Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial

Conics in Baer subplanes

Susan G. Barwick, Wen-Ai Jackson and Peter Wild

msp

Vol. 17 No. 2 2019

Innovations in Incidence Geometry
Algebraic, Topological and Combinatorial

vol. 17, no. 2, 2019
dx.doi.org/10.2140/iig.2019.17.85

msp

Conics in Baer subplanes

Susan G. Barwick, Wen-Ai Jackson and Peter Wild

This article studies conics and subconics of $PG(2, q^2)$ and their representation in the André/Bruck–Bose setting in $PG(4, q)$. In particular, we investigate their relationship with the transversal lines of the regular spread. The main result is to show that a conic in a tangent Baer subplane of $PG(2, q^2)$ corresponds in $PG(4, q)$ to a normal rational curve that meets the transversal lines of the regular spread. Conversely, every 3- and 4-dimensional normal rational curve in $PG(4, q)$ that meets the transversal lines of the regular spread corresponds to a conic in a tangent Baer subplane of $PG(2, q^2)$.

1. Introduction

This article investigates the representation of conics and subconics of $PG(2, q^2)$ in the Bruck–Bose representation in $PG(4, q)$. The Bruck–Bose representation of $PG(2, q^2)$ uses a regular spread $\mathcal{S}$ in the hyperplane at infinity of $PG(4, q)$. The regular spread $\mathcal{S}$ has two unique transversal lines g, g^q in the quadratic extension $PG(4, q^2)$. There are several known characterizations of objects of $PG(4, q)$ in terms of their relationship with these transversal lines. Firstly, a conic $\mathcal{C}$ in $PG(4, q)$ corresponds to a Baer subline of $PG(2, q^2)$ if and only if the extension of $\mathcal{C}$ to a conic of $PG(4, q^2)$ contains a point of g and a point of g^q [Casse and Quinn 2002]. A ruled cubic surface $\mathcal{V}$ in $PG(4, q)$ corresponds to a Baer subplane of $PG(2, q^2)$ if and only if the extension of $\mathcal{V}$ to $PG(4, q^2)$ contains g and g^q [Casse and Quinn 2002]. Further, an orthogonal cone $\mathcal{U}$ corresponds to a classical unital of $PG(2, q^2)$ if and only if the extension of $\mathcal{U}$ to $PG(4, q^2)$ contains g and g^q [Metsch 1997]. Hence the interaction of certain objects with the transversals of $\mathcal{S}$ is intrinsic to their characterization in $PG(2, q^2)$. In this article we study conics and subconics of $PG(2, q^2)$ and determine their relationship with the transversals of $\mathcal{S}$ in the Bruck–Bose setting in $PG(4, q)$. In particular, we characterize normal rational curves of $PG(4, q)$ whose extension meets the transversals as subconics of $PG(2, q^2)$.

MSC2010: 51E20.
Keywords: Bruck–Bose representation, Baer subplanes, conics, subconics.

The article is set out as follows. Section 2 gives background and proves some preliminary results. In particular, in order to study how objects of the Bruck–Bose representation relate to the transversals of the regular spread $\mathcal{S}$, we formally define the notion of g-special sets, or special sets in PG$(4, q)$ (page 90). In the last subsection (pages 93–95) we consider a Baer subplane $\mathcal{B}$ tangent to ℓ_∞, and give a geometric construction via PG$(4, q)$ that partitions the affine points of $\mathcal{B}$ into q conics, one of which is degenerate.

In Section 3, we discuss how the notion of specialness relates to the known Bruck–Bose representation of Baer sublines and Baer subplanes.

In Section 4, we investigate nondegenerate conics of PG$(2, q^2)$ in the PG$(4, q)$ Bruck–Bose representation, and specifically the structure in the quadratic extension to PG$(4, q^2)$. We show that in PG$(4, q^2)$, the (extended) structure corresponding to a nondegenerate conic $\mathcal{O}$ is the intersection of two quadrics which meet g in the two points (possibly repeated or in an extension) corresponding to $\mathcal{O} \cap \ell_\infty$.

In Section 5 we characterize the Bruck–Bose representation of conics contained in Baer subplanes. In PG$(2, q^2)$, let $\mathcal{B}$ be a Baer subplane tangent to ℓ_∞, and $\mathcal{C}$ a nondegenerate conic contained in $\mathcal{B}$. We show that in PG$(4, q)$, $\mathcal{C}$ corresponds to a normal rational curve that meets the transversals of the regular spread. Conversely, we characterize every normal rational curve in PG$(4, q)$ that meets the transversals of the regular spread as corresponding to a nondegenerate conic in a Baer subplane of PG$(2, q^2)$.

While the proofs in Section 4 are largely coordinate-based, the proofs in Section 5 use geometrical arguments.

2. Background and preliminary results

In this section we give the necessary background, introduce the notation we use in this article, and prove a number of preliminary results.

Conjugate points. For q a prime power, we denote the unique finite field of order q by $\mathbb{F}_q$. We use the phrase conjugate points in different settings. Firstly, consider the automorphism $x \mapsto x^q$ for $x \in \mathbb{F}_{q^r}$ and the induced automorphic collineation of PG(n, q^r) given by $X = (x_0, \dots, x_n) \mapsto X^q = (x_0^q, \dots, x_n^q)$. The points $X, X^q, \dots, X^{q^{n-1}}$ are called *conjugate*. Secondly, let $\mathcal{B}$ be a Baer subplane of PG$(2, q^2)$; there is a unique involutory collineation that fixes $\mathcal{B}$ pointwise, and we call this map *conjugacy with respect to* $\mathcal{B}$. Note that $P, Q \in \ell_\infty$ are conjugate with respect to the secant Baer subplane $\mathcal{B}$ if and only if P, Q are conjugate with respect to the Baer subline $\mathcal{B} \cap \ell_\infty$.

***Spreads in* PG(3, *q*).** The following construction of a regular spread of PG$(3, q)$ will be needed, see [Hirschfeld and Thas 1991] for more information on spreads.

Embed PG$(3, q)$ in PG$(3, q^2)$ and let g be a line of PG$(3, q^2)$ disjoint from PG$(3, q)$. The line g has a conjugate line g^q with respect to the map $x \mapsto x^q$, $x \in \mathbb{F}_{q^2}$, and g^q is also disjoint from PG$(3, q)$. Let P_i be a point on g; then the line $\langle P_i, P_i^q \rangle$ meets PG$(3, q)$ in a line. As P_i ranges over all the points of g, we obtain $q^2 + 1$ lines of PG$(3, q)$ that partition PG$(3, q)$. These lines form a regular spread $\mathcal{S}$ of PG$(3, q)$. The lines g, g^q are called the (conjugate skew) *transversal lines* of the regular spread $\mathcal{S}$. Conversely, given a regular spread $\mathcal{S}$ in PG$(3, q)$, there is a unique pair of transversal lines in PG$(3, q^2)$ that generate $\mathcal{S}$ in this way.

The Bruck–Bose representation. We will use the linear representation of a finite translation plane of dimension at most two over its kernel, introduced independently in [André 1954] and [Bruck and Bose 1964; 1966]. Let Σ_∞ be a hyperplane of PG$(4, q)$ and let $\mathcal{S}$ be a spread of Σ_∞. The phrase *a subspace of* PG$(4, q) \setminus \Sigma_\infty$ will be used to mean a subspace of PG$(4, q)$ that is not contained in Σ_∞. Consider the following incidence structure: the *points* of $\mathcal{A}(\mathcal{S})$ are the points of PG$(4, q) \setminus \Sigma_\infty$; the *lines* of $\mathcal{A}(\mathcal{S})$ are the planes of PG$(4, q) \setminus \Sigma_\infty$ that contain an element of $\mathcal{S}$; and *incidence* in $\mathcal{A}(\mathcal{S})$ is induced by incidence in PG$(4, q)$. Then the incidence structure $\mathcal{A}(\mathcal{S})$ is an affine plane of order q^2. We can complete $\mathcal{A}(\mathcal{S})$ to a projective plane $\mathcal{P}(\mathcal{S})$; the points on the line at infinity ℓ_∞ have a natural correspondence to the elements of the spread $\mathcal{S}$. We call this the *Bruck–Bose representation* of $\mathcal{P}(\mathcal{S})$ in PG$(4, q)$. The projective plane $\mathcal{P}(\mathcal{S})$ is the Desarguesian plane PG$(2, q^2)$ if and only if $\mathcal{S}$ is a regular spread of $\Sigma_\infty \cong$ PG$(3, q)$ (see [Bruck 1969]). We use the following notation in the Bruck–Bose setting:

- $\mathcal{S}$ is a regular spread with transversal lines g, g^q.
- An affine point of PG$(2, q^2) \setminus \ell_\infty$ is denoted with a capital letter, A say, and $[A]$ denotes the corresponding point of PG$(4, q) \setminus \Sigma_\infty$.
- A point on ℓ_∞ in PG$(2, q^2)$ is denoted with an over-lined capital letter, $\bar{T}$ say, and the corresponding spread line is denoted $[T]$.
- The points of ℓ_∞ are in one-to-one correspondence with the points of g; for a point $\bar{T} \in \ell_\infty$, we denote the corresponding point of g by T.
- A set of points $\mathcal{X}$ in PG$(2, q^2)$ corresponds to a set of points denoted $[\mathcal{X}]$ in PG$(4, q)$.

We will work in the extension of PG$(4, q)$ to PG$(4, q^2)$ and to PG$(4, q^4)$. Let $\mathcal{K}$ be a primal of PG$(4, q)$, so $\mathcal{K}$ is the set of points of PG$(4, q)$ satisfying a homogeneous equation $f(x_0, \ldots, x_4) = 0$, with coefficients in $\mathbb{F}_q$. We define $\mathcal{K}^{\star}$ to be the (unique) primal of PG$(4, q^2)$ which is the set of points of PG$(4, q^2)$ satisfy the same homogeneous equation $f = 0$. Note that if $\mathcal{K} = \Pi$ is an r-dimensional subspace of PG$(4, q)$, then $\Pi^{\star}$ is the (unique) r-dimensional subspace of PG$(4, q^2)$ containing Π. Further, if $\mathcal{V}$ is a variety of PG$(4, q)$, the intersection of primals

$\mathcal{K}_1, \dots, \mathcal{K}_s$, then we define $\mathcal{V}^{\star} = \mathcal{K}_1^{\star} \cap \cdots \cap \mathcal{K}_s^{\star}$. Similarly, we can extend a primal $\mathcal{K}$ to $\mathrm{PG}(4, q^4)$, and we denote the resulting set by $\mathcal{K}^{\bigstar}$. The transversals g, g^q of the regular spread $\mathcal{S}$ lie in $\mathrm{PG}(4, q^2)$, and we denote their extensions to lines of $\mathrm{PG}(4, q^4)$ by $g^{\bigstar}$, $g^{q\bigstar}$ respectively.

***Ruled cubic surfaces in* PG(4, *q*).** A ruled cubic surface $\mathcal{V}$ of $\mathrm{PG}(4, q)$ consists of a line directrix t, a conic directrix $\mathcal{C}$ lying in a plane disjoint from t, and a set of $q + 1$ pairwise disjoint generator lines joining the points of t and $\mathcal{C}$ according to a projectivity $\omega \in \mathrm{PGL}(2, q)$. Specifically, if $\theta, \phi \in \mathbb{F}_q \cup \{\infty\}$ are the nonhomogeneous coordinates of t and $\mathcal{C}$, then ω maps $(1, \theta)$ to $(1, \phi)$. The generators of $\mathcal{V}$ are the lines joining points of t to the corresponding point of $\mathcal{C}$ under ω. We will need the following result, which shows how hyperplanes of $\mathrm{PG}(4, q)$ meet a ruled cubic surface.

Result 2.1 [Quinn 2002]. *A hyperplane of* $\mathrm{PG}(4, q)$ *meets a ruled cubic surface in one of the following*:

- *The line directrix*; $(q^2 - q)/2$ *hyperplanes do this.*
- *The line directrix and one generator line*; $q + 1$ *hyperplanes do this.*
- *The line directrix and two generator lines*; $(q^2 + q)/2$ *hyperplanes do this.*
- *A conic and a generator line*; $q^3 + q^2$ *hyperplanes do this.*
- *A twisted cubic curve* (*which meets the line directrix in a unique point*); $q^4 - q^2$ *hyperplanes do this.*

Corollary 2.2. *Let* Π *be a hyperplane of* $\mathrm{PG}(4, q)$ *that meets a ruled cubic surface* $\mathcal{V}$ *in a twisted cubic* $\mathcal{N}$. *Then* $\mathcal{N}$ *meets each generator line of* $\mathcal{V}$ *in a unique point.*

Proof. If $\mathcal{N}$ meets a generator line ℓ of $\mathcal{V}$ in two points, then the 3-space Π containing $\mathcal{N}$ also contains ℓ, which is not possible by Result 2.1. Hence $\mathcal{N}$ meets each generator line in at most one point. As $\mathcal{N}$ contains $q + 1$ points, each generator of $\mathcal{V}$ contains a unique point of $\mathcal{N}$. □

There are two ways to extend the ruled cubic surface to $\mathrm{PG}(4, q^2)$, we show that they are equivalent. The ruled cubic surface $\mathcal{V}$ is a variety whose points are the exact intersection of three quadrics, $\mathcal{V} = \mathcal{Q}_0 \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$ (see for example [Barwick and Jackson 2012]). So extending this variety to $\mathrm{PG}(4, q^2)$ yields $\mathcal{V}^{\star} = \mathcal{Q}_0^{\star} \cap \mathcal{Q}_1^{\star} \cap \mathcal{Q}_2^{\star}$. Alternatively, we can extend $\mathcal{V}$ to $\mathrm{PG}(4, q^2)$ as in [Casse and Quinn 2002]: namely extending the line directrix t and conic directrix $\mathcal{C}$ to $\mathrm{PG}(4, q^2)$ by taking $\theta, \phi \in \mathbb{F}_{q^2} \cup \{\infty\}$, and extending the projectivity ω to act over $\mathbb{F}_{q^2}$. We denote this extension by $\mathcal{V}'$, thus $\mathcal{V}'$ is the ruled cubic surface with line directrix $t^{\star}$, conic directrix $\mathcal{C}^{\star}$, and ruled using the (extended) projectivity ω. We show that these two extensions $\mathcal{V}^{\star}$, $\mathcal{V}'$ are the same. The surface $\mathcal{V}$ contains exactly q^2 conics $\mathcal{C}_1, \dots, \mathcal{C}_{q^2}$, and these conics cover each point of $\mathcal{V} \setminus t$ q-times (see [Barwick and Ebert 2008] for more details). Hence both sets $\mathcal{V}^{\star}$, $\mathcal{V}'$ contain the extended conics $\mathcal{C}_1^{\star}, \dots, \mathcal{C}_{q^2}^{\star}$. Moreover, these conics together with $t^{\star}$ cover all the points of $\mathcal{V}'$, and so $\mathcal{V}^{\star}$ contains $\mathcal{V}'$. However, $\mathcal{V}^{\star}$ is the intersection of three quadrics over $\mathbb{F}_{q^2}$, whose intersection over $\mathbb{F}_q$ is a ruled cubic surface. By [Bernasconi and Vincenti 1981], all ruled cubic surfaces are projectively equivalent, hence $\mathcal{V}^{\star}$ and $\mathcal{V}'$ are the same ruled cubic surface of $\mathrm{PG}(4, q^2)$.

Coordinates in Bruck–Bose. We now show the relation between the coordinates of points in $\mathrm{PG}(2, q^2)$ and the coordinates of the corresponding points in the Bruck–Bose representation of $\mathrm{PG}(4, q)$. See [Barwick and Ebert 2008, Section 3.4] for more details. Let τ be a primitive element in $\mathbb{F}_{q^2}$ with primitive polynomial $x^2 - t_1 x - t_0$ over $\mathbb{F}_q$. Then every element $\alpha \in \mathbb{F}_{q^2}$ can be uniquely written as $\alpha = a_0 + a_1\tau$ with $a_0, a_1 \in \mathbb{F}_q$. That is, $\mathbb{F}_{q^2} = \{x_0 + x_1\tau \mid x_0, x_1 \in \mathbb{F}_q\}$. It is useful to keep in mind the relationships: $\tau\tau^q = -t_0$, $\tau + \tau^q = t_1$, $t_0/\tau = -\tau^q = \tau - t_1$ and $\tau^{q^2} = 1$. Points in $\mathrm{PG}(2, q^2)$ have homogeneous coordinates (x, y, z) with $x, y, z \in \mathbb{F}_{q^2}$, not all zero. We let the line at infinity ℓ_∞ have equation $z = 0$, so affine points of $\mathrm{PG}(2, q^2)$ have coordinates $(x, y, 1)$, with $x, y \in \mathbb{F}_{q^2}$. Points in $\mathrm{PG}(4, q)$ have homogeneous coordinates (x_0, x_1, y_0, y_1, z) with $x_0, x_1, y_0, y_1, z \in \mathbb{F}_q$, not all zero. We let the hyperplane at infinity Σ_∞ have equation $z = 0$, so the affine points of $\mathrm{PG}(4, q)$ have coordinates $(x_0, x_1, y_0, y_1, 1)$, with $x_0, x_1, y_0, y_1 \in \mathbb{F}_q$. Let A be an affine point in $\mathrm{PG}(2, q^2)$ with coordinates $A = (x_0 + x_1\tau, y_0 + y_1\tau, z)$, where $x_0, x_1, y_0, y_1, z \in \mathbb{F}_q$, $z \neq 0$. The map

$$\varphi : \mathrm{PG}(2, q^2) \setminus \ell_\infty \to \mathrm{PG}(4, q) \setminus \Sigma_\infty,$$
$$A = (x_0 + x_1\tau, y_0 + y_1\tau, z) \mapsto [A] = (x_0, x_1, y_0, y_1, z),$$

is a bijection from the affine points of $\mathrm{PG}(2, q^2)$ to the affine points of $\mathrm{PG}(4, q)$, called the *Bruck–Bose map*. We can extend this to a projective map: for a point $\bar{T} = (\delta, 1, 0) \in \ell_\infty$, write $\delta = d_0 + d_1\tau \in \mathbb{F}_{q^2}$, $d_0, d_1 \in \mathbb{F}_q$; then

$$\bar{T} = (\delta, 1, 0) \mapsto [T] = \langle (d_0, d_1, 1, 0, 0), (t_0 d_1, d_0 + t_1 d_1, 0, 1, 0) \rangle.$$

The transversal lines g, g^q of $\mathcal{S}$ have coordinates given by

$$g = \langle A_0 = (\tau^q, -1, 0, 0, 0), A_1 = (0, 0, \tau^q, -1, 0) \rangle,$$
$$g^q = \langle A_0^q = (\tau, -1, 0, 0, 0), A_1^q = (0, 0, \tau, -1, 0) \rangle.$$

Recall that each line of the regular spread $\mathcal{S}$ meets the transversal g of $\mathcal{S}$ in a point. The one-to-one correspondence between points of ℓ_∞ and points of g is

$$\bar{T} = (\delta, 1, 0) \in \ell_\infty \leftrightarrow T = \delta A_0 + A_1 \in g, \quad \delta \in \mathbb{F}_{q^2} \cup \{\infty\},$$

that is, $T = [T]^{\star} \cap g$ and $[T]^{\star} = TT^q$.

Coordinates and the quartic extension PG$(4, q^4)$. We will be interested in nondegenerate conics of PG$(2, q^2)$, and one of the cases to consider is when a conic $\mathcal{C}$ is exterior to ℓ_∞, and so meets ℓ_∞ in two points which lie in the quadratic extension of PG$(2, q^2)$ to PG$(2, q^4)$. That is, $\mathcal{C}$ meets ℓ_∞ in two points $\bar{P}$, $\bar{Q}$ over $\mathbb{F}_{q^4}$. Note that $\bar{P}$, $\bar{Q}$ are conjugate with respect to the map $x \mapsto x^{q^2}$, $x \in \mathbb{F}_{q^4}$, that is $\bar{Q} = \bar{P}^{q^2}$. There is no direct representation for the point $\bar{P}$ in the Bruck–Bose representation in PG$(4, q)$. However, there is a related point in the quartic extension PG$(4, q^4)$. We can extend the one-to-one correspondence between points ℓ_∞ and points of g to a one-to-one correspondence between points of the quadratic extension of ℓ_∞ and points of the extended transversal $g^\star$ in PG$(4, q^4)$, so

$$\bar{P} = (\alpha, 1, 0) \leftrightarrow P = \alpha A_0 + A_1 \in g^\star, \quad \alpha \in \mathbb{F}_{q^4} \cup \{\infty\}.$$

If $\bar{P} = (\alpha, 1, 0)$ for some $\alpha \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, that is $\bar{P} \in \mathrm{PG}(2, q^4) \setminus \mathrm{PG}(2, q^2)$, then in PG$(4, q^4)$ the corresponding point P lies in $g^\star \setminus g$, and the conjugate point $P^q = \alpha^q A_0^q + A_1^q$ lies on $g^{q\star} \setminus g^q$. As $\bar{P} \notin \mathrm{PG}(2, q^2)$, the line PP^q is not a line of the spread $\mathcal{S}$; PP^q is a line of PG$(4, q^4)$ that does not meet Σ_∞.

g-special sets. When studying curves of PG$(2, q^2)$ in the PG$(4, q)$ Bruck–Bose setting, the transversals g, g^q of the regular spread $\mathcal{S}$ play an important role in characterizations. Let $\mathcal{V}$ be a variety or rational curve in PG$(4, q)$, we are interested in how $\mathcal{V}^{\star}$ meets g, g^q in the extension to PG$(4, q^2)$. Note that if $\mathcal{V}^{\star}$ meets g in a point P, then as $\mathcal{V}$ is defined over $\mathbb{F}_q$, $\mathcal{V}^{\star}$ also meets g^q in the point P^q. A nondegenerate conic $\mathcal{C}$ in PG$(4, q)$ is called a *g-special conic* if in PG$(4, q^2)$, $\mathcal{C}^{\star}$ contains one point of g, and one point of g^q. A twisted cubic $\mathcal{N}$ in PG$(4, q)$ is called a *g-special twisted cubic* if in PG$(4, q^2)$, $\mathcal{N}^{\star}$ contains one point of g, and one point of g^q. A 4-dimensional normal rational curve $\mathcal{N}$ in PG$(4, q)$ is called a *g-special normal rational curve* if in PG$(4, q^2)$, $\mathcal{N}^{\star}$ contains two points of g (possibly repeated) and two points of g^q. Further, $\mathcal{N}$ is called *$g^\star$-special* if in the quartic extension PG$(4, q^4)$, $\mathcal{N}^\star$ contains two points of the extended transversal $g^\star \setminus g$. A ruled cubic surface $\mathcal{V}$ in PG$(4, q)$ is called a *g-special ruled cubic surface* if in PG$(4, q^2)$, $\mathcal{V}^{\star}$ contains g and g^q.

Representations of Baer sublines and subplanes. We use the following representations of Baer sublines and subplanes of PG$(2, q^2)$ in PG$(4, q)$, see [Barwick and Ebert 2008] for more details.

Result 2.3. *Let $\mathcal{S}$ be a regular spread in a* 3*-space Σ_∞ in* PG$(4, q)$ *and consider the representation of the Desarguesian plane $\mathcal{P}(\mathcal{S})$ =* PG$(2, q^2)$ *defined by the Bruck–Bose construction.*

1. *A Baer subline of ℓ_∞ in* PG$(2, q^2)$ *corresponds to a regulus of $\mathcal{S}$.*

2. *A Baer subline of* $\mathrm{PG}(2, q^2)$ *that meets* ℓ_∞ *in a point corresponds to a line of* $\mathrm{PG}(4, q) \setminus \Sigma_\infty$.
3. *A Baer subplane of* $\mathrm{PG}(2, q^2)$ *secant to* ℓ_∞ *corresponds to a plane, not containing a spread line, of* $\mathrm{PG}(4, q) \setminus \Sigma_\infty$.
4. *A Baer subline of* $\mathrm{PG}(2, q^2)$ *that is disjoint from* ℓ_∞ *corresponds in* $\mathrm{PG}(4, q)$ *to a* g*-special conic.*
5. *A Baer subplane tangent to* ℓ_∞ *at a point* $\bar{T}$ *corresponds in* $\mathrm{PG}(4, q)$ *to a* g*-special ruled cubic surface containing the corresponding spread line* $[T]$.

Moreover, the converse of each of these correspondences holds.

Remark 2.4. The correspondences in parts 2 and 3 are not exact at infinity. The exact at infinity representation of a Baer subline that meets ℓ_∞ in a point T is an affine line that meets the spread line $[T]$ *union* with the spread line $[T]$. Similarly, the exact at infinity representation of a secant Baer subplane is a plane α not containing a spread line, *union* the lines of $\mathcal{S}$ that α meets.

Representations of subconics. The representation of nondegenerate conics contained in a Baer subplane was considered in [Quinn 2002].

Result 2.5 [Quinn 2002]. *Let* $\mathcal{C}$ *be a nondegenerate conic contained in a Baer subplane* $\mathcal{B}$ *of* $\mathrm{PG}(2, q^2)$.

1. *Suppose* $\mathcal{B}$ *is secant to* ℓ_∞*. Then* $\mathcal{C}$ *corresponds to a nondegenerate conic in the plane* $[\mathcal{B}]$ *of* $\mathrm{PG}(4, q)$.
2. *Suppose* $\mathcal{B}$ *is tangent to* ℓ_∞, $\mathcal{B} \cap \ell_\infty \in \mathcal{C}$, *and* $q \geq 3$*. Then* $\mathcal{C}$ *corresponds to a twisted cubic on the ruled cubic surface* $[\mathcal{B}]$ *of* $\mathrm{PG}(4, q)$.
3. *Suppose* $\mathcal{B}$ *is tangent to* ℓ_∞, $\mathcal{B} \cap \ell_\infty \notin \mathcal{C}$, *and* $q \geq 4$*. Then* $\mathcal{C}$ *corresponds to a* 4*-dimensional normal rational curve on the ruled cubic surface* $[\mathcal{B}]$ *of* $\mathrm{PG}(4, q)$.

In Section 5, we show that the 3- and 4-dimensional normal rational curves of Result 2.5 are g-special. Conversely, we show that every g-special normal rational curve in $\mathrm{PG}(4, q)$ corresponds to a nondegenerate conic contained in a tangent Baer subplane.

Remark 2.6. The correspondence in Result 2.5 parts 1 and 2 is not exact at infinity (compare with Remark 2.4). For example, in part 2, the point $\bar{T} = \mathcal{B} \cap \ell_\infty$ is in $[\mathcal{C}]$, and the twisted cubic $[\mathcal{C}]$ meets Σ_∞ in a point of $[T]$. The exact-at-infinity representation is: the set $[\mathcal{C}]$ is a twisted cubic *union* the spread line $[T]$. We use the simpler, not exact-at-infinity correspondence given in Result 2.5 as it does not lead to any confusion.

***The circle geometry* CG(2, *q*).** Circle geometries $\mathrm{CG}(d,q)$, $d \geq 2$, were introduced in [Bruck 1973a; 1973b], and we summarize the results we need here. Note that $\mathrm{CG}(2,q)$ is an inversive plane. We can construct $\mathrm{CG}(2,q)$ from the line $\mathrm{PG}(1,q^2)$, in this case the circles are the Baer sublines of $\mathrm{PG}(1,q^2)$. Equivalently, we can construct $\mathrm{CG}(2,q)$ from the lines of a regular spread $\mathcal{S}$ of $\mathrm{PG}(3,q)$, in this case the circles are the reguli contained in $\mathcal{S}$. Using the representation of $\mathrm{CG}(2,q)$ as $\ell_\infty \cong \mathrm{PG}(1,q^2)$, we can use properties of the circle geometry to deduce several properties of the projective plane $\mathrm{PG}(2,q^2)$. If $\bar{P}$, $\bar{Q}$ are two distinct points on ℓ_∞ in $\mathrm{PG}(2,q^2)$, then there is a unique partition of ℓ_∞ into $\bar{P}$, $\bar{Q}$ and $q-1$ Baer sublines $\ell_1,\dots,\ell_{q-1}$, where the points $\bar{P}$, $\bar{Q}$ are conjugate with respect to each Baer subline ℓ_i. Further, if $\mathcal{B}$ is a Baer subplane secant to ℓ_∞, such that $\bar{P}$, $\bar{Q}$ are conjugate with respect to $\mathcal{B}$, then $\mathcal{B}$ meets ℓ_∞ in one of the Baer sublines ℓ_i. Of particular interest is an application to conics.

Result 2.7. *Let $\mathcal{O}$ be a nondegenerate conic of* $\mathrm{PG}(2,q^2)$ *that meets ℓ_∞ in $\{\bar{P},\bar{Q}\}$. Then there is a unique partition of the q^2-1 affine points of $\mathcal{O}$ into $q-1$ subconics $\mathcal{C}_1,\dots,\mathcal{C}_{q-1}$, lying in Baer subplanes $\mathcal{B}_1,\dots,\mathcal{B}_{q-1}$ which are secant to ℓ_∞. Further, the Baer sublines $\mathcal{B}_i \cap \ell_\infty$ are either equal or disjoint*

The properties of the circle geometry also lead to properties of a regular spread $\mathcal{S}$ in $\mathrm{PG}(3,q)$. Let g, g^q be the transversals of $\mathcal{S}$, so g and g^q lie in $\mathrm{PG}(3,q^2)$. Consider the set of lines of $\mathrm{PG}(3,q^2)$ that meet both g and g^q. This set is called the *hyperbolic congruence* of g and g^q in [Hirschfeld 1985]. Note that if two distinct lines in the hyperbolic congruence meet, then they meet on g or g^q. The hyperbolic congruence contains the extended spread lines $[P]^{\star} = PP^q$ for $P \in g$ and the lines PQ^q for distinct $P, Q \in g$. The lines PQ^q have an interesting relationship with the regular spread $\mathcal{S}$.

Result 2.8 [Bruck 1973b]. *Let $[P]$, $[Q]$ be two lines of a regular spread $\mathcal{S}$ in* $\mathrm{PG}(3,q)$, *and denote their intersections with the transversal g of $\mathcal{S}$ by P, Q. There is a unique partition of $\mathcal{S}$ into $[P]$, $[Q]$ and $q-1$ reguli $\mathcal{R}_1,\dots,\mathcal{R}_{q-1}$. Denote the opposite regulus of $\mathcal{R}_i$ by $\mathcal{R}_i'$. Then the set $\{[P],[Q],\mathcal{R}_1',\dots,\mathcal{R}_{q-1}'\}$ is a regular spread with transversals PQ^q, P^qQ.*

We will show that the lines in the hyperbolic congruence of g, g^q are related to the Bruck–Bose representation of nondegenerate conics of $\mathrm{PG}(2,q^2)$ in $\mathrm{PG}(4,q)$.

Normal rational curves contained in quadrics. Next, we show that if a normal rational curve is contained in a quadric in $\mathrm{PG}(4,q)$, then the containment also holds in the quadratic extension $\mathrm{PG}(4,q^2)$, provided q is not small.

Lemma 2.9. *In* $\mathrm{PG}(4,q)$, $q>7$, *let $\mathcal{N}$ be a* 4*-dimensional normal rational curve and $\mathcal{Q}$ a quadric, with $\mathcal{N} \subset \mathcal{Q}$. Then in the quadratic extension* $\mathrm{PG}(4,q^2)$, $\mathcal{N}^{\star} \subset \mathcal{Q}^{\star}$.

Proof. Without loss of generality, let $\mathcal{N} = \{P_\theta = (1, \theta, \theta^2, \theta^3, \theta^4) \mid \theta \in \mathbb{F}_q \cup \{\infty\}\}$. Let $\mathcal{Q}$ have equation $g(x_0, x_1, x_2, x_3, x_4) = 0$. Consider $g(P_\theta) = g(1, \theta, \theta^2, \theta^3, \theta^4) = f(\theta)$. As $\mathcal{Q}$ is a quadric, $f(\theta)$ is a polynomial in θ of degree at most 8. Now as $\mathcal{N} \subset \mathcal{Q}$, $f(P_\theta) = 0$ for all $\theta \in \mathbb{F}_q \cup \{\infty\}$. So if $q + 1 > 8$, f is identically 0, and so $f(P_\theta) = 0$ for all $\theta \in \mathbb{F}_{q^2}$. Using $\theta = \infty$, this implies that the coefficient of θ^8 is zero, thus the degree of f is at most 7. As $f(\theta) = 0$ for the q values of $\theta \in \mathbb{F}_q$, it follows that f has q roots, so if $q > 7$ then f is the zero polynomial, thus $f(\theta) = 0$ for all θ in any extension of $\mathbb{F}_q$, and so $g(P_\theta) = 0$ for all θ in any extension of $\mathbb{F}_q$. So if $q > 7$, the point P_θ, $\theta \in \mathbb{F}_{q^2} \cup \{\infty\}$, lies on $\mathcal{Q}^{\star}$, and so $\mathcal{N}^{\star} \subset \mathcal{Q}^{\star}$. □

The bound on q in Lemma 2.9 is tight as shown by the following example. In PG(4, 7), let $\mathcal{N}$ be the normal rational curve $\mathcal{N} = \{P_\theta = (1, \theta, \theta^2, \theta^3, \theta^4) \mid \theta \in \mathrm{GF}(7) \cup \{\infty\}\}$ and let $\mathcal{Q}$ be the quadric with equation $f(x_0, x_1, x_2, x_3, x_4) = -x_0x_1 - x_3^2 + x_2x_4 + x_3x_4$. First note that $f(P_\theta) = \theta^7 - \theta = 0$ for all $\theta \in \mathrm{GF}(7)$. Further, $P_\infty = (0, 0, 0, 0, 1)$, so $f(P_\infty) = 0$. Hence $\mathcal{N} \subset \mathcal{Q}$ in PG(4, 7). Now extend GF(7) to GF(7^2) using a primitive element τ. The point $P_\tau = (1, \tau, \tau^2, \tau^3, \tau^4)$ lies in the extended curve $\mathcal{N}^{\star}$. However $f(P_\tau) = \tau^7 - \tau \neq 0$ as $\tau \notin \mathrm{GF}(7)$, and so P_τ does not lie on the extended quadric $\mathcal{Q}^{\star}$, that is $\mathcal{N}^{\star} \not\subset \mathcal{Q}^{\star}$.

Baer pencils and partitions of Baer subplanes. In this section we investigate the representation in PG$(2, q^2)$ of a 3-space of PG$(4, q)$. We use this to partition tangent Baer subplanes into conics.

Definition 2.10. A *Baer pencil* $\mathcal{K}$ in PG$(2, q^2)$ is the cone of $q + 1$ lines joining a vertex point P to a Baer subline base b. An *ℓ_∞-Baer pencil* $\mathcal{K}$ is a Baer pencil with vertex in ℓ_∞ and base b meeting ℓ_∞ in a point.

Let $\mathcal{K}$ be a Baer pencil; then every line of PG$(2, q^2)$ not through the vertex of $\mathcal{K}$ meets $\mathcal{K}$ in a Baer subline. Also note that an ℓ_∞-Baer pencil $\mathcal{K}$ contains ℓ_∞ and a further q^3 affine points. It is straightforward to characterize the ℓ_∞-Baer pencils of PG$(2, q^2)$ in PG$(4, q)$.

Lemma 2.11. *Let Π be a 3-space in* PG$(4, q)$ *distinct from Σ_∞. Then Π corresponds in* PG$(2, q^2)$ *to an ℓ_∞-Baer pencil with vertex corresponding to the unique spread line in Π. Conversely, any ℓ_∞-Baer pencil in* PG$(2, q^2)$ *corresponds to a 3-space of* PG$(4, q)$.

We look at how ℓ_∞-Baer pencils meet a tangent Baer subplane.

Theorem 2.12. *Let $\mathcal{B}$ be a Baer subplane in* PG$(2, q^2)$ *tangent to ℓ_∞ at the point $\bar{T} = \mathcal{B} \cap \ell_\infty$. An ℓ_∞-Baer pencil with vertex $\bar{P} \neq \bar{T}$ meets $\mathcal{B}$ in either a nondegenerate conic through $\bar{T}$ or in two lines, namely the unique line of $\mathcal{B}$ whose extension contains $\bar{P}$, and one line through $\bar{T}$. Of the ℓ_∞-Baer pencils with vertex $\bar{P}$, there are $q^2 - 1$ of the first type, and $q + 1$ of the second type (each containing one of the $q + 1$ lines of $\mathcal{B}$ through $\bar{T}$).*

Proof. In PG(4, q), let X be a point on the spread line $[T]$ and let $\alpha = \langle X, [P]\rangle$. Label the 3-spaces of PG(4, q) (not equal to Σ_∞) that contain the plane α by $\mathcal{L} = \{\Pi_1, \ldots, \Pi_q\}$. By Lemma 2.11, each 3-space in $\mathcal{L}$ corresponds to an ℓ_∞-Baer pencil of PG(2, q^2) with vertex P. Result 2.1 describes how a 3-space meets the ruled cubic surface $[\mathcal{B}]$. As each 3-space in $\mathcal{L}$ meets $[T]$ in one point, and the 3-spaces in $\mathcal{L}$ partition the affine points, we deduce that one of the 3-spaces in $\mathcal{L}$, Π_1 say, meets $[\mathcal{B}]$ in a conic and the generator line of $[\mathcal{B}]$ through X, and the remaining 3-spaces in $\mathcal{L}$ meet $[\mathcal{B}]$ in a twisted cubic $\mathcal{N}_i = \Pi_i \cap [\mathcal{B}]$, $i = 2, \ldots, q$. By Result 2.5, the twisted cubics $\mathcal{N}_i$ each correspond in PG(2, q^2) to nondegenerate conics in $\mathcal{B}$ that contains $\bar{T}$. Note that there is a unique plane of PG(4, q) $\setminus \Sigma_\infty$ that contains the spread line $[P]$ and meets $[\mathcal{B}]$ in a conic; namely the plane that corresponds in PG(2, q^2) to the unique line m_P through $\bar{P}$ that meets $\mathcal{B}$ in a Baer subline. Hence $\Pi_1 \cap [\mathcal{B}]$ contains the generator line $[m]$ of $[\mathcal{B}]$ through the point X and a conic in the plane $[m_P]$. This corresponds in PG(2, q^2) to an ℓ_∞-Baer pencil with vertex $\bar{P}$ that meets $[\mathcal{B}]$ in the two Baer sublines $m_P \cap \mathcal{B}$ and m.

As there are $q + 1$ choices for the point X on $[T]$, there are $(q + 1)(q - 1)$ 3-spaces about $[P]$ that meets $[\mathcal{B}]$ in a twisted cubic, and $q + 1$ that meet $[\mathcal{B}]$ in a line and a conic, giving the required number of Baer pencils. □

The next result shows that a nondegenerate conic in $\mathcal{B}$ lies in a unique ℓ_∞-Baer pencil, and describes the relationship between the conic and the vertex of the pencil.

Theorem 2.13. *Let $\mathcal{B}$ be a Baer subplane in* PG(2, q^2) *tangent to ℓ_∞ at the point $\bar{T} = \mathcal{B} \cap \ell_\infty$ and let $\mathcal{C}$ be a nondegenerate conic in $\mathcal{B}$ with $\bar{T} \in \mathcal{C}$. Then $\mathcal{C}$ lies in a unique ℓ_∞-Baer pencil $\mathcal{K}$. Moreover, the vertex of $\mathcal{K}$ lies in the extension of $\mathcal{C}$ to* PG(2, q^2).

Proof. Let $\mathcal{C}$ be a nondegenerate conic in $\mathcal{B}$, with $\bar{T} = \mathcal{B} \cap \ell_\infty \in \mathcal{C}$. As ℓ_∞ is not a line of $\mathcal{B}$, it is not the tangent line of $\mathcal{C}$ at the point $\bar{T}$. Let $\mathcal{C}^+$ be the extension of $\mathcal{C}$ to PG(2, q^2); then ℓ_∞ is a secant to $\mathcal{C}^+$, so $\mathcal{C}^+ \cap \ell_\infty = \{\bar{T}, \bar{L}\}$. We will show that $\mathcal{C}$ lies in a unique ℓ_∞-Baer pencil $\mathcal{K}$ which has vertex $\bar{L}$.

We first show that any point $X \in \mathcal{C}^+$ projects $\mathcal{C}$ onto a Baer subline. Without loss of generality, let $\mathcal{C} = \{P_\theta = (1, \theta, \theta^2) \mid \theta \in \mathbb{F}_{q^2} \cup \{\infty\}\}$, so $\mathcal{C}^+ = \{(1, \theta, \theta^2) \mid \theta \in \mathbb{F}_q \cup \{\infty\}\}$. Let $\omega \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, so the point $X = (1, \omega, \omega^2)$ lies in $\mathcal{C}^+ \setminus \mathcal{C}$. The projection of the point P_θ, $\theta \in \mathbb{F}_q \cup \{\infty\}$ from X onto the line ℓ with equation $x = 0$ is $P'_\theta = (0, 1, \theta + \omega)$. That is, the projection of $\mathcal{C}$ from X onto ℓ is the set $\{(0, 1, \omega) + \theta(0, 0, 1) \mid \theta \in \mathbb{F}_q \cup \{\infty\}\}$, which is a Baer subline.

We next show that $\mathcal{C}$ lies in a unique ℓ_∞-Baer pencil. By Result 2.5, in PG(4, q), $[\mathcal{C}]$ is a twisted cubic meeting the spread line $[T]$ in one point and $[\mathcal{C}]$ lies in a 3-space Π that meets $[T]$ in exactly one point. Hence Π contains a unique spread line $[P]$, with $\bar{P} \neq \bar{T}$. By Lemma 2.11, Π corresponds to an ℓ_∞-Baer pencil $\mathcal{K}$

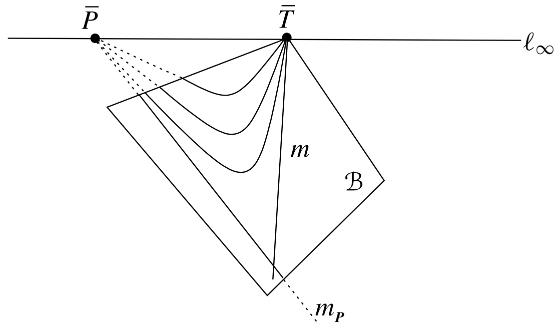

Figure 1. A partition of $\mathcal{B} \setminus \bar{T}$ into q conics through $\bar{T}$.

with vertex $\bar{P}$, so $\mathcal{C}$ lies in the pencil $\mathcal{K}$. If $\mathcal{C}$ were in two ℓ_∞-Baer pencils $\mathcal{K}, \mathcal{K}'$, then $[\mathcal{C}]$ would lie in two 3-spaces $\Pi_{\mathcal{K}}$, $\Pi_{\mathcal{K}'}$, which is not possible.

Hence $\mathcal{C}$ lies in a unique ℓ_∞-Baer pencil $\mathcal{K}$ with some vertex $\bar{P} \in \ell_\infty$. Further, as argued above, the point $\bar{L} \in \mathcal{C}^+ \cap \ell_\infty$ projects $\mathcal{C}$ onto a Baer subline, and so $\mathcal{C}$ lies in an ℓ_∞-Baer pencil with vertex $\bar{L}$. Thus $\bar{P} = \bar{L}$ as required. □

The ℓ_∞-Baer pencils give rise to partitions of the affine points of a tangent Baer subplane into q conics: one degenerate and $q-1$ nondegenerate.

Corollary 2.14. *Let $\mathcal{B}$ be a Baer subplane in* PG$(2, q^2)$ *tangent to ℓ_∞ at the point $\bar{T} = \mathcal{B} \cap \ell_\infty$. For each line m of $\mathcal{B}$ through $\bar{T}$ and point $\bar{P} \in \ell_\infty$, $\bar{P} \neq \bar{T}$, there is a set of q ℓ_∞-Baer pencils with vertex $\bar{P}$ that partition the affine points of* PG$(2, q^2)$ *and partition the affine points of $\mathcal{B}$ into q conics through $\bar{T}$, one being degenerate. Moreover, the extension of each of these conics to* PG$(2, q^2)$ *contains the point $\bar{P}$* (see Figure 1).

Proof. The proof of Theorem 2.12 gives a construction for these partitions. The line m corresponds in PG$(4, q)$ to a line $[m]$ that meets the spread line $[T]$ in a point X. Let $\mathcal{L}$ be the set of q 3-spaces of PG$(4, q) \setminus \Sigma_\infty$ containing the plane $\alpha = \langle X, [P]\rangle$. These 3-spaces partition the affine points of PG$(4, q)$ and hence partition the affine points of $[\mathcal{B}]$. As argued in the proof of Theorem 2.12, one of the 3-spaces in $\mathcal{L}$ gives rise in PG$(2, q^2)$ to two lines in $\mathcal{B}$, and the remaining $q-1$ give rise to nondegenerate conics of $\mathcal{B}$ containing $\bar{T}$. By Theorem 2.13, the extension of these conics to PG$(2, q^2)$ contains the point $\bar{P}$. □

3. Specialness and Baer sublines and subplanes

Parts 4 and 5 of Result 2.3 illustrate that the concept of g-specialness is important in the Bruck–Bose representation of Baer substructures. In this section we discuss how parts 1 and 3 of Result 2.3 relate to the notion of specialness.

Let b be a Baer subline of ℓ_∞; then by Result 2.3(1), in PG$(4, q)$, $[b]$ is a regulus contained in the regular spread $\mathcal{S}$. Hence in PG(4, q^2), the transversals g, g^q of $\mathcal{S}$ are lines of the regulus opposite to $[b]^{\star}$. That is, the regulus $[b]$ is closely related to the transversals of $\mathcal{S}$. There is another way to express this relationship.

Theorem 3.1. 1. *Let b be a Baer subline of ℓ_∞ in* PG(2, q^2). *Then in the Bruck–Bose representation in* PG(4, q), *each nondegenerate conic contained in the regulus $[b]$ is a g-special conic.*

2. *Conversely, every g-special conic in Σ_∞ lies in a unique regulus of $\mathcal{S}$, and so corresponds to a Baer subline of ℓ_∞.*

Proof. Let b be a Baer subline of ℓ_∞ in PG(2, q^2). By Result 2.3(1), in PG(4, q), $[b]$ is a regulus contained in the regular spread $\mathcal{S}$. There are q^3-q planes of Σ_∞ that meet the regulus $[b]$ is a nondegenerate conic, namely the planes that do not contain a line of $[b]$. Let α be such a plane, so α contains a unique spread line $[L]$, and $\mathcal{C}=[b]\cap\alpha$ is a nondegenerate conic. In PG(4, q^2), the transversal g meets each extended spread line, and so g meets at least three lines of the extended regulus $[b]^{\star}$, hence g is a line of the opposite regulus. In particular, each point of g lies on one line of $[b]^{\star}$. Now $\mathcal{C}^{\star}$ is the exact intersection $[b]^{\star}\cap\alpha^{\star}$, and $\alpha^{\star}$ meets g in one point, hence $\mathcal{C}^{\star}$ contains the points $g\cap\alpha^{\star}$, $g^q\cap\alpha^{\star}$, and so $\mathcal{C}$ is a g-special conic.

Conversely, let $\mathcal{C}$ be a g-special conic in Σ_∞. So $\mathcal{C}$ lies in a plane α; moreover, α contains a spread line $[L]$, and in PG(4, q^2), $\mathcal{C}^{\star}$ contains the points $X=g\cap[L]^{\star}$ and $X^q=g^q\cap[L]^{\star}$. Let $\mathcal{K}$ be the set of lines of $\mathcal{S}$ that meet $\mathcal{C}$, we need to show that $\mathcal{K}$ is a regulus. Let $[P_1]$, $[P_2]$, $[P_3]$ be three lines of $\mathcal{K}$ and let $\mathcal{R}$ be the unique regulus containing the three lines. By the argument above, $\mathcal{D}=\mathcal{R}\cap\alpha$ is a g-special conic, and $\mathcal{D}^{\star}$ contains the points X and X^q. So $\mathcal{C}^{\star}$ and $\mathcal{D}^{\star}$ have five points in common, namely X, X^q, $[P_i]\cap\alpha$, $i=1,2,3$. Hence $\mathcal{C}^{\star}=\mathcal{D}^{\star}$ and so $\mathcal{K}=\mathcal{R}$. That is, the points of $\mathcal{C}$ lie on lines of a regulus of $\mathcal{S}$, which by Result 2.3 corresponds to a Baer subline of ℓ_∞ in PG(2, q^2). □

Furthermore, the regulus $[b]$ has a relationship to the lines in the hyperbolic congruence of g, g^q.

Theorem 3.2. *Let b be a Baer subline of ℓ_∞, and let $\bar{P}$, $\bar{Q}\in\ell_\infty$ be conjugate with respect to b. Then in* PG(4, q^2), *the lines PQ^q, P^qQ are lines of the regulus $[b]^{\star}$.*

Proof. Let $\bar{P}$, $\bar{Q}$ be two points of ℓ_∞ that are conjugate with respect to a Baer subline $b\subset\ell_\infty$. By Result 2.3, in PG(4, q), $[b]$ is a regulus of $\mathcal{S}$. By Result 2.8, the unique partition of $\mathcal{S}\setminus\{[P],[Q]\}$ into reguli contains the regulus $[b]$; and in PG(4, q^2), the lines PQ^q, P^qQ meet each line of the regulus opposite to $[b]^{\star}$. Hence the lines PQ^q, P^qQ are lines of the regulus $[b]^{\star}$. □

Remark 3.3. Given a Baer subline b of ℓ_∞, the points of $\ell_\infty\setminus\{b\}$ can be partitioned into pairs of points $\{\bar{P}_i,\bar{Q}_i\}$ which are conjugate with respect to b. Hence the q^2-q

lines $P_i Q_i^q$ and $(P_i Q_i^q)^q$ are exactly the lines of $\mathrm{PG}(4, q^2)$ in the regulus $[b]^{\star}$ that are not lines of $\mathrm{PG}(4, q)$.

We now consider a Baer subplane $\mathcal{B}$ of $\mathrm{PG}(2, q^2)$ secant to ℓ_∞. By Result 2.3, $[\mathcal{B}]$ is a plane of $\mathrm{PG}(4, q)$, and the line $[\mathcal{B}] \cap \Sigma_\infty$ meets $q+1$ lines of $\mathcal{S}$ which form a regulus denoted by $\mathcal{R}$. As noted above, in $\mathrm{PG}(4, q^2)$, the transversals g, g^q are lines of the regulus opposite to $\mathcal{R}$. Moreover, by Theorem 3.2 the extended regulus $\mathcal{R}^{\star}$ contains the line PQ^q where the corresponding points $\bar{P}, \bar{Q} \in \ell_\infty$ are conjugate with respect to $\mathcal{B}$.

Corollary 3.4. *Let $\mathcal{B}$ be a Baer subplane of* $\mathrm{PG}(2, q^2)$ *that is secant to ℓ_∞, and let $\bar{P}, \bar{Q} \in \ell_\infty$ be conjugate with respect to $\mathcal{B}$. Then in* $\mathrm{PG}(4, q^2)$, *the lines PQ^q, $P^q Q$ meet the plane* $[\mathcal{B}]^{\star}$.

4. Conics of $\mathrm{PG}(2, q^2)$

In [Barwick et al. 2011], it is shown that a nondegenerate conic $\mathcal{O}$ in $\mathrm{PG}(2, q^2)$ corresponds in $\mathrm{PG}(4, q)$ to the intersection of two quadrics. Moreover, this correspondence is exact-at-infinity: that is, an affine point $A \in \mathrm{PG}(2, q^2) \setminus \ell_\infty$ lies in $\mathcal{O}$ if and only if the affine point $[A] \in \mathrm{PG}(4, q) \setminus \Sigma_\infty$ lies in $[\mathcal{O}] = \mathcal{Q}_1 \cap \mathcal{Q}_2$ and a point $\bar{T} \in \ell_\infty$ lies in $\mathcal{O}$ if and only if the spread line $[T]$ is contained in $[\mathcal{O}] = \mathcal{Q}_1 \cap \mathcal{Q}_2$. So the set $[\mathcal{O}] = \mathcal{Q}_1 \cap \mathcal{Q}_2$ meets Σ_∞ either in the empty set, or in 1 or 2 spread lines. We determine the relationship of $[\mathcal{O}]$ with the transversals g, g^q of the regular spread $\mathcal{S}$.

The arguments used are coordinate-based. A conic $\mathcal{O}$ has equation $f(x, y, z) = 0$ where f is a homogeneous equation of degree two over $\mathbb{F}_{q^2}$. Using the Bruck–Bose map, this can be written as $f_\infty(x_0, x_1, y_0, y_1, z) + \tau f_0(x_0, x_1, y_0, y_1, z) = 0$, where $f_\infty = 0$ and $f_0 = 0$ are homogeneous quadratic equations over $\mathbb{F}_q$, which is to say equations of quadrics $\mathcal{Q}_\infty$, $\mathcal{Q}_0$ in $\mathrm{PG}(4, q)$; hence $[\mathcal{O}] = \mathcal{Q}_\infty \cap \mathcal{Q}_0$. Moreover, $[\mathcal{O}]$ is contained in the pencil of quadrics $\{\mathcal{Q}_t = t\mathcal{Q}_\infty + \mathcal{Q}_0, t \in \mathbb{F}_q \cup \{\infty\}\}$ where $\mathcal{Q}_t$ has equation $f_t = tf_\infty + f_0 = 0$. There is a natural extension to $\mathrm{PG}(4, q^2)$ and to $\mathrm{PG}(4, q^4)$, namely $[\mathcal{O}]^{\star} = \mathcal{Q}_\infty^{\star} \cap \mathcal{Q}_0^{\star}$ and $[\mathcal{O}]^{\bigstar} = \mathcal{Q}_\infty^{\bigstar} \cap \mathcal{Q}_0^{\bigstar}$. In order to study subconics in Baer subplanes, we will need a full analysis of how these sets meet the hyperplane at infinity, which we give in this section. We first show that none of the quadrics $\mathcal{Q}_t^{\star}$, $t \in \mathbb{F}_q \cup \{\infty\}$, contain g, and so $[\mathcal{O}]^{\star}$ does not contain g.

Theorem 4.1. *Let $\mathcal{O}$ be a nondegenerate conic in* $\mathrm{PG}(2, q^2)$, *so* $[\mathcal{O}] = \mathcal{Q}_\infty \cap \mathcal{Q}_0$. *In* $\mathrm{PG}(4, q^2)$, *the quadric* $\mathcal{Q}_t^{\star} = t\mathcal{Q}_\infty^{\star} + \mathcal{Q}_0^{\star}$, $t \in \mathbb{F}_q \cup \{\infty\}$, *meets g in* 0, 1 *or* 2 *points, according to whether $\mathcal{O}$ meets ℓ_∞ in* 0, 1 *or* 2 *points respectively.*

Proof. Consider first the case when $\mathcal{O}$ is tangent to ℓ_∞. The group $\mathrm{PGL}(3, q^2)$ is transitive on nondegenerate conics, and the subgroup fixing a nondegenerate conic $\mathcal{O}$ is transitive on the tangent lines of $\mathcal{O}$. Hence we can without loss of generality, prove the result for the conic $\mathcal{O}$ of equation $y^2 = xz$ in $\mathrm{PG}(2, q^2)$, which meets ℓ_∞ in one (repeated) point $\overline{T}=(1,0,0)$. The affine point $(x, y, 1)=(x_0+x_1\tau, y_0+y_1\tau, 1)$ is on $\mathcal{O}$ if $(y_0+y_1\tau)^2=x_0+x_1\tau$, that is $(y_0^2+y_1^2t_0-x_0)+(y_1^2t_1+2y_0y_1-x_1)\tau=0$. The solutions $(x_0, x_1, y_0, y_1, 1)\in \mathrm{PG}(4,q)$ to this are the affine points in $[\mathcal{O}]$. That is, $[\mathcal{O}]$ is the intersection of the two quadrics $\mathcal{Q}_\infty$, $\mathcal{Q}_0$ with homogeneous equations $f_\infty=0$, $f_0=0$ respectively, where

$$f_\infty=y_0^2+y_1^2t_0-x_0z \quad \text{and} \quad f_0=y_1^2t_1+2y_0y_1-x_1z. \tag{1}$$

Note that the intersection $[\mathcal{O}]=\mathcal{Q}_\infty\cap\mathcal{Q}_0$ is exact on Σ_∞; both $\mathcal{Q}_\infty$ and $\mathcal{Q}_0$ contain the spread line $[T]=\{(a,b,0,0,0)\mid a,b\in\mathbb{F}_q\}$, and these are the only points of Σ_∞ contained in both $\mathcal{Q}_\infty$ and $\mathcal{Q}_0$. Also note that in $\mathrm{PG}(4,q^2)$, $\mathcal{Q}_\infty^{\star}$ and $\mathcal{Q}_0^{\star}$ both contain the extended spread line $[T]^{\star}$, and so both contain at least one point of g, namely $[T]^{\star}\cap g=A_0$. Also, $[\mathcal{O}]$ lies in the pencil of quadrics $\{\mathcal{Q}_t=t\mathcal{Q}_\infty+\mathcal{Q}_0\mid t\in\mathbb{F}_q\cup\{\infty\}\}$ where $\mathcal{Q}_t$ has equation $f_t=tf_\infty+f_0=0$. Recall that the transversal g of $\mathcal{S}$ consists of the points $G_\beta=\beta A_0+A_1=(\beta\tau^q, -\beta, \tau^q, -1, 0)$ for $\beta\in\mathbb{F}_{q^2}\cup\{\infty\}$. For $\beta\in\mathbb{F}_{q^2}$, we have $f_\infty(G_\beta)=\tau^q(\tau^q-\tau)$ and $f_0(G_\beta)=\tau-\tau^q$. Let $f_t=tf_\infty+f_0$; then $f_t(G_\beta)=(\tau^q-\tau)(t\tau^q-1)$ which is never zero when $t\in\mathbb{F}_q$. Hence $G_\infty=A_0$ is the only point of g contained in the quadric $\mathcal{Q}_t$. Similarly, A_0^q is the only point of the (other) transversal g^q contained in the quadric $\mathcal{Q}_t$. That is, when $\mathcal{O}$ is tangent to ℓ_∞, the quadrics $\mathcal{Q}_t^{\star}$ each meet g in one point, namely $[T]^{\star}\cap g=A_0$. A similar argument using the conic with equation $f(x,y,z)=x^2-\delta y^2+z^2$, $\delta\in\mathbb{F}_{q^2}\setminus\{0\}$ for q odd, and $\delta x^2+y^2+z^2+yx=0$, $\delta\in\mathbb{F}_{q^2}$ for q even completes the other cases. □

The proof of Theorem 4.1, and the one-to-one correspondence between points $\overline{P}$ of ℓ_∞ and points $P=[P]^{\star}\cap g$ of the transversal g, allow us to identify the points of the quadric $\mathcal{Q}_t^{\star}$ on g.

Corollary 4.2. *Let $\mathcal{O}$ be a nondegenerate conic in* $\mathrm{PG}(2,q^2)$; *then*

1. $\overline{P}\in\mathcal{O}\cap\ell_\infty$ *if and only if in* $\mathrm{PG}(4,q^2)$, $P\in g$;
2. $\overline{P}$ *is a point in the intersection of the extension of* $\mathcal{O}$ *and the extension of* ℓ_∞ *to* $\mathrm{PG}(2,q^4)$ *if and only if in* $\mathrm{PG}(4,q^4)$, $P\in g^{\bigstar}\setminus g$.

Next we consider the set $[\mathcal{O}]$ extended to $\mathrm{PG}(4,q^2)$ and $\mathrm{PG}(4,q^4)$, and determine the exact intersection with the hyperplane at infinity.

Theorem 4.3. *Let $\mathcal{O}$ be a nondegenerate conic in* $\mathrm{PG}(2,q^2)$.

1. *Suppose $\mathcal{O}$ is secant to ℓ_∞, so $\mathcal{O}\cap\ell_\infty=\{\overline{P},\overline{Q}\}$. Then*
   (a) *in* $\mathrm{PG}(4,q)$, $[\mathcal{O}]\cap\Sigma_\infty=\{[P],[Q]\}$;
   (b) *in* $\mathrm{PG}(4,q^2)$, $[\mathcal{O}]^{\star}\cap\Sigma_\infty^{\star}=\{[P]^{\star},[Q]^{\star}, PQ^q, P^qQ\}$;
   (c) *in* $\mathrm{PG}(4,q^4)$, $[\mathcal{O}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=\{[P]^{\bigstar},[Q]^{\bigstar},(PQ^q)^{\bigstar},(P^qQ)^{\bigstar}\}$.
2. *Suppose $\mathcal{O}$ is tangent to ℓ_∞, so $\mathcal{O}\cap\ell_\infty=\{\overline{P}\}$. Then*
   (a) *in* $\mathrm{PG}(4,q)$, $[\mathcal{O}]\cap\Sigma_\infty=\{[P]\}$;

(b) *in* $\mathrm{PG}(4, q^2)$, $[\mathcal{O}]^{\star} \cap \Sigma_\infty^{\star} = \{[P]^{\star}\}$,
(c) *in* $\mathrm{PG}(4, q^4)$, $[\mathcal{O}]^{\bigstar} \cap \Sigma_\infty^{\bigstar} = \{[P]^{\bigstar}\}$.

3. *Suppose* $\mathcal{O}$ *is exterior to* ℓ_∞, *so in the extension to* $\mathrm{PG}(2, q^4)$, *the extension of* $\mathcal{O}$ *meets the extension of* ℓ_∞ *in two points* $\{\bar{P}, \bar{P}^{q^2}\}$. *Then*
   (a) *in* $\mathrm{PG}(4, q)$, $[\mathcal{O}] \cap \Sigma_\infty = \varnothing$;
   (b) *in* $\mathrm{PG}(4, q^2)$, $[\mathcal{O}]^{\star} \cap \Sigma_\infty^{\star} = \varnothing$;
   (c) *in* $\mathrm{PG}(4, q^4)$, $[\mathcal{O}]^{\bigstar} \cap \Sigma_\infty^{\bigstar} = \{\ell_P, \ell_P^q, \ell_P^{q^2}, \ell_P^{q^3}\}$, *where* $\ell_P = PP^q$.

Proof. As noted above, $[\mathcal{O}] = \mathcal{Q}_\infty \cap \mathcal{Q}_0$ for quadrics $\mathcal{Q}_\infty$, $\mathcal{Q}_0$, and this correspondence is exact, so $[\mathcal{O}]$ meets Σ_∞ in either the empty set, or in 1 or 2 spread lines (corresponding respectively to $\mathcal{O}$ meeting ℓ_∞ in 0, 1 or 2 points). The cases $\mathcal{O}$ tangent, secant and exterior to ℓ_∞, q odd and even, are proved separately using the same conic equations as in the proof of Theorem 4.1. We omit the calculations, noting that we rely on [Bruen and Hirschfeld 1988, Table 2] to show that the intersection of the two quadrics in the 3-space $\Sigma_\infty^{\star}$ is a set of four lines, possibly repeated. □

We have shown that in $\mathrm{PG}(4, q^2)$, the set $[\mathcal{O}]^{\star}$ contains an extended spread line $[P]^{\star}$ if and only if in $\mathrm{PG}(2, q^2)$, the point $\bar{P} \in \mathcal{O} \cap \ell_\infty$. We will need the next corollary which considers whether the set $[\mathcal{O}]^{\star}$ can contain a point of any other extended spread line.

Corollary 4.4. *Let* $\mathcal{O}$ *be a nondegenerate conic in* $\mathrm{PG}(2, q^2)$. *Let* $\bar{L}$ *be a point of* ℓ_∞ *not in* $\mathcal{O}$. *In* $\mathrm{PG}(4, q^2)$, *the corresponding extended spread line* $[L]^{\star}$ *is disjoint from* $[\mathcal{O}]^{\star}$.

Proof. If $\mathcal{O}$ is secant to ℓ_∞, so $\mathcal{O} \cap \ell_\infty = \{\bar{P}, \bar{Q}\}$, then by Theorem 4.3, $[\mathcal{O}]^{\star} \cap \Sigma_\infty^{\star}$ consists of the four lines $[P]^{\star}$, $[Q]^{\star}$, PQ^q, P^qQ. Let $[L]^{\star}$ be an extended spread line, $\bar{L} \neq \bar{P}, \bar{Q}$. Then $[L]^{\star}$, $[P]^{\star}$, $[Q]^{\star}$, PQ^q, P^qQ are all lines of the hyperbolic congruence of g and g^q, and so do not meet off g, g^q, and hence are mutually skew. So $[L]^{\star} \cap [\mathcal{O}]^{\star} = \varnothing$. If $\mathcal{O}$ is tangent to ℓ_∞, then by Theorem 4.3, $[\mathcal{O}]^{\star} \cap \Sigma_\infty^{\star} = [P]^{\star}$. Hence $[\mathcal{O}]^{\star}$ meets no other spread line. If $\mathcal{O}$ is exterior to ℓ_∞, then by Theorem 4.3, $[\mathcal{O}]^{\star} \cap \Sigma_\infty^{\star} = \varnothing$, so $[\mathcal{O}]^{\star}$ contains no point on any extended spread line, as required. □

5. Conics of Baer subplanes

In this section we improve Result 2.5 by characterizing the normal rational curves of $\mathrm{PG}(4, q)$ that correspond to conics of a Baer subplane of $\mathrm{PG}(2, q^2)$. In particular, we show that if $\mathcal{C}$ is a conic contained in a tangent Baer subplane $\mathcal{B}$ of $\mathrm{PG}(2, q^2)$, then in $\mathrm{PG}(4, q)$, the corresponding 3- or 4-dimensional normal rational curve $[\mathcal{C}]$ is g-special. Further, we show that any g-special 3- or 4-dimensional normal rational curve in $\mathrm{PG}(4, q)$ corresponds to a conic in a Baer subplane of $\mathrm{PG}(2, q^2)$.

$\mathbb{F}_{q^2}$-conics and $\mathbb{F}_q$-conics. In this section we show that the notion of specialness is also intrinsic to the Bruck–Bose representation of conics in Baer subplanes. First we introduce some notation to easily distinguish between conics in $\mathrm{PG}(2, q^2)$ and conics contained in a Baer subplane. An *$\mathbb{F}_{q^2}$-conic* in $\mathrm{PG}(2, q^2)$ is a nondegenerate conic of $\mathrm{PG}(2, q^2)$. Note that an $\mathbb{F}_{q^2}$-conic meets a Baer subplane $\mathcal{B}$ in either 0, 1, 2, 3 or 4 points, or in a nondegenerate conic of $\mathcal{B}$. We define an *$\mathbb{F}_q$-conic* of $\mathrm{PG}(2, q^2)$ to be a nondegenerate conic in a Baer subplane of $\mathrm{PG}(2, q^2)$. For the remainder of this article, $\mathcal{C}$ will denote an $\mathbb{F}_q$-conic. Further, we denote the *unique* $\mathbb{F}_{q^2}$-conic containing $\mathcal{C}$ by $\mathcal{C}^+$. An $\mathbb{F}_{q^2}$-conic contains many $\mathbb{F}_q$-conics.

Lemma 5.1. *Let $\mathcal{O}$ be an $\mathbb{F}_{q^2}$-conic in* $\mathrm{PG}(2, q^2)$. *Any three points of $\mathcal{O}$ lie in a unique $\mathbb{F}_q$-conic contained in $\mathcal{O}$, so there are $q(q^2+1)$ $\mathbb{F}_q$-conics contained in $\mathcal{O}$.*

Proof. The $\mathbb{F}_{q^2}$-conic $\mathcal{O}$ is equivalent to the line $\ell \cong \mathrm{PG}(1, q^2)$, and subconics of $\mathcal{O}$ are equivalent to Baer sublines of ℓ. Since three points of ℓ lie in a unique Baer subline of ℓ, three points of $\mathcal{O}$ lie in a unique subconic $\mathcal{C}$. As $\mathcal{C}$ is a normal rational curve over $\mathbb{F}_q$, by [Hirschfeld and Thas 1991, Theorem 21.1.1] there is a homography ϕ that maps $\mathcal{C}$ to $\mathcal{C}' = \phi(\mathcal{C}) = \{(1, \theta, \theta^2) \mid \theta \in \mathbb{F}_q \cup \{\infty\}\}$. As $\mathcal{C}'$ lies in the Baer subplane $\mathcal{B}' = \mathrm{PG}(2, q)$, $\mathcal{C}$ lies in the Baer subplane $\phi^{-1}(\mathcal{B}')$, that is, $\mathcal{C}$ is an $\mathbb{F}_q$-conic. Straightforward counting shows that the number of $\mathbb{F}_q$-conics in $\mathcal{O}$ is $(q^2+1)q^2(q^2-1)/(q+1)q(q-1) = q(q^2+1)$. □

Remark 5.2. Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in $\mathrm{PG}(2, q^2)$, $q > 4$, so there is a unique $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ with $\mathcal{C} \subset \mathcal{C}^+$. Then in $\mathrm{PG}(4, q)$, $[\mathcal{C}] \subset [\mathcal{C}^+]$. This is clearly true for the affine points. For the points at infinity, we recall Remark 2.6, if $\bar{T} \in \mathcal{C} \cap \ell_\infty \subseteq \mathcal{C}^+ \cap \ell_\infty$, then $[\mathcal{C}]$ meets the spread line $[T]$ in a point, while $[\mathcal{C}^+]$ contains the spread line $[T]$.

Conics in secant Baer subplanes. In this section we consider the Bruck–Bose representation of $\mathbb{F}_q$-conics in secant Baer subplanes of $\mathrm{PG}(2, q^2)$, in particular looking at the relationship with the lines of the hyperbolic congruence of g, g^q.

Theorem 5.3. *Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in a Baer subplane $\mathcal{B}$ secant to ℓ_∞. The $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ meets ℓ_∞ in two (possibly equal) points $\bar{P}$, $\bar{Q}$. In* $\mathrm{PG}(4, q)$, *$[\mathcal{C}]$ is a nondegenerate conic in the plane $[\mathcal{B}]$, and $[\mathcal{C}^+] \cap \Sigma_\infty$ is the two spread lines $[P]$, $[Q]$.*

1. *If $\bar{P} = \bar{Q}$, then $\bar{P} \in \mathcal{B}$, and $[\mathcal{C}]$ meets Σ_∞ in one point $[P] \cap [\mathcal{B}]$.*
2. *If $\bar{P} \neq \bar{Q}$ and $\bar{P}, \bar{Q} \in \mathcal{B}$, then $[\mathcal{C}]$ meets Σ_∞ in two points $[P] \cap [\mathcal{B}]$ and $[Q] \cap [\mathcal{B}]$.*
3. *If $\bar{P} \neq \bar{Q}$ and $\bar{P}, \bar{Q} \notin \mathcal{B}$, then $[\mathcal{C}]$ is a (PQ^q)-special conic.*

Proof. By Results 2.3 and 2.5, in $\mathrm{PG}(4, q)$, $[\mathcal{B}]$ is a plane, and $[\mathcal{C}]$ is a conic in $[\mathcal{B}]$. Parts 1 and 2 follow immediately from the Bruck–Bose definition. For part 3, the set $[\mathcal{C}^+]$ contains the spread lines $[P]$ and $[Q]$. The set $[\mathcal{B}]$ is a plane, and the line $m = [\mathcal{B}] \cap \Sigma_\infty$ meets $q+1$ spread lines, but does not meet the spread lines $[P]$, $[Q]$.

The set $[\mathcal{C}]$ is a nondegenerate conic in $[\mathcal{B}]$ which does not meet m, and in the extension to PG(4, q^2), $[\mathcal{C}]^{\star}$ meets $\Sigma_\infty^{\star}$ in two points of the line $m^{\star} = [\mathcal{B}]^{\star} \cap \Sigma_\infty^{\star}$. In PG(2, q^2), we have $\mathcal{C} = \mathcal{B} \cap \mathcal{C}^{+}$, and in PG(4, q), $[\mathcal{C}] = [\mathcal{B}] \cap [\mathcal{C}^{+}]$. Moreover, in PG(4, q^2), $[\mathcal{C}]^{\star} = [\mathcal{B}]^{\star} \cap [\mathcal{C}^{+}]^{\star}$, hence $[\mathcal{C}]^{\star} \cap \Sigma_\infty^{\star} = \{[\mathcal{B}]^{\star} \cap \Sigma_\infty^{\star}\} \cap \{[\mathcal{C}^{+}]^{\star} \cap \Sigma_\infty^{\star}\}$. By Theorem 4.3, this is equal to $\{m^{\star}\} \cap \{g, g^q, PQ^q, P^qQ\}$. Now $m^{\star}$ does not meet g (or g^q) as the only lines of Σ_∞ whose extension meets g are the lines of $\mathcal{S}$. Hence the two points of $[\mathcal{C}]^{\star} \cap \Sigma_\infty^{\star}$ lie in PQ^q and P^qQ, that is, $[\mathcal{C}]$ is a (PQ^q)-special conic of PG(4, q). □

We now characterize $\mathbb{F}_q$-conics in secant Baer subplanes by showing that the converse is true.

Theorem 5.4. *In* PG(4, q), *let α be a plane not containing a spread line, and let $\mathcal{N}$ be a nondegenerate conic in α.*

1. *In* PG(2, q^2), *there is a secant Baer subplane $\mathcal{B}$ with* $[\mathcal{B}] = \alpha$, *and an $\mathbb{F}_q$-conic $\mathcal{C}$ in $\mathcal{B}$ with* $[\mathcal{C}] = \mathcal{N}$.
2. *If $\mathcal{N}$ meets Σ_∞ in a point of the spread line $[T]$, then $\bar{T} \in \mathcal{C}$.*
3. *If $\mathcal{N}$ is a (PQ^q)-special conic, then the $\mathbb{F}_{q^2}$-conic $\mathcal{C}^{+}$ containing $\mathcal{C}$ meets ℓ_∞ in the points $\bar{P}$, $\bar{Q}$.*

Proof. Parts 1 and 2 follow from Result 2.3. For part 3, in PG(4, q), let $\mathcal{N}$ be a (PQ^q)-special conic of PG(4, q) lying in a plane α that does not contain a spread line. By part 1, $[\mathcal{B}] = \alpha$ and $[\mathcal{C}] = \mathcal{N}$ where in PG(2, q^2), $\mathcal{B}$ is a secant Baer subplane containing the $\mathbb{F}_q$-conic $\mathcal{C}$. As $\mathcal{N}$ is a (PQ^q)-special conic, $\mathcal{N} \cap \Sigma_\infty = \varnothing$, and in PG(4, q^2), $\mathcal{N}^{\star}$ is a conic which meets the line $\alpha \cap \Sigma_\infty^{\star}$ is two points, one lying on each of PQ^q and P^qQ. As $\mathcal{N} \cap \Sigma_\infty = \varnothing$, in PG(2, q^2), the $\mathbb{F}_q$-conic $\mathcal{C}$ does not meet ℓ_∞, so the $\mathbb{F}_{q^2}$-conic $\mathcal{C}^{+}$ meets ℓ_∞ in two points $\bar{A}, \bar{B} \notin \mathcal{B}$. By Theorem 5.3, $[\mathcal{C}] = \mathcal{N}$ is a (AB^q)-special conic. Hence $\{\bar{A}, \bar{B}\} = \{\bar{P}, \bar{Q}\}$, so $\mathcal{C}^{+} \cap \ell_\infty = \{\bar{P}, \bar{Q}\}$ as required. □

Conics in tangent Baer subplanes. We now consider a Baer subplane $\mathcal{B}$ that is tangent to ℓ_∞ and look at $\mathbb{F}_q$-conics in $\mathcal{B}$. There are two cases to consider, namely whether the $\mathbb{F}_q$-conic contains the point $\mathcal{B} \cap \ell_\infty$ or not. In each case we generalize Result 2.5 by showing that the corresponding normal rational curve of PG(4, q) is g-special. Further, we characterize all g-special normal rational curves in PG(4, q) as corresponding to $\mathbb{F}_q$-conics in a tangent Baer subplane.

Conics in $\mathcal{B}$ containing the point $\bar{T} = \mathcal{B} \cap \ell_\infty$. We first look at an $\mathbb{F}_q$-conic $\mathcal{C}$ in a tangent Baer subplane $\mathcal{B}$, with $\mathcal{B} \cap \ell_\infty$ in $\mathcal{C}$.

Theorem 5.5. *In* PG(2, q^2), $q > 5$, *let $\mathcal{B}$ be a tangent Baer subplane and $\mathcal{C}$ an $\mathbb{F}_q$-conic in $\mathcal{B}$ containing the point $\mathcal{B} \cap \ell_\infty$. Then in* PG(4, q), $[\mathcal{C}]$ *is a g-special twisted cubic.*

Proof. Let $\mathcal{B}$ be a Baer subplane of $\mathrm{PG}(2, q^2)$ that is tangent to ℓ_∞ in the point $\bar{T} = \mathcal{B} \cap \ell_\infty$. Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic of $\mathcal{B}$ that contains $\bar{T}$. By Result 2.5, in $\mathrm{PG}(4, q)$, $\mathcal{C}$ corresponds to a twisted cubic $[\mathcal{C}]$ that lies in a 3-space denoted $\Pi_{\mathcal{C}}$. By Result 2.1, $\Pi_{\mathcal{C}}$ meets the ruled cubic surface $[\mathcal{B}]$ in exactly the twisted cubic $[\mathcal{C}]$. We show that $[\mathcal{C}]$ is g-special. By Lemma 2.11, the 3-space $\Pi_{\mathcal{C}}$ corresponds to an ℓ_∞-Baer pencil $\mathcal{K}$ of $\mathrm{PG}(2, q^2)$ that meets $\mathcal{B}$ in $\mathcal{C}$. By Theorem 2.13, $\mathcal{K}$ has vertex $\bar{P} \in \mathcal{C}^+$. Hence $\Pi_{\mathcal{C}}$ contains the spread line $[P]$ (and this is the only spread line in $\Pi_{\mathcal{C}}$). Consider the extension of $\mathrm{PG}(4, q)$ to $\mathrm{PG}(4, q^2)$. Note that Lemma 2.9 can be generalized to a 3-dimensional normal rational curve when $q > 5$. Hence as $[\mathcal{B}]$ is the intersection of three quadrics [Barwick and Jackson 2012], we have $[\mathcal{C}]^{\star} \subset [\mathcal{B}]^{\star}$ in $\mathrm{PG}(4, q^2)$. Thus by Corollary 2.2, the twisted cubic $[\mathcal{C}]^{\star}$ contains a unique point of each generator line of the ruled cubic surface $[\mathcal{B}]^{\star}$. By Result 2.3, $[\mathcal{B}]$ is g-special, so the transversal lines g, g^q of the regular spread $\mathcal{S}$ are generator lines of the extended ruled cubic surface $[\mathcal{B}]^{\star}$. Hence $[\mathcal{C}]^{\star}$ contains a point of g and g^q. Thus $[\mathcal{C}]^{\star}$ contains the points corresponding to the vertex of $\mathcal{K}$, that is, the point $g \cap \Pi_{\mathcal{C}}{}^{\star} = g \cap [P]^{\star} = P$ and P^q. That is, the twisted cubic $[\mathcal{C}]$ is g-special. □

The converse of Theorem 5.5 is also true.

Theorem 5.6. *A g-special twisted cubic in* $\mathrm{PG}(4, q)$ *corresponds to an $\mathbb{F}_q$-conic in some tangent Baer subplane of* $\mathrm{PG}(2, q^2)$.

Proof. Let $\mathcal{N}$ be a g-special twisted cubic in $\mathrm{PG}(4, q)$, so in $\mathrm{PG}(4, q^2)$, $\mathcal{N}^{\star}$ meets the transversal g of $\mathcal{S}$ in a point R, and meets g^q in the point R^q. The line RR^q meets Σ_∞ in a spread line denoted $[R]$, corresponding to the point $\bar{R} \in \ell_\infty$. Let $\Pi_{\mathcal{N}}$ be the 3-space containing $\mathcal{N}$, and recall that a twisted cubic meets a plane in three points, possibly repeated, or in an extension. As $\mathcal{N}$ meets the plane $\pi = \Pi_{\mathcal{N}} \cap \Sigma_\infty$ in two points R, R^q over $\mathbb{F}_{q^2}$, $\mathcal{N}$ meets π in one point X over $\mathbb{F}_q$. Let $[T]$ be the spread line containing the point X, so $[T] \notin \Pi_{\mathcal{N}}$. Let $[A]$, $[B]$, $[C]$ be three affine points of $\mathcal{N}$, and let $\alpha = \langle [A], [B], [C] \rangle$.

As α lies in the 3-space $\Pi_{\mathcal{N}}$, if α contained a spread line, it would contain $[R]$. However, if α contains $[R]$, then the plane $\langle [A], [B], [R] \rangle^{\star}$ would contain four points of $\mathcal{N}^{\star}$, namely $[A]$, $[B]$, R, R^q, a contradiction. If α contained the point X, then α would contain four points of $\mathcal{N}$, namely X, $[A]$, $[B]$, $[C]$, a contradiction. Hence α corresponds to a Baer subplane $\mathcal{B}_\alpha$ of $\mathrm{PG}(2, q^2)$ that is secant to ℓ_∞, with $\bar{T} \notin \mathcal{B}_\alpha$. Hence the points $\{\bar{T}, A, B, C\}$ form a quadrangle and so lie in a unique Baer subplane denoted $\mathcal{B}$. As $\mathcal{B}_\alpha$ is the unique Baer subplane containing A, B, C and secant to ℓ_∞, we have $\mathcal{B} \neq \mathcal{B}_\alpha$, and $\mathcal{B}$ is tangent to ℓ_∞ at the point $\bar{T}$.

In $\mathrm{PG}(4, q)$, $[\mathcal{B}]$ is a ruled cubic surface with line directrix $[T]$. As X, $[A]$, $[B]$, $[C]$ are points of $\mathcal{N}$, no three are collinear, so $[A]$, $[B]$, $[C]$ lie on distinct generators of $[\mathcal{B}]$. Recall that $\Pi_{\mathcal{N}}$ does not contain $[T]$, so by Result 2.1, $\Pi_{\mathcal{N}}$ meets $[\mathcal{B}]$ in a twisted cubic, denoted $\mathcal{N}_1$. The argument in the proof of Theorem 5.5 shows that in the quadratic extension, $\mathcal{N}_1^{\star}$ contains the points R and R^q. Hence $\mathcal{N}^{\star}$ and $\mathcal{N}_1^{\star}$ share six points, and so are equal. That is, $\mathcal{N}$ is a g-special twisted cubic contained in a g-special ruled cubic surface, and $\mathcal{N}$ meets Σ_∞ in one point.

Straightforward counting shows that in $\mathrm{PG}(2, q^2)$, the number of $\mathbb{F}_q$-conics in $\mathcal{B}$ that contain $\bar{T}$ is $q^4 - q^2$. By Result 2.1, the number of 3-spaces of $\mathrm{PG}(4, q)$ that meet the ruled cubic surface $[\mathcal{B}]$ in a twisted cubic is $q^4 - q^2$. Hence they are in one to one correspondence. That is, $\mathcal{N}$ corresponds to an $\mathbb{F}_q$-conic in the tangent Baer subplane $\mathcal{B}$ as required. □

The proofs of Theorems 5.5 and 5.6 show that the points of g on a g-special twisted cubic correspond to the points on ℓ_∞ contained in the associated $\mathbb{F}_{q^2}$-conic.

Corollary 5.7. *Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in a tangent Baer subplane $\mathcal{B}$ in* $\mathrm{PG}(2, q^2)$, $q > 5$, *with $\bar{T} = \mathcal{B} \cap \ell_\infty \in \mathcal{C}$. The $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ meets ℓ_∞ in a point $\bar{P} \neq \bar{T}$ if and only if in* $\mathrm{PG}(4, q^2)$ *the twisted cubic $[\mathcal{C}]^{\star}$ meets the transversals of $\mathcal{S}$ in the points P, P^q.*

Conics of $\mathcal{B}$ not containing the point $\bar{T} = \mathcal{B} \cap \ell_\infty$. We now look at an $\mathbb{F}_q$-conic $\mathcal{C}$ in a tangent Baer subplane $\mathcal{B}$, with $\mathcal{B} \cap \ell_\infty$ not in $\mathcal{C}$. The $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ meets ℓ_∞ in two distinct points (which may lie in $\mathrm{PG}(2, q^4)$). We show that if these two points lie in $\mathrm{PG}(2, q^2)$, then $[\mathcal{C}]$ is a g-special normal rational curve. Further, if the two points lie in the quadratic extension of $\mathrm{PG}(2, q^2)$ to $\mathrm{PG}(2, q^4)$, then $[\mathcal{C}]$ is an $g^{\bigstar}$-special normal rational curve.

Theorem 5.8. *In* $\mathrm{PG}(2, q^2)$, $q > 7$, *let $\mathcal{B}$ be a Baer subplane tangent to ℓ_∞ with $\bar{T} = \mathcal{B} \cap \ell_\infty$. Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in $\mathcal{B}$, $\bar{T} \notin \mathcal{C}$. In* $\mathrm{PG}(4, q)$, *$[\mathcal{C}]$ is a g-special or $g^{\bigstar}$-special* 4*-dimensional normal rational curve.*

Proof. Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in $\mathcal{B}$ not through $\bar{T} = \mathcal{B} \cap \ell_\infty$, and consider the $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ containing $\mathcal{C}$. Then either (i) $\mathcal{C}^+$ is secant to ℓ_∞ and $\mathcal{C}^+ \cap \ell_\infty$ consists of two distinct points $\bar{P}$, $\bar{Q}$, or (ii) $\mathcal{C}^+$ is tangent to ℓ_∞ and $\mathcal{C}^+ \cap \ell_\infty$ is a repeated point $\bar{P} = \bar{Q}$, or (iii) $\mathcal{C}^+$ is exterior to ℓ_∞ and in $\mathrm{PG}(2, q^4)$ the extension of $\mathcal{C}^+$ meets the extension of ℓ_∞ in two points $\bar{P}$, $\bar{Q}$ which are conjugate with respect to this extension from $\mathrm{PG}(2, q^2)$ to $\mathrm{PG}(2, q^4)$, that is, $\bar{Q} = \bar{P}^{q^2}$. By Result 2.5, as $\bar{T} \notin \mathcal{C}$, in $\mathrm{PG}(4, q)$, $[\mathcal{C}]$ is a 4-dimensional normal rational curve lying on the g-special ruled cubic surface $[\mathcal{B}]$, and $[\mathcal{C}]$ does not meet Σ_∞. Thus it remains to show that in $\mathrm{PG}(4, q)$ $[\mathcal{C}]$ is a g-special or $g^{\bigstar}$-special. We will show that in an appropriate extension of $\mathrm{PG}(4, q)$, the extension of the normal rational curve $[\mathcal{C}]$ contains the points P, Q of the (possibly extended) transversal g, giving the g-special property. Recall that a 4-dimensional normal rational curve of $\mathrm{PG}(4, q)$ meets the 3-space Σ_∞ in four points, possibly repeated or in an extension. As $[\mathcal{C}]$ is disjoint from Σ_∞, either (a) in $\mathrm{PG}(4, q^2)$, $[\mathcal{C}]^{\star}$ meets $\Sigma_\infty^{\star}$ in four points of the form X, X^q, Y, Y^q, possibly $X = Y$; or (b) in $\mathrm{PG}(4, q^4)$, $[\mathcal{C}]^{\bigstar}$ meets $\Sigma_\infty^{\bigstar}$ in four points of form X, X^q, X^{q^2}, X^{q^3}.

In $\mathrm{PG}(2,q^2)$, we have $\mathcal{C}\subset\mathcal{C}^+$, so as discussed in Remark 5.2, in $\mathrm{PG}(4,q)$, $[\mathcal{C}]\subset[\mathcal{C}^+]$. By [Barwick et al. 2011, Corollary 3.3], $[\mathcal{C}^+]$ is the exact intersection of two quadrics, so by Lemma 2.9 $[\mathcal{C}]^{\star}\subset[\mathcal{C}^+]^{\star}$ in $\mathrm{PG}(4,q^2)$ and $[\mathcal{C}]^{\bigstar}\subset[\mathcal{C}^+]^{\bigstar}$ in $\mathrm{PG}(4,q^4)$. Similarly, as $[\mathcal{C}]\subset[\mathcal{B}]$ and $[\mathcal{B}]$ is the intersection of three quadrics [Barwick and Jackson 2012], we have $[\mathcal{C}]^{\star}\subset[\mathcal{B}]^{\star}$ and $[\mathcal{C}]^{\bigstar}\subset[\mathcal{B}]^{\bigstar}$ by Lemma 2.9. In $\mathrm{PG}(2,q^2)$, we have $\mathcal{C}=\mathcal{B}\cap\mathcal{C}^+$. As $[\mathcal{C}]$ is disjoint from Σ_∞, in $\mathrm{PG}(4,q)$, we have $[\mathcal{C}]=[\mathcal{B}]\cap[\mathcal{C}^+]$. We need to determine $[\mathcal{C}]^{\star}\cap\Sigma_\infty^{\star}=[\mathcal{C}^+]^{\star}\cap[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}$ in $\mathrm{PG}(4,q^2)$ and $[\mathcal{C}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=[\mathcal{C}^+]^{\bigstar}\cap[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}$ in $\mathrm{PG}(4,q^4)$.

First we determine $[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}$ and $[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}$. In $\mathrm{PG}(2,q^2)$, $\bar{T}\in\mathcal{B}$, so in $\mathrm{PG}(4,q)$, $[T]\subset[\mathcal{B}]$, and $[\mathcal{B}]\cap\Sigma_\infty=[T]$. Hence in $\mathrm{PG}(4,q^2)$, $[T]^{\star}\subset[\mathcal{B}]^{\star}$, and in $\mathrm{PG}(4,q^4)$, $[T]^{\bigstar}\subset[\mathcal{B}]^{\bigstar}$. By Result 2.3, $[\mathcal{B}]$ is a g-special ruled cubic surface, so the transversal lines g,g^q lie in $[\mathcal{B}]^{\star}$. That is, $\{[T]^{\star},g,g^q\}$ lie in $[\mathcal{B}]^{\star}$, and using Result 2.1 in $\mathrm{PG}(4,q^2)$, the 3-space $\Sigma_\infty^{\star}$ meets the ruled cubic surface $[\mathcal{B}]^{\star}$ in exactly these three lines, so $[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}=\{[T]^{\star},g,g^q\}$. Similarly, in $\mathrm{PG}(4,q^4)$, the 3-space $\Sigma_\infty^{\bigstar}$ meets the ruled cubic surface $[\mathcal{B}]^{\bigstar}$ in the three lines $\{[T]^{\bigstar},g^{\bigstar},g^{q\bigstar}\}$.

Recall that Theorem 4.3 determines the intersection $[\mathcal{C}^+]^{\star}\cap\Sigma_\infty^{\star}$ and $[\mathcal{C}^+]^{\bigstar}\cap\Sigma_\infty^{\bigstar}$ for the three cases where $\mathcal{C}^+$ is (i) secant, (ii) tangent or (iii) exterior to ℓ_∞ in $\mathrm{PG}(2,q^2)$. For each case we determine $[\mathcal{C}^+]^{\star}\cap[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}$ in $\mathrm{PG}(4,q^2)$ and $[\mathcal{C}^+]^{\bigstar}\cap[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}$ in $\mathrm{PG}(4,q^2)$.

In case (i), $\mathcal{C}^+$ is secant to ℓ_∞, so $[\mathcal{C}^+]^{\star}\cap\Sigma_\infty^{\star}=\{[P]^{\star},[Q]^{\star},PQ^q,P^qQ\}$, by Theorem 4.3. Now $[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}=\{[T]^{\star},g,g^q\}$, and by Corollary 4.4, $[\mathcal{C}^+]^{\star}\cap\Sigma_\infty^{\star}$ does not meet $[T]^{\star}$. Hence $[\mathcal{C}^+]^{\star}\cap[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}$ consists of the four points P, Q, P^q and Q^q. Similarly, $[\mathcal{C}^+]^{\bigstar}\cap[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=\{P,Q,P^q,Q^q\}$. As $[\mathcal{C}]^{\star}\cap\Sigma_\infty^{\star}=[\mathcal{C}^+]^{\star}\cap[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}$, $[\mathcal{C}]^{\star}$ meets g in two distinct points, namely P, Q and so $[\mathcal{C}]$ is a g-special normal rational curve.

In case (ii), $\mathcal{C}^+$ is tangent to ℓ_∞, so by Theorem 4.3, $\{[\mathcal{C}^+]^{\star}\cap\Sigma_\infty^{\star}\}\cap\{[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}\}=\{[P]^{\star}\}\cap\{[T]^{\star},g,g^q\}=\{P,P^q\}$. Similarly, $[\mathcal{C}^+]^{\bigstar}\cap[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=\{P,P^q\}$. Hence $[\mathcal{C}]^{\star}$ meets g in the repeated point P, and so $[\mathcal{C}]$ is a g-special normal rational curve.

In case (iii), $\mathcal{C}^+$ is exterior to ℓ_∞, so in $\mathrm{PG}(2,q^4)$, the extension of $\mathcal{C}^+$ meets the extension of ℓ_∞ in two points $\bar{P}$, $\bar{Q}$, where $\bar{Q}=\bar{P}^{q^2}$. By Theorem 4.3, $[\mathcal{C}^+]^{\star}\cap\Sigma_\infty^{\star}=\varnothing$ and $[\mathcal{C}^+]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=\{\ell_P,\ell_P^q,\ell_P^{q^2},\ell_P^{q^3}\}$, where $\ell_P=PP^q$. Hence $[\mathcal{C}^+]^{\star}\cap[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}=\varnothing$, and $[\mathcal{C}^+]^{\bigstar}\cap[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=\{P,P^q,P^{q^2},P^{q^3}\}$. So in this case the normal rational curve $[\mathcal{C}]$ meets Σ_∞ in four points over $\mathbb{F}_{q^4}$. As $[\mathcal{C}]^{\bigstar}$ meets $g^{\bigstar}$ in two points (namely P and $P^{q^2}=Q$) $[\mathcal{C}]$ is an $g^{\bigstar}$-special normal rational curve. □

Conversely, every g-special or $g^{\bigstar}$-special normal rational curve corresponds to an $\mathbb{F}_q$-conic:

Theorem 5.9. *Let $\mathcal{N}$ be a g-special or $g^{\bigstar}$-special 4-dimensional normal rational curve in $\mathrm{PG}(4,q)$. Then $\mathcal{N}=[\mathcal{C}]$ where $\mathcal{C}$ is an $\mathbb{F}_q$-conic in a tangent Baer subplane of $\mathrm{PG}(2,q^2)$.*

Proof. Let $\mathcal{N}$ be a g-special 4-dimensional normal rational curve in PG(4, q). So there are two (possibly equal) spread lines $[P]$, $[Q]$ such that $\mathcal{N}^{☆} \cap \Sigma_\infty^{☆}$ consists of the four points $P = g \cap [P]^{☆}$, $P^q = g^q \cap [P]^{☆}$, $Q = g \cap [Q]^{☆}$, $Q^q = g^q \cap [Q]^{☆}$. Note that as $\mathcal{N}^{☆}$ meets $\Sigma_\infty^{☆} \setminus \Sigma_\infty$ in four points, $\mathcal{N}$ is disjoint from Σ_∞. There are three cases to consider.

Case (i): Suppose first that $[P] \neq [Q]$. Let $[A], [B], [C]$ be three points of $\mathcal{N}$, so $[A], [B], [C] \notin \Sigma_\infty$. If the plane $\alpha = \langle [A], [B], [C] \rangle$ contained a point of the spread line $[P]$, then the 3-space $\langle \alpha, [P] \rangle^{☆}$ contains five points of $\mathcal{N}^{☆}$, namely $[A], [B], [C], P, P^q$, a contradiction. So α is disjoint from the spread lines $[P]$ and $[Q]$. If α contained a spread line $[X]$, then in PG(4, q^2), $\langle \alpha^{☆}, g \rangle$ is a 3-space that contains five points of $\mathcal{N}^{☆}$, namely $[A], [B], [C], P, Q$, a contradiction. So α corresponds to a Baer subplane $\mathcal{B}_\alpha$ of PG(2, q^2) that is secant to ℓ_∞, and does not contain $\bar{P}$ or $\bar{Q}$.

Consider the corresponding points $\bar{P}, \bar{Q}, A, B, C$ in PG(2, q^2). So $\bar{P}, \bar{Q} \in \ell_\infty$ and $A, B, C \in \text{PG}(2, q^2) \setminus \ell_\infty$. Now A, B, C are not collinear as α does not contain a spread line. So $\mathcal{B}_\alpha$ is the unique Baer subplane that contains A, B, C and is secant to ℓ_∞. As $\bar{P}, \bar{Q} \in \ell_\infty \setminus \mathcal{B}$ and $A, B, C \in \mathcal{B} \setminus \ell_\infty$, no three of $\bar{P}, \bar{Q}, A, B, C$ are collinear, hence they lie on a unique $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$. By Lemma 5.1, A, B, C lie in a unique $\mathbb{F}_q$-conic $\mathcal{C}$ contained in $\mathcal{C}^+$, and $\mathcal{C}$ lies in a Baer subplane $\mathcal{B}$.

Suppose $\mathcal{B} = \mathcal{B}_\alpha$; then by Corollary 3.4, in PG(4, q^2), the plane $\alpha^{☆}$ meets PQ^q. Note that the line PQ^q contains two points of $\mathcal{N}^{☆}$, namely P, Q^q. Hence $\langle \alpha^{☆}, PQ^q \rangle$ is a 3-space of PG(4, q^2) that contains five points of $\mathcal{N}^{☆}$, namely $[A], [B], [C], P, Q^q$, a contradiction. Thus $\mathcal{B} \neq \mathcal{B}_\alpha$.

Hence the Baer subplane $\mathcal{B}$ is tangent to ℓ_∞. As $\mathcal{C}^+$ is secant to ℓ_∞, we are in case (i) of the proof of Theorem 5.8, hence in PG(4, q), $[\mathcal{C}]$ is a g-special 4-dimensional normal rational curve and $[\mathcal{C}]^{☆}$ contains the seven points A, B, C, P, P^q, Q, Q^q. As seven points lie on a unique 4-dimensional normal rational curve, we have $\mathcal{N}^{☆} = [\mathcal{C}]^{☆}$ and so $\mathcal{N} = [\mathcal{C}]$. That is, the normal rational curve $\mathcal{N}$ corresponds in PG(2, q^2) to an $\mathbb{F}_q$-conic $\mathcal{C}$ in the tangent Baer subplane $\mathcal{B}$ as required.

Case (ii): Suppose $[P] = [Q]$, the proof is very similar to case (i). Let $\mathcal{N}$ be a 4-dimensional normal rational curve of PG(4, q) such that $\mathcal{N} \cap \Sigma_\infty = \varnothing$, and $\mathcal{N}^{☆} \cap \Sigma_\infty^{☆}$ consists of two repeated points P, P^q. Let $[A], [B], [C] \in \mathcal{N}$ and $\alpha = \langle [A], [B], [C] \rangle$. Similarly to case (i), α corresponds to a Baer subplane $\mathcal{B}_\alpha$ of PG(2, q^2) that is secant to ℓ_∞, and does not contain $\bar{P}$. The points $\bar{P}, A, B, C$ lie in a unique $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ that is tangent to ℓ_∞ at $\bar{P}$. By Lemma 5.1, A, B, C lie in a unique $\mathbb{F}_q$-conic $\mathcal{C}$ contained in $\mathcal{C}^+$, and $\mathcal{C}$ lies in a Baer subplane $\mathcal{B}$. If $\mathcal{B} = \mathcal{B}_\alpha$, then $\bar{P} \notin \mathcal{C}$, and so $\mathcal{C}^+$ meets ℓ_∞ in two points, a contradiction. Hence $\mathcal{B} \neq \mathcal{B}_\alpha$ and $\mathcal{B}$ is tangent to ℓ_∞. As $\mathcal{C}^+$ is tangent to ℓ_∞, we are in case (ii) of the proof of Theorem 5.8, hence in PG(4, q), $[\mathcal{C}]$ is a g-special 4-dimensional normal rational curve containing A, B, C, and $[\mathcal{C}]^{\star}$ meets $\Sigma_\infty^{\star}$ twice at P and twice at P^q. These conditions define a unique normal rational curve of PG$(4, q^2)$, and so $\mathcal{N} = \mathcal{C}$ as required.

Case (iii): Suppose $\mathcal{N}$ is an $g^\star$-special 4-dimensional normal rational curve. As $\mathcal{N}$ is a normal rational curve over $\mathbb{F}_q$, $\mathcal{N}$ meets $\Sigma_\infty^\star \setminus \Sigma_\infty^{\star}$ in four points which are conjugate with respect to the map $x \mapsto x^q$, $x \in \mathbb{F}_q$. That is, points of form X, X^q, X^{q^2}, X^{q^3} with $X, X^{q^2} \in g^\star$ and $X^q, X^{q^3} \in g^{q\star}$. Recalling the one-to-one correspondence between points of $g^\star$ and points of the quadratic extension of ℓ_∞ to PG$(2, q^4)$, there are points $\bar{P}$, $\bar{Q}$ on the quadratic extension of ℓ_∞ such that $P = X$ and $Q = X^{q^2}$. The argument of case (i) now generalizes by working in the quadratic extension of PG$(2, q^2)$ to PG$(2, q^4)$ and the quartic extension of PG$(4, q)$ to PG$(4, q^4)$. □

Moreover, the proofs of Theorems 5.8 and 5.9 show that the normal rational curve corresponding to an $\mathbb{F}_q$-conic $\mathcal{C}$ meets the transversal g of the regular spread $\mathcal{S}$ in points corresponding to the points $\mathcal{C}^+ \cap \ell_\infty$. The three cases when $\mathcal{C}^+$ is tangent, secant or exterior to ℓ_∞ are summarized in the next result.

Theorem 5.10. *In* PG$(2, q^2)$, $q > 7$, *let* $\mathcal{B}$ *be a Baer subplane tangent to* ℓ_∞. *Let* $\mathcal{C}$ *be an* $\mathbb{F}_q$*-conic in* $\mathcal{B}$ *with* $\mathcal{B} \cap \ell_\infty \notin \mathcal{C}$, *so* $[\mathcal{C}]$ *is a* 4*-dimensional normal rational curve. The* $\mathbb{F}_{q^2}$*-conic* $\mathcal{C}^+$ *meets* ℓ_∞ *in two points denoted* $\bar{P}$, $\bar{Q}$, *possibly equal or in an extension. The three possibilities when* $\mathcal{C}^+$ *is tangent, secant or exterior to* ℓ_∞ *are as follows.*

(1) $\bar{P} = \bar{Q}$ *if and only if, in* PG$(4, q^2)$, $[\mathcal{C}]^{\star}$ *meets the transversal* g *of* $\mathcal{S}$ *in the point* P.

(2) $\bar{P}, \bar{Q} \in \ell_\infty$ *if and only if, in* PG$(4, q^2)$, $[\mathcal{C}]^{\star}$ *meets the transversal* g *of* $\mathcal{S}$ *in the two points* P, Q.

(3) $\bar{P}$, $\bar{Q}$ *lie in the extension* PG$(2, q^4)$ *if and only if, in* PG$(4, q^4)$, $[\mathcal{C}]^\star$ *meets the extended transversal* $g^\star$ *in the two points* P *and* Q.

References

[André 1954] J. André, "Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe", *Math. Z.* **60** (1954), 156–186. MR Zbl

[Barwick and Ebert 2008] S. Barwick and G. Ebert, *Unitals in projective planes*, Springer, 2008. MR Zbl

[Barwick and Jackson 2012] S. G. Barwick and W.-A. Jackson, "Sublines and subplanes of PG$(2, q^3)$ in the Bruck–Bose representation in PG$(6, q)$", *Finite Fields Appl.* **18**:1 (2012), 93–107. MR Zbl

[Barwick et al. 2011] S. G. Barwick, W.-A. Jackson, and C. T. Quinn, "Conics and caps", *J. Geom.* **100**:1-2 (2011), 15–28. MR Zbl

[Bernasconi and Vincenti 1981] C. Bernasconi and R. Vincenti, "Spreads induced by varieties V_2^3 of PG$(4, q)$ and Baer subplanes", *Boll. Un. Mat. Ital. B* (5) **18**:3 (1981), 821–830. MR Zbl

[Bruck 1969] R. H. Bruck, "Construction problems of finite projective planes", pp. 426–514 in *Combinatorial Mathematics and its Applications* (Proc. Conf., Univ. North Carolina, Chapel Hill, N.C., 1967)), Univ. North Carolina Press, Chapel Hill, N.C., 1969. MR Zbl

[Bruck 1973a] R. H. Bruck, "Circle geometry in higher dimensions", pp. 69–77 in *A survey of combinatorial theory* (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971)), edited by G.-C. Rota and S. S. Shrikhande, North-Holland, Amsterdam, 1973. MR Zbl

[Bruck 1973b] R. H. Bruck, "Circle geometry in higher dimensions, II", *Geometriae Dedicata* **2** (1973), 133–188. MR Zbl

[Bruck and Bose 1964] R. H. Bruck and R. C. Bose, "The construction of translation planes from projective spaces", *J. Algebra* **1** (1964), 85–102. MR Zbl

[Bruck and Bose 1966] R. H. Bruck and R. C. Bose, "Linear representations of projective planes in projective spaces", *J. Algebra* **4** (1966), 117–172. MR Zbl

[Bruen and Hirschfeld 1988] A. A. Bruen and J. W. P. Hirschfeld, "Intersections in projective space, II: Pencils of quadrics", *European J. Combin.* **9**:3 (1988), 255–270. MR Zbl

[Casse and Quinn 2002] R. Casse and C. T. Quinn, "Ruled cubic surfaces in PG(4, q), Baer subplanes of PG(2, q^2) and Hermitian curves", *Discrete Math.* **248**:1-3 (2002), 17–25. MR Zbl

[Hirschfeld 1985] J. W. P. Hirschfeld, *Finite projective spaces of three dimensions*, Oxford University Press, New York, 1985. MR Zbl

[Hirschfeld and Thas 1991] J. W. P. Hirschfeld and J. A. Thas, *General Galois geometries*, Oxford University Press, New York, 1991. MR Zbl

[Metsch 1997] K. Metsch, "A note on Buekenhout–Metz unitals", pp. 177–180 in *Geometry, combinatorial designs and related structures* (Spetses, 1996), edited by J. W. P. Hirschfeld et al., London Math. Soc. Lecture Note Ser. **245**, Cambridge Univ. Press, 1997. MR Zbl

[Quinn 2002] C. T. Quinn, "The André/Bruck and Bose representation of conics in Baer subplanes of PG(2, q^2)", *J. Geom.* **74**:1-2 (2002), 123–138. MR Zbl

Received 4 Jul 2018. Revised 4 Dec 2018.

SUSAN G. BARWICK:

susan.barwick@adelaide.edu.au
School of Mathematical Sciences, University of Adelaide, Australia

WEN-AI JACKSON:

wen.jackson@adelaide.edu.au
School of Mathematical Sciences, University of Adelaide, Australia

PETER WILD:

peterrwild@gmail.com
University of London, United Kingdom