = p_0([k], [1]) + p_1([k\tau], [\tau]) + p_2([k\tau^2], [\tau^2])$ of PG(5, q^3), and note that $P \in [S_k]^*$. Using the matrix M_k defined in Section 2B, we have

$$P = p_0(M_k[1], [1]) + p_1(M_k[\tau], [\tau]) + p_2(M_k[\tau^2], [\tau^2]) = (M_k A, A) = (kA, A)$$

by (1). Hence $P = kA_1 + A_2$ which lies in $g_{\mathbb{S}} = \langle A_1, A_2\rangle$, that is, P is the intersection of $g_{\mathbb{S}}$ and $[S_k]^*$ proving part (i).

Consider the homography β defined in (7), acting on PG(5, q^3). The proof of Lemma 5.1 shows that β maps $g_{\mathbb{S}}$ to $g_{\mathbb{T}}$, and maps $g_{\mathbb{T}}$ to $g_{\mathbb{C}}$. Each element $y \in \mathbb{F}'_{q^3}$ can be considered as a point $[y]$ in PG(2, q). The collineation of PG(2, q) mapping the point $[y]$ to $[y^q]$ is a homography, and can be represented using a matrix N with entries in $\mathbb{F}_q$. We omit the transpose notation, and write $N[y] = [y^q]$. Hence we can write the collineation β as $\beta([x], [y]) = ([x], N[y])$. Clearly $\beta(A_1) = A_1$, and we show that $\beta(A_2) = A_2^{q^2}$. Recall the point $A = (p_0, p_1, p_2) = p_0[1] + p_1[\tau] + p_2[\tau^2]$, so $NA = p_0[1] + p_1[\tau^q] + p_2[\tau^{2q}]$. Using the matrix M_k from Section 2B, it is straightforward to write this as $NA = (p_0^{q^2} I + p_1^{q^2} M_\tau + p_2^{q^2} M_{\tau^2})^q\,[1]$. Now

$$(p_0^{q^2} I + p_1^{q^2} M_\tau + p_2^{q^2} M_{\tau^2})[1] = A^{q^2} \quad \text{and} \quad (p_0^{q^2} I + p_1^{q^2} M_\tau + p_2^{q^2} M_{\tau^2})A^{q^2} = \eta^{q^2} A^{q^2}$$

by (1). So repeated use of (1) yields $NA = \eta^{q^2(q-1)} A^{q^2} = \eta^{1-q^2} A^{q^2}$. Further, as N is over $\mathbb{F}_q$, we have

$$NA = \eta^{1-q^2} A^{q^2}, \quad NA^q = \eta^{q-1} A, \quad NA^{q^2} = \eta^{q^2-q} A^q. \tag{8}$$

Hence $\beta(kA_1 + A_2) = kA_1 + \eta^{1-q^2} A_2^{q^2}$. As $\beta : g_{\mathbb{S}} \mapsto g_{\mathbb{T}}$, we have $g_{\mathbb{T}} \cap [T_k]^* = kA_1 + \eta^{1-q^2} A_2^{q^2}$ and $g_{\mathbb{T}} = \langle A_1, A_2^{q^2}\rangle$, proving part (ii). Similarly, calculating

$$\beta(kA_1 + \eta^{1-q^2} A_2^{q^2}) = kA_1 + \eta^{1-q^2+q^2-q} A_2^q = kA_1 + \eta^{1-q} A_2^q,$$

and using $\beta : g_{\mathbb{T}} \mapsto g_{\mathbb{C}}$, we get $g_{\mathbb{C}} \cap [C_k]^* = kA_1 + \eta^{1-q} A_2^q$ and $g_{\mathbb{C}} = \langle A_1, A_2^q\rangle$. □

We can use the transversals of the covers $\mathbb{T}$ and $\mathbb{C}$ to generalise the notion of $\mathcal{S}$-special conics and twisted cubics in $\mathrm{PG}(6,q)$ defined in Definition 2.2. We define $\mathbb{C}$-special here, $\mathbb{T}$-special is similarly defined.

Definition 6.4. (i) A $\mathbb{C}$-*special conic* is a nondegenerate conic $\mathcal{C}$ contained in a plane of $\mathbb{C}$, such that the extension of $\mathcal{C}$ to $\mathrm{PG}(6,q^3)$ meets the transversals of $\mathbb{C}$.

(ii) A $\mathbb{C}$-*special twisted cubic* is a twisted cubic $\mathcal{N}$ in a 3-space of $\mathrm{PG}(6,q)\backslash\Sigma_\infty$ about a plane of $\mathbb{C}$, such that the extension of $\mathcal{N}$ to $\mathrm{PG}(6,q^3)$ meets the transversals of $\mathbb{C}$.

6B. ***Characterising the carriers in*** $\mathbf{PG(6,}\,\boldsymbol{q}\mathbf{)}$**.** Letting $\mathcal{S}$ be a regular 2-spread of $\mathrm{PG}(5,q)$, and $\mathbb{S}$ be an exterior splash contained in $\mathcal{S}$, with covers $\mathbb{C}$ and $\mathbb{T}$, we can then characterise the carriers of $\mathbb{S}$ in terms of the nine transversals of $\mathbb{S}$, $\mathbb{C}$ and $\mathbb{T}$.

Theorem 6.5. *Let $\mathcal{S}$ be a regular 2-spread of* $\mathrm{PG}(5,q)$, *and let $\mathbb{S}\subset\mathcal{S}$ be an exterior splash with covers $\mathbb{C}$, $\mathbb{T}$, whose corresponding triples of transversal lines are $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}$, $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$, and $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$, respectively. Then the carriers of $\mathbb{S}$ are the only two planes of $\mathcal{S}$ whose extension to* $\mathrm{PG}(5,q^3)$ *meets all nine transversal lines.*

Proof. By Theorem 2.4, we can without loss of generality show this for the exterior splash $\mathbb{S}$ of the exterior order-q-subplane $\mathcal{B}$ coordinatised in Section 3, with carriers $E_1=(1,0,0)$, $E_2=(0,1,0)$. In $\mathrm{PG}(6,q)$, the transversal lines $g_{\mathbb{S}}, g_{\mathbb{S}}^q, g_{\mathbb{S}}^{q^2}$ each meet the carriers $[E_1]$, $[E_2]$ of $\mathbb{S}$. We use the notation for planes $[S_k]\in\mathcal{S}$, $[T_k]\in\mathbb{T}$ and $[C_k]\in\mathbb{C}$ from Lemma 5.1. By Corollary 6.2, in the cubic extension $\mathrm{PG}(5,q^3)$, the transversal lines $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$ meet each plane $[T_k]$, $k\in\mathbb{F}_{q^3}\cup\{\infty\}$; and the transversal lines $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$ meet each plane $[C_k]$, $k\in\mathbb{F}_{q^3}\cup\{\infty\}$. The carriers of $\mathbb{S}$ satisfy $[E_2]=[S_0]=[T_0]=[C_0]$ and $[E_1]=[S_\infty]=[T_\infty]=[C_\infty]$. Hence in the cubic extension $\mathrm{PG}(5,q^3)$, all nine transversal lines meet the carriers of $\mathbb{S}$.

We now show that no other plane of the regular 2-spread $\mathcal{S}$ meets all nine transversal lines. We use the homography with matrix M_k defined in Section 2B. A plane of the regular 2-spread $\mathcal{S}$ distinct from $[E_1]$, $[E_2]$ has the form $[S_k]=\{([kx],[x],0):x\in\mathbb{F}'_{q^3}\}$, for some $k\in\mathbb{F}'_{q^3}$. This plane is spanned by the three points

$$S_{0,k}=([k],[1],0)=(M_k(1,0,0),(1,0,0)),$$
$$S_{1,k}=([k\tau],[\tau],0)=(M_k(0,1,0),(0,1,0))$$
$$S_{2,k}=([k\tau^2],[\tau^2],0)=(M_k(0,0,1),(0,0,1)).$$

Hence the extension $[S_k]^*$ to $\mathrm{PG}(5,q^3)$ contains the points

$$S_{k,j}=c_0S_{0,j}+c_1S_{1,j}+c_2S_{2,j},$$

where $c_i \in \mathbb{F}_{q^3}$, not all zero. By Theorem 6.3, a general point X on the transversal line $g_{\mathbb{T}}$ has coordinates $X = rA_1 + A_2^{q^2} = (rp_0, rp_1, rp_2, p_0^{q^2}, p_1^{q^2}, p_2^{q^2})$, for some $r \in \mathbb{F}_{q^3} \cup \{\infty\}$. Now $S_{j,k} = X$ if and only if $c_i = p_i^{q^2}$, $i = 0, 1, 2$, and $M_k(c_0, c_1, c_2) = r(p_0, p_1, p_2)$. That is, $M_k A^{q^2} = rA$. However, $M_k A^{q^2} = k^{q^2} A^{q^2}$, by (1), so there are no solutions to c_0, c_1, c_2. Hence the transversal line $g_{\mathbb{T}}$ does not meet any further plane of the regular 2-spread $\mathcal{S}$, and so $g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$ do not meet any further plane of $\mathcal{S}$. A similar argument shows that the lines $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$ do not meet any further plane of the regular 2-spread $\mathcal{S}$. □

6C. *Transversal lines of exterior splashes with common carriers.* As exterior splashes are equivalent to covers of the circle geometry CG(3, q), there are $q-1$ disjoint exterior splashes on ℓ_∞ with common carriers E_1, E_2. We show that in PG(6, q), the covers of these disjoint exterior splashes have common transversals.

Theorem 6.6. *Let $\mathbb{S}_0, \dots, \mathbb{S}_{q-1}$ be $q-1$ disjoint exterior splashes on ℓ_∞ with common carriers E_1, E_2, and let exterior splash $\mathbb{S}_j$ have covers $\mathbb{C}_j$, $\mathbb{T}_j$. Then the covers $\mathbb{C}_0, \dots, \mathbb{C}_{q-1}$ have common transversal lines $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$, and the covers $\mathbb{T}_0, \dots, \mathbb{T}_{q-1}$ have common transversal lines $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$.*

Proof. By Theorem 2.4, we can without loss of generality prove this for the order-q-subplane $\mathcal{B}$ coordinatised in Section 3. Let $\mathcal{K} = \{k \in \mathbb{F}'_{q^3} : k^{q^2+q+1} = 1\} = \{k = \tau^{i(q-1)} : 0 \le i < q^2+q+1\}$. Recall that $\mathcal{B}$ has carriers $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$, and exterior splash $\mathbb{S}_0 = \{S_{k,0} = (k, 1, 0) : k \in \mathcal{K}\}$. Let $\mathcal{K}_j = \tau^j \mathcal{K}$, for $j = 0, \dots, q-2$, be the $q-1$ cosets of $\mathcal{K}$ in $\mathbb{F}'_{q^3}$. Let $\mathbb{S}_j = \{S_{k,j} = (k, 1, 0) : k \in \mathcal{K}_j\}$, $0 \le j \le q-2$. Consider the homography ξ acting on ℓ_∞ that maps the point $(x, y, 0)$ to $(\tau x, y, 0)$. Then ξ fixes E_1, E_2, maps $\mathbb{S}_j$ to $\mathbb{S}_{j+1}$ $(0 \le j \le q-3)$, and maps $\mathbb{S}_{q-2}$ to $\mathbb{S}_0$. Hence $\mathbb{S}_0, \dots, \mathbb{S}_{q-1}$ are the $q-1$ disjoint exterior splashes on ℓ_∞ with carriers $(1, 0, 0)$ and $(0, 1, 0)$.

In $\Sigma_\infty \cong \mathrm{PG}(5, q)$, we have planes $[S_{k,j}] = \{([kx], [x]) : x \in \mathbb{F}'_{q^3}\} \in \mathbb{S}$, and define the planes $[T_{k,j}] = \{([kx], [x^q]) : x \in \mathbb{F}'_{q^3}\}$, and $[C_{k,j}] = \{([kx], [x^{q^2}]) : x \in \mathbb{F}'_{q^3}\}$, for $k \in \mathcal{K}_j$. So $\mathbb{S}_j = \{[S_{k,j}],\ k \in \mathcal{K}_j\}$, and define $\mathbb{T}_j = \{[T_{k,j}], k \in \mathcal{K}_j\}$ and $\mathbb{C}_j = \{[C_{k,j}], k \in \mathcal{K}_j\}$. Note that $\mathbb{T}_0$, $\mathbb{C}_0$ are the covers of the exterior splash $\mathbb{S}_0$ of $\mathcal{B}$. Now consider the map θ_τ of PG(5, q) acting on Σ_∞ defined in Section 2B; it maps $\mathbb{S}_j$ to $\mathbb{S}_{j+1}$, $\mathbb{T}_j$ to $\mathbb{T}_{j+1}$, and $\mathbb{C}_j$ to $\mathbb{C}_{j+1}$. Hence $\mathbb{T}_j$ and $\mathbb{C}_j$ are covers for $\mathbb{S}_j$. By Theorem 6.3, the transversal line of $\mathbb{T}_0$ is $g_{\mathbb{T}} = \langle A_1, A_2^{q^2} \rangle$. Using (1), we see that the homography θ_τ fixes $g_{\mathbb{T}}$, and so $g_{\mathbb{T}}$ is a transversal for all $\mathbb{T}_j$. So $g_{\mathbb{T}}, g_{\mathbb{T}}^q, g_{\mathbb{T}}^{q^2}$ are transversal lines of $\mathbb{T}_j$ for each $j = 0, \dots, q-2$. Similarly, $g_{\mathbb{C}}, g_{\mathbb{C}}^q, g_{\mathbb{C}}^{q^2}$ are transversal lines of $\mathbb{C}_j$ for each $j = 0, \dots, q-2$. □

Remark 6.7. We can interpret this result using the terminology of [Culbert and Ebert 2005]. We can partition the planes of a regular 2-spread into $q-1$ disjoint hyper-reguli with common carriers. Each hyper-regulus has two replacement hyper-reguli, which correspond to our conic and tangent covers. If we replace all $q-1$

hyper-reguli of $\mathcal{S}$ with hyper-reguli of the *same type* (that is, all belonging to $\mathbb{C}$, or all belonging to $\mathbb{T}$), then the resulting 2-spread has transversals either $g_{\mathbb{C}}, g_{\mathbb{C}}^{q}, g_{\mathbb{C}}^{q^2}$ or $g_{\mathbb{T}}, g_{\mathbb{T}}^{q}, g_{\mathbb{T}}^{q^2}$, and so is regular. Hence the resulting André plane is Desarguesian. If we replace all the hyper-reguli of $\mathcal{S}$ with a combination of hyper-reguli from each type, then the resulting 2-spread is not regular, and so the resulting André plane is non-Desarguesian.

7. Sublines of an exterior splash

In this section we characterise the order-q-sublines of $\mathbb{S}$ with respect to the covers of $\mathbb{S}$ and their transversal lines.

7A. ***Background.*** Let π be an exterior order-q-subplane of $\mathrm{PG}(2, q^3)$ with exterior splash $\mathbb{S}$ on ℓ_∞. There are $2(q^2+q+1)$ order-q-sublines in an exterior splash which lie in two families of size q^2+q+1. These families are studied in [Lavrauw and Van de Voorde 2010; Barwick and Jackson 2016].

We first describe properties of the two families given in [Lavrauw and Van de Voorde 2010]; here the two families are called regular and irregular with respect to a plane in one of the covers. That is, let $\mathbb{S}$ be an exterior splash in $\mathrm{PG}(5, q)$, and let α be a plane that meets each plane of $\mathbb{S}$ in a point, so α lies in one of the covers $\mathbb{X}$ or $\mathbb{Y}$ of $\mathbb{S}$. In $\mathrm{PG}(2, q^3)$, let b be an $\mathbb{F}_q$-subline contained in $\mathbb{S}$, so by Theorem 2.3, in $\mathrm{PG}(6, q)$, $[b]$ is a 2-regulus. The subline b is called regular with respect to α if $\alpha \cap [b]$ is a line, otherwise b is irregular. Suppose α lies in the cover $\mathbb{X}$, and $\alpha \cap [b]$ is a line, then each plane in the cover $\mathbb{X}$ meets $[b]$ in a line, and each plane in the cover $\mathbb{Y}$ meets $[b]$ in a set of points which is not collinear. We adapt the phrases regular and irregular with respect to α in terms of the covers of $\mathbb{S}$. We say b is both $\mathbb{X}$-*regular* and $\mathbb{Y}$-*irregular* if each plane in $\mathbb{X}$ meets $[b]$ in a line. In particular, we note that if we start with a scattered $\mathbb{F}_q$-linear set of rank 3 of $\mathrm{PG}(1, q^3)$, then an $\mathbb{F}_q$-subline b contained in the linear set can be categorised as both regular and irregular (by choosing α in different covers).

In [Lunardon and Polverino 2004], it is shown that if $\mathbb{S}$ is an exterior splash of ℓ_∞ in $\mathrm{PG}(2, q^3)$, then there is an order-q-subplane β and point P such that $\mathbb{S}$ is the projection of β from P onto ℓ_∞. In [Barwick and Jackson 2016, Theorem 5.2], the projection and splash constructions are compared, and it is shown that in almost all cases, the projection and exterior splash of β are distinct. In [Lavrauw and Van de Voorde 2010], the two families of sublines of $\mathbb{S}$ are characterised in relation to a point P and subplane β which project $\mathbb{S}$: one family arises from projecting the sublines of β, the other arises from projecting certain conics of β. The latter family are described as irregular in [Lavrauw and Van de Voorde 2010], although it is not specified which cover these sublines are irregular with respect to.

Now we describe properties of the two families given in [Barwick and Jackson 2016]. Here the two families of order-q-sublines of $\mathbb{S}$ are characterised with respect to geometric objects of an exterior π with exterior splash $\mathbb{S}$. If A is a point of π, then the pencil of $q+1$ lines of π through A meets ℓ_∞ in an order-q-subline of $\mathbb{S}$, called a *π-pencil-subline* of $\mathbb{S}$. Recall from Section 2D that a (π, ℓ_∞)-carrier-dual conic of π is a dual conic that contains the three lines fixed by the subgroup I fixing π and ℓ. If Γ is a (π, ℓ_∞)-carrier-dual conic of π, then the lines of Γ meet ℓ_∞ in an order-q-subline of $\mathbb{S}$, called a *π-dual-conic-subline* of $\mathbb{S}$. Note that in [Barwick and Jackson 2016, Theorem 4.4], we show that it is possible to switch the roles of the two families by considering different associated order-q-subplanes.

7B. ***A characterisation of the sublines of an exterior splash.*** We now consider the interaction in PG(6, q) of the two families of order-q-sublines of $\mathbb{S}$ with the two covers of $\mathbb{S}$. We show in Theorem 7.1 that each family meets planes from one cover in lines, and planes from the other cover in conics. Theorem 7.2 shows that the converse is true, and so we have a characterisation of the order-q-sublines of $\mathbb{S}$. This allows us to relate the families from [Barwick and Jackson 2016] and [Lavrauw and Van de Voorde 2010]. Theorem 7.4 shows that the conics concerned in each case are special with respect to the conic cover.

Suppose $\mathcal{R}$ is a 2-regulus in PG(5, q), and consider a plane α that meets $\mathcal{R}$ in a set of $q+1$ points. Then an easy counting argument shows that these points form either a line or a conic in α. We abbreviate this to "$\mathcal{R}$ meets α in a line or a conic".

Theorem 7.1. *Let π be an exterior order-q-subplane with exterior splash $\mathbb{S}$, conic cover $\mathbb{C}$, and tangent cover $\mathbb{T}$.*

(i) *A π-pencil-subline of $\mathbb{S}$ corresponds in* PG(6, q) *to a 2-regulus that meets each plane of $\mathbb{T}$ in a distinct line, and meets each plane of $\mathbb{C}$ in a conic.*

(ii) *A π-dual-conic-subline of $\mathbb{S}$ corresponds in* PG(6, q) *to a 2-regulus that meets each plane of $\mathbb{T}$ in a conic, and meets each plane of $\mathbb{C}$ in a distinct line.*

Proof. Let P be a point in the exterior order-q-subplane π, and let d be the corresponding π-pencil-subline of $\mathbb{S}$. By Theorem 2.3, in PG(6, q), $[d]$ is a 2-regulus contained in $\mathbb{S}$. Consider the tangent plane $\mathcal{T}_P$ to $[\pi]$ at P. By Theorem 4.2, the lines of $\mathcal{T}_P$ through P meet Σ_∞ in points that lie in distinct planes of the 2-regulus $[d]$. Hence $\mathcal{T}_P \cap \Sigma_\infty$ is a ruling line of the 2-regulus $[d]$. By Theorem 5.3, this ruling line $\mathcal{T}_P \cap \Sigma_\infty$ lies in a tangent cover plane. The homography Θ of Lemma 5.2 fixes the planes of $[b]$ and is transitive on the cover planes of $\mathbb{T}$. Hence each ruling line of $[b]$ meets a unique cover plane of $\mathbb{T}$.

A straightforward geometric argument shows that planes of $\mathbb{T}, \mathbb{C}$ meet a 2-regulus of $\mathbb{S}$ in a line or a conic. Hence a conic cover plane meets the 2-regulus $[d]$ in a conic. As there are q^2+q+1 π-pencil-sublines of $\mathbb{S}$, every line in a plane of $\mathbb{T}$ is a ruling line for some 2-regulus corresponding to a π-pencil-subline. Hence if $[d']$ is a 2-regulus of $\mathbb{S}$ corresponding to a π-dual-conic-subline, then planes of $\mathbb{T}$ meet $[d']$ in conics, and so planes of $\mathbb{C}$ meet $[d']$ in ruling lines of $[d']$. Moreover, applying the homography of Lemma 5.2 shows that each ruling line of $[d']$ lies in a unique conic cover plane. □

By Theorem 2.3, there is a one-to-one correspondence between the order-q-sublines of $\mathbb{S}$ in $\mathrm{PG}(2, q^3)$, and the 2-reguli contained in $\mathbb{S}$ in $\mathrm{PG}(6, q)$. Hence the converse of Theorem 7.1 is also true, and so we have a characterisation of order-q-sublines of $\mathbb{S}$ relating to the cover planes of the associated order-q-subplane.

Theorem 7.2. *Let π be an exterior order-q-subplane with exterior splash $\mathbb{S}$, conic cover $\mathbb{C}$, and tangent cover $\mathbb{T}$.*

(i) *A 2-regulus contained in $\mathbb{S}$ that meets some plane of $\mathbb{T}$ in a line corresponds to a π-pencil-subline of $\mathbb{S}$.*

(ii) *A 2-regulus contained in $\mathbb{S}$ that meets some plane of $\mathbb{C}$ in a conic corresponds to a π-pencil-subline of $\mathbb{S}$.*

(iii) *A 2-regulus contained in $\mathbb{S}$ that meets some plane of $\mathbb{T}$ in a conic corresponds to a π-dual-conic-subline of $\mathbb{S}$.*

(iv) *A 2-regulus contained in $\mathbb{S}$ that meets some plane of $\mathbb{C}$ in a line corresponds to a π-dual-conic-subline of $\mathbb{S}$.*

This allows us to determine the relationship between the different family naming used in [Barwick and Jackson 2016] and [Lavrauw and Van de Voorde 2010].

Corollary 7.3. *Let π be an exterior order-q-subplane with exterior splash $\mathbb{S}$, conic cover $\mathbb{C}$, and tangent cover $\mathbb{T}$.*

(i) *Let b be a π-pencil-subline of $\mathbb{S}$, then b is $\mathbb{T}$-regular and $\mathbb{C}$-irregular.*

(ii) *Let d be a π-dual-conic-subline of $\mathbb{S}$, then d is $\mathbb{C}$-regular and $\mathbb{T}$-irregular.*

In fact, we can give a stronger characterisation of the order-q-sublines of $\mathbb{S}$, namely that the conics of Theorem 7.1 are *special* with respect to the associated cover. In order to prove that the conics are special, we need to introduce coordinates, and the proof is calculation intensive.

Theorem 7.4. *Let π be an exterior order-q-subplane with exterior splash $\mathbb{S}$, conic cover $\mathbb{C}$, and tangent cover $\mathbb{T}$.*

(i) *A 2-regulus of $\mathbb{S}$ corresponding to a π-pencil-subline of $\mathbb{S}$ meets each plane of $\mathbb{C}$ in a $\mathbb{C}$-special conic.*

(ii) *A 2-regulus of $\mathbb{S}$ corresponding to a π-dual-conic-subline of $\mathbb{S}$ meets each plane of $\mathbb{T}$ in a $\mathbb{T}$-special conic.*

Proof. By Theorem 2.4, we can without loss of generality prove this for the exterior order-q-subplane $\mathcal{B}$ coordinatised in Section 3. We start with the order-q-subplane $\pi_0 = \mathrm{PG}(2, q)$ and the line $\ell = [-\tau\tau^q, \tau + \tau^q, -1]$ which is exterior to π_0. Note that using the notation for p_0, p_1, p_2 given in Theorem 6.3, we have $\ell = [p_0^{q^2}, p_1^{q^2}, p_2^{q^2}]$. A line of π_0 has coordinates $[l, m, n]$ for $l, m, n \in \mathbb{F}_q$, and meets ℓ in the point $W'_{l,m,n} = (-n(\tau + \tau^q) - m,\ l - n\tau\tau^q,\ m\tau\tau^q + l(\tau + \tau^q))$. We apply the homography σ of Section 3 with matrix K to map π_0 and ℓ to $\mathcal{B}$ and ℓ_∞, respectively. The point $W'_{l,m,n}$ of ℓ maps to the point $W_{l,m,n} = KW'_{l,m,n} = (l + m\tau + n\tau^2,\ l + m\tau^q + n\tau^{2q},\ 0)$ of ℓ_∞. Writing $\varepsilon = \varepsilon_{l,m,n} = l + m\tau + n\tau^2$, we have $W_\varepsilon = W_{l,m,n} = (\varepsilon, \varepsilon^q, 0) \equiv (\varepsilon^{1-q}, 1, 0)$. Using the notation from Lemma 5.1, this is the point $S_{\varepsilon^{1-q}} \in \ell_\infty$. In PG(6, q), W_ε corresponds to the spread plane $[W_\varepsilon] = [W_{l,m,n}] = \{([\varepsilon x], [\varepsilon^q x], 0) \equiv ([\varepsilon^{1-q}x], [x], 0) : x \in \mathbb{F}'_{q^3}\} = [S_{\varepsilon^{1-q}}]$.

Fix a point $P = (a, b, c)$ of π_0, so $a, b, c, \in \mathbb{F}_q$, not all zero. Let

$$\mathcal{L} = \{(l, m, n) : l, m, n \in \mathbb{F}_q \text{ not all zero, and } la + mb + nc = 0\}.$$

The $q + 1$ lines of π_0 through P have coordinates $[l, m, n] \in \mathcal{L}$. These $q + 1$ lines meet the exterior line ℓ of π_0 in a π_0-pencil-subline which, under the collineation σ, maps to a $\mathcal{B}$-pencil-subline d of ℓ_∞. By Theorem 2.3, in PG(6, q), d corresponds to the 2-regulus $[d]$ which we denote by $\mathcal{R}$, so $\mathcal{R} = [d] = \{[W_\varepsilon] = [S_{\varepsilon^{1-q}}] : \varepsilon \in \mathcal{W}\}$, where $\mathcal{W} = \{\varepsilon = \varepsilon_{l,m,n} = l + m\tau + n\tau^2 : (l, m, n) \in \mathcal{L}\}$. For each $\alpha \in \mathbb{F}'_{q^3}$, consider the set of points $t_\alpha = \{([\varepsilon\alpha], [\varepsilon^q\alpha], 0) : \varepsilon \in \mathcal{W}\}$. As $\mathcal{W}$ is closed under addition, t_α is a line of $\Sigma_\infty \cong \mathrm{PG}(5, q)$; further t_α meets every plane in $\mathcal{R}$. Hence t_α is a ruling line of the 2-regulus $\mathcal{R}$.

By Theorem 7.2(ii), the 2-regulus $\mathcal{R}$ meets a cover plane of the conic cover $\mathbb{C}$ in a conic $\mathcal{C}_k = [C_k] \cap \mathcal{R}$ for $k \in \mathcal{K}$. To show that the conic $\mathcal{C}_k$ is $\mathbb{C}$-special, we need to extend it to PG(5, q^3), and show that it meets the three transversal lines of $\mathbb{C}$. To do this, we extend the 2-regulus $\mathcal{R}$ of $\Sigma_\infty \cong \mathrm{PG}(5, q)$ to a 2-regulus $\mathcal{R}^*$ of PG(5, q^3), so $\mathcal{C}^*_k = [C_k]^* \cap \mathcal{R}^*$. We then use coordinates to show that one of the planes of $\mathcal{R}^*$ contains the transversal line $g_{\mathbb{C}}^{q^2}$ of $\mathbb{C}$, and then deduce that $\mathcal{C}^*_k$ meets $g_{\mathbb{C}}^{q^2}$.

To extend $\mathcal{R}$ to a 2-regulus $\mathcal{R}^*$ of PG(5, q^3), we find four lines in PG(5, q^3) that meet each extended plane of $\mathcal{R}$. As a 2-regulus is uniquely determined by four ruling lines in general position, we can use these four lines to define the 2-regulus $\mathcal{R}^*$. The transversal line $g_{\mathbb{S}}$ of the regular 2-spread $\mathcal{S}$ can be used as one of our ruling lines; for the other three ruling lines, we use the extended lines t^*_1, t^*_τ, $t^*_{\tau^2}$, which each meet every plane of $\mathcal{R}$. So $\mathcal{R}^*$ is the 2-regulus of PG(5, q^3) determined by the four ruling lines t^*_1, t^*_τ, $t^*_{\tau^2}$, $g_{\mathbb{S}}$ (which are in general position), and further $\mathcal{R}^* \cap \Sigma_\infty = \mathcal{R}$.

We now exhibit a plane γ of $\mathcal{R}^*$ that contains the transversal line $g_{\mathbb{C}}^{q^2}$ of the conic cover $\mathbb{C}$. Extend the set $\mathcal{L}$ to

$$\mathcal{L}^* = \{(l, m, n) : l, m, n \in \mathbb{F}_{q^3} \text{ not all zero, and } la + mb + nc = 0\}.$$

We use the matrix M_τ defined in Section 2B, and write $M = M_\tau$. The ruling line $t^*_{\tau^i}$, $i = 0, 1, 2$, has points $P_{\tau^i,l,m,n}$ with $(l, m, n) \in \mathcal{L}^*$, where $P_{\tau^i,l,m,n} = l(M^i[1], M^i[1], 0) + m(M^i[\tau], M^i[\tau^q], 0) + n(M^i[\tau^2], M^i[\tau^{2q}], 0)$. Recall that the order-$q$-subline d corresponds to the fixed point $P = (a, b, c) \in \pi_0$. Consider the following $(l, m, n) \in \mathcal{L}^*$:

$$l = c\tau - b\tau^2, \quad m = a\tau^2 - c, \quad n = b - a\tau. \tag{9}$$

Note that for these l, m, n we have

$$l + m\tau + n\tau^2 = 0. \tag{10}$$

For l, m, n as in (9), consider the plane γ spanned by the three points $P_{1,l,m,n} \in t^*_1$, $P_{\tau,l,m,n} \in t^*_\tau$, $P_{\tau^2,l,m,n} \in t^*_{\tau^2}$. We first show that γ is a plane of the 2-regulus $\mathcal{R}^*$ by showing that the fourth ruling line $g_{\mathbb{S}}$ of $\mathcal{R}^*$ also meets γ. By Theorem 6.3, $g_{\mathbb{S}} = \langle A_1, A_2\rangle$, and we show that $g_{\mathbb{S}}$ meets γ by showing that the point A_2 lies in γ. With l, m, n given by (9), consider the point $F = p_0 P_{1,l,m,n} + p_1 P_{\tau,l,m,n} + p_2 P_{\tau^2,l,m,n}$ of γ. To simplify the notation, we use the point $A = (p_0, p_1, p_2)^t$, and matrix $U_0 = p_0 I + p_1 M + p_1 M^2$ defined in Section 2B, and note that $U_0[\alpha] = \alpha A$. We have

$$\begin{aligned} F &= (lU_0[1] + mU_0[\tau] + nU_0[\tau^2],\ lU_0[1] + mU_0[\tau^q] + nU_0[\tau^{2q}],\ 0) \\ &= (lA + m\tau A + n\tau^2 A,\ lA + m\tau^q A + n\tau^{2q} A,\ 0). \end{aligned}$$

By (10), $F \equiv ([0], A, 0) = A_2$, and by Lemma 2.1, $g_{\mathbb{S}} = \langle A_1, A_2\rangle$, so $F \in g_{\mathbb{S}} \cap \gamma$. That is, the four ruling lines $t^*_1, t^*_\tau, t^*_{\tau^2}, g_{\mathbb{S}}$ of the 2-regulus $\mathcal{R}^*$ all meet the plane γ, and so γ is a plane of $\mathcal{R}^*$.

We now show that the transversal line $g_{\mathbb{C}}^{q^2}$ of $\mathbb{C}$ lies in the plane γ of $\mathcal{R}^*$. Let $G = p_0^{q^2} P_{1,l,m,n} + p_1^{q^2} P_{\tau,l,m,n} + p_2^{q^2} P_{\tau^2,l,m,n}$, and note that $G \in \gamma$. We use the matrix

$$U_2 = p_0^{q^2} I + p_1^{q^2} M + p_1^{q^2} M^2$$

defined in Section 2B, and note that $U_2[\alpha] = \alpha^{q^2} A^{q^2}$, so we have

$$\begin{aligned} G &= (lU_2[1] + mU_2[\tau] + n^2U_2[\tau^2],\ lU_2[1] + mU_2[\tau^q] + nU_2[\tau^{2q}],\ 0) \\ &= (lA^{q^2} + m\tau^{q^2} A^{q^2} + n\tau^{2q^2} A^{q^2},\ lA^{q^2} + m\tau A^{q^2} + n\tau^2 A^{q^2},\ 0). \end{aligned}$$

By (10), $G \equiv (A^{q^2}, [0], 0) = A_1^{q^2}$, so γ contains the points $G = A_1^{q^2}$ and $F = A_2$. Hence by Theorem 6.3, γ contains the transversal line $g_{\mathbb{C}}^{q^2} = \langle A_1^{q^2}, A_2\rangle$ of $\mathbb{C}$.

We showed above that the 2-regulus $[d] = \mathcal{R}$ meets a cover plane $[C_i]$ of $\mathbb{C}$ in a conic $\mathcal{C}_i$. We want to show that $\mathcal{C}_i$ is a $\mathbb{C}$-special conic, that is, we want to show that in $\mathrm{PG}(6, q^3)$, the extended conic $\mathcal{C}_i^* = [C_i]^* \cap \mathcal{R}^*$ contains the three points $g_{\mathbb{C}} \cap [C_i]^*$, $g_{\mathbb{C}}^q \cap [C_i]^*$, $g_{\mathbb{C}}^{q^2} \cap [C_i]^*$. We have shown that the transversal line $g_{\mathbb{C}}^{q^2}$ of $\mathbb{C}$ lies in a plane γ of $\mathcal{R}^*$. As the extended cover plane $[C_i]^*$ meets the transversal line $g_{\mathbb{C}}^{q^2}$ in a unique point denoted P_i, we have

$$P_i = [C_i]^* \cap g_{\mathbb{C}}^{q^2} = [C_i]^* \cap \gamma \in [C_i]^* \cap \mathcal{R}^* = \mathcal{C}_i^*.$$

Hence $\mathcal{C}_i^*$ contains the point $g_{\mathbb{C}}^{q^2} \cap [C_i]^*$, and hence it also contains the conjugate points $g_{\mathbb{C}}^{q} \cap [C_i]^*$, $g_{\mathbb{C}} \cap [C_i]^*$. That is, the conic $\mathcal{C}_i = [C_i] \cap \mathcal{R}$ is a $\mathbb{C}$-special conic, completing the proof of part (i). As $\mathbb{C}$ and $\mathbb{T}$ are projectively equivalent by [Lavrauw et al. 2015, Theorem 16], part (ii) holds by symmetry. □

8. Conclusion

An investigation into the interaction between an exterior order-q-subplane π of PG(2, q^3), and its exterior splash on ℓ_∞ began in [Barwick and Jackson 2016]. The main focus of that paper was to show that exterior splashes are projectively equivalent to scattered $\mathbb{F}_q$-linear sets of rank 3, covers of circle geometries, Sherk sets of size q^2+q+1. Further, we investigated the geometric relationship between the order-q-sublines of $\mathbb{S}$ and the points of π. The current article focusses on using the Bruck–Bose representation in PG(6, q) to continue the study of exterior splashes, in particular their interplay with order-q-subplanes. The notion of special conics and special twisted cubics is closely tied with this interplay.

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Ruled quintic surfaces in PG(6, q)

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We look at a scroll of PG$(6, q)$ that uses a projectivity to rule a conic and a twisted cubic. We show this scroll is a ruled quintic surface $\mathcal{V}_2^5$, and study its geometric properties. The motivation in studying this scroll lies in its relationship with an $\mathbb{F}_q$-subplane of PG$(2, q^3)$ via the Bruck–Bose representation.

1. Introduction

In this article we consider a scroll of PG$(6, q)$ that rules a conic and a twisted cubic according to a projectivity. The motivation in studying this scroll lies in its relationship with an $\mathbb{F}_q$-subplane of PG$(2, q^3)$ via the Bruck–Bose representation as described in Section 3. In PG$(6, q)$, let $\mathcal{C}$ be a nondegenerate conic in a plane α; $\mathcal{C}$ is called the *conic directrix*. Let $\mathcal{N}_3$ be a twisted cubic in a 3-space Π_3 with $\alpha \cap \Pi_3 = \varnothing$; $\mathcal{N}_3$ is called the *twisted cubic directrix*. Let ϕ be a projectivity from the points of $\mathcal{C}$ to the points of $\mathcal{N}_3$. By this we mean that if we write the points of $\mathcal{C}$ and $\mathcal{N}_3$ using a nonhomogeneous parameter, so $\mathcal{C} = \{C_\theta = (1, \theta, \theta^2) \mid \theta \in \mathbb{F}_q \cup \{\infty\}\}$ and $\mathcal{N}_3 = \{N_\epsilon = (1, \epsilon, \epsilon^2, \epsilon^3) \mid \epsilon \in \mathbb{F}_q \cup \{\infty\}\}$, then $\phi \in \mathrm{PGL}(2, q)$ is a projectivity mapping $(1, \theta)$ to $(1, \epsilon)$. Let $\mathcal{V}$ be the set of points of PG$(6, q)$ lying on the $q+1$ lines joining each point of $\mathcal{C}$ to the corresponding point (under ϕ) of $\mathcal{N}_3$. These $q+1$ lines are called the *generators* of $\mathcal{V}$. As the two subspaces α and Π_3 are disjoint, $\mathcal{V}$ is not contained in a 5-space. We note that this construction generalises the ruled cubic surface $\mathcal{V}_2^3$ in PG$(4, q)$, a variety that has been well studied; see [Vincenti 1983].

We work with normal rational curves in PG$(6, q)$. Suppose that $\mathcal{N}$ is a normal rational curve that generates an i-dimensional space. Then we call $\mathcal{N}$ an *i-dim nrc*, and often use the notation $\mathcal{N}_i$. See [Hirschfeld and Thas 1991] for details on normal rational curves. As we will be looking at 5-dim nrcs contained in $\mathcal{V}$, we assume $q \geq 6$ throughout.

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This article studies the geometric structure of $\mathcal{V}$. In Section 2, we show that $\mathcal{V}$ is a variety $\mathcal{V}_2^5$ of order 5 and dimension 2, and that all such scrolls are projectively equivalent. Further, we show that $\mathcal{V}$ contains exactly $q+1$ lines and one nondegenerate conic. In Section 3, we describe the Bruck–Bose representation of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(6, q)$, and discuss how $\mathcal{V}$ corresponds to an $\mathbb{F}_q$-subplane of $\mathrm{PG}(2, q^3)$. We use the Bruck–Bose setting to show that $\mathcal{V}$ contains exactly q^2 twisted cubics, and that each can act as a directrix of $\mathcal{V}$. In Section 4, we count the number of 4- and 5-dim nrcs contained in $\mathcal{V}$. Further, we determine how 5-spaces meet $\mathcal{V}$, and count the number of 5-spaces of each intersection type. The main result is Theorem 4.8. In Section 5, we determine how 5-spaces meet $\mathcal{V}$ in relation to the regular 2-spread in the Bruck–Bose setting.

2. Simple properties of $\mathcal{V}$

Theorem 2.1. *Let $\mathcal{V}$ be a scroll of* $\mathrm{PG}(6, q)$ *that rules a conic and a twisted cubic according to a projectivity. Then $\mathcal{V}$ is a variety of dimension* 2 *and order* 5, *denoted $\mathcal{V}_2^5$ and called a ruled quintic surface. Further, any two ruled quintic surfaces are projectively equivalent.*

Proof. Let $\mathcal{V}$ be a scroll of $\mathrm{PG}(6, q)$ with conic directrix $\mathcal{C}$ in a plane α, twisted cubic directrix $\mathcal{N}_3$ in a 3-space Π_3, and ruled by a projectivity as described in Section 1. The group of collineations of $\mathrm{PG}(6, q)$ is transitive on planes, and transitive on 3-spaces. Further, all nondegenerate conics in a projective plane are projectively equivalent, and all twisted cubics in a 3-space are projectively equivalent. Hence, without loss of generality, we can coordinatise $\mathcal{V}$ as follows.

Let α be the plane which is the intersection of the four hyperplanes $x_0 = 0$, $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$. Let $\mathcal{C}$ be the nondegenerate conic in α with points $C_\theta = (0, 0, 0, 0, 1, \theta, \theta^2)$ for $\theta \in \mathbb{F}_q \cup \{\infty\}$. Note that the points of $\mathcal{C}$ are the exact intersection of α with the quadric of equation $x_5^2 = x_4x_6$. Let Π_3 be the 3-space which is the intersection of the three hyperplanes $x_4 = 0$, $x_5 = 0$, and $x_6 = 0$. Let $\mathcal{N}_3$ be the twisted cubic in Π_3 with points $N_\theta = (1, \theta, \theta^2, \theta^3, 0, 0, 0)$ for $\theta \in \mathbb{F}_q \cup \{\infty\}$. Note that the points of $\mathcal{N}_3$ are the exact intersection of Π_3 with the three quadrics with equations $x_1^2 = x_0x_2$, $x_2^2 = x_1x_3$, and $x_0x_3 = x_1x_2$. A projectivity in $\mathrm{PGL}(2, q)$ is uniquely determined by the image of three points, so without loss of generality, let $\mathcal{V}$ have generator lines $\ell_\theta = \{V_{\theta,t} = N_\theta + tC_\theta,\ t \in \mathbb{F}_q \cup \{\infty\}\}$ for $\theta \in \mathbb{F}_q \cup \{\infty\}$. That is, $V_{\theta,t} = (1, \theta, \theta^2, \theta^3, t, t\theta, t\theta^2)$. Equivalently, $\mathcal{V}$ consists of the points

$$V_{x,y,z} = (x^3,\ x^2y, xy^2,\ y^3,\ zx^2,\ zxy,\ zy^2)$$

for $x, y \in \mathbb{F}_q$ not both 0 and $z \in \mathbb{F}_q \cup \{\infty\}$. It is straightforward to verify that the pointset of $\mathcal{V}$ is the exact intersection of the following ten quadrics:

$$x_0x_5 = x_1x_4, \quad x_0x_6 = x_1x_5 = x_2x_4, \quad x_1x_6 = x_2x_5 = x_3x_4, \quad x_2x_6 = x_3x_5,$$
$$x_1^2 = x_0x_2, \quad x_2^2 = x_1x_3, \quad x_5^2 = x_4x_6, \quad x_0x_3 = x_1x_2.$$

Hence the points of $\mathcal{V}$ form a variety.

We follow [Semple and Roth 1949] to calculate the dimension and order of $\mathcal{V}$. The following map defines an algebraic one-to-one correspondence between the plane π of PG(3, q) with points $(x, y, z, 0)$, $x, y, z \in \mathbb{F}_q$ not all 0, and the points of $\mathcal{V}$:

$$\sigma : \pi \to \mathcal{V}, \quad (x, y, z, 0) \mapsto (x^3, x^2y, xy^2, y^3, x^2z, xyz, y^2z).$$

Thus $\mathcal{V}$ is an absolutely irreducible variety of dimension 2 and so we are justified in calling it a surface. Now consider a generic 4-space of PG(6, q) with equation given by the two hyperplanes $\Sigma_1 : a_0x_0+\cdots+a_6x_6=0$ and $\Sigma_2 : b_0x_0+\cdots+b_6x_6=0$ for $a_i, b_i \in \mathbb{F}_q$. The point $V_{x,y,z} = (x^3, x^2y, xy^2, y^3, x^2z, xyz, y^2z)$ lies on Σ_1 if $a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + a_4x^2z + a_5xyz + a_6y^2z = 0$. This corresponds to a cubic $\mathcal{K}$ in the plane π. Moreover, $\mathcal{K}$ contains the point $P = (0, 0, 1, 0)$, and P is a double point of $\mathcal{K}$. Similarly the set of points $V_{x,y,z} \in \Sigma_2$ corresponds to a cubic in π with a double point $(0, 0, 1, 0)$. Two cubics in a plane meet generically in nine points. As $(0, 0, 1, 0)$ lies in the kernel of σ, in PG(6, q) the 4-space $\Sigma_1 \cap \Sigma_2$ meets $\mathcal{V}$ in five points, and so $\mathcal{V}$ has order 5. □

Theorem 2.2. *Let $\mathcal{V}_2^5$ be a ruled quintic surface in* PG(6, q).

(1) *No two generators of $\mathcal{V}_2^5$ lie in a plane.*

(2) *No three generators of $\mathcal{V}_2^5$ lie in a* 4*-space.*

(3) *No four generators of $\mathcal{V}_2^5$ lie in a* 5*-space.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface of PG(6, q) with conic directrix $\mathcal{C}$ in a plane α, and twisted cubic directrix $\mathcal{N}_3$ lying in a 3-space Π_3. Suppose two generator lines ℓ_0, ℓ_1 of $\mathcal{V}_2^5$ lie in a plane. Let m be the line in α joining the distinct points $\ell_0 \cap \alpha$, $\ell_1 \cap \alpha$. Let m' be the line in Π_3 joining the distinct points $\ell_0 \cap \Pi_3$, $\ell_1 \cap \Pi_3$. The lines m, m' lie in the plane $\langle \ell_0, \ell_1 \rangle$ and so meet in a point, contradicting disjointness of α and Π_3. Hence the generator lines of $\mathcal{V}_2^5$ are pairwise skew.

For (2), suppose a 4-space Π_4 contains three distinct generators of $\mathcal{V}_2^5$. As distinct generators meet $\mathcal{C}$ in distinct points, Π_4 contains three distinct points of $\mathcal{C}$, and so contains the plane α. Further, distinct generators meet $\mathcal{N}_3$ in distinct points, hence Π_4 contains three points of $\mathcal{N}_3$, and so $\Pi_4 \cap \Pi_3$ has dimension at least 2. Hence $\langle \Pi_4, \Pi_3 \rangle$ has dimension at most $4 + 3 - 2 = 5$. However, $\mathcal{V}_2^5 \subseteq \langle \Pi_4, \Pi_3 \rangle$, a contradiction as $\mathcal{V}_2^5$ is not contained in a 5-space.

For (3), suppose a 5-space Π_5 contains four distinct generators of $\mathcal{V}_2^5$. Distinct generators meet Π_3 in distinct points of $\mathcal{N}_3$, so Π_5 contains four points of $\mathcal{N}_3$ which do not lie in a plane. Hence Π_5 contains Π_3. Similarly Π_5 contains α, and so Π_5 contains $\mathcal{V}_2^5$, a contradiction as $\mathcal{V}_2^5$ is not contained in a 5-space. □

Corollary 2.3. *No two generators of $\mathcal{V}_2^5$ lie in a 3-space containing α.*

Proof. Suppose a 3-space Π_3 contained α and two generators of $\mathcal{V}_2^5$. Let P be a point of $\mathcal{V}_2^5$ not in Π_3 and ℓ the generator of $\mathcal{V}_2^5$ through P. Then $\Pi_4 = \langle \Pi_3, P\rangle$ contains two distinct points of ℓ, namely P and $\ell \cap \mathcal{C}$, and so Π_4 contains ℓ. That is, Π_4 is a 4-space containing three generators, contradicting Theorem 2.2. □

We now show that the only lines on $\mathcal{V}_2^5$ are the generators, and the only nondegenerate conic on $\mathcal{V}_2^5$ is the conic directrix. We show later in Theorem 3.2 that there are exactly q^2 twisted cubics on $\mathcal{V}_2^5$, and that each is a directrix.

Theorem 2.4. *Let $\mathcal{V}_2^5$ be a ruled quintic surface in* PG$(6, q)$. *A line of* PG$(6, q)$ *meets $\mathcal{V}_2^5$ in* $0, 1, 2,$ *or* $q+1$ *points. Further, $\mathcal{V}_2^5$ contains exactly $q+1$ lines, namely the generator lines.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface of PG$(6, q)$ with conic directrix $\mathcal{C}$ lying in a plane α, and twisted cubic directrix $\mathcal{N}_3$ lying in the 3-space Π_3. Let m be a line of PG$(6, q)$ that is not a generator of $\mathcal{V}_2^5$, and suppose m meets $\mathcal{V}_2^5$ in three points P, Q, R. As m is not a generator of $\mathcal{V}_2^5$, the points P, Q, R lie on distinct generator lines denoted ℓ_P, ℓ_Q, ℓ_R, respectively. As $\mathcal{C}$ is a nondegenerate conic, m is not a line of α and so at most one of the points P, Q, R lie in $\mathcal{C}$. Suppose firstly that $P, Q, R \notin \mathcal{C}$. Then $\langle \alpha, m\rangle$ is a 3- or 4-space that contains the three generators ℓ_P, ℓ_Q, ℓ_R, contradicting Theorem 2.2. Now suppose $P \in \mathcal{C}$ and $Q, R \notin \mathcal{C}$. Then $\Sigma_3 = \langle \alpha, m\rangle$ is a 3-space which contains the two generator lines ℓ_Q, ℓ_R. So $\Sigma_3 \cap \Pi_3$ contains the distinct points $\ell_R \cap \mathcal{N}_3$, $\ell_Q \cap \mathcal{N}_3$, and so has dimension at least 1. Hence $\langle \Sigma_3, \Pi_3\rangle$ has dimension at most $3+3-1=5$, a contradiction as $\mathcal{V}_2^5 \subset \langle \Sigma_3, \Pi_3\rangle$, but $\mathcal{V}_2^5$ is not contained in a 5-space. Hence a line of PG$(6, q)$ is either a generator line of $\mathcal{V}_2^5$, or meets $\mathcal{V}_2^5$ in 0, 1, or 2 points. □

Theorem 2.5. *The ruled quintic surface $\mathcal{V}_2^5$ contains exactly one nondegenerate conic.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface with conic directrix $\mathcal{C}$ in a plane α. Suppose $\mathcal{V}_2^5$ contains another nondegenerate conic $\mathcal{C}'$ in a plane $\alpha' \neq \alpha$. If $\mathcal{C}'$ contains two points on a generator ℓ of $\mathcal{V}_2^5$, then $\alpha' \cap \mathcal{V}_2^5$ contains $\mathcal{C}'$ and ℓ. However, by the proof of Theorem 2.1, $\mathcal{V}_2^5$ is the intersection of quadrics, and the configuration $\mathcal{C}' \cup \ell$ is not contained in any planar quadric. Hence $\mathcal{C}'$ contains exactly one point on each generator of $\mathcal{V}_2^5$.

We consider the three cases where $\alpha \cap \alpha'$ is either empty, a point, or a line. Suppose $\alpha \cap \alpha' = \varnothing$. Then $\langle \alpha, \alpha'\rangle$ is a 5-space that contains $\mathcal{C}$ and $\mathcal{C}'$, and so contains two distinct points on each generator of $\mathcal{V}_2^5$. Hence $\langle \alpha, \alpha'\rangle$ contains each generator of $\mathcal{V}_2^5$ and so contains $\mathcal{V}_2^5$, a contradiction as $\mathcal{V}_2^5$ is not contained in a 5-space. Suppose $\alpha \cap \alpha'$ is a point P. Then $\langle \alpha, \alpha' \rangle$ is a 4-space that contains at least q generators of $\mathcal{V}_2^5$, contradicting Theorem 2.2 as $q \geq 6$. Finally, suppose $\alpha \cap \alpha'$ is a line. Then $\langle \alpha, \alpha' \rangle$ is a 3-space that contains at least $q - 1$ generators, contradicting Theorem 2.2 as $q \geq 6$. So $\mathcal{V}_2^5$ contains exactly one nondegenerate conic. □

We aim to classify how 5-spaces meet $\mathcal{V}_2^5$, so we begin with a simple description.

Remark 2.6. Let Π_5 be a 5-space. Then $\Pi_5 \cap \mathcal{V}_2^5$ contains a set of $q + 1$ points, one on each generator.

Lemma 2.7. *A* 5*-space meets* $\mathcal{V}_2^5$ *in either* (a) *a* 5*-dim nrc*, (b) *a* 4*-dim nrc and* 0 *or* 1 *generators*, (c) *a* 3*-dim nrc and* 0, 1, *or* 2 *generators*, *or* (d) *the conic directrix and* 0, 1, 2, *or* 3 *generators.*

Proof. Using properties of varieties (see, for example, [Semple and Roth 1949]) we have $\mathcal{V}_2^5 \cap \mathcal{V}_5^1 = \mathcal{V}_1^5$, that is, the variety $\mathcal{V}_2^5$ meets a 5-space $\mathcal{V}_5^1$ in a curve of degree 5. Denote this curve of PG(6, q) by $\mathcal{K}$. The degree of $\mathcal{K}$ can be partitioned as

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

By Theorem 2.4, the only lines on $\mathcal{V}_2^5$ are the generators. By Theorem 2.2, $\mathcal{K}$ does not contain more than 3 generators. By Remark 2.6, $\mathcal{K}$ contains at least one point on each generator. Hence $\mathcal{K}$ is not empty, and is not the union of 1, 2, or 3 generators, so the partition $1+1+1+1+1$ for the degree of $\mathcal{K}$ does not occur.

Suppose that the degree of $\mathcal{K}$ is partitioned as either (a) $2+2+1$ or (b) $2+1+1+1$. By Remark 2.6, $\mathcal{K}$ contains a point on each generator, so $\mathcal{K}$ contains an irreducible conic. By Theorem 2.5, this conic is the conic directrix $\mathcal{C}$ of $\mathcal{V}_2^5$, and case (a) does not occur. Hence $\mathcal{K}$ consists of $\mathcal{C}$ and 0, 1, 2, or 3 generators of $\mathcal{V}_2^5$.

Suppose that the degree of $\mathcal{K}$ is partitioned as $3+1+1$. So $\mathcal{K}$ consists of at most 2 generators, and an irreducible cubic $\mathcal{K}'$. By Remark 2.6, $\mathcal{K}$ contains a point on each generator, so $\mathcal{K}'$ contains a point on at least $q - 1$ generators. If $\mathcal{K}'$ generates a 3-space, then it is a 3-dim nrc of PG(6, q). If not, $\mathcal{K}'$ is an irreducible cubic contained in a plane Π_2. By the proof of Theorem 2.1, $\mathcal{K}'$ is contained in a quadric, so $\mathcal{K}'$ is not an irreducible planar cubic. Thus $\mathcal{K}'$ is a 3-dim nrc of PG(6, q). Hence $\mathcal{K}$ consists of a 3-dim nrc and 0, 1, or 2 generators of $\mathcal{V}_2^5$.

Suppose that the degree of $\mathcal{K}$ is partitioned as $2+3$. By Remark 2.6, $\mathcal{K}$ contains a point on each generator. As argued above, $\mathcal{K}$ does not contain an irreducible planar cubic. Suppose $\mathcal{K}$ contained both an irreducible conic $\mathcal{C}$ and a twisted cubic $\mathcal{N}_3$. Then there is at least one generator ℓ that meets $\mathcal{C}$ and $\mathcal{N}_3$ in distinct points. In this case ℓ lies in the 5-space and so lies in $\mathcal{K}$, a contradiction. So $\mathcal{K}$ is not the union of an irreducible conic and a twisted cubic.

Suppose that the degree of $\mathcal{K}$ is partitioned as $4+1$. So $\mathcal{K}$ consists of at most 1 generator, and an irreducible quartic $\mathcal{K}'$. By Remark 2.6, $\mathcal{K}$ contains a point on each generator, so $\mathcal{K}'$ contains a point on at least q generators. If $\mathcal{K}'$ generates a 4-space, then it is a 4-dim nrc of PG$(6, q)$. If not, $\mathcal{K}'$ is an irreducible quartic contained in a 3-space Π_3. Let ℓ, m be two generators not in $\mathcal{K}$. Then by Remark 2.6 they meet $\mathcal{K}'$. So $\langle \Pi_3, \ell, m\rangle$ has dimension at most 5, and meets $\mathcal{V}_2^5$ in an irreducible quartic and 2 lines, which is a curve of degree 6, a contradiction. Thus $\mathcal{K}'$ is a 4-dim nrc of PG$(6, q)$. That is, $\mathcal{K}$ consists of a 4-dim nrc and 0 or 1 generators of $\mathcal{V}_2^5$.

Suppose the curve $\mathcal{K}$ is irreducible. By Remark 2.6, $\mathcal{K}$ contains a point on each generator. So either $\mathcal{K}$ is a 5-dim nrc of PG$(6, q)$, or $\mathcal{K}$ lies in a 4-space. Suppose $\mathcal{K}$ lies in a 4-space Π_4, and let ℓ be a generator. Then $\langle \Pi_4, \ell\rangle$ has dimension at most 5 and meets $\mathcal{V}_2^5$ in a curve of degree 6, a contradiction. So $\mathcal{K}$ is a 5-dim nrc of PG$(6, q)$. □

Corollary 2.8. *Let Π_r be an r-space for $r = 3, 4, 5$ that contains an r-dim nrc of $\mathcal{V}_2^5$. Then Π_r contains 0 generators of $\mathcal{V}_2^5$.*

Proof. First suppose $r = 3$. By Lemma 2.7, a 5-space containing a twisted cubic $\mathcal{N}_3$ of $\mathcal{V}_2^5$ contains at most two generators of $\mathcal{V}_2^5$. Hence a 4-space containing $\mathcal{N}_3$ contains at most one generator of $\mathcal{V}_2^5$. Hence the 3-space Π_3 containing $\mathcal{N}_3$ contains no generator of $\mathcal{V}_2^5$.

If $r = 4$, by Lemma 2.7, a 5-space containing a 4-dim nrc $\mathcal{N}_4$ of $\mathcal{V}_2^5$ contains at most one generator of $\mathcal{V}_2^5$. Hence the 4-space Π_4 containing $\mathcal{N}_4$ contains no generators of $\mathcal{V}_2^5$. If $r = 5$, then by Lemma 2.7, Π_5 contains 0 generators of $\mathcal{V}_2^5$. □

Theorem 2.9. *Let $\mathcal{N}_r$ be an r-dim nrc lying on $\mathcal{V}_2^5$ for $r = 3, 4, 5$. Then $\mathcal{N}_r$ contains exactly one point on each generator of $\mathcal{V}_2^5$.*

Proof. Let $\mathcal{N}_r$ be an r-dim nrc lying on $\mathcal{V}_2^5$ for $r = 3, 4, 5$, and denote the r-space containing $\mathcal{N}_r$ by Π_r. If Π_r contained 2 points of a generator of $\mathcal{V}_2^5$, then it contains the whole generator, so by Corollary 2.8, the $q + 1$ points of $\mathcal{N}_r$ consist of one on each generator of $\mathcal{V}_2^5$. □

3. $\mathcal{V}_2^5$ and $\mathbb{F}_q$-subplanes of PG$(2, q^3)$

To study $\mathcal{V}_2^5$ in more detail, we use the linear representation of PG$(2, q^3)$ in PG$(6, q)$ developed independently by André [1954] and Bruck and Bose [1964; 1966]. Let $\mathcal{S}$ be a regular 2-spread of PG$(6, q)$ in a 5-space Σ_∞. Let $\mathcal{I}$ be the incidence structure with the points of PG$(6, q) \setminus \Sigma_\infty$ as *points*, the 3-spaces of PG$(6, q)$ that contain a plane of $\mathcal{S}$ and are not in Σ_∞ as *lines*, and inclusion as *incidence*. Then $\mathcal{I}$ is isomorphic to AG$(2, q^3)$. We can uniquely complete $\mathcal{I}$ to PG$(2, q^3)$, the points on ℓ_∞ correspond to the planes of $\mathcal{S}$. We call this the *Bruck–Bose representation* of PG$(2, q^3)$ in PG$(6, q)$; see [Barwick and Jackson 2012] for a detailed discussion on this representation. Of particular interest is the relationship between the ruled quintic surface of PG$(6, q)$ and the $\mathbb{F}_q$-subplanes of PG$(2, q^3)$.

To describe this relationship, we need to use the cubic extension of PG(6, q) to PG(6, q^3). The regular 2-spread $\mathcal{S}$ has a unique set of three conjugate *transversal* lines in this cubic extension, denoted g, g^q, g^{q^2}, which meet each extended plane of $\mathcal{S}$; for more details on regular spreads and transversals, see [Hirschfeld and Thas 1991, Section 25.6]. An r-space Π_r of PG(6, q) lies in a unique r-space of PG(6, q^3), denoted $\Pi_r^\star$. An nrc $\mathcal{N}$ of PG(6, q) lies in a unique nrc of PG(6, q^3), denoted $\mathcal{N}^\star$. Let $\mathcal{V}_2^5$ be a ruled quintic surface with conic directrix $\mathcal{C}$, twisted cubic directrix $\mathcal{N}_3$, and associated projectivity ϕ. Then we can extend $\mathcal{V}_2^5$ to a unique ruled quintic surface $\mathcal{V}_2^{5\star}$ of PG(6, q^3) with conic directrix $\mathcal{C}^\star$, twisted cubic directrix $\mathcal{N}_3^\star$, and the same associated projectivity, that is, extend ϕ from acting on PG(1, q) to acting on PG(1, q^3). We need the following characterisations.

Result 3.1 [Barwick and Jackson 2012; 2014]. *Let $\mathcal{S}$ be a regular 2-spread in a 5-space Σ_∞ in* PG(6, q) *and consider the Bruck–Bose plane* PG(2, q^3).

(1) *An $\mathbb{F}_q$-subline of* PG(2, q^3) *that meets ℓ_∞ in a point corresponds in* PG(6, q) *to a line not in Σ_∞.*

(2) *An $\mathbb{F}_q$-subline of* PG(2, q^3) *that is disjoint from ℓ_∞ corresponds in* PG(6, q) *to a twisted cubic $\mathcal{N}_3$ lying in a 3-space about a plane of $\mathcal{S}$ such that the extension $\mathcal{N}_3^\star$ to* PG(6, q^3) *meets each transversal of $\mathcal{S}$ in a point.*

(3) *An $\mathbb{F}_q$-subplane of* PG(2, q^3) *tangent to ℓ_∞ at the point T corresponds in* PG(6, q) *to a ruled quintic surface $\mathcal{V}_2^5$ with conic directrix in the spread plane corresponding to T such that in the cubic extension* PG(6, q^3), *the transversals g, g^q, g^{q^2} of $\mathcal{S}$ are generators of $\mathcal{V}_2^{5\star}$.*

Moreover, the converse of each is true.

We use this characterisation to show that $\mathcal{V}_2^5$ contains exactly q^2 twisted cubics.

Theorem 3.2. *The ruled quintic surface $\mathcal{V}_2^5$ contains exactly q^2 twisted cubics, and each is a directrix of $\mathcal{V}_2^5$.*

Proof. By Theorem 2.1, all ruled quintic surfaces are projectively equivalent. So without loss of generality, we can position a ruled quintic surface so that it corresponds to an $\mathbb{F}_q$-subplane of PG(2, q^3), which we denote by $\mathcal{B}$. That is, by Result 3.1, $\mathcal{S}$ is a regular 2-spread in a hyperplane Σ_∞, $\mathcal{V}_2^5 \cap \Sigma_\infty$ is the conic directrix $\mathcal{C}$ of $\mathcal{V}_2^5$, $\mathcal{C}$ lies in a plane of $\mathcal{S}$, and in the cubic extension PG(6, q^3), the transversals g, g^q, g^{q^2} of $\mathcal{S}$ are generators of $\mathcal{V}_2^{5\star}$.

Let $\mathcal{N}_3$ be a twisted cubic contained in $\mathcal{V}_2^5$, and denote the 3-space containing $\mathcal{N}_3$ by Π_3. As $\mathcal{V}_2^5 \cap \Sigma_\infty = \mathcal{C}$, Π_3 meets Σ_∞ in a plane; we show this is a plane of $\mathcal{S}$. In PG(6, q^3), $\mathcal{V}_2^{5\star}$ is a ruled quintic surface that contains the twisted cubic $\mathcal{N}_3^\star$. Moreover, the transversals g, g^q, g^{q^2} of $\mathcal{S}$ are generators of $\mathcal{V}_2^{5\star}$. So by Theorem 2.9, $\mathcal{N}_3^\star$ contains one point on each of g, g^q, and g^{q^2}. Hence the 3-space $\Pi_3^\star$ contains an extended plane of $\mathcal{S}$, and so Π_3 meets Σ_∞ in a plane of $\mathcal{S}$. Hence

$\Pi_3 \cap \alpha = \emptyset$. Further, by Theorem 2.9, $\mathcal{N}_3$ contains one point on each generator of $\mathcal{V}_2^5$, and thus $\mathcal{N}_3$ is a directrix of $\mathcal{V}_2^5$.

By Result 3.1, $\mathcal{N}_3$ corresponds in $\mathrm{PG}(2, q^3)$ to an $\mathbb{F}_q$-subline of $\mathcal{B}$ disjoint from ℓ_∞. Conversely, every $\mathbb{F}_q$-subline of $\mathcal{B}$ disjoint from ℓ_∞ corresponds to a twisted cubic on $\mathcal{V}_2^5$. Thus the twisted cubics in $\mathcal{V}_2^5$ are in one-to-one correspondence with the $\mathbb{F}_q$-sublines of $\mathcal{B}$ that are disjoint from ℓ_∞. As there are q^2 such $\mathbb{F}_q$-sublines, there are q^2 twisted cubics on $\mathcal{V}_2^5$. □

Suppose we position $\mathcal{V}_2^5$ so that it corresponds via the Bruck–Bose representation to a tangent $\mathbb{F}_q$-subplane $\mathcal{B}$ of $\mathrm{PG}(2, q^3)$. So we have a regular 2-spread $\mathcal{S}$ in a hyperplane Σ_∞, and the conic directrix of $\mathcal{V}_2^5$ lies in a plane $\alpha \in \mathcal{S}$. We define the *splash* of $\mathcal{B}$ to be the set of q^2+1 points on ℓ_∞ that lie on an extended line of $\mathcal{B}$. The *splash* of $\mathcal{V}_2^5$ is defined to be the corresponding set of q^2+1 planes of $\mathcal{S}$. We denote the splash of $\mathcal{V}_2^5$ by $\mathbb{S}$. Note that α is a plane of $\mathbb{S}$. We show that the remaining q^2 planes of $\mathbb{S}$ are related to the q^2 twisted cubics of $\mathcal{V}_2^5$.

Corollary 3.3. *Let $\mathcal{S}$ be a regular 2-spread in a hyperplane Σ_∞ of $\mathrm{PG}(6, q)$. Without loss of generality, we can position $\mathcal{V}_2^5$ so that it corresponds via the Bruck–Bose representation to a tangent $\mathbb{F}_q$-subplane of $\mathrm{PG}(2, q^3)$. Then the conic directrix of $\mathcal{V}_2^5$ lies in a plane $\alpha \in \mathcal{S}$, the q^2 3-spaces containing a twisted cubic of $\mathcal{V}_2^5$ meet Σ_∞ in distinct planes of $\mathcal{S}$, and these planes together with α form the splash $\mathbb{S}$ of $\mathcal{V}_2^5$.*

Proof. By Theorem 2.1, all ruled quintic surfaces are projectively equivalent, so without loss of generality, let $\mathcal{V}_2^5$ be positioned so that it corresponds to an $\mathbb{F}_q$-subplane $\mathcal{B}$ of $\mathrm{PG}(2, q^3)$ which is tangent to ℓ_∞. Let b be an $\mathbb{F}_q$-subline of $\mathcal{B}$ disjoint from ℓ_∞, so the extension of b meets ℓ_∞ in a point R which lies in the splash of $\mathcal{B}$. By Result 3.1, b corresponds in $\mathrm{PG}(6, q)$ to a twisted cubic of $\mathcal{V}_2^5$ which lies in a 3-space that meets Σ_∞ in the plane of $\mathbb{S}$ corresponding to the point R. □

Using this Bruck–Bose setting, we describe the 3-spaces of $\mathrm{PG}(6, q)$ that contain a plane of the regular 2-spread $\mathcal{S}$.

Corollary 3.4. *Position $\mathcal{V}_2^5$ as in Corollary 3.3, so $\mathcal{S}$ is a regular 2-spread in the hyperplane Σ_∞, and the conic directrix of $\mathcal{V}_2^5$ lies in a plane α contained in the splash $\mathbb{S} \subset \mathcal{S}$ of $\mathcal{V}_2^5$.*

(1) *Let $\beta \in \mathbb{S} \setminus \alpha$. Then there exists a unique 3-space containing β that meets $\mathcal{V}_2^5$ in a twisted cubic. The remaining 3-spaces containing β (and not in Σ_∞) meet $\mathcal{V}_2^5$ in 0 or 1 point.*

(2) *Let $\gamma \in \mathcal{S} \setminus \mathbb{S}$. Then each 3-space containing γ and not in Σ_∞ meets $\mathcal{V}_2^5$ in 0 or 1 point.*

Proof. By Corollary 3.3, we can position $\mathcal{V}_2^5$ so that it corresponds to an $\mathbb{F}_q$-subplane $\mathcal{B}$ of $\mathrm{PG}(2, q^3)$ which is tangent to ℓ_∞. The 3-spaces that contain a plane of $\mathcal{S}$ (and do not lie in Σ_∞) correspond to lines of PG(2, q^3). Each point on ℓ_∞ not in $\mathcal{B}$ but in the splash of $\mathcal{B}$ lies on a unique line that meets $\mathcal{B}$ in an $\mathbb{F}_q$-subline. By Result 3.1, this corresponds to a twisted cubic in $\mathcal{V}_2^5$. The remaining lines meet $\mathcal{B}$ in 0 or 1 point, so the remaining 3-spaces meet $\mathcal{V}_2^5$ in 0 or 1 point. □

As $\mathcal{V}_2^5$ corresponds to an $\mathbb{F}_q$-subplane, we have the following result.

Theorem 3.5. *Let $\mathcal{V}_2^5$ be a ruled quintic surface in* PG(6, q).

(1) *Two twisted cubics on $\mathcal{V}_2^5$ meet in a unique point.*

(2) *Let P, Q be points lying on different generators of $\mathcal{V}_2^5$, and not in the conic directrix. Then P, Q lie on a unique twisted cubic of $\mathcal{V}_2^5$.*

Proof. Without loss of generality, let $\mathcal{V}_2^5$ be positioned as described in Corollary 3.3. So the conic directrix lies in a plane α contained in a regular 2-spread $\mathcal{S}$ in Σ_∞, and $\mathcal{V}_2^5$ corresponds to an $\mathbb{F}_q$-subplane $\mathcal{B}$ of PG(2, q^3) tangent to ℓ_∞. Let $\mathcal{N}_1, \mathcal{N}_2$ be two twisted cubics contained in $\mathcal{V}_2^5$. By Result 3.1, they correspond in PG(2, q^3) to two $\mathbb{F}_q$-sublines of $\mathcal{B}$ not containing $\mathcal{B} \cap \ell_\infty$, and so meet in a unique affine point P. This corresponds to a unique point $P \in \mathcal{V}_2^5 \setminus \alpha$ lying in both $\mathcal{N}_1$ and $\mathcal{N}_2$, proving (1).

For (2), let P, Q be points lying on distinct generators of $\mathcal{V}_2^5$, $P, Q \notin \mathcal{C}$. If the line PQ met α, then $\langle \alpha, P, Q \rangle$ is a 3-space that contains α and the generators of $\mathcal{V}_2^5$ containing P and Q, contradicting Corollary 2.3. Hence the line PQ is skew to α. In PG(2, q^3), P, Q correspond to two affine points in the tangent $\mathbb{F}_q$-subplane $\mathcal{B}$, so they lie on a unique $\mathbb{F}_q$-subline b of $\mathcal{B}$. By Result 3.1, the generators of $\mathcal{V}_2^5$ correspond to the $\mathbb{F}_q$-sublines of $\mathcal{B}$ through the point $\mathcal{B} \cap \ell_\infty$. As PQ is skew to α, we have $b \cap \ell_\infty = \varnothing$. Hence, by Result 3.1, in PG(6, q) the points P, Q lie on a unique twisted cubic of $\mathcal{V}_2^5$. □

4. Intersection types for 5-spaces meeting $\mathcal{V}_2^5$

In this section we determine how 5-spaces meet $\mathcal{V}_2^5$ and count the different intersection types. A series of lemmas is used to prove the main result which is stated in Theorem 4.8.

Lemma 4.1. *Let $\mathcal{V}_2^5$ be a ruled quintic surface of* PG(6, q) *with conic directrix $\mathcal{C}$. Of the $q^3 + q^2 + q + 1$ 5-spaces of* PG(6, q) *containing $\mathcal{C}$, r_i of them meet $\mathcal{V}_2^5$ in precisely $\mathcal{C}$ and i generators, where*

$$r_3 = \frac{q^3 - q}{6}, \quad r_2 = q^2 + q, \quad r_1 = \frac{q^3}{2} + \frac{q}{2} + 1, \quad r_0 = \frac{q^3 - q}{3}.$$

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface of PG(6, q) with conic directrix $\mathcal{C}$ lying in a plane α. By Lemma 2.7, a 5-space containing $\mathcal{C}$ contains at most three generator lines of $\mathcal{V}_2^5$. By Theorem 2.2, three generators of $\mathcal{V}_2^5$ lie in a unique 5-space. Hence there are

$$r_3 = \binom{q+1}{3}$$

5-spaces that contain three generators of $\mathcal{V}_2^5$. Such a 5-space contains three points of $\mathcal{C}$, and so contains $\mathcal{C}$ and α.

Denote the generator lines of $\mathcal{V}_2^5$ by $\ell_0, \ldots, \ell_q$ and consider two generators, ℓ_0, ℓ_1 say. By Corollary 2.3, $\Sigma_4 = \langle \alpha, \ell_0, \ell_1 \rangle$ is a 4-space. By Theorem 2.2, $\langle \Sigma_4, \ell_i \rangle$ for $i = 2, \ldots, q$ are distinct 5-spaces. That is, $q-1$ of the 5-spaces about Σ_4 contain 3 generators, and hence the remaining two contain ℓ_0, ℓ_1 and no further generator of $\mathcal{V}_2^5$. Hence, by Lemma 2.7, $q-1$ of the 5-spaces about Σ_4 meet $\mathcal{V}_2^5$ in exactly $\mathcal{C}$ and 3 generators; and the remaining two 5-spaces about Σ_4 meet $\mathcal{V}_2^5$ in exactly $\mathcal{C}$ and two generators. There are $\binom{q+1}{2}$ choices for Σ_4, and hence the number of 5-spaces that meet $\mathcal{V}_2^5$ in precisely $\mathcal{C}$ and two generators is

$$r_2 = 2 \times \binom{q+1}{2} = (q+1)q.$$

Next, let r_1 be the number of 5-spaces that meet $\mathcal{V}_2^5$ in precisely $\mathcal{C}$ and one generator. We count in two ways ordered pairs (ℓ, Π_5) where ℓ is a generator of $\mathcal{V}_2^5$, and Π_5 is a 5-space that contains ℓ and α, giving

$$(q+1)(q^2+q+1) = 3r_3 + 2r_2 + r_1.$$

Hence $r_1 = q^3/2 + q/2 + 1$. Finally, the number of 5-spaces containing $\mathcal{C}$ and zero generators is $r_0 = (q^3+q^2+q+1) - r_3 - r_2 - r_1 = (q^3-q)/3$, as required. □

Lemma 4.2. *Let $\mathcal{V}_2^5$ be a ruled quintic surface of* PG(6, q) *and let $\mathcal{N}_3$ be a twisted cubic directrix of $\mathcal{V}_2^5$.*

(1) *Of the q^2+q+1 5-spaces of* PG(6, q) *containing $\mathcal{N}_3$, s_i of them meet $\mathcal{V}_2^5$ in precisely $\mathcal{N}_3$ and i generators, where*

$$s_2 = \frac{q^2+q}{2}, \quad s_1 = q+1, \quad s_0 = \frac{q^2-q}{2}.$$

(2) *The total number of 5-spaces that meet $\mathcal{V}_2^5$ in a twisted cubic and i generators is $q^2 s_i$, for $i = 0, 1, 2$.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface of PG(6, q) with a twisted cubic directrix $\mathcal{N}_3$ lying in the 3-space Π_3. By Lemma 2.7, a 5-space containing $\mathcal{N}_3$ contains at most two generators of $\mathcal{V}_2^5$, so the number of 5-spaces that contain Π_3 and exactly two generator lines is $s_2 = \binom{q+1}{2}$. Let ℓ be a generator of $\mathcal{V}_2^5$ and consider the 4-space $\Pi_4 = \langle \Pi_3, \ell \rangle$. For each generator $m \neq \ell$, $\langle \Pi_4, m \rangle$ is a 5-space about Π_4 that meets $\mathcal{V}_2^5$ in $\mathcal{N}_3$, ℓ, and m, and in no further point by Lemma 2.7. This accounts for

q of the 5-spaces containing Π_4. Hence the remaining 5-space containing Π_4 meets $\mathcal{V}_2^5$ in exactly $\mathcal{N}_3$ and ℓ. That is, exactly one of the 5-spaces about $\Pi_4 = \langle \Pi_3, \ell \rangle$ meets $\mathcal{V}_2^5$ in precisely $\mathcal{N}_3$ and ℓ. There are $q+1$ choices for the generator ℓ, and hence $s_1 = q+1$. Finally $s_0 = (q^2+q+1) - s_2 - s_1 = (q^2-q)/2$, as required.

For (2), by Theorem 3.2, $\mathcal{V}_2^5$ contains q^2 twisted cubics, so the total number of 5-spaces meeting $\mathcal{V}_2^5$ in a twisted cubic and i generators is $q^2 s_i$, $i = 0, 1, 2$. □

The next result looks at properties of 4-dim nrcs contained in $\mathcal{V}_2^5$. In particular, we show that there are no 5-spaces that meet $\mathcal{V}_2^5$ in a 4-dim nrc and 0 generator lines.

Lemma 4.3. *Let $\mathcal{V}_2^5$ be a ruled quintic surface of* PG(6, q) *with conic directrix $\mathcal{C}$ in the plane α, and let $\mathcal{N}_4$ be a 4-dim nrc contained in $\mathcal{V}_2^5$.*

(1) *The $q+1$ 5-spaces containing $\mathcal{N}_4$ each contain a distinct generator line of $\mathcal{V}_2^5$.*
(2) *The 4-space containing $\mathcal{N}_4$ meets α in a point P, and either $P = \mathcal{C} \cap \mathcal{N}_4$ or q is even and P is the nucleus of $\mathcal{C}$.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface in PG(6, q) with conic directrix $\mathcal{C}$ lying in a plane α. Let $\mathcal{N}_4$ be a 4-dim nrc contained in $\mathcal{V}_2^5$, so $\mathcal{N}_4$ lies in a 4-space, which we denote Π_4. By Corollary 2.8, Π_4 does not contain a generator of $\mathcal{V}_2^5$. By Lemma 2.7, a 5-space containing $\mathcal{N}_4$ can contain at most one generator of $\mathcal{V}_2^5$. Hence each of the $q+1$ 5-spaces containing $\mathcal{N}_4$ contains a distinct generator. In particular, if we label the points of $\mathcal{C}$ by $Q_0, \ldots, Q_q$, and the generator through Q_i by ℓ_{Q_i}, then the $q+1$ 5-spaces containing $\mathcal{N}_4$ are $\Sigma_i = \langle \Pi_4, \ell_{Q_i} \rangle$, for $i = 0, \ldots, q$.

If Π_4 met the plane α in a line, then $\langle \Pi_4, \alpha \rangle$ is a 5-space whose intersection with $\mathcal{V}_2^5$ contains $\mathcal{N}_4$ and $\mathcal{C}$, contradicting Lemma 2.7. Hence Π_4 meets α in a point P. There are three possibilities for the point $P = \Pi_4 \cap \alpha$, namely $P \in \mathcal{C}$, q even and P the nucleus of $\mathcal{C}$, or q even, $P \notin \mathcal{C}$, and P not the nucleus of $\mathcal{C}$.

Case 1. Suppose $P \in \mathcal{C}$. For $i = 0, \ldots, q$, the 5-space $\Sigma_i = \langle \Pi_4, \ell_{Q_i} \rangle$ meets α in a line m_i. Label $\mathcal{C}$ so that $P = Q_0$, so the line m_0 is the tangent to $\mathcal{C}$ at P, and m_i for $i = 1, \ldots, q$, is the secant line PQ_i. We now show that $P = Q_0$ is a point of $\mathcal{N}_4$. Let $i \in \{1, \ldots, q\}$. Then by Lemma 2.7, Σ_i meets $\mathcal{V}_2^5$ in precisely $\mathcal{N}_4 \cup \ell_{Q_i}$, and $\Sigma_i \cap \mathcal{V}_2^5 \cap \alpha$ is the two points P, Q_i. As $P \notin \ell_{Q_i}$ we have $P \in \mathcal{N}_4$. That is, $P = \mathcal{C} \cap \mathcal{N}_4$.

Case 2. Suppose q is even and $P = \Pi_4 \cap \alpha$ is the nucleus of $\mathcal{C}$. For $i = 0, \ldots, q$, the 5-space $\Sigma_i = \langle \Pi_4, \ell_{Q_i} \rangle$ meets α in the tangent to $\mathcal{C}$ through Q_i. In this case, $\mathcal{C} \cap \mathcal{N}_4 = \varnothing$.

Case 3. Suppose $P = \Pi_4 \cap \alpha$ is not in $\mathcal{C}$, and P is not the nucleus of $\mathcal{C}$. Now P lies on some secant $m = QR$ of $\mathcal{C}$, for some points $Q, R \in \mathcal{C}$. The intersection of the 5-space $\langle \Pi_4, m \rangle$ with $\mathcal{V}_2^5$ contains $\mathcal{N}_4$ and two points R, Q of $\mathcal{C}$. As R, Q lie on distinct generators and are not in $\mathcal{N}_4$, this contradicts Lemma 2.7. Hence this case cannot occur. □

We can now describe how an nrc of $\mathcal{V}_2^5$ meets the conic directrix, and note that Theorem 5.1 shows that each possibility in (3) below can occur.

Corollary 4.4. *Let $\mathcal{V}_2^5$ be a ruled quintic surface of* PG(6, q) *with conic directrix $\mathcal{C}$.*

1. *A twisted cubic $\mathcal{N}_3 \subseteq \mathcal{V}_2^5$ contains* 0 *points of $\mathcal{C}$.*
2. *A* 4*-dim nrc $\mathcal{N}_4 \subseteq \mathcal{V}_2^5$ contains either* 1 *point of $\mathcal{C}$, or* 0 *points of $\mathcal{C}$, in which case q is even and the* 4*-space containing $\mathcal{N}_4$ contains the nucleus of $\mathcal{C}$.*
3. *A* 5*-dim nrc $\mathcal{N}_5 \subseteq \mathcal{V}_2^5$ contains* 0, 1, *or* 2 *points of $\mathcal{C}$.*

Proof. Let $\mathcal{V}_2^5$ be a ruled quintic surface of PG(6, q) with conic directrix $\mathcal{C}$ in a plane α. Let $\mathcal{N}_3$ be a twisted cubic of $\mathcal{V}_2^5$, so by Theorem 3.2, $\mathcal{N}_3$ is a directrix of $\mathcal{V}_2^5$, and so is disjoint from α, proving (1). Next let $\mathcal{N}_4$ be a 4-dim nrc on $\mathcal{V}_2^5$, and let Π_4 be the 4-space containing $\mathcal{N}_4$. By Lemma 4.3, $\Pi_4 \cap \alpha$ is a point P, and either $P = \mathcal{C} \cap \mathcal{N}_4$, or q is even and P is the nucleus of $\mathcal{C}$. Thus, $P \notin \mathcal{V}_2^5$ and so $P \notin \mathcal{N}_4$, proving (2). Let Π_5 be a 5-space containing a 5-dim nrc of $\mathcal{V}_2^5$. By Lemma 2.7, Π_5 cannot contain α. Hence Π_5 meets α in a line, and so contains at most two points of $\mathcal{C}$, proving (3). □

We now use the Bruck–Bose setting to count the 4-dim nrcs contained in $\mathcal{V}_2^5$.

Lemma 4.5. *Let $\mathcal{S}$ be a regular* 2*-spread in a* 5*-space Σ_∞ in* PG(6, q). *Position $\mathcal{V}_2^5$ as in Corollary 3.3, so $\mathcal{V}_2^5$ has splash $\mathbb{S} \subset \mathcal{S}$. Then a* 4*- or* 5*-space about a plane $\beta \in \mathbb{S}$ cannot contain a* 4*-dim nrc of $\mathcal{V}_2^5$.*

Proof. Position $\mathcal{V}_2^5$ as described in Corollary 3.3, so $\mathcal{S}$ is a regular 2-spread in a 5-space Σ_∞, the conic directrix of $\mathcal{V}_2^5$ lies in a plane $\alpha \in \mathcal{S}$, and $\mathbb{S} \subset \mathcal{S}$ denotes the splash of $\mathcal{V}_2^5$. By Lemma 2.7, a 4-space containing α cannot contain a 4-dim nrc of $\mathcal{V}_2^5$. Let $\beta \in \mathbb{S} \setminus \alpha$. Then by Corollary 3.4, β lies in exactly one 3-space that contains a twisted cubic of $\mathcal{V}_2^5$. Denote these by Π_3 and $\mathcal{N}_3$, respectively. By Theorem 3.2, $\mathcal{N}_3$ is a directrix of $\mathcal{V}_2^5$, and so Π_3 is disjoint from α. So if ℓ_P is a generator of $\mathcal{V}_2^5$, then $\Pi_4 = \langle \Pi_3, \ell_P \rangle$ is a 4-space and $\Pi_4 \cap \alpha$ is the point $P = \ell_P \cap \mathcal{C}$. Let ℓ be a line of α through P and let $\Pi_5 = \langle \Pi_3, \ell \rangle$. If ℓ is tangent to $\mathcal{C}$, then $\Pi_5 \cap \mathcal{V}_2^5$ is exactly $\mathcal{N}_3 \cup \ell_P$. If ℓ is a secant of $\mathcal{C}$, so $\ell \cap \mathcal{C} = \{P, Q\}$, then $\Pi_5 \cap \mathcal{V}_2^5$ consists of $\mathcal{N}_3$, ℓ_P, and the generator ℓ_Q through Q. Varying ℓ_P and ℓ, we get all the 5-spaces that contain β and contain 1 or 2 generators of $\mathcal{V}_2^5$. That is, each 5-space containing β and 1 or 2 generators of $\mathcal{V}_2^5$ also contains $\mathcal{N}_3$. The remaining 5-spaces about β hence contain 0 generators of $\mathcal{V}_2^5$ and meet α in an exterior line of $\mathcal{C}$. Hence, by Lemma 4.3, none of the 5-spaces about β contain a 4-dim nrc of $\mathcal{V}_2^5$. □

Lemma 4.6. (1) *The number of* 4*-dim nrcs contained in $\mathcal{V}_2^5$ is $q^4 - q^2$.*

(2) *The number of* 5*-spaces that meet $\mathcal{V}_2^5$ in a* 4*-dim nrc and one generator is $q^5 + q^4 - q^3 - q^2$.*

Proof. Without loss of generality, position $\mathcal{V}_2^5$ as described in Corollary 3.3. That is, let $\mathcal{S}$ be a regular 2-spread in a 5-space Σ_∞, let the conic directrix of $\mathcal{V}_2^5$ lie in a plane $\alpha \in \mathcal{S}$, and let $\mathbb{S} \subset \mathcal{S}$ be the splash of $\mathcal{V}_2^5$. Straightforward counting shows that a 5-space distinct from Σ_∞ contains a unique spread plane. If this plane is in the splash $\mathbb{S}$, then by Lemma 4.5, the 5-space does not contain a 4-dim nrc of $\mathcal{V}_2^5$. So a 5-space containing a 4-dim nrc of $\mathcal{V}_2^5$ contains a unique plane of $\mathcal{S} \setminus \mathbb{S}$. Consider a plane $\gamma \in \mathcal{S} \setminus \mathbb{S}$. Let $P \in \mathcal{C}$, let ℓ_P be the generator of $\mathcal{V}_2^5$ through P, and consider the 4-space $\Pi_4 = \langle \gamma, \ell_P \rangle$. Suppose first that Π_4 contains two generators of $\mathcal{V}_2^5$. Then there is a 5-space Π_5 containing γ and two generators. By Lemma 2.7, Π_5 contains either $\mathcal{C}$ or a twisted cubic of $\mathcal{V}_2^5$. A 5-space distinct from Σ_∞ cannot contain two planes of $\mathcal{S}$, so Π_5 does not contain $\mathcal{C}$. Moreover, by Corollary 3.3, Π_5 does not contain a twisted cubic of $\mathcal{V}_2^5$. Hence Π_4 contains exactly one generator of $\mathcal{V}_2^5$. If every generator of $\mathcal{V}_2^5$ contained at least one point of Π_4, then the intersection of Π_4 with $\mathcal{V}_2^5$ contains at least ℓ_P and q further points, one on each generator. By Lemma 2.7 and Corollary 2.8, the only possibility is that $\Pi_4 \cap \mathcal{V}_2^5$ contains a twisted cubic, which is not possible by Corollary 3.3. Hence there is at least one generator which is disjoint from Π_4; denote this ℓ_Q. Label the points of ℓ_Q by $X_0, \ldots, X_q$. Then the $q+1$ 5-spaces containing Π_4 are $\Sigma_i = \langle \gamma, \ell_P, X_i \rangle$. For each $i = 0, \ldots, q$, the intersection of Σ_i with $\mathcal{V}_2^5$ contains the generator ℓ_P and the point X_i. By Corollary 3.3, Σ_i does not contain a twisted cubic of $\mathcal{V}_2^5$. Hence, by Lemma 2.7, $\Sigma_i \cap \mathcal{V}_2^5$ is ℓ_P and a 4-dim nrc.

That is, there are $(q+1)^2$ 5-spaces containing γ and one generator of $\mathcal{V}_2^5$. Each contains a 4-dim nrc of $\mathcal{V}_2^5$. Further, if Π_5 is a 5-space containing γ and zero generators of $\mathcal{V}_2^5$, then by Lemma 4.3, Π_5 does not contain a 4-dim nrc of $\mathcal{V}_2^5$. Hence, as there are $q^3 - q^2$ choices for γ, there are

$$(q+1)^2 \times (q^3 - q^2) = q^5 + q^4 - q^3 - q^2$$

5-spaces that meet $\mathcal{V}_2^5$ in one generator and a 4-dim nrc. By Lemma 4.3, every 4-dim nrc in $\mathcal{V}_2^5$ lies in $q+1$ such 5-spaces. Hence the number of 4-dim nrcs contained in $\mathcal{V}_2^5$ is $(q^5 + q^4 - q^3 - q^2)/(q+1)$ as required. □

We now count the number of 5-dim nrcs contained in $\mathcal{V}_2^5$.

Lemma 4.7. *The number of 5-spaces meeting $\mathcal{V}_2^5$ in a 5-dim nrc is $q^6 - q^4$.*

Proof. We show that the number of 5-spaces meeting $\mathcal{V}_2^5$ in a 5-dim nrc is $q^6 - q^4$ by counting in two ways the number x of incident pairs (A, Π_5) where A is a point of $\mathcal{V}_2^5$ and Π_5 is a 5-space containing A. The number of ways to choose a point A of $\mathcal{V}_2^5$ is $(q+1)^2$. The point A lies in $q^5 + q^4 + q^3 + q^2 + q + 1$ 5-spaces. So

$$x = (q+1)^2 \times (q^5+q^4+q^3+q^2+q+1) = q^7+3q^6+4q^5+4q^4+4q^3+4q^2+3q+1.$$

Alternatively, we count the 5-spaces first; there are several possibilities for Π_5. By Lemma 2.7, $\Pi_5 \cap \mathcal{V}_2^5$ is either empty, or contains an r-dim nrc for some $r \in \{2, \ldots, 5\}$. Let n_r be the number of pairs (A, Π_5) with $A \in \mathcal{V}_2^5 \cap \Pi_5$ and Π_5 containing an r-dim nrc of $\mathcal{V}_2^5$. Note that

$$x = n_2 + n_3 + n_4 + n_5. \tag{1}$$

We now calculate n_2, n_3, and n_4, and then use (1) to determine the number of 5-spaces meeting $\mathcal{V}_2^5$ in a 5-dim nrc.

For n_2, consider a 5-space Π_5 that contains the conic directrix $\mathcal{C}$, so by Lemma 4.1, Π_5 contains 0, 1, 2, or 3 generators of $\mathcal{V}_2^5$, and the number of 5-spaces meeting $\mathcal{V}_2^5$ in exactly the conic directrix and i generators is r_i. In this case the number of ways to pick a point of $\Pi_5 \cap \mathcal{V}_2^5$ is $iq + q + 1$. Hence the total number of pairs (A, Π_5) with Π_5 containing the conic directrix is

$$n_2 = \sum_{i=0}^{3} r_i(iq + q + 1) = 2q^4 + 4q^3 + 4q^2 + 3q + 1.$$

For n_3, consider a 5-space Π_5 that contains a twisted cubic. Then by Lemma 4.2, Π_5 contains 0, 1, or 2 generators of $\mathcal{V}_2^5$, and the number of 5-spaces meeting $\mathcal{V}_2^5$ in a given twisted cubic and i generators is s_i. In this case the number of ways to pick A in $\mathcal{V}_2^5 \cap \Pi_5$ is $iq + q + 1$. Hence the number of pairs (A, Π_5) with Π_5 containing a twisted cubic of $\mathcal{V}_2^5$ is

$$n_3 = q^2 \sum_{i=0}^{2} s_i(iq + q + 1) = 2q^5 + 4q^4 + 3q^3 + q^2.$$

For n_4, consider a 5-space Π_5 that contains a 4-dim nrc of $\mathcal{V}_2^5$. By Lemma 4.3, Π_5 contains 1 generator of $\mathcal{V}_2^5$. By Lemma 4.6, the number of 5-spaces meeting $\mathcal{V}_2^5$ in exactly a 4-dim nrc and one generator is $q^5 + q^4 - q^3 - q^2$. The number of ways to pick A in $\mathcal{V}_2^5 \cap \Pi_5$ is $2q + 1$. So

$$n_4 = (q^5 + q^4 - q^3 - q^2) \times (2q + 1) = 2q^6 + 3q^5 - q^4 - 3q^3 - q^2.$$

Finally, denote the number of 5-spaces containing a 5-dim nrc of $\mathcal{V}_2^5$ by y. Then the number of pairs (A, Π_5) with Π_5 containing a 5-dim nrc of $\mathcal{V}_2^5$ is

$$n_5 = y \times (q + 1).$$

Substituting the calculated values for x, n_2, n_3, n_4, n_5 into (1) and rearranging gives $y = q^6 - q^4$ as required. □

Summarising the preceding lemmas gives the following theorem describing $\mathcal{V}_2^5$.

Theorem 4.8. *Let $\mathcal{V}_2^5$ be the ruled quintic surface in* PG(6, q), $q \geq 6$.

(1) *$\mathcal{V}_2^5$ contains exactly*

$q+1$	*lines,*
1	*nondegenerate conic,*
q^2	*twisted cubics,*
q^4-q^2	4-*dim nrcs,*
q^6-q^4	5-*dim nrcs.*

(2) *A* 5-*space meets $\mathcal{V}_2^5$ in one of the following configurations*:

number of 5-*spaces*	*meeting $\mathcal{V}_2^5$ in the configuration*
q^6-q^4	5-*dim nrc,*
$q^5+q^4-q^3-q^2$	4-*dim nrc and* 1 *generator,*
$(q^4-q^3)/2$	*twisted cubic,*
q^3+q^2	*twisted cubic and* 1 *generator,*
$(q^4+q^3)/2$	*twisted cubic and* 2 *generators,*
$(q^3-q)/3$	*conic,*
$q^3/2+q/2+1$	*conic and* 1 *generator,*
q^2+q	*conic and* 2 *generators,*
$(q^3-q)/6$	*conic and* 3 *generators.*

5. The Bruck–Bose spread and 5-spaces

Let $\mathcal{S}$ be a regular 2-spread in a 5-space Σ_∞ in PG(6, q), and position $\mathcal{V}_2^5$ so that it corresponds to a tangent $\mathbb{F}_q$-subplane of PG(2, q^3). So $\mathcal{V}_2^5$ has splash $\mathbb{S} \subset \mathcal{S}$, the conic directrix $\mathcal{C}$ lies in a plane $\alpha \in \mathbb{S}$, and each of the q^2 3-spaces containing a twisted cubic directrix of $\mathcal{V}_2^5$ meets Σ_∞ in a distinct plane of $\mathbb{S} \setminus \alpha$. In Corollary 3.4, we looked at how 3-spaces containing a plane of $\mathcal{S}$ meet $\mathcal{V}_2^5$. In Lemma 4.5, we looked at how 4-spaces containing a plane of $\mathcal{S}$ meet $\mathcal{V}_2^5$. Next we look at how 5-spaces containing a plane of $\mathcal{S}$ meet $\mathcal{V}_2^5$. Note that straightforward counting shows that a 5-space distinct from Σ_∞ contains a unique plane π of $\mathcal{S}$, and meets every other plane of $\mathcal{S}$ in a line. If $\pi = \alpha$, then Lemma 4.1 describes the possible intersections with $\mathcal{V}_2^5$. The next theorem describes the possible intersections with $\mathcal{V}_2^5$ for the remaining cases $\pi \in \mathbb{S} \setminus \alpha$ and $\pi \in \mathcal{S} \setminus \mathbb{S}$.

Theorem 5.1. *Position $\mathcal{V}_2^5$ as in Corollary 3.3, so $\mathcal{S}$ is a regular* 2-*spread in a hyperplane Σ_∞, the conic directrix $\mathcal{C}$ lies in a plane $\alpha \in \mathcal{S}$, and $\mathcal{V}_2^5$ has splash $\mathbb{S} \subset \mathcal{S}$. Let ℓ be a line of α with $|\ell \cap \mathcal{C}| = i$ and let $\pi \in \mathcal{S}$, $\pi \neq \alpha$. Then the q* 5-*spaces containing π, ℓ and distinct from Σ_∞ meet $\mathcal{V}_2^5$ as follows.*

(1) *If $\pi \in \mathbb{S} \setminus \alpha$, then $q-1$ meet $\mathcal{V}_2^5$ in a* 5-*dim nrc, and* 1 *meets $\mathcal{V}_2^5$ in a twisted cubic and i generators.*

(2) *If* $\pi \in \mathcal{S} \setminus \mathbb{S}$, *then* $q - i$ *meet* $\mathcal{V}_2^5$ *in a* 5*-dim nrc, and* i *meet* $\mathcal{V}_2^5$ *in a* 4*-dim nrc and* 1 *generator.*

Proof. By [Barwick and Jackson 2012], the group of collineations of $\mathrm{PG}(6, q)$ fixing $\mathcal{S}$ and $\mathcal{V}_2^5$ is transitive on the planes of $\mathbb{S} \setminus \alpha$ and on the planes of $\mathcal{S} \setminus \mathbb{S}$. As this group fixes the conic directrix $\mathcal{C}$, it is transitive on the lines of α tangent to $\mathcal{C}$, the lines of α secant to $\mathcal{C}$, and the lines of α exterior to $\mathcal{C}$. So without loss of generality let ℓ_0 be a line of α exterior to $\mathcal{C}$, let ℓ_1 be a line of α tangent to $\mathcal{C}$, let ℓ_2 be a line of α secant to $\mathcal{C}$, let β be a plane in $\mathbb{S} \setminus \alpha$, and let γ be a plane of $\mathcal{S} \setminus \mathbb{S}$. For $i = 0, 1, 2$, label the 4-spaces $\Sigma_{4,i} = \langle \beta, \ell_i \rangle$ and $\Pi_{4,i} = \langle \gamma, \ell_i \rangle$. By Corollary 3.4, as $\beta \in \mathbb{S} \setminus \alpha$, there is a unique twisted cubic of $\mathcal{V}_2^5$ that lies in a 3-space about β. Denote this 3-space by Π_3. Hence for $i = 0, 1, 2$, there is a unique 5-space containing $\Sigma_{4,i}$ whose intersection with $\mathcal{V}_2^5$ contains a twisted cubic, namely the 5-space $\langle \Pi_3, \ell_i \rangle$.

First consider the line ℓ_0 which is exterior to $\mathcal{C}$. A 5-space meeting α in ℓ_0 contains 0 points of $\mathcal{C}$, and so contains 0 generators of $\mathcal{V}_2^5$. The 4-space $\Sigma_{4,0} = \langle \beta, \ell_0 \rangle$ lies in q 5-spaces distinct from Σ_∞, each containing 0 generators of $\mathcal{V}_2^5$. Exactly one of these 5-spaces, namely $\langle \Pi_3, \ell_0 \rangle$, contains a twisted cubic of $\mathcal{V}_2^5$. The remaining $q - 1$ 5-spaces about $\Sigma_{4,0}$ contain 0 generators, and do not contain a conic or twisted cubic of $\mathcal{V}_2^5$, so by Theorem 4.8, they meet $\mathcal{V}_2^5$ in a 5-dim nrc, proving (1) for $i = 0$. For (2), let $\Pi_5 \neq \Sigma_\infty$ be any 5-space containing $\Pi_{4,0} = \langle \gamma, \ell_0 \rangle$. As $\gamma \notin \mathbb{S}$, by Corollary 3.3, Π_5 cannot contain a twisted cubic of $\mathcal{V}_2^5$. As Π_5 contains 0 generator lines of $\mathcal{V}_2^5$ and does not contain a conic or twisted cubic of $\mathcal{V}_2^5$, by Theorem 4.8, Π_5 meets $\mathcal{V}_2^5$ in a 5-dim nrc. That is, the q 5-spaces (distinct from Σ_∞) containing $\Pi_{4,0}$ meet $\mathcal{V}_2^5$ in a 5-dim nrc, proving (2) for $i = 0$.

Next consider the line ℓ_1 which is tangent to $\mathcal{C}$. Let $P = \ell_1 \cap \mathcal{C}$ and denote the generator of $\mathcal{V}_2^5$ through P by ℓ_P. A 5-space meeting α in a tangent line contains 1 point of $\mathcal{C}$, and so contains at most one generator of $\mathcal{V}_2^5$. So exactly one 5-space contains $\Sigma_{4,1}$ and a generator, namely the 5-space $\langle \Sigma_{4,1}, \ell_P \rangle$. Consider the 5-space $\langle \Pi_3, \ell_1 \rangle$. It contains P and a twisted cubic of $\mathcal{V}_2^5$, which by Corollary 4.4 is disjoint from α. Hence $\langle \Pi_3, \ell_1 \rangle$ contains the generator ℓ_P. That is, $\langle \Pi_3, \ell_1 \rangle$ contains β, ℓ_1, ℓ_P and so $\langle \Pi_3, \ell_1 \rangle = \langle \Sigma_{4,1}, \ell_P \rangle$. That is, the intersection of $\langle \Sigma_{4,1}, \ell_P \rangle$ with $\mathcal{V}_2^5$ is a twisted cubic and one generator. Let $\Pi_5 \neq \Sigma_\infty$ be one of the remaining $q - 1$ 5-spaces (distinct from Σ_∞) that contains $\Sigma_{4,1}$, so Π_5 contains 0 generators of $\mathcal{V}_2^5$ and does not contain a conic or twisted cubic of $\mathcal{V}_2^5$. So by Theorem 4.8, Π_5 meets $\mathcal{V}_2^5$ in a 5-dim nrc, proving (1) for $i = 1$. For (2), we consider $\Pi_{4,1} = \langle \gamma, \ell_1 \rangle$. By Corollary 3.3, as $\gamma \notin \mathbb{S}$, no 5-space containing $\Pi_{4,1}$ contains a twisted cubic of $\mathcal{V}_2^5$. The 5-space $\langle \Pi_{4,1}, \ell_P \rangle$ contains one generator of $\mathcal{V}_2^5$, so by Theorem 4.8, it meets $\mathcal{V}_2^5$ in exactly a 4-dim nrc and the generator ℓ_P. Let $\Pi_5 \neq \Sigma_\infty$ be one of the remaining $q - 1$ 5-spaces containing $\Pi_{4,1}$. Then Π_5 contains 0 generators of $\mathcal{V}_2^5$. So by Theorem 4.8, Π_5 meets $\mathcal{V}_2^5$ in a 5-dim nrc, proving (2) for $i = 1$.

Finally, consider the line ℓ_2 which is secant to $\mathcal{C}$. Let $\mathcal{C} \cap \ell_2 = \{P, Q\}$ and let ℓ_P, ℓ_Q be the generators of $\mathcal{V}_2^5$ through P, Q, respectively. The intersection of the 5-space $\langle \Pi_3, \ell_2 \rangle$ and $\mathcal{V}_2^5$ contains a twisted cubic, and P and Q. By Corollary 4.4, this twisted cubic is disjoint from α, so $\langle \Pi_3, \ell_2 \rangle$ contains the two generators ℓ_P, ℓ_Q. Thus $\langle \Pi_3, \ell_2 \rangle = \langle \Sigma_{4,2}, \ell_P \rangle = \langle \Sigma_{4,2}, \ell_Q \rangle = \langle \Sigma_{4,2}, \ell_P, \ell_Q \rangle$. The remaining $q-1$ 5-spaces (distinct from Σ_∞) about $\Sigma_{4,2}$ contain 0 generators and two points of $\mathcal{C}$. By Lemma 4.3 they cannot contain a 4-dim nrc of $\mathcal{V}_2^5$. So by Theorem 4.8, they meet $\mathcal{V}_2^5$ in a 5-dim nrc, proving (1) for $i = 2$. For (2), let $\Pi_5 \neq \Sigma_\infty$ be a 5-space containing $\Pi_{4,2} = \langle \gamma, \ell_2 \rangle$. By Corollary 3.3, Π_5 does not contain a twisted cubic of $\mathcal{V}_2^5$, as $\gamma \notin \mathbb{S}$. So by Theorem 4.8, Π_5 contains at most one generator of $\mathcal{V}_2^5$. Hence $\langle \Pi_{4,2}, \ell_P \rangle, \langle \Pi_{4,2}, \ell_Q \rangle$ are distinct 5-spaces about $\Pi_{4,2}$, and by Theorem 4.8, they each meet $\mathcal{V}_2^5$ in a 4-dim nrc and one generator. Let $\Sigma_5 \neq \Sigma_\infty$ be one of the remaining $q-2$ 5-spaces about $\Pi_{4,2}$. Then Σ_5 contains 0 generators of $\mathcal{V}_2^5$, and so by Theorem 4.8, meets $\mathcal{V}_2^5$ in a 5-dim nrc, proving (2) for $i = 2$. □

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A characterization of Clifford parallelism by automorphisms

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Betten and Riesinger have shown that Clifford parallelism on real projective space is the only topological parallelism that is left invariant by a group of dimension at least 5. We improve the bound to 4. Examples of different parallelisms admitting a group of dimension ≤ 3 are known, so 3 is the "critical dimension".

Consider $\mathbb{R}^4$ as the quaternion skew field $\mathbb{H}$. Then the orthogonal group $\mathrm{SO}(4, \mathbb{R})$ may be described as the product of two commuting copies $\tilde{\Lambda}$, $\tilde{\Phi}$ of the unitary group $\mathrm{U}(2, \mathbb{C})$, consisting of the maps $q \mapsto aq$ and $q \mapsto qb$, respectively, where a, b are quaternions of norm one and multiplication is quaternion multiplication. The intersection of the two factors is of order two, containing the map $-\mathrm{id}$. Thus, passing to projective space, we get $\mathrm{PSO}(4, \mathbb{R}) = \Lambda \times \Phi$, a direct product of two copies of $\mathrm{SO}(3, \mathbb{R})$. The left and right Clifford parallelisms are defined as the equivalence relations on the line space of $\mathrm{PG}(3, \mathbb{R})$ formed by the orbits of Λ and Φ, respectively.

The two Clifford parallelisms are equivalent under quaternion conjugation $q \to \bar{q}$; this is immediate from their definition in view of the fact that conjugation does not change the norm and is an antiautomorphism, i.e., that $\overline{pq} = \bar{q}\,\bar{p}$. Note that both Λ and Φ are transitive on the point set of projective space. Since they centralize one another, each acts transitively on the parallelism defined by the other, and the group $\mathrm{PSO}(4, \mathbb{R})$ leaves both parallelisms invariant (we say that it consists of *automorphisms* of these parallelisms). For more information on Clifford parallels, see [Berger 1987; Klingenberg 1984; Betten and Riesinger 2012]. For generalizations to other dimensions, compare also [Tyrrell and Semple 1971].

The notion of a *topological parallelism* on real projective 3-space $\mathrm{PG}(3, \mathbb{R})$ generalizes this example. A *spread* is a set $\mathcal{C}$ of lines such that every point is incident with exactly one of them, and a topological parallelism may be defined

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as a compact set Π of compact spreads such that every line belongs to exactly one of them; see, e.g., [Betten and Riesinger 2014b] for details. Many examples of different topological parallelisms have been constructed in a series of papers by Betten and Riesinger, see, e.g., [Betten and Riesinger 2009].

The group $\Sigma = \mathrm{Aut}\,\Pi$ of automorphisms of a topological parallelism is a closed subgroup of the Lie group $\mathrm{PGL}(4, \mathbb{R})$, hence it is a Lie group, as well. In particular, the identity component Σ^1 is an open subgroup of Σ and has the same (manifold) dimension as Σ. We know that Σ^1 is compact [Betten and Löwen 2017], and hence (conjugate to) a subgroup of $\mathrm{PSO}(4, \mathbb{R}) \cong \mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R})$. The group $\mathrm{SO}(3, \mathbb{R})$ does not have any 2-dimensional closed subgroups, because its Lie algebra is $\mathbb{R}^3$ with the vector product $\times$ and $x \times y$ is always orthogonal to both x and y. Moreover, the 1-dimensional closed subgroups of $\mathrm{SO}(3, \mathbb{R})$ form a single conjugacy class. It follows easily that there are no closed 5-dimensional subgroups of $\mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R})$ and all 4-dimensional ones are isomorphic to $\mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(2, \mathbb{R})$.

We see that in the case of the Clifford parallelism, Σ^1 is the 6-dimensional group $\mathrm{PSO}(4, \mathbb{R})$ that we used to define the parallelism. Betten and Riesinger [2014b] proved that no other topological parallelism has a group of dimension $\dim \Sigma \geq 5$. Examples of parallelisms with 1-, 2- or 3-dimensional automorphism groups are known; see [Betten and Riesinger 2014a; 2009; 2011] . Here we consider parallelisms with a 4-dimensional group.

Theorem 1. *Let Σ be the automorphism group of a topological parallelism Π on* $\mathrm{PG}(3, \mathbb{R})$. *If* $\dim \Sigma \geq 4$, *then Π is equivalent to the Clifford parallelism.*

Proof. Recall that a topological parallelism Π is homeomorphic to the real projective plane in the Hausdorff topology on the space of compact sets of lines, and that every equivalence class is a compact spread and homeomorphic to the 2-sphere; compare [Betten and Riesinger 2014b].

The remarks preceding the theorem show that a group Σ of dimension at least 4 contains a 4-dimensional connected closed subgroup Δ, and it will suffice for our proof to use this group. Further, up to equivalence, we may assume that $\Delta = \Lambda \cdot \Gamma$, where $\Gamma \leq \Phi$ is the subgroup defined by restricting the factor b to be a complex number (here we use the notation of the introduction). Since Λ does not have any one-dimensional coset spaces, we know that Λ acts on Π either transitively or trivially. If it acts trivially, then the classes of Π are the Λ-orbits of lines, and we have the Clifford parallelism. Observe here that every Λ-orbit is contained in a single class, and both the orbit and the class are 2-spheres.

In what follows, assume therefore that Λ acts transitively on Π. There is only one possibility for this action, namely, the standard transitive action of $\mathrm{SO}(3, \mathbb{R})$ on the real projective plane. Every 2-dimensional subgroup of Δ contains Γ. Hence, there is no effective action of Δ on the projective plane Π, and the kernel can only be Γ since the only other proper normal subgroup is Λ, which is transitive. If $\mathcal{C} \in \Pi$ is any equivalence class, then the stabilizer $\Lambda_{\mathcal{C}}$ is a product of a 1-torus and a group of order two. Hence $\Delta_{\mathcal{C}}$ contains a 2-torus T. There is only one conjugacy class of 2-tori in Δ, represented by the group

$$T_0 = \{\langle q \rangle \mapsto \langle aqb \rangle \mid a, b \in \mathbb{C}, |a| = |b| = 1\}.$$

Here, $\langle q \rangle$ denotes the 1-dimensional real vector space spanned by q. We may assume that $T = T_0$. Write quaternions as pairs of complex numbers with multiplication $(x, y)(u, v) = (xu - \bar{v}y, vx + y\bar{u})$; see 11.1 of [Salzmann et al. 1995]. Then complex numbers become pairs $(a, 0)$, and the elements of T are now given by

$$\langle (z, w) \rangle \mapsto \langle (azb, aw\bar{b}) \rangle.$$

The kernel of ineffectivity of T on the 2-sphere $\mathcal{C}$ must be a 1-torus Ξ, and the elements of the kernel other than the identity cannot have eigenvalue 1 — otherwise they would be axial collineations of the translation plane defined by the spread $\mathcal{C}$ and would act nontrivially on $\mathcal{C}$. There are only two subgroups of the 2-torus satisfying these conditions, given by $b = 1$ and by $a = 1$, respectively. In other words, the kernel Ξ is a subgroup either of Λ or of Φ. In both cases, $\mathcal{C}$ consists of the fixed lines of Ξ. If $\Xi \leq \Phi$, then Λ permutes these lines, contrary to the transitivity of Λ on Π. If $\Xi \leq \Lambda$, then Φ permutes the fixed lines, which means that $\mathcal{C}$ is a Φ-orbit. Now Λ is transitive both on Π and on the set of Φ-orbits, hence Π equals the Clifford parallelism formed by the Φ-orbits. □

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Generalized quadrangles, Laguerre planes and shift planes of odd order

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We characterize the Miquelian Laguerre planes, and thus the classical orthogonal generalized quadrangles $Q(4, q)$, of odd order q by the existence of shift groups in affine derivations.

Introduction

A finite Laguerre plane $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ of order n consists of a set P of $n(n+1)$ points, a set $\mathcal{C}$ of n^3 circles and a set $\mathcal{G}$ of $n+1$ generators, where both circles and generators are subsets of P, such that the following three axioms are satisfied:

(G) $\mathcal{G}$ partitions P and each generator contains n points.

(C) Each circle intersects each generator in precisely one point.

(J) Three points of which no two are on the same generator are joined by a unique circle.

Circles through x are called *touching in* x if they are equal or have no other point in common. The set of all circles through a given point x is denoted by $\mathcal{C}_x$. The *derived affine plane* $\mathbb{A}_x = (P \setminus [x], \mathcal{C}_x \cup \mathcal{G} \setminus \{[x]\})$ at a point $x \in P$ has the collection of all points not on the generator $[x]$ through x as its point set and, as lines, all circles passing through x (without the point x) and all generators apart from $[x]$. The axioms above easily yield that $\mathbb{A}_x$ is an affine plane. We refer to the generators as *vertical lines* in $\mathbb{A}_x$. Circles that touch each other in x give parallel lines in $\mathbb{A}_x$. A line W is introduced to obtain the projective completion $\mathbb{P}_x$ of $\mathbb{A}_x$; the common point of the verticals will be denoted by $v \in W$.

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The group $\mathrm{Aut}(\mathcal{L})$ of all automorphisms of a Laguerre plane $\mathcal{L}$ acts on the set $\mathcal{G}$ of generators. We call $\mathcal{L}$ an *elation Laguerre plane* if the kernel Δ of that action acts transitively on the set $\mathcal{C}$ of circles. It is known (see [Steinke 1991, 1.3]) that in every finite elation Laguerre plane the group Δ has a (unique) regular normal subgroup E called the *elation group*. For more details on elation Laguerre planes, we refer the reader to the introduction in [Steinke and Stroppel 2013].

In the present note, we only use a weaker transitivity assumption on Δ but combine this with additional assumptions. Our results can (and will) be applied to elation Laguerre planes with additional homogeneity assumptions, e.g., in [Steinke and Stroppel 2018] (see Theorem 2.3 below).

Finite Laguerre planes of *odd* order q are equivalent to antiregular generalized quadrangles of order q (i.e., with parameters (q, q)); see [Thas et al. 2006, Theorem 2.4.2]. Derivation at an antiregular point of a generalized quadrangle of odd order q produces a Laguerre plane of order q. Conversely, the Lie geometry of a Laguerre plane of odd order yields a generalized quadrangle with an antiregular point. Thus this generalized quadrangle is antiregular; see [Thas et al. 2006, Theorem 2.4.6]. However, this construction does not work when q is even.

On the other hand, a finite elation Laguerre plane of order q (regardless of whether q is even or odd) is equivalent to a generalized oval (or pseudo-oval) with $q+1$ points and thus to a translation generalized quadrangle of order q; see [Casse et al. 1985] or [Thas et al. 2006].

The elation group E is a $3m$-dimensional vector space over some field $\mathbb{F}$, and the stabilizer E_x of each point x is a $2m$-dimensional vector subspace of E. Under a duality the E_x yield a family of $q+1$ vector subspaces of dimension m in $\mathbb{F}^{3m}$. Changing to projective notation one sees that, geometrically, a finite elation Laguerre plane of order q is equivalent to a $(q+1)$-set of $(m-1)$-dimensional subspaces in the $(3m-1)$-dimensional projective space over $\mathbb{F}$; compare [Casse et al. 1985]. In [Thas et al. 2006] such a set is called a generalized oval. In fact, a generalized oval is just a 4-gonal family of type (q, q) in an abelian group; see [Thas et al. 2006, 3.2.2]. One obtains a translation generalized quadrangle of order q from a generalized oval, and on the other hand, every translation generalized quadrangle of order q arises from a generalized oval in this way; see [Thas et al. 2006, Theorem 3.5.1] or [Payne and Thas 2009, 8.7.1].

With the correspondence between Laguerre planes and certain generalized quadrangles as above, our results on Laguerre planes have corresponding formulations in generalized quadrangles, but we mainly use the language of Laguerre planes.

1. Translation planes

Theorem 1.1. *Let $\mathbb{P}$ be a finite projective plane of order n. Assume that a subgroup $D \leq \mathrm{Aut}(\mathbb{P})$ fixes each point on some line L. If n^2 divides the order of D then D*

contains a subgroup T of order n^2 consisting of elations with axis L. In particular, the plane $\mathbb{P}$ is a translation plane, and the order n is a prime power.

Proof. For each nontrivial element $\delta \in D$ there is a (unique) center c_δ, i.e., a point c_δ such that δ fixes each line through c_δ ([Baer 1946], see [Hughes and Piper 1973, Theorem 4.9]). The elations in D are just those in the set

$$T := \{\mathrm{id}\} \cup \{\tau \in D \setminus \{\mathrm{id}\} \mid c_\tau \in L\};$$

that set forms a normal subgroup of D (see [Hughes and Piper 1973, Theorem 4.13]).

For any point x outside L, the stabilizer D_x consists of id and elements with center x. The order of any element of D_x divides $n-1$. So the order of D_x and the number n^2 of points outside L are coprime, and D acts transitively on the set A of points outside L. For each $\delta \in D \setminus T$ we have $c_\delta \notin L$, and $\delta \in D_{c_\delta}$ yields that the order of δ divides $n-1$, and is coprime to n^2.

Let $\mathcal{B}$ denote the set of T-orbits in A. Then D acts on $\mathcal{B}$, and so does D/T because $T \trianglelefteq D$ acts trivially on $\mathcal{B}$. Transitivity of D on A implies that D/T is transitive on $\mathcal{B}$. Now $|\mathcal{B}| = n^2/|T|$ divides $|D/T|$. The latter order is coprime to n^2 because each member (distinct from T) of the quotient has a representative of order coprime to n^2. So $|\mathcal{B}| = 1$, and transitivity of T is proved. □

Theorem 1.2. *Let $\mathcal{L}$ be a Laguerre plane of finite order n with kernel Δ. If ∞ is a point such that n^2 divides the order of the stabilizer Δ_∞ then the derived projective plane $\mathbb{P}_\infty$ is a dual translation plane, and the order n is a prime power.*

Proof. The group D induced by Δ_∞ on the dual $\mathbb{P}$ of $\mathbb{P}_\infty$ satisfies the assumptions of Theorem 1.1. □

Theorem 1.3. *Let $\mathcal{L}$ be a Laguerre plane of finite order n, and assume that there is a point ∞ such that n^2 divides the order of the stabilizer Δ_∞. If there exist a circle $K \in \mathcal{C}_\infty$ and a subgroup $H \leq \mathrm{Aut}(\mathcal{L})_\infty$ such that H fixes each circle touching K in ∞ and H acts transitively on $K \setminus \{\infty\}$, then $\mathbb{P}_\infty$ has Lenz type V (at least), and is coordinatized by a semifield.*

Proof. From Theorem 1.2 we know that $\mathbb{P}_\infty$ is a dual translation plane. The translation axis in the dual of $\mathbb{P}_\infty$ is the common point v for the generators in the projective closure of $\mathbb{A}_\infty$. The elations of $\mathbb{P}_\infty$ with center v and axis W form a group of order n; we denote that group by V and note that V is a group of translations of $\mathbb{A}_\infty$.

Our assumptions on H secure that H induces a group of translations of $\mathbb{A}_\infty$; the common center is the point at infinity for the "horizontal line" $K \setminus \{\infty\}$. We obtain a transitive group HV of translations on $\mathbb{A}_\infty$. So $\mathbb{P}_\infty$ is also a translation plane, and has Lenz type V at least. □

2. Shift groups

Recall that a shift group on a projective plane is a group of automorphisms fixing an incident point-line pair (x, Y) and acting regularly both on the set of points outside Y and on the set of lines not through x.

Theorem 2.1. *Let $\mathcal{L}$ be a finite Laguerre plane of odd order, and assume that there exists a point u and a subgroup $S \leq \mathrm{Aut}(\mathcal{L})_u$ such that S induces a transitive group of translations on the affine plane $\mathbb{A}_u$.*

(1) *If $s \in [u] \setminus \{u\}$ is fixed by S then S induces a shift group on $\mathbb{P}_s$.*

(2) *If S fixes a point t of $\mathcal{L}$ and induces a transitive group of translations on $\mathbb{A}_t$ then $t = u$.*

Proof. Let n denote the order of $\mathcal{L}$. Assume that $s \in [u] \setminus \{u\}$ is fixed by S. Then S induces a group of automorphisms of $\mathbb{P}_s$; we have to exhibit an incident point-line pair (x, Y) such that S acts regularly both on the set of points outside Y and on the set of lines not through x.

It is obvious that S acts regularly on the set of affine points in $\mathbb{P}_s$ because that set coincides with the set of points of $\mathbb{A}_u$. We let the line W at infinity play the role of Y. Also, the set of vertical lines (induced by generators) is invariant under S, so we let their point at infinity play the role of x (so $x \in W$ is the point v at infinity of vertical lines).

It remains to show that S acts regularly on the set of nonvertical lines of $\mathbb{A}_s$; these lines are induced by the circles through s. Assume that $\tau \in S$ fixes a circle C through s. Our assumption that n be odd implies that the translation of $\mathbb{A}_u$ induced by τ does not have any orbit of length 2, and we obtain that τ is trivial if there is a set of one or two points outside $[u]$ invariant under τ.

Note that no vertical line distinct from $[u]$ is fixed by τ when τ is not the identity. As τ induces a translation on $\mathbb{A}_u$, there exists $D \in \mathcal{C}_u$ such that τ fixes each circle touching D in u (these circles induce the parallels to the line induced by D on $\mathbb{A}_u$). Pick a point $z \in C \setminus \{s\}$, and let D' be the circle through z touching D in u. Then τ leaves the intersection $D' \cap C$ invariant. This is a set with one or two elements, and we find that τ is trivial. So the orbit of C under S has length $|S| = n^2$, and fills all of $\mathcal{C}_s$. Thus S acts regularly on the set of nonvertical lines of $\mathbb{A}_s$, as required.

Now assume that S fixes t and induces a transitive group of translations on $\mathbb{A}_t$. Then $t \in [u]$ because S acts regularly on the set of points outside $[u]$. For any circle $C \in \mathcal{C}_t$, we pick two points $a, b \in C \setminus \{t\}$. Then there exists $\tau \in S$ such that $\tau(a) = b$. As τ is a translation both of $\mathbb{A}_u$ and of $\mathbb{A}_t$, the orbit of a under $\langle\tau\rangle$ is contained both in the line C of $\mathbb{A}_t$ and in some line B of $\mathbb{A}_u$, that is, in some circle B through u. Since n is odd, that orbit has at least three points, and $B = C$. This yields $t = u$, as claimed. □

Theorem 2.2. *Assume that $\mathcal{L}$ is a finite Laguerre plane of odd order n, and let ∞ be a point. Let U denote the set of all points $u \in [\infty] \setminus \{\infty\}$ such that there exists a subgroup $S_u \leq \mathrm{Aut}(\mathcal{L})$ of order n^2 fixing both ∞ and u and acting as a group of translations on $\mathbb{A}_u$. Then the following hold*:

(1) *There are at least $|U|$ many different shift groups on $\mathbb{P}_\infty$.*

(2) *If $|U| > 1$ then $\mathbb{A}_\infty$ is a translation plane.*

(3) *If $\mathbb{A}_\infty$ is a translation plane and U is not empty then $\mathbb{P}_\infty$ has Lenz type V at least and can be coordinatized by a commutative semifield, and the middle nucleus of such a coordinatizing semifield has order at least $|U| + 1$.*

(4) *If $|U| > \sqrt{n}$ then $\mathbb{P}_\infty$ is Desarguesian.*

Proof. Using Theorem 2.1 we see for any $u \in U$ that S_u is a shift group on $\mathbb{P}_\infty$, and different points $t, u \in U$ yield different groups S_t and S_u. This gives the first assertion. All these shift groups have the same fixed flag in $\mathbb{P}_\infty$.

If a finite projective plane admits more than one shift group, it is a translation plane; see [Knarr and Stroppel 2009, 10.2]. If a translation plane admits at least one shift group then it can be coordinatized by a commutative semifield ([Knarr and Stroppel 2009, 9.12], [Spille and Pieper-Seier 1998]) and the different shift groups with the same fixed flag are parameterized by the nonzero elements of the middle nucleus of such a semifield; see [Knarr and Stroppel 2009, 9.4].

The additive group of the coordinatizing semifield forms a vector space over the middle nucleus (see [Hughes and Piper 1973, p. 170]). If the middle nucleus has more than $\sqrt{n}$ elements then that vector space has dimension 1, and the middle nucleus coincides with the semifield. This means that the semifield is a field, and the plane is Desarguesian. □

Theorem 2.2 is used in [Steinke and Stroppel 2018] to prove the following:

Theorem 2.3. *Let $\mathcal{L}$ be a finite elation Laguerre plane of odd order. If there exists a point ∞ such that $\mathrm{Aut}(\mathcal{L})_\infty$ acts two-transitively on $\mathcal{G} \setminus \{[\infty]\}$ then the affine plane $\mathbb{A}_\infty$ is Desarguesian, and $\mathcal{L}$ is Miquelian.* □

Remark 2.4. If $\mathbb{P}$ is a projective plane of even order then a shift group on $\mathbb{P}$ will never be elementary abelian; see [Knarr and Stroppel 2009, 1.5, 5.8]. Thus a shift group on such a plane will not act as a transitive group of translations on any other affine plane (of the same order).

With the correspondence between Laguerre planes and certain generalized quadrangles as mentioned in the introduction, Theorem 2.3 yields the following. Here we use the standard notation of $x^\perp$ for all points collinear to x in a generalized quadrangle $\mathcal{Q}$ and $\pi(x, y)$ for the affine plane obtained at an antiregular point x; see [Thas et al. 2006, Theorem 2.4.1] for a definition).

Corollary 2.5. *Let $\mathcal{Q}$ be a finite translation generalized quadrangle of odd order q with an antiregular base point x. If there exists a point y collinear to x such that the stabilizer $\mathrm{Aut}(\mathcal{Q})_{x,y}$ acts two-transitively on $x^\perp \setminus \{x, y\}^{\perp\perp}$, then the affine plane $\pi(x, y)$ is Desarguesian, and $\mathcal{Q}$ is the classical orthogonal generalized quadrangle $Q(4, q)$.*

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A new family of 2-dimensional Laguerre planes that admit $\mathrm{PSL}_2(\mathbb{R}) \times \mathbb{R}$ as a group of automorphisms

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We construct a new family of 2-dimensional Laguerre planes that differ from the classical real Laguerre plane only in the circles that meet a given circle in precisely two points. These planes share many properties with but are non-isomorphic to certain semiclassical Laguerre planes pasted along a circle in that they admit 4-dimensional groups of automorphisms that contain $\mathrm{PSL}_2(\mathbb{R})$ and are of Kleinewillinghöfer type I.G.1.

1. Introduction

A 2-dimensional Laguerre plane is an incidence structure on the cylinder $Z = \mathbb{S}^1 \times \mathbb{R}$ determined by a collection of graphs of continuous functions $\mathbb{S}^1 \to \mathbb{R}$; see the following section for a definition of and facts about Laguerre planes. The collection of all automorphisms of a 2-dimensional Laguerre plane is a Lie group of dimension at most 7. All 2-dimensional Laguerre planes whose automorphism groups have dimension at least 5 are known; see [Löwen and Pfüller 1987, Theorem 1]. The classification of 2-dimensional Laguerre planes whose automorphism groups are 4-dimensional is almost complete except when the automorphism group fixes no parallel class but is not transitive on the point set. Examples of 2-dimensional Laguerre planes which exhibit such groups of automorphisms can be found in [Steinke 1987; Löwen and Steinke 2007].

In this paper we contribute to the investigation of 2-dimensional Laguerre planes whose automorphism groups are 4-dimensional, and construct a new family of such planes that admit a group of automorphisms isomorphic to $\mathrm{PSL}_2(\mathbb{R}) \times \mathbb{R}$. It shares many circles with the classical real Laguerre plane (and the semiclassical Laguerre planes of group dimension 4 from [Steinke 1987]; see Section 5 for a brief description). Its full automorphism group fixes a distinguished circle and is 3-transitive on it. Derived projective planes at points on the distinguished circle are dual to

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the derived projective planes at corresponding points in the semiclassical Laguerre planes of group dimension 4 pasted along a circle. However, our Laguerre planes are not semiclassical. The new planes and the semiclassical Laguerre planes of group dimension 4 will play a prominent role in the classification of 2-dimensional Laguerre planes of group dimension 4 whose automorphism groups fix a circle.

Section 2 summarizes facts about 2-dimensional Laguerre planes. Section 3 describes the new family of 2-dimensional Laguerre planes. Section 4 proves that these are indeed 2-dimensional Laguerre planes. In the last section we determine isomorphism classes, full automorphism groups and Kleinewillinghöfer types of our planes. We further show that the Laguerre planes are not semiclassical and investigate the associated compact 3-dimensional generalized quadrangles.

2. Laguerre planes

A *Laguerre plane* $\mathscr{L} = (P, \mathscr{C}, \|)$ is an incidence structure consisting of a point set P, a circle set $\mathscr{C}$ and an equivalence relation $\|$ (parallelism) defined on the point set such that

- three mutually nonparallel points can be joined by a unique circle,
- given a point p on a circle C and a point q not parallel to p, there is a unique circle that contains both points and *touches C geometrically* at p, that is, intersects C only in p or coincides with C,
- each parallel class meets each circle in a unique point (parallel projection), and
- there are four points not on a circle and there is a circle that contains at least three points (richness);

compare [Groh 1968; 1969b].

In this paper we are only concerned with Laguerre planes whose common point set is the cylinder $Z = \mathbb{S}^1 \times \mathbb{R}$ (where the 1-sphere $\mathbb{S}^1$ usually is represented as $\mathbb{R} \cup \{\infty\}$), whose circles are graphs of functions $\mathbb{S}^1 \to \mathbb{R}$ and whose parallel classes of points are the generators of the cylinder. Notice that for an incidence structure on the cylinder with circles and parallel classes like this, the axioms of parallel projection and richness are automatically satisfied. In particular, we are interested in 2*-dimensional or flat Laguerre planes* on the cylinder. These Laguerre planes are characterized by the fact that all their circles are graphs of continuous functions from $\mathbb{S}^1$ to $\mathbb{R}$; cf. [Groh 1968; 1969b]. The axiom of joining and touching show that the collection of circle-describing functions of a 2-dimensional Laguerre plane solves the Hermite interpolation problem of rank 3.

The *classical real Laguerre plane* $\mathscr{L}_{\text{cl}}$ is obtained as the geometry of nontrivial plane sections of a cylinder in $\mathbb{R}^3$ with an ellipse in $\mathbb{R}^2$ as base, or equivalently, as the geometry of nontrivial plane sections of an elliptic cone, in real projective three-space, with its vertex removed. The parallel classes are the generators of the cylinder or cone. By replacing the ellipse in this construction by arbitrary ovals in $\mathbb{R}^2$ (i.e., convex, differentiable simply closed curves), we also obtain 2-dimensional Laguerre planes. These are the so-called *2-dimensional ovoidal Laguerre planes*.

Circles of a 2-dimensional Laguerre plane, as described above, are homeomorphic to the unit circle $\mathbb{S}^1$. When the circle set is topologized by the Hausdorff metric with respect to a metric that induces the topology of the point set, then the plane is *topological* in the sense that the operations of joining three points by a circle, intersecting two circles, and touching are continuous with respect to the induced topologies on their respective domains of definition. For more information on topological Laguerre planes we refer to [Groh 1968; 1969b].

For each point p of $\mathscr{L}$ we form the incidence structure $\mathscr{A}_p = (A_p, \mathscr{L}_p)$ whose point set A_p consists of all points of $\mathscr{L}$ that are not parallel to p and whose line set $\mathscr{L}_p$ consists of all restrictions to A_p of circles of $\mathscr{L}$ passing through p and of all parallel classes not passing through p. It readily follows that $\mathscr{A}_p$ is an affine plane. We call $\mathscr{A}_p$ the *derived affine plane at* p. In fact, the axioms of a Laguerre plane are equivalent to each derived incidence structure being an affine plane. For example, each derived affine plane of an ovoidal Laguerre plane is Desarguesian.

Each derived affine plane $\mathscr{A}_p$ of a 2-dimensional Laguerre plane is even a topological affine plane and extends to a 2-dimensional compact projective plane $\mathscr{P}_p$, which we call the *derived projective plane at* p; see [Salzmann 1967], [Salzmann et al. 1995] or [Polster and Steinke 2001, Chapter 2] for more information on topological 2-dimensional compact projective planes. Circles not passing through the distinguished point p induce closed ovals in $\mathscr{P}_p$ by removing the point parallel to p and adding in $\mathscr{P}_p$ the point ω at infinity of lines that come from parallel classes of $\mathscr{L}$. The line at infinity of $\mathscr{P}_p$ (relative to $\mathscr{A}_p$) is a tangent to this oval. According to [Polster and Steinke 1994, Proposition 2] there is a unique topology extending the natural topology of the affine plane such that one obtains a 2-dimensional Laguerre plane.

An *automorphism* of a Laguerre plane is a permutation of the point set such that parallel classes are mapped to parallel classes and circles are mapped to circles. Every automorphism of a 2-dimensional Laguerre plane is continuous and thus a homeomorphism of Z. The collection of all automorphisms of a 2-dimensional Laguerre plane $\mathscr{L}$ forms a group with respect to composition, the automorphism group Γ of $\mathscr{L}$. This group is a Lie group of dimension at most 7 with respect to the compact-open topology; see [Steinke 1986]. We call the dimension of Γ the *group dimension* of $\mathscr{L}$.

The maximum dimension is attained precisely in the classical real Laguerre plane. In fact, group dimension 6 does not occur. Furthermore, 2-dimensional

Laguerre planes of group dimension 5 must be special ovoidal Laguerre planes; see [Löwen and Pfüller 1987, Theorem 1].

We investigated 2-dimensional Laguerre planes admitting 4-dimensional point-transitive groups of automorphisms in [Steinke 1993]. It was shown that such planes must be classical. The 2-dimensional Laguerre planes admitting 4-dimensional groups of automorphisms that fix a parallel class were completely determined in [Steinke 2015]. These planes are covered by the families of Laguerre planes of generalized shear type, Laguerre planes of translation type and Laguerre planes of shift type; see [Steinke 2015, Corollary 3.5] for details and references to the various types of Laguerre planes.

The remaining open case is when a closed connected 4-dimensional group of automorphisms fixes a circle but no parallel class. Then the automorphism group contains a subgroup isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ or its universal (simply connected) covering group $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$; compare [Steinke 1990, Theorem B]. Examples of 2-dimensional Laguerre planes which admit such groups of automorphisms can be found in [Steinke 1987; Löwen and Steinke 2007].

The collection of all automorphisms of $\mathscr{L}$ that fix each parallel class is a closed normal subgroup of Γ, called the *kernel* of $\mathscr{L}$. The kernel of a 2-dimensional Laguerre plane has dimension at most 4. Furthermore, a kernel of dimension 4 characterizes the ovoidal Laguerre planes among 2-dimensional Laguerre planes, that is, a 2-dimensional Laguerre plane $\mathscr{L}$ is ovoidal if and only if its kernel is 4-dimensional; see [Groh 1969a].

3. The new models of 2-dimensional Laguerre planes

We construct a class of 2-dimensional Laguerre planes that admit a 4-dimensional group of automorphisms fixing a circle. This class depends on a real positive parameter k. To begin with, it is readily seen that a multiplicative homeomorphism of $\mathbb{R}$ is of the form

$$h_k(x) = x\,|x|^{k-1},$$

where $k > 0$. Furthermore, h_k is differentiable for all $x \neq 0$ and has derivative $h'_k(x) = k|x|^{k-1}$. We use h_k also when $k \leq 0$. Of course, in this case, h_k is not defined at 0, but still multiplicative on $\mathbb{R} \setminus \{0\}$.

Description of the models $\mathscr{L}_k$. We consider the following incidence structures $\mathscr{L}_k$, where $0 < k < 2$. For each such k we let $k' = 2 - k$, so that $0 < k' < 2$. The point set is the cylinder $Z = (\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$. Two points $(x_1, y_1), (x_2, y_2) \in Z$ are parallel if and only if $x_1 = x_2$, and parallel classes in $\mathscr{L}$ are the sets $\{u\} \times \mathbb{R}$ for $u \in \mathbb{R} \cup \{\infty\}$. Circles are of one of the following forms:

- $C_{a,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = ax^2 + bx + c\} \cup \{(\infty, a)\}$, where $b^2 \leq 4ac$; these are circles of the classical real Laguerre plane and precisely those that do not meet $C_0 = C_{0,0,0}$ in exactly two points;

- $D_{0,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = bh_k(x-c)\} \cup \{(\infty, 0)\}$, where $b > 0$;
- $D_{0,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = bh_{k'}(x-c)\} \cup \{(\infty, 0)\}$, where $b < 0$; and
- $D_{a,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = ah_k(x-b)h_{k'}(x-c)\} \cup \{(\infty, a)\}$, where $a(b-c) > 0$.

We call a circle of the form $C_{a,b,c}$ a C-*circle* and a circle of the form $D_{a,b,c}$ a D-*circle*; see Figure 1 for the shape of D-circles. Note that unless $k = k' = 1$, the graph of $D_{a,b,c}$ for $a \neq 0$ has a vertical tangent line at one of its points on the x-axis.

The set of all circles (C- and D-circles as above) is denoted by $\mathscr{C}_k$. Then $\mathscr{L}_k = (Z, \mathscr{C}_k, \parallel)$ is the incidence structure with point set Z, set of circles $\mathscr{C}_k$ and equivalence relation $\parallel$ on Z as given above.

Sometimes it will be more convenient to use a slightly different parametrization of C-circles. We define

$$C'_{a,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = a(x-b)^2 + c\} \cup \{(\infty, a)\},$$

where $ac \geq 0$, $a \neq 0$. This uniquely covers all C-circles except the circles $C_{0,0,c}$ where $c \in \mathbb{R}$, the circles that touch C_0 at $(\infty, 0)$. (Extending the definition of $C'_{a,b,c}$ to include $a = 0$ would yield multiple descriptions of the latter touching circles.) Note that when the parameter c tends to b in a D-circle $D_{a,b,c}$ one just obtains $C'_{a,b,0}$. This is due to the fact that $h_k(x)h_{k'}(x) = x^2$ for all $x \in \mathbb{R}$.

We show in the next section that $\mathscr{L}_k$ is indeed a Laguerre plane. C-circles are the same as in the classical real Laguerre plane $\mathscr{L}_{\mathrm{cl}}$, which is obviously isomorphic to $\mathscr{L}_1$. So only the circles meeting C_0 in precisely two points have been replaced in $\mathscr{L}_{\mathrm{cl}}$ by the D-circles.

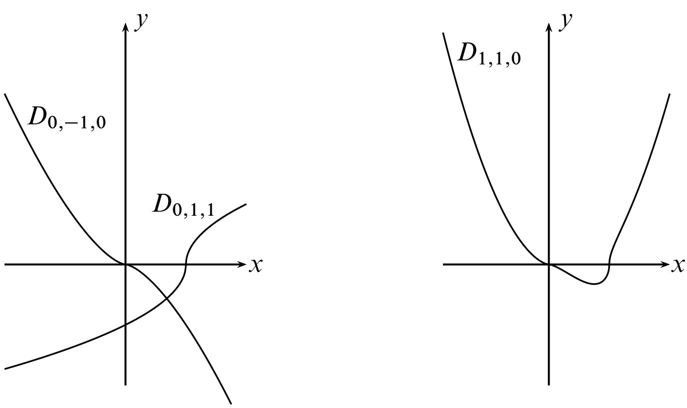

Figure 1. The circles $D_{0,1,1}$, $D_{0,-1,0}$ and $D_{1,1,0}$ in $\mathscr{L}_{1/2}$.

In [Polster and Steinke 1995, Proposition 6] it was proved that the set of circles that meet a given circle in exactly two points can be exchanged by a corresponding set of circles from a different 2-dimensional Laguerre plane so long as the two planes share the circles that touch the distinguished circle. However, the planes $\mathscr{L}_k$ are not examples for this construction as we do not have a 2-dimensional Laguerre plane (other than $\mathscr{L}_k$) that contains all D-circles of $\mathscr{L}_k$ and all circles touching C_0.

It is readily verified that the permutations

$$\gamma_{a,b,c,d,r} : (x, y) \mapsto \begin{cases} \left(\dfrac{ax+b}{cx+d}, \dfrac{r(ad-bc)y}{(cx+d)^2}\right) & \text{if } x \in \mathbb{R},\ cx+d \neq 0, \\ \left(\infty, \dfrac{rc^2y}{ad-bc}\right) & \text{if } c \neq 0,\ x = -\dfrac{d}{c}, \\ \left(\dfrac{a}{c}, \dfrac{r(ad-bc)y}{c^2}\right) & \text{if } c \neq 0,\ x = \infty, \\ \left(\infty, \dfrac{rdy}{a}\right) & \text{if } c = 0,\ x = \infty \end{cases}$$

of the cylinder Z, where $a, b, c, d, r \in \mathbb{R}$, $ad-bc \neq 0$ and $r > 0$, are automorphisms of $\mathscr{L}_k$ (i.e., take circles to circles). Indeed, since each $\gamma_{a,b,c,d,r}$ is an automorphism of the classical real Laguerre plane, a C-circle is taken to a C-circle. For D-circles it suffices to consider the generating transformations $\gamma_{1,t,0,1,1}$ with $t \in \mathbb{R}$, $\gamma_{s,0,0,1,1}$ with $s \neq 0$, $\gamma_{1,0,0,1,r}$ with $r > 0$, and $\gamma_{0,-1,1,0,1}$. For example, in case $a \neq 0$ one finds that

$$\begin{aligned} \gamma_{1,t,0,1,1}(D_{a,b,c}) &= D_{a,b+t,c+t}, \\ \gamma_{s,0,0,1,1}(D_{a,b,c}) &= D_{a/s,bs,cs}, \\ \gamma_{1,0,0,1,r}(D_{a,b,c}) &= D_{ra,b,c}, \\ \gamma_{0,-1,1,0,1}(D_{a,b,c}) &= D_{ah_k(b)h_{k'}(c),-1/b,-1/c}, \end{aligned}$$

where also $bc \neq 0$ in the last case.

Let

$$\Gamma = \{\gamma_{a,b,c,d,r} \mid a, b, c, d, r \in \mathbb{R}, ad-bc \neq 0, r > 0\}.$$

Then Γ is a group of automorphisms of $\mathscr{L}_k$. Obviously,

$$\Sigma = \{\gamma_{a,b,c,d,1} \mid a, b, c, d \in \mathbb{R}, ad-bc \neq 0\}$$

is a subgroup of Γ. Furthermore, Σ is isomorphic to $\mathrm{PGL}_2(\mathbb{R})$ and Γ is isomorphic to $\mathrm{PGL}_2(\mathbb{R}) \times \mathbb{R}$. The action of Σ on C_0 is equivalent to the standard action of $\mathrm{PGL}_2(\mathbb{R})$ on $\mathbb{R} \cup \{\infty\}$. In particular, Σ is sharply 3-transitive on C_0. The subgroup $\{\gamma_{1,0,0,1,r} \mid r > 0\}$ of Γ comprises the kernel of Γ. Moreover, Σ and Γ have two orbits on Z, namely C_0 and $Z \setminus C_0$. On the circle space, Γ has four orbits: $\{C_0\}$, $\{C_{a,b,c} \mid b^2 = 4ac\}$, $\{C_{a,b,c} \mid b^2 < 4ac\}$, and the set of all D-circles.

We equip the cylinder Z with the natural Euclidean topology of $\mathbb{S}^1 \times \mathbb{R}$. On $\mathbb{R}^2 \subset Z$, the usual Euclidean topology is induced. In our representation, a neighbourhood of a point (∞, a) consists of all (x, y) such that either $x = \infty$ and y is sufficiently close to a, or $x \in \mathbb{R}$ is of sufficiently large modulus and y/x^2 is sufficiently close to a. It is readily checked that in this topology, circles of $\mathscr{L}_k$ are closed subsets of Z (in fact, are homeomorphic to $\mathbb{S}^1$) and that all transformations in Γ are continuous.

4. The geometric axioms

Since Γ has precisely two orbits on Z it suffices to verify that the derived incidence structures at $(\infty, 0)$ and $(\infty, 1)$ are affine planes in order to show that $\mathscr{L}_k$ is a Laguerre plane.

We first deal with the derived incidence structure $\mathscr{A}_0$ at $(\infty, 0)$. The point set of $\mathscr{A}_0$ is $\mathbb{R}^2$ and nonvertical lines come from $C_{0,0,c}$, $c \in \mathbb{R}$, and $D_{0,b,c}$, $b \neq 0$. Hence, nonvertical and nonhorizontal lines are given by

$$\begin{aligned} y &= bh_k(x-c), \quad b > 0, \quad \text{and} \\ y &= bh_{k'}(x-c), \quad b < 0. \end{aligned}$$

Lemma 4.1. *The derived incidence structure $\mathscr{A}_0$ of $\mathscr{L}_k$ at $(\infty, 0)$ is an affine plane. Furthermore, $\mathscr{A}_0$ is Desarguesian if and only if $k = 1$.*

Proof. We make the coordinate transformation

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (h_k^{-1}(y), x).$$

Then the nonvertical and nonhorizontal lines in the new (u, v)-coordinates become

$$\begin{aligned} v &= Bu + c, & &\text{where } B > 0 \quad (B = 1/h_k^{-1}(b)), \quad \text{and} \\ v &= Bh_{k/k'}(u) + c, & &\text{where } B < 0 \quad (B = 1/h_{k'}^{-1}(b)). \end{aligned}$$

One also has the vertical and horizontal lines $u = c$ and $v = c$, respectively. Since $h_{k/k'}$ is an orientation preserving homeomorphism of $\mathbb{R}$, one sees that A_0 is an affine plane; compare [Steinke 1985, Proposition 2.1]. In the notation of [Steinke 1985] the plane described above in the (u, v)-coordinates is the affine plane $\mathscr{A}_{h_{k/k'},\mathrm{id}}$. It is a plane over a Cartesian field — see [Salzmann et al. 1995, Section 37] — the affine part of the plane $\mathscr{P}_{1,k/k',1}$ in the notation of [Salzmann et al. 1995, 37.3]. Such a plane is Desarguesian if and only if $k/k' = 1$; compare [Steinke 1985, Corollary 3.2] or [Salzmann et al. 1995, 37.3 and Theorem 37.4]. However, $k = k'$ implies $k = 1$. □

Before we consider the derived incidence structure $\mathscr{A}_1$ at $(\infty, 1)$, we deal with the intersection of two general distinct circles in $\mathscr{L}_k$.

Lemma 4.2. *Two distinct circles in $\mathscr{L}_k$ have at most two points in common.*

Proof. The statement is obviously true for two distinct C-circles. Consider a C-circle and D-circle. By applying the group Γ we may assume that the D-circle is $D_{0,m,t}$, where $m \neq 0$ and the C-circle is $C_{1,0,c}$, where $c \geq 0$. The x-coordinates of points of intersection are found from the equation

$$x^2 + c = mh_k(x-t). \tag{1}$$

We apply $h_k^{-1} = h_{1/k}$ on both sides to obtain

$$h_{1/k}(x^2 + c) = Ax + B,$$

where $A = h_k^{-1}(m) \neq 0$ and $B = -h_k^{-1}(m)t$. However, the function $f_c : x \mapsto h_{1/k}(x^2 + c)$ on the left-hand side is strictly convex. This can be seen from the second derivative of f_c given by $f_c''(x) = \frac{2}{k}(x^2+c)^{\frac{1}{k}-2}\big(\frac{k'}{k}x^2 + c\big)$, which is positive except possibly when $x = 0$. Hence, (1) has at most two solutions and thus $C_{1,0,c}$ and $D_{0,m,t}$ have at most two points of intersection.

In the last case we consider two D-circles. By applying the group Γ and Lemma 4.1 we may assume that one circle is $D_{0,m,t}$, where $m \neq 0$ and the other circle is $D_{1,1,0}$. We first assume that $m > 0$. Then x-coordinates of points of intersection are found from the equation

$$h_k(x-1)h_{k'}(x) = mh_k(x-t). \tag{2}$$

We apply h_k^{-1} on both sides to obtain

$$(x-1)h_{k'/k}(x) = Ax + B,$$

where $A = h_k^{-1}(m) > 0$ and $B = -h_k^{-1}(m)t$. The function $f_+ : x \mapsto (x-1)h_l(x)$, where $l = k'/k$, on the left-hand side of the above equation has derivative

$$f_+'(x) = h_l(x) + l(x-1)|x|^{l-1} = ((l+1)x - l)|x|^{l-1}$$

and second derivative

$$\begin{aligned} f_+''(x) &= (l+1)|x|^{l-1} + (l-1)((l+1)x-l)h_{l-2}(x) \\ &= h_{l-2}(x)((l+1)x + (l-1)((l+1)x-l)) \\ &= lh_{l-2}(x)((l+1)x - l + 1). \end{aligned}$$

Hence f_+ is strictly decreasing on $(-\infty, x_{\min})$, where $x_{\min} = l/(l+1) > 0$, strictly increasing on $(x_{\min}, +\infty)$ and has an absolute minimum at $x_{\min}$. Furthermore, f_+ is strictly convex on the interval $(x_{\min}, +\infty)$; compare the diagram on the left in Figure 2. Since the restriction of f_+ to $(x_{\min}, +\infty)$ (the increasing branch of the graph of f_+) is convex, a Euclidean line of positive slope can meet the increasing branch in at most two points and the decreasing branch (the graph of

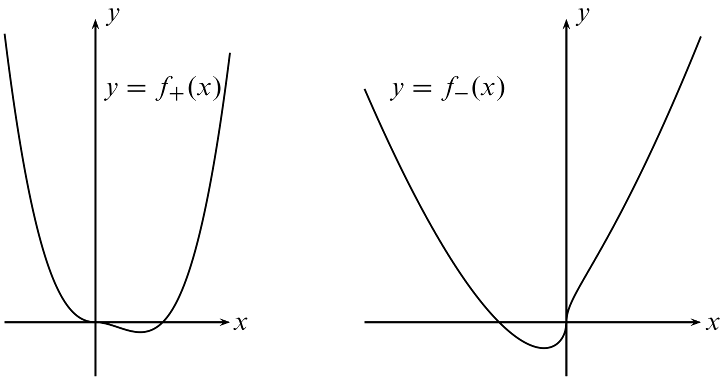

Figure 2. The graphs of $f_+(x) = (x-1)h_2(x)$ and $f_-(x) = (x+1)h_{1/2}(x)$.

the restriction of f_+ to $(-\infty, x_{\min})$) in at most one point. If such a line meets the increasing branch in two points, then because $\lim_{x\to+\infty} f_+(x)/x = +\infty$ the point $(x_{\min}, f_+(x_{\min}))$ lies above this line, so that the line cannot meet the graph of f_+ in any more points. In any case, we see that a Euclidean line of positive slope intersects the graph of f_+ at most twice. This shows that (2) has at most two solutions and thus that $D_{0,m,t}$, where $m > 0$, and $D_{1,1,0}$ have at most two points in common.

When $m < 0$ one similarly considers the equation

$$h_k(x-1)h_{k'}(x) = mh_{k'}(x-t), \tag{3}$$

from which one obtains

$$(x+1)h_{k/k'}(x) = Ax + B,$$

where $A = h_{k'}^{-1}(m) < 0$ and $B = h_{k'}^{-1}(m)(1-t)$. A similar straightforward analysis of the function $f_- : x \mapsto (x+1)h_l(x)$ on the left-hand side, where now $l = k/k'$, shows that the decreasing branch is strictly convex, so that a Euclidean line of negative slope intersects the graph of f_- at most twice; compare the diagram on the right in Figure 2. Therefore, (3) has at most two solutions and thus $D_{0,m,t}$, where $m < 0$, and $D_{1,1,0}$ have at most two points in common. This shows that in any case two distinct D-circles intersect in at most two points. □

We are now ready to deal with the derived incidence structure $\mathscr{A}_1$ at $(\infty, 1)$. The point set of $\mathscr{A}_1$ is $\mathbb{R}^2$ and nonvertical lines are induced by $C_{1,b,c}$, where $b^2 \le 4c$, and $D_{1,b,c}$, where $b > c$. Explicitly, these lines are given by

$$y = x^2 + bx + c, \qquad b^2 \le 4c, \quad \text{and}$$
$$y = h_k(x-b)h_{k'}(x-c), \qquad b > c.$$

We call them C-lines and D-lines, respectively, as they come from C- and D-circles of $\mathscr{L}_k$.

Lemma 4.3. *The derived incidence structure $\mathscr{A}_1$ of $\mathscr{L}_k$ at $(\infty, 1)$ is a linear space.*

Proof. By Lemma 4.2 we know that two different lines in $\mathscr{A}_1$ intersect in at most one point. This yields the uniqueness of a line joining two points if it exists.

Let $p_i = (x_i, y_i)$, $i = 1, 2$, be two distinct points of $\mathscr{A}_1$. If $x_1 = x_2$, then the vertical line $x = x_1$ (coming from a parallel class of the Laguerre plane) joins the two points. We therefore assume that $x_1 \neq x_2$. By the transitivity properties of the stabilizer $\Gamma_{(\infty,1)}$ we may assume that without loss of generality $x_1 = 0$ and $x_2 = 1$. Finally, because $\mathscr{A}_0$ (and thus each $\mathscr{A}_{(u,0)}$ where $u \in \mathbb{R}$) is an affine plane by Lemma 4.1, we may further assume that $y_1, y_2 \neq 0$.

In case $2(y_1 + y_2) \geq (y_2 - y_1)^2 + 1$ there is a unique C-line through p_1 and p_2. Indeed, the Euclidean parabola given by $y = x^2 + (y_2 - y_1 - 1)x + y_1$ passes through the two points, and this is a line of $\mathscr{A}_1$ if and only if

$$0 \geq (y_2 - y_1 - 1)^2 - 4y_1 = (y_2 - y_1)^2 + 1 - 2(y_1 + y_2).$$

In this case, the two points cannot be on a D-line by Lemma 4.2.

So now assume that $2(y_1 + y_2) < (y_2 - y_1)^2 + 1$. We must show that p_1 and p_2 are on a D-line $D_{1,b,c}$. The two parameters $b > c$ satisfy the equations

$$y_1 = h_k(b)h_{k'}(c), \qquad y_2 = h_k(b-1)h_{k'}(c-1).$$

After application of $h_{k'}^{-1}$ on both sides we obtain

$$v_1 := h_{k'}^{-1}(y_1) = h_l(b)c, \tag{4}$$

$$v_2 := h_{k'}^{-1}(y_2) = h_l(b-1)(c-1), \tag{5}$$

where $l = k/k'$. Hence

$$g(b) := h_l(b)h_l(b-1) - v_1 h_l(b-1) + v_2 h_l(b) = 0.$$

First note that $g(b) = (h_l(b) - v_1)(h_l(b-1) + v_2) + v_1 v_2$. From this equation one sees that $\lim_{b\to\pm\infty} g(b) = +\infty$.

When $y_2 < 0$, then $g(1) = v_2 < 0$. Thus, by the intermediate value theorem, there is a $b > 1$ such that $g(b) = 0$. From (5) it follows that $c < 1$. Similarly, when $y_1 < 0 < y_2$, then $g(0) = v_1 < 0$ and $g(1) = v_2 > 0$ so that there is some b, $0 < b < 1$, such that $g(b) = 0$. From (4) it then follows that $c < 0$. Hence, in these two cases, $b > c$ and we have a D-line through p_1 and p_2.

We finally assume that $y_1, y_2 > 0$. We compute

$$\begin{aligned} v_i &= h_{1/k'}(y_i) = (y_i)^{1/k'} = (\sqrt{y_i})^{2/k'} \\ &= (\sqrt{y_i})^{(k+k')/k'} = (\sqrt{y_i})^{l+1} = \sqrt{y_i}\, h_l(\sqrt{y_i}), \end{aligned}$$

where $i = 1, 2$. Hence,

$$\begin{aligned} g(\sqrt{y_1}) &= h_l(\sqrt{y_1})[h_l(\sqrt{y_1}-1)(1-\sqrt{y_1})+v_2] \\ &= h_l(\sqrt{y_1})((\sqrt{y_2})^{l+1} - |\sqrt{y_1}-1|^{l+1}). \end{aligned}$$

One similarly obtains that

$$\begin{aligned} g(-\sqrt{y_1}) &= h_l(-\sqrt{y_1})[h_l(-\sqrt{y_1}-1)(1+\sqrt{y_1})+v_2] \\ &= h_l(-\sqrt{y_1})(\sqrt{y_2}^{l+1} - |\sqrt{y_1}+1|^{l+1}). \end{aligned}$$

The inequality $2(y_1+y_2) < (y_2-y_1)^2+1$ can be rewritten as

$$(y_2-y_1-1)^2 - 4y_1 > 0$$

from which we see that either $y_2 > (\sqrt{y_1}+1)^2$ or $y_2 < (\sqrt{y_1}-1)^2$. In the former case, $g(-\sqrt{y_1}) < 0$, and in the latter case, $g(\sqrt{y_1}) < 0$. Since $g(0) = v_1 > 0$ and $\lim_{b\to+\infty} g(b) = +\infty$, there must be a $b \in (-\sqrt{y_1}, 0)$ or $b \in (\sqrt{y_1}, +\infty)$, respectively, such that $g(b) = 0$. Finally, because

$$y_1 = (\sqrt{y_1})^2 = h_k(\sqrt{y_1})h_{k'}(\sqrt{y_1}),$$

one obtains from (4) that

$$h_{k'}(c/\sqrt{y_1}) = h_k(\sqrt{y_1}/b).$$

Hence $c < -\sqrt{y_1} < b$ when $b < 0$, and $0 < c < \sqrt{y_1} < b$ when $b > \sqrt{y_1}$. Hence, in any case, $b > c$ and we have a D-line through p_1 and p_2.

This proves that any two distinct points of $\mathscr{A}_1$ can be joined by a unique line, that is, $\mathscr{A}_1$ is a linear space as claimed. □

Lemma 4.4. *The derived incidence structure $\mathscr{A}_1$ of $\mathscr{L}_k$ at $(\infty, 1)$ is an affine plane.*

Proof. By Lemma 4.3 it only remains to show that through each point there is a unique line that is parallel to a given line. This is clearly the case for vertical lines.

For a nonvertical line we define its slope s by

$$s(C_{1,b,c}) = -b \quad \text{and} \quad s(D_{1,b,c}) = kb + k'c.$$

We claim that two nonvertical lines of $\mathscr{A}_1$ are parallel if and only if they have the same slope. To see this and where the definition of s comes from, we apply the coordinate transformation induced by $\gamma_{0,1,-1,0,1}$, that is, $(x, y) \mapsto (-1/x, y/x^2)$ for x real and nonzero, suitably extended to Z. Then $C_{1,b,c}$ and $D_{1,b,c}$ are described by $v = cu^2 - bu + 1$ and $v = h_k(1+bu)h_{k'}(1+cu)$, respectively. Differentiation at $u = 0$ yields $-b$ and $kb + k'c$, that is, the slope of the corresponding line. Now, if the slopes of two nonvertical lines are different, then after the above coordinate transformation the resulting circles intersect transversally at $(0, 1)$. Hence these circles intersect in a second point in $Z \setminus \{0\} \times \mathbb{R}$. Therefore the original lines meet in a point of $\mathscr{A}_1$ and so are not parallel.

We now assume that two nonvertical lines of $\mathscr{A}_1$ have the same slope s. In case of two C-lines C_{1,b_1,c_1} and C_{1,b_2,c_2} this means that $b_1 = b_2 = -s$, and the two lines are clearly parallel.

A D-line of slope s is described by the function $f(c,x) = h_k(x-b)h_{k'}(x-c)$, where $b = (s-k'c)/k$. Differentiation with respect to c yields

$$\begin{aligned}\frac{\partial f(c,x)}{\partial c} &= k'|x-b|^{k-1}h_{k'}(x-c) - k'h_k(x-b)|x-c|^{k'-1} \\ &= k'|x-b|^{k-1}|x-c|^{k'-1}(b-c) \\ &= \frac{k'}{k'}|x-b|^{k-1}|x-c|^{k'-1}(s-2c).\end{aligned}$$

But $b > c$ if and only if $s-2c > 0$. Thus $\frac{\partial}{\partial c}f(c,x) > 0$, and $c \mapsto f(c,x)$ is strictly increasing on $\left(-\infty, \frac{s}{2}\right)$ for all $x \in \mathbb{R}$. It now follows that two D-lines D_{1,b_1,c_1} and D_{1,b_2,c_2} of the same slope $kb_1 + k'c_1 = kb_2 + k'c_2$ are parallel.

Note that $c < \frac{s}{2} < b$ for a D-line of slope s. Furthermore, as c tends to $\frac{s}{2}$, the D-line $D_{1,b,c}$, $kb + k'c = s$, converges to $D_{1,s/2,s/2} = C'_{1,s/2,0}$. In particular, $C'_{1,s/2,0}$ and $D_{1,b,c}$ are parallel, and $D_{1,b,c}$ lies below $C'_{1,s/2,0}$. Finally, a C-line $C'_{1,s/2,c}$, $c \geq 0$, of slope s lies above or coincides with $C'_{1,s/2,0}$. Hence, a C-line and a D-line of slope s are parallel.

Finally, given a point $p = (x_0, y_0)$ and a line of slope s there is a unique line of slope through p. Indeed, when $y_0 \geq \left(x_0 - \frac{s}{2}\right)^2$, the parallel through p must be a C-line $C'_{1,s/2,c}$, and c is uniquely determined by

$$c = y_0 - \left(x_0 - \frac{s}{2}\right)^2 \geq 0.$$

When $y_0 < \left(x_0 - \frac{s}{2}\right)^2$, the parallel through p must be a D-line $D_{1,b,c}$, $kb + k'c = s$. Since

$$\begin{aligned}&\lim_{c\to-\infty}(y_0 - h_k(x-b)h_{k'}(x-c)) = +\infty \quad \text{and} \\ &\lim_{c\to s/2}(y_0 - h_k(x-b)h_{k'}(x-c)) = y_0 - \left(x_0 - \tfrac{s}{2}\right)^2 < 0,\end{aligned}$$

there is a c such that $D_{1,b,c}$ passes through p.

This shows that $\mathscr{A}_1$ satisfies the parallel axiom and that $\mathscr{A}_1$ is an affine plane. □

The following is a direct consequence of Lemmata 4.1 and 4.4, together with the transitivity properties of Γ and the fact that each derived plane of an ovoidal Laguerre plane is Desarguesian.

Corollary 4.5. *The incidence structure $\mathscr{L}_k$ where $0 < k < 2$ is a Laguerre plane. Furthermore, $\mathscr{L}_k$ is ovoidal if and only if $k = 1$. In this case the Laguerre plane is classical.*

Since in the topology on Z circles of $\mathscr{L}_k$ are closed Jordan curves on Z we have the following; compare [Groh 1969b, 3.10].

Theorem 4.6. *Each $\mathscr{L}_k$ where $0 < k < 2$ is a 2-dimensional Laguerre plane.*

5. Isomorphisms and other properties

Lemma 5.1. *Let ψ be an isomorphism from $\mathscr{L}_k$ to $\mathscr{L}_l$. If $k \neq 1$, then ψ takes C_0 in $\mathscr{L}_k$ to C_0 in $\mathscr{L}_l$.*

Proof. Suppose that $\psi(C_0) \neq C_0$. Then $\psi\Gamma_k\psi^{-1}$ is a group of automorphisms of $\mathscr{L}_l$ that has $Z \setminus \psi(C_0)$ and $\psi(C_0)$ as orbits. However, Γ_l has $Z \setminus C_0$ and C_0 as orbits, and it follows that the automorphism group of $\mathscr{L}_l$ must be transitive on Z. Hence, $\mathscr{L}_l$ is classical by [Steinke 1993]. But then $\mathscr{L}_k$ is also classical and $k = 1$ — a contradiction to our assumption. This shows that $\psi(C_0) = C_0$. □

Proposition 5.2. *Two Laguerre planes $\mathscr{L}_k$ and $\mathscr{L}_l$ are isomorphic if and only if $l \in \{k, k'\}$. In particular, each plane is isomorphic to exactly one plane $\mathscr{L}_k$, where $0 < k \leq 1$.*

Proof. Note that $\mu : Z \to Z$ given by $\mu(x, y) = (x, -y)$ is an automorphism of the classical real Laguerre plane; circles $C_{a,b,c}$ are taken to $C_{-a,-b,-c}$. In fact, μ induces an isomorphism from $\mathscr{L}_k$ onto $\mathscr{L}_{k'}$: one has

$$\mu\big(D^{(k)}_{a,b,c}\big) = D^{(k')}_{a,c,b}$$

when $a \neq 0$, and

$$\mu\big(D^{(k)}_{0,b,c}\big) = D^{(k')}_{0,-b,c}.$$

(Here the superscripts refer to the Laguerre planes the circles are from.) This verifies that $\mathscr{L}_k$ and $\mathscr{L}_{k'}$ are isomorphic.

Assume that $\mathscr{L}_k$ and $\mathscr{L}_l$ are isomorphic. If $k = 1$, then $\mathscr{L}_k$ is classical and so is $\mathscr{L}_l$. Thus $l = 1$, and $l = k = k'$.

Suppose that $k \neq 1$. Let ψ be an isomorphism from $\mathscr{L}_k$ to $\mathscr{L}_l$. By Lemma 5.1 we know that $\psi(C_0) = C_0$. Using the transitivity properties of Γ_l on $\mathscr{L}_l$ we may further assume that ψ takes $(\infty, 0)$, $(\infty, 1)$ and $(0, 0)$ in $\mathscr{L}_k$ to the corresponding points with the same coordinates in $\mathscr{L}_l$. Hence the derived affine planes $\mathscr{A}^{(k)}_0$ and $\mathscr{A}^{(l)}_0$ are isomorphic. As seen in the proof of Lemma 4.1 the projective extensions of $\mathscr{A}^{(k)}_0$ and $\mathscr{A}^{(l)}_0$ are isomorphic to cartesian planes $\mathscr{P}_{1,k/k',1}$ and $\mathscr{P}_{1,l/l',1}$, respectively. By [Salzmann et al. 1995, Theorem 37.3 and Proposition 37.6] we thus have that $l/l' = k/k'$ or $l/l' = k'/k$. In the former case $\frac{l}{2-l} = \frac{k}{2-k}$ so that $l = k$. In the latter case we similarly obtain $l = k'$. □

Proposition 5.3. *The group* Γ *from Section 3 is the full automorphism group of* $\mathcal{L}_k$ *when* $k \neq 1$.

Proof. Let $k \neq 1$ and let α be an automorphism of $\mathcal{L}_k$. By Lemma 5.1 the automorphism leaves C_0 invariant. The 3-transitivity of Σ on C_0 implies that there is a $\sigma \in \Sigma$ such that $\sigma\alpha$ fixes each of $(\infty, 0)$, $(0, 0)$ and $(1, 0)$. By using an automorphism $\gamma_{1,0,0,1,r}$, $r > 0$, we can furthermore achieve that $\gamma = \gamma_{1,0,0,1,r}\sigma\alpha$ fixes $(\infty, 0)$, $(0, 0)$, $(1, 0)$ and takes $(\infty, 1)$ to $(\infty, 1)$ or $(\infty, -1)$. In the former case γ fixes each of the four points $(\infty, 0)$, $(0, 0)$, $(1, 0)$, $(\infty, 1)$. Hence γ must be the identity by [Steinke 1990, Lemma 2.10] or [Salzmann 1967, Corollary 3.6]. Thus $\alpha = \gamma_{1,0,0,1,1/r}\sigma^{-1} \in \Gamma$.

In the latter case there is an $s > 0$ such that $\gamma_{1,0,0,1,s}\gamma^2$ fixes each of the four points $(\infty, 0)$, $(0, 0)$, $(1, 0)$, $(\infty, 1)$. Therefore $\gamma_{1,0,0,1,s}\gamma^2 = \mathrm{id}$ so that γ^2 acts trivially on C_0. But γ fixes the three points $(\infty, 0)$, $(0, 0)$, $(1, 0)$ on C_0 and thus is an orientation preserving homeomorphism of C_0. This implies that γ is the identity on C_0.

Given a point p in the open upper half-cylinder Z^+ not parallel to $(\infty, 0)$, there are exactly two circles through $(\infty, 1)$ and p that touch C_0. Indeed, if $p = (x_0, y_0)$, where $y_0 > 0$, the two touching circles are $C'_{1,x_0+\sqrt{y_0},0}$ and $C'_{1,x_0-\sqrt{y_0},0}$. Since the point of touching on C_0 is fixed by γ and because $\gamma(\infty, 1) = (\infty, -1)$, these circles are taken to $C'_{-1,x_0+\sqrt{y_0},0}$ and $C'_{-1,x_0-\sqrt{y_0},0}$, respectively. Therefore, $\gamma(x_0, y_0) = (x_0, -y_0)$.

Now, the trace of a D-circle $D_{0,1,0}$ on Z^+ is taken by γ to the set

$$\{(x, -h_k(x)) \mid x > 0\},$$

which must be part of a D-circle through $(\infty, 0)$ and $(0, 0)$. Therefore, there must be an $m < 0$ such that $-h_k(x) = mh_{k'}(x)$ for all $x > 0$. When $x = 1$ we obtain $m = -1$. But then $x^k = x^{k'}$ for all $x > 0$, so that $k = k'$ — a contradiction to our assumption that $k \neq 1$. This shows that the latter case cannot occur, and we have $\alpha \in \Gamma$. □

Kleinewillinghöfer [1979; 1980] classified Laguerre planes with respect to *central automorphisms*, that is, automorphisms of the Laguerre plane such that at least one point is fixed and central collineations are induced in the derived projective plane at one of the fixed points. A subgroup of central automorphisms with the same "centre" and "axis" is said to be linearly transitive if the induced subgroup of central collineations of the derived projective plane is linearly transitive, that is, transitive on the points of each central line except the centre and its intersection with the axis. In [Polster and Steinke 2004], 2-dimensional Laguerre planes were considered and their so-called Kleinewillinghöfer types were investigated, that is, the Kleinewillinghöfer types with respect to the full automorphism group. The classification of those types that can occur in 2-dimensional Laguerre planes is almost complete except for two open cases; see [Steinke 2012] and the references to models of various types given there.

It turns out that the planes $\mathscr{L}_k$ constructed here are of type I.G.1 when $k \neq 1$, the same type as some semiclassical Laguerre planes pasted along a circle; see [Polster and Steinke 2004, Section 6] and below for a description of these semiclassical planes. This means that there is no circle for which the automorphism group of $\mathscr{L}_k$ is linearly transitive with respect to Laguerre homologies (type I, a Laguerre homology fixes a circle pointwise), that there is a circle C such that for each point p on C the group of Laguerre translations fixing p and the bundle of circles touching C at p is linearly transitive (type G, a Laguerre translation fixes a parallel class pointwise and induces a translation in a derived projective plane at one of the fixed points), and that there is no group of Laguerre homotheties that is linearly transitive (type 1, a Laguerre homothety fixes two nonparallel points and each circle through them). In type VII.K.13 all possible subgroups of central automorphisms with given centre and axis are linearly transitive. We refer to [Kleinewillinghöfer 1979] or [Polster and Steinke 2004] for a description of all types.

Proposition 5.4. *The Laguerre plane $\mathscr{L}_k$ is of Kleinewillinghöfer type* I.G.1 *when $k \neq 1$ and of type* VII.K.13 *when $k = 1$.*

Proof. When $k = 1$ we have the classical real Laguerre plane, which is of type VII.K.13; see [Polster and Steinke 2004, Corollaries 3.2 and 4.2,] and [Hartmann 1982, Satz 7]. Assume that $k \neq 1$. Then every automorphism of $\mathscr{L}_k$ fixes C_0, so that C_0 is the only possible axis of a Laguerre homology. Similarly, points on C_0 are the only possible centres of Laguerre homotheties, and Laguerre translations must be in direction of a tangent bundle to C_0. Hence, together with the 3-transitivity of Γ on C_0, only types I or II with respect to Laguerre homologies, types A or G with respect to Laguerre translations and types 1 or 6 with respect to Laguerre homotheties are possible as the types of $\mathscr{L}_k$. See [Kleinewillinghöfer 1979] or [Polster and Steinke 2004] for a full list of Kleinewillinghöfer types.

Now $\{\gamma_{1,t,0,1,1} \mid t \in \mathbb{R}\}$ is a linearly transitive group of Laguerre translations in direction of the tangent bundle to C_0 at $(\infty, 0)$. Conjugation by elements in Γ then shows that $\mathscr{L}_k$ has type G with respect to Laguerre translations. The automorphisms of $\mathscr{L}_k$ that fix each point of C_0 are $\gamma_{a,0,0,a,r}$, where $a \neq 0$, $r > 0$. However, the collection of these Laguerre homologies is not linearly transitive (because the open upper half-cylinder Z^+ is left invariant). Thus $\mathscr{L}_k$ has type I with respect to Laguerre homologies.

Similarly, the automorphisms of $\mathscr{L}_k$ that fix $(\infty, 0)$ and $(0, 0)$ are $\gamma_{a,0,0,d,r}$, where $ad \neq 0$, $r > 0$. Explicitly, these are the maps $(x, y) \mapsto (sx, rsy)$ extended to the parallel class at infinity, where $0 \neq s(= a/d)$, $r > 0$. A D-circle $D_{0,b,0}$

is taken to $D_{0,rb/|s|^k,0}$ when $b > 0$ and $D_{0,rb/|s|^{k'},0}$ when $b < 0$. However, a Laguerre homothety with centres $(\infty, 0)$ and $(0, 0)$ must fix each circle through the two centres, so that

$$r = |s|^k = |s|^{k'}$$

for all $s \neq 0$. This implies $k = k'$ — a contradiction to $k \neq 1$. This shows that $\mathscr{L}_k$ has type 1 with respect to Laguerre homotheties. □

In [Steinke 1987; 1988], semiclassical Laguerre planes were introduced. These are 2-dimensional Laguerre planes which are composed of two classical half-planes. By a half-plane we mean the closure of a connected component of the complement of two parallel classes or of a circle. Such a half-plane is called classical if, with its induced geometry, it is isomorphic to a half-plane of the same kind in the classical real Laguerre plane.

Some of the semiclassical planes also admit $\mathrm{PSL}_2(\mathbb{R}) \times \mathbb{R}$ as a group of automorphisms and are of Kleinewillinghöfer type I.G.1. These are the planes $\mathscr{L}(h_m, \mathrm{id})$, where $m > 0$, in the notation of [Steinke 1987]. They are obtained by pasting along a circle. According to [Steinke 1987, Theorem 4.8] in this case circles are of the form

$$K_{a,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = ax^2 + bx + c\} \cup \{(\infty, a)\},$$

where $a, b, c \in \mathbb{R}$, $b^2 \leq 4ac$ and

$$\begin{aligned} K_{a,b,c} = \{&(x, y) \in \mathbb{R}^2 \mid y = ax^2 + bx + c \geq 0\} \\ &\cup \{(x, y) \in \mathbb{R}^2 \mid y = (b^2 - 4ac)^{(m-1)/2}(ax^2 + bx + c) \leq 0\} \cup \{(\infty, \bar{a})\}, \end{aligned}$$

where $a, b, c \in \mathbb{R}$, $b^2 > 4ac$, $m > 0$ and

$$\bar{a} = \begin{cases} a, & \text{if } a \geq 0, \\ (b^2 - 4ac)^{(m-1)/2} a, & \text{if } a < 0. \end{cases}$$

(In case $m = 1$ one just obtains the classical real Laguerre plane $\mathscr{L}_{\mathrm{cl}}$.)

These planes are semiclassical because the geometries and topologies on the closed upper half-cylinder $\overline{Z}_+ = \mathbb{S}^1 \times [0, +\infty)$ and the closed lower half-cylinder $\overline{Z}_- = \mathbb{S}^1 \times (-\infty, 0]$ are the same as on the corresponding subsets of the (topological) classical real Laguerre plane $\mathscr{L}_{\mathrm{cl}}$. The two classical geometries are pasted together along the circle $K_0 := K_{0,0,0}$.

Those permutations $\gamma_{a,b,c,d,r}$ of Z from Section 3 where $ad - bc = 1$ and $r > 0$ are in fact also automorphisms of $\mathscr{L}(h_m, \mathrm{id})$; see [Steinke 1987, 4.3 and Lemmata 4.4 and 4.5]. The collection of all these transformations is a group with respect to composition and is isomorphic to $\mathrm{PSL}_2(\mathbb{R}) \times \mathbb{R}$.

Note that the circles that do not meet K_0 in precisely two points are the same as in $\mathscr{L}_{\mathrm{cl}}$ and thus as in our planes $\mathscr{L}_k$. However, our planes are not semiclassical except for the classical plane itself.

Proposition 5.5. *No Laguerre plane $\mathscr{L}_k$, $k \neq 1$, is semiclassical.*

Proof. By [Steinke 1988, Proposition 5.1] an automorphism of a semiclassical Laguerre plane pasted along two parallel classes leaves invariant the union of the two parallel classes along which the pasting occurs, provided the Laguerre plane is nonclassical. Since the automorphism group of $\mathscr{L}_k$ is transitive on the set of parallel classes, $\mathscr{L}_k$ cannot be isomorphic to a semiclassical Laguerre plane of this kind unless $k = 1$.

Regarding semiclassical Laguerre planes pasted along a circle, only the plane $\mathscr{L}(h_m, \mathrm{id})$, where $m > 0$, pasted along the circle K_0, needs to be considered because other planes have lower group dimension; see [Steinke 1987, Theorem 4.8]. One first notes as in the proof of Lemma 5.1 that an isomorphism ψ from $\mathscr{L}(h_m, \mathrm{id})$ to $\mathscr{L}_k$, where $k \neq 1$, must take K_0 as in the description above to C_0.

As in the proof of Proposition 5.3 we may without loss of generality assume that ψ fixes $(\infty, 0)$, $(0, 0)$, $(1, 0)$ and takes $(\infty, 1)$ to $(\infty, 1)$ or $(\infty, -1)$. In the former case ψ fixes each of the four points $(\infty, 0)$, $(0, 0)$, $(1, 0)$, $(\infty, 1)$. Hence the circles $K_{1,0,0}$ and $K_{1,-2,1}$, which pass through $(\infty, 1)$ and touch K_0 at $(0, 0)$ and $(1, 0)$, respectively, are taken to the corresponding circles in $\mathscr{L}_k$, that is, to $C_{1,0,0}$ and $C_{1,-2,1}$. Therefore the point $\left(\frac{1}{2}, \frac{1}{4}\right)$ in the intersection of $K_{1,0,0}$ and $K_{1,-2,1}$ is taken to the point $\left(\frac{1}{2}, \frac{1}{4}\right)$ in the intersection of $C_{1,0,0}$ and $C_{1,-2,1}$. Moreover, the circle $K_{1,-1,0}$ through $(0, 0)$, $(1, 0)$, $(\infty, 1)$ is taken to the corresponding circle $D_{1,1,0}$ in $\mathscr{L}_k$. Finally, there is a unique circle through $(\infty, 0)$ that touches K_0 and $K_{1,-1,0}$. The latter point of touching is calculated to be $\left(\frac{1}{2}, -\frac{1}{4}\right)$. In $\mathscr{L}_k$ one calculates that the unique circle through $(\infty, 0)$ that touches C_0 and $D_{1,1,0}$ is $C_{0,0,c}$, where

$$c = -\tfrac{1}{2} h_k(k) h_{k'}(k'),$$

and that the common point between the latter two circles is $\left(\frac{k'}{2}, c\right)$. However, ψ preserves parallelity of points so that

$$\left(\tfrac{1}{2}, \tfrac{1}{4}\right) = \psi\left(\tfrac{1}{2}, \tfrac{1}{4}\right) \parallel \psi\left(\tfrac{1}{2}, -\tfrac{1}{4}\right) = \left(\tfrac{k'}{2}, c\right).$$

This shows that $\frac{k'}{2} = \frac{1}{2}$, that is, $k' = 1$ — a contradiction to our assumption $k \neq 1$.

In the case that ψ takes $(\infty, 1)$ to $(\infty, -1)$, we may apply the isomorphism $\mu : \mathscr{L}_k \to \mathscr{L}_{k'}$ from the proof of Proposition 5.2. Then the map $\mu\psi$ fixes each of the four points $(\infty, 0)$, $(0, 0)$, $(1, 0)$, $(\infty, 1)$. Hence we conclude as before that $k = (k')' = 1$ — again a contradiction.

This proves that $\mathscr{L}_k$, $k \neq 1$, is not semiclassical. □

Remark 5.6. In the proof of Lemma 4.1 we already mentioned that the derived projective plane of $\mathscr{L}(k)$ at $(\infty, 0)$ is isomorphic to a cartesian plane $\mathscr{P}_{1,k/k',1}$. It is readily seen that the derived projective plane of a semiclassical plane $\mathscr{L}(h_m, \mathrm{id})$ at $(\infty, 0)$ is isomorphic to a cartesian plane $\mathscr{P}_{m,1,1}$. As mentioned in [Salzmann et al.

1995, Proof of 37.6] the plane $\mathscr{P}_{\alpha,\beta,c}$ is dual to $\mathscr{P}_{\beta,\alpha,c}$. Hence, when $m = k/k'$, the derived projective plane at $(\infty, 0)$ of a Laguerre plane $\mathscr{L}(k)$ and of a semiclassical plane $\mathscr{L}(h_m, \mathrm{id})$ are dual to each other. However, there does not seem to be an extension of this duality to the level of the Laguerre planes (for example, via associated generalized quadrangles, see below).

It is well known that 2-dimensional Laguerre planes correspond to certain compact 3-dimensional generalized quadrangles, compare [Schroth 1993a], [Schroth 1993b] or [Schroth 1995b]. In a compact 3-dimensional generalized quadrangle the point and line spaces are compact and 3-dimensional. These generalized quadrangles are also characterized by having topological parameter 1 (so that all lines and line pencils are homeomorphic to the 1-dimensional sphere $\mathbb{S}^1$). More precisely, the Lie geometry associated with a 2-dimensional Laguerre plane is an antiregular compact generalized quadrangle with topological parameter 1. Up to duality, every compact 3-dimensional generalized quadrangle is the Lie geometry of a 2-dimensional Laguerre plane; see [Schroth 1995b, Corollary 2.16 and Chapter 3]. Recall that the *Lie geometry* of a Laguerre plane $\mathscr{L}$ has as points the points of $\mathscr{L}$ plus the circles of $\mathscr{L}$ plus one additional point at infinity, denoted by $\overline{\infty}$. (The bar helps distinguish this from other uses of the symbol ∞.) The lines of the Lie geometry are the augmented parallel classes, that is, the parallel classes to which the point $\overline{\infty}$ is adjoined, and the augmented tangent pencils, that is, the collections of all circles that touch a given circle at a given point p together with the point p, called the support of the tangent pencil. Incidence is the natural one. So "collinear" in the Lie geometry corresponds to "on the same parallel class or incident or touching" in the Laguerre plane. The generalized quadrangle obtained from the classical real Laguerre plane $\mathscr{L}_{\mathrm{cl}}$ is the real orthogonal quadrangle $Q(4, \mathbb{R})$ over $\mathbb{R}$. Points are the 1-dimensional isotropic subspaces of $\mathbb{R}^5$, with respect to a symmetric form of Witt index 2; lines are the 2-dimensional totally isotropic subspaces of $\mathbb{R}^5$.

Conversely, for every point p of an antiregular generalized quadrangle $\mathfrak{Q}$, one obtains a Laguerre plane $\mathfrak{Q}'_p$, called the *derivation of* $\mathfrak{Q}$ *at* p, whose points are the points of $\mathfrak{Q}$ that are collinear with p except p itself and whose circles are of the form $p^\perp \cap q^\perp$ for points q not collinear with p, where $x^\perp$ denotes the set of all points collinear with the point x. See also [Joswig 1999, Theorem 3.1], where it is shown that it suffices to have a strongly antiregular point of the generalized quadrangle in order to obtain a Laguerre plane as derivation at that point. Each derived Laguerre plane of the real orthogonal quadrangle $Q(4, \mathbb{R})$ over $\mathbb{R}$ is isomorphic to the classical real Laguerre plane.

Starting with a 2-dimensional Laguerre plane $\mathscr{L}$ one obtains an antiregular compact 3-dimensional generalized quadrangle $\mathfrak{Q}(\mathscr{L})$. One can then derive at any point p of $\mathfrak{Q}(\mathscr{L})$ to obtain another 2-dimensional Laguerre plane $\mathscr{L}'_p = (\mathfrak{Q}(\mathscr{L}))'_p$. In [Schroth 1995a] and [Schroth 1995b, Chapter 6] this Laguerre plane $\mathscr{L}'_p$ is called a *sister* of $\mathscr{L}$. The process of going from $\mathscr{L}$ to its sister $\mathscr{L}'_p$ can be completely described within $\mathscr{L}$ without explicitly using the associated generalized quadrangle; see [Schroth 1995a, Section 3]. In case one derives $\mathfrak{Q}(\mathscr{L})$ at a point that comes from a circle K of $\mathscr{L}$, the points of $\mathscr{L}'_K$ are the circles of $\mathscr{L}$ that touch K and the points of $\mathscr{L}$ on K. The parallel classes of $\mathscr{L}'_K$ are obtained from the tangent pencils with support on K.

Circles of $\mathscr{L}'_K$ correspond to the points of $\mathscr{L}$ not on K (more precisely, such a point q represents the collection of all circles of $\mathscr{L}$ through q that touch K) and to the circles of $\mathscr{L}$ not touching K (more precisely, such a circle C represents the collection of all circles of $\mathscr{L}$ that touch C and K), and the extra point $\overline{\infty}$. Incidence is the natural one; compare [Schroth 1995a, Section 3].

Note that an automorphism α of $\mathscr{L}$ extends to an automorphism $\bar{\alpha}$ of $\mathfrak{Q}(\mathscr{L})$. Furthermore, $\bar{\alpha}$ fixes $\overline{\infty}$. If α fixes a point or circle of $\mathscr{L}$, then $\bar{\alpha}$ induces an automorphism in the derived Laguerre plane of $\mathfrak{Q}(\mathscr{L})$ at that point or circle.

We carry out the above procedure for the Laguerre planes $\mathscr{L}_k$ and the distinguished circle C_0. Since C_0 is fixed by Γ, this group is again a group of automorphisms of $(\mathscr{L}_k)'_{C_0}$. Note that $\mathscr{L}_k$ shares many circles with the classical real Laguerre plane $\mathscr{L}_{\text{cl}}$ and, in particular, all the circles that touch C_0. So we expect that $(\mathscr{L}_k)'_{C_0}$ has many circles in common with $\mathscr{L}_{\text{cl}}$, and looks like one of the Laguerre planes constructed in this paper or a semiclassical Laguerre plane obtained by pasting along a circle. In fact, we have the following.

Proposition 5.7. *The Laguerre plane $(\mathscr{L}_k)'_{C_0}$ obtained by deriving the generalized quadrangle $\mathfrak{Q}(\mathscr{L}_k)$ at C_0 is isomorphic to $\mathscr{L}_k$.*

Proof. A circle of $\mathscr{L}_k$ touching C_0 is $C'_{a,b,0}$, where $a, b \in \mathbb{R}$, $a \neq 0$, or $C_{0,0,c}$, where $c \in \mathbb{R}$, $c \neq 0$. We identify such a circle with $\left(b, \frac{1}{a}\right) \in Z$ and $\left(\infty, \frac{1}{c}\right)$, respectively. A point $(x, 0)$ on C_0 is identified with $(x, 0) \in Z$. This coordinatization maps all points of $(\mathscr{L}_k)'_{C_0}$ onto the cylinder Z. Parallel classes are still the generators of Z.

The point $\overline{\infty}$ gives rise to the set C_0, which thus is again a circle of $(\mathscr{L}_k)'_{C_0}$. If (x_0, y_0), $y_0 \neq 0$, is a point not on C_0, then for each $b \in \mathbb{R}$, $b \neq x_0$, there is a unique circle through (x_0, y_0) that touches C_0 at $(b, 0)$; this circle is $C'_{y_0/(x_0-b)^2,b,0}$, which yields the point $(b, (x_0 - b)^2/y_0)$ according to the above rule. One further obtains $(x_0, 0)$ (from the parallel class through (x_0, y_0)) and $(\infty, 1/y_0)$ (from the circle $C_{0,0,y_0}$ touching C_0 at $(\infty, 0)$). Put together we thus obtain all the points on $C'_{1/y_0,x_0,0}$, so that this is again a circle of $(\mathscr{L}_k)'_{C_0}$.

Next consider a circle not meeting C_0. Such a circle is of the form $C'_{a,b,c}$, where $ac > 0$. The circle of $\mathscr{L}_k$ touching C_0 at $(u, 0)$ and also touching $C'_{a,b,c}$ is $C'_{\tilde{a},u,0}$, where $u \in \mathbb{R}$ and $\tilde{a} = ac/(a(u - b)^2 + c$. Hence we obtain the point

$$\left(u, \frac{1}{c}(u - b)^2 + \frac{1}{a}\right)$$

in $(\mathscr{L}_k)'_{C_0}$. When $u=\infty$ we find the circle $C_{0,0,c}$, which yields the point $\left(\infty,\frac{1}{c}\right)$. Thus we have recovered the C-circle $C'_{1/c,b,1/a}$ as a circle of $(\mathscr{L}_k)'_{C_0}$.

Finally consider a circle meeting C_0 in two points. Such a circle is a D-circle. In this case the calculations are a bit more involved. To find the circle $C'_{v,u,0}$ that touches $D_{a,b,c}$, $a\neq 0$, and also touches C_0 at $(u,0)$, where $u\neq b,c,\infty$, it is necessary that the equations

$$v(x-u)^2 = ah_k(x-b)h_{k'}(x-c), \tag{6}$$

$$\begin{aligned} 2v(x-u) &= a(h_k(x-b)h_{k'}(x-c))' \\ &= a(2x-k'b-kc)|x-b|^{k-1}|x-c|^{k'-1} \end{aligned} \tag{7}$$

are satisfied. Dividing (6) by (7) one finds that

$$x=\frac{u(k'b+kc)-2bc}{2u-kb-k'c}.$$

Substitution into (6) then yields

$$\frac{1}{v}=-\frac{4}{a(b-c)^2h_k(k)h_{k'}(k')}h_k(u-c)h_{k'}(u-b).$$

In the coordinates of $(\mathscr{L}_k)'_{C_0}$ as introduced above the two points $(b,0)$ and $(c,0)$ of intersection of C_0 and $D_{a,b,c}$ yield the points $(b,0)$ and $(c,0)$ on the circle induced by $D^{\perp}_{a,b,c}$. When $u=\infty$ one similarly obtains from $(h_k(x-b)h_{k'}(x-c))'=0$ that $x=\frac{1}{2}(k'b+kc)$ and thus $v=-\frac{1}{4}a(b-c)^2h_k(k)h_{k'}(k')$. In total we have recovered all the points of $D_{\tilde{a},c,b}$, where

$$\tilde{a}=-\frac{4}{a(b-c)^2h_k(k)h_{k'}(k')}.$$

The cases when $a=0$ are dealt with in a similar way. □

In case one derives the generalized quadrangle $\mathfrak{Q}(\mathscr{L})$ at a point that comes from a point p of $\mathscr{L}$ then the points of $\mathscr{L}'_p$ are the circles of $\mathscr{L}$ that pass through p, the points of $\mathscr{L}$ on the parallel class $|p|$ of p but not p itself, and the extra point $\overline{\infty}$. The parallel classes of $\mathscr{L}'_p$ are obtained from the parallel class $|p|$ and the tangent pencils with support p. The circles of $\mathscr{L}'_p$ correspond to the points of $\mathscr{L}$ not on $|p|$ (more precisely, such a point q represents the collection of all circles of $\mathscr{L}$ through p and q) and to the circles of $\mathscr{L}$ not passing through p (more precisely, such a circle C represents the collection of all circles of $\mathscr{L}$ through p that touch C). Thus the affine part of $\mathscr{L}'_p$ with respect to the parallel class containing $\overline{\infty}$ is made up of the nonvertical lines of the derived affine plane $\mathscr{A}_p$ of $\mathscr{L}$ at p, and points of $\mathscr{A}_p$ represent circles of $\mathscr{L}'_p$ through $\overline{\infty}$. Hence the derived projective plane $\mathscr{P}'_{\overline{\infty}}$ of $\mathscr{L}'_p$ at $\overline{\infty}$ is the dual of $\mathscr{P}_p$, the derived projective plane of $\mathscr{L}$ at p. A circle of $\mathscr{L}$ not passing through p induces an oval $\mathbb{O}$ in $\mathscr{P}_p$. Since this circle also represents a circle of $\mathscr{L}'_p$, we just obtain the dual oval $\mathbb{O}^*$ of $\mathbb{O}$ in $\mathscr{P}'_{\overline{\infty}}$. Hence, the whole process involves forming the dual of the derived projective plane $\mathscr{P}_p$ plus all duals of the ovals in $\mathscr{P}_p$ that are induced by circles of $\mathscr{L}$; we then remove one line to obtain the affine part of the sister $\mathscr{L}'_p$ and add one parallel class at infinity in order to complete the Laguerre plane. Although applying this process to a point p on K_0 of a nonclassical semiclassical Laguerre plane $\mathscr{L}(h_m, \mathrm{id})$ yields the dual of the derived plane at p, other circles of $\mathscr{L}(h_m, \mathrm{id})'_p$ do not match circles of $\mathscr{L}_k$. Since the point p has a 1-dimensional orbit we also expect the automorphism group of $\mathscr{L}(h_m, \mathrm{id})'_p$ to be at most 3-dimensional, and so $\mathscr{L}(h_m, \mathrm{id})'_p$ cannot be isomorphic to a plane $\mathscr{L}_k$.

Schroth [2000] used a provisional classification of 2-dimensional Laguerre planes of group dimension 4 to show that a compact 3-dimensional generalized quadrangle is the real orthogonal quadrangle $Q(4, \mathbb{R})$, or its dual if the group of automorphisms of the quadrangle has dimension at least 6. Since the new Laguerre planes $\mathscr{L}_k$ do not appear on the list used in [Schroth 2000], this can potentially affect Schroth's result. However, as noted in [Schroth 2000, Section, 3.7], in case of a 4-dimensional group of automorphisms of a 2-dimensional Laguerre plane such that a circle is fixed, the information on the groups involved is enough to see that the dimension of the automorphism group of the associated quadrangle does not become larger; see also [Schroth 2000, Section 4.6]. The automorphism group of $\mathscr{L}_k$ has at most as many orbits on the circle set and point set as the automorphism group of semiclassical Laguerre planes pasted along a circle. This implies that the same dimensions of orbits occur as stated in [Schroth 2000, Section 4.6]. Hence we have the following result; compare [Schroth 2000, Theorem 4.8].

Corollary 5.8. *The automorphism group of the* 3*-dimensional compact generalized quadrangle* $\mathfrak{Q}(\mathscr{L}_k)$ *is* 4*-dimensional when* $k \neq 1$.

As a consequence, the planes constructed here are not counterexamples to the main theorem of [Schroth 2000].

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