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Regular pseudo-hyperovals and regular pseudo-ovals in even characteristic

Joseph A. Thas

S. Rottey and G. Van de Voorde characterized regular pseudo-ovals of $\boldsymbol{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ and n prime. Here an alternative proof is given and slightly stronger results are obtained.

1. Introduction

Pseudo-ovals and pseudo-hyperovals were introduced in [Thas 1971]; see also [Thas et al. 2006]. These objects play a key role in the theory of translation generalized quadrangles [Payne and Thas 2009; Thas et al. 2006]. Pseudo-hyperovals only exist in even characteristic. A characterization of regular pseudo-ovals in odd characteristic was given in [Casse et al. 1985]; see also [Thas et al. 2006]. In [Rottey and Van de Voorde 2015] a characterization of regular pseudo-ovals and regular pseudo-hyperovals in $\boldsymbol{PG}(3n-1, q)$, q even, $q \neq 2$ and n prime, is obtained. Here a shorter proof is given and slightly stronger results are obtained.

2. Ovals and hyperovals

A *k-arc* in $\boldsymbol{PG}(2, q)$ is a set of k points, $k \geq 3$, no three of which are collinear. Any nonsingular conic of $\boldsymbol{PG}(2, q)$ is a $(q+1)$-arc. If $\mathcal{K}$ is any k-arc of $\boldsymbol{PG}(2, q)$, then $k \leq q+2$. For q odd $k \leq q+1$, and for q even a $(q+1)$-arc extends to a $(q+2)$-arc; see [Hirschfeld 1998]. A $(q+1)$-arc is an *oval*; a $(q+2)$-arc, q even, is a *complete oval* or *hyperoval*.

A famous theorem of B. Segre [1954] tells us that for q odd every oval of $\boldsymbol{PG}(2, q)$ is a nonsingular conic. For q even, there are many ovals that are not conics [Hirschfeld 1998]; also, there are many hyperovals that do not contain a conic [loc. cit.].

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3. Generalized ovals and hyperovals

Arcs, ovals and hyperovals can be generalized by replacing their points with m-dimensional subspaces to obtain generalized k-arcs, generalized ovals and generalized hyperovals. These objects have strong connections to generalized quadrangles, projective planes, circle geometries, flocks and other structures. See [Payne and Thas 2009; Thas et al. 2006; Thas 1971; 2011; Casse et al. 1985; Penttila and Van de Voorde 2013]. Below, some basic definitions and results are formulated; for an extensive study, many applications and open problems, see [Thas et al. 2006].

A *generalized k-arc* of $\boldsymbol{PG}(3n-1, q)$, $n \geq 1$, is a set of k $(n-1)$-dimensional subspaces of $\boldsymbol{PG}(3n-1, q)$, every three of which generate $\boldsymbol{PG}(3n-1, q)$. If q is odd, then $k \leq q^n + 1$; if q is even, then $k \leq q^n + 2$. Every generalized $(q^n + 1)$-arc of $\boldsymbol{PG}(3n-1, q)$, q even, can be extended to a generalized $(q^n + 2)$-arc.

If $\mathcal{O}$ is a generalized $(q^n + 1)$-arc in $\boldsymbol{PG}(3n-1, q)$, then it is a *pseudo-oval* or *generalized oval* or $[n-1]$-*oval* of $\boldsymbol{PG}(3n-1, q)$. For $n = 1$, a [0]-oval is just an oval of $\boldsymbol{PG}(2, q)$. If $\mathcal{O}$ is a generalized $(q^n + 2)$-arc in $\boldsymbol{PG}(3n-1, q)$, q even, then it is a *pseudo-hyperoval* or *generalized hyperoval* or $[n-1]$-*hyperoval* of $\boldsymbol{PG}(3n-1, q)$. For $n = 1$, a [0]-hyperoval is just a hyperoval of $\boldsymbol{PG}(2, q)$.

If $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n}\}$ is a pseudo-oval of $\boldsymbol{PG}(3n-1, q)$, then π_i is contained in exactly one $(2n-1)$-dimensional subspace τ_i of $\boldsymbol{PG}(3n-1, q)$ which has no point in common with $(\pi_0 \cup \pi_1 \cup \cdots \cup \pi_{q^n}) \setminus \pi_i$, with $i = 0, 1, \ldots, q^n$; the space τ_i is the *tangent space* of $\mathcal{O}$ at π_i. For q even the $q^n + 1$ tangent spaces of $\mathcal{O}$ contain a common $(n-1)$-dimensional space π_{q^n+1}, the *nucleus* of $\mathcal{O}$; also, $\mathcal{O} \cup \{\pi_{q^n+1}\}$ is a pseudo-hyperoval of $\boldsymbol{PG}(3n-1, q)$. For q odd, the tangent spaces of a pseudo-oval $\mathcal{O}$ are the elements of a pseudo-oval $\mathcal{O}^*$ in the dual space of $\boldsymbol{PG}(3n-1, q)$.

4. Regular pseudo-ovals and pseudo-hyperovals

In the extension $\boldsymbol{PG}(3n-1, q^n)$ of $\boldsymbol{PG}(3n-1, q)$, we consider n planes ξ_i, $i = 1, 2, \ldots, n$, that are conjugate in the extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ and which span $\boldsymbol{PG}(3n-1, q^n)$. This means that they form an orbit of the Galois group corresponding to this extension and span $\boldsymbol{PG}(3n-1, q^n)$.

In ξ_1 consider an oval $\mathcal{O}_1 = \{x_0^{(1)}, x_1^{(1)}, \ldots, x_{q^n}^{(1)}\}$. Further, let $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n)}$, with $i = 0, 1, \ldots, q^n$, be conjugate in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. The points $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n)}$ define an $(n-1)$-dimensional subspace π_i over $\mathbb{F}_q$ for $i = 0, 1, \ldots, q^n$. Then, $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n}\}$ is a generalized oval of $\boldsymbol{PG}(3n-1, q)$. These objects are the *regular* or *elementary pseudo-ovals*. If $\mathcal{O}_1$ is replaced by a hyperoval, and so q is even, then the corresponding $\mathcal{O}$ is a *regular* or *elementary pseudo-hyperoval*.

All known pseudo-ovals and pseudo-hyperovals are regular.

5. Characterizations

Let $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n}\}$ be a pseudo-oval in $\boldsymbol{PG}(3n-1, q)$. The tangent space of $\mathcal{O}$ at π_i will be denoted by τ_i, with $i = 0, 1, \ldots, q^n$. Choose π_i, $i \in \{0, 1, \ldots, q^n\}$, and let $\boldsymbol{PG}(2n-1, q) \subseteq \boldsymbol{PG}(3n-1, q)$ be skew to π_i. Let $\tau_i \cap \boldsymbol{PG}(2n-1, q) = \eta_i$ and $\langle \pi_i, \pi_j \rangle \cap \boldsymbol{PG}(2n-1, q) = \eta_j$, with $j \neq i$. Then $\{\eta_0, \eta_1, \ldots, \eta_{q^n}\} = \Delta_i$ is an $(n-1)$-spread of $\boldsymbol{PG}(2n-1, q)$.

Now, let q be even and π the nucleus of $\mathcal{O}$. Let $\boldsymbol{PG}(2n-1, q) \subseteq \boldsymbol{PG}(3n-1, q)$ be skew to π. If $\zeta_j = \boldsymbol{PG}(2n-1, q) \cap \langle \pi, \pi_j \rangle$, then $\{\zeta_0, \zeta_1, \ldots, \zeta_{q^n}\} = \Delta$ is an $(n-1)$-spread of $\boldsymbol{PG}(2n-1, q)$.

Next, let q be odd. Choose τ_i, with $i \in \{0, 1, \ldots, q^n\}$. If $\tau_i \cap \tau_j = \delta_j$, with $j \neq i$, then $\{\delta_0, \delta_1, \ldots, \delta_{i-1}, \pi_i, \delta_{i+1}, \ldots, \delta_{q^n}\} = \Delta_i^\star$ is an $(n-1)$-spread of τ_i.

Finally, let q be even and let $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n+1}\}$ be a pseudo-hyperoval in $\boldsymbol{PG}(3n-1, q)$. Choose π_i, with $i \in \{0, 1, \ldots, q^n+1\}$, and let $\boldsymbol{PG}(2n-1, q) \subseteq \boldsymbol{PG}(3n-1, q)$ be skew to π_i. Let $\langle \pi_i, \pi_j \rangle \cap \boldsymbol{PG}(2n-1, q) = \eta_j$, with $j \neq i$. Then $\{\eta_0, \eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_{q^n+1}\} = \Delta_i$ is an $(n-1)$-spread of $\boldsymbol{PG}(2n-1, q)$.

Theorem 5.1 [Casse et al. 1985]. *Consider a pseudo-oval $\mathcal{O}$ with q odd. Then at least one of the $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}, \Delta_0^\star, \Delta_1^\star, \ldots, \Delta_{q^n}^\star$ is regular if and only if they all are regular if and only if the pseudo-oval $\mathcal{O}$ is regular. In such a case $\mathcal{O}$ is essentially a conic over $\mathbb{F}_{q^n}$.*

Theorem 5.2 [Rottey and Van de Voorde 2015]. *Consider a pseudo-oval $\mathcal{O}$ in $\boldsymbol{PG}(3n-1, q)$ with $q = 2^h$, $h > 1$ and n prime. Then $\mathcal{O}$ is regular if and only if all $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}$ are regular.*

6. Alternative proof and improvements

Theorem 6.1. *Consider a pseudo-hyperoval $\mathcal{O}$ in $\boldsymbol{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ and n prime. Then $\mathcal{O}$ is regular if and only if all $(n-1)$-spreads Δ_i, with $i = 0, 1, \ldots, q^n+1$, are regular.*

Proof. If $\mathcal{O}$ is regular, then clearly all $(n-1)$-spreads Δ_i, with $i = 0, 1, \ldots, q^n+1$, are regular.

Conversely, assume that the $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n+1}$ are regular. Let $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n+1}\}$, and let $\widehat{\mathcal{O}} = \{\beta_0, \beta_1, \ldots, \beta_{q^n+1}\}$ be the dual of $\mathcal{O}$, with β_i being the dual of π_i.

Choose β_i, $i \in \{0, 1, \ldots, q^n+1\}$, and let $\beta_i \cap \beta_j = \alpha_{ij}$, $j \neq i$. Then

$$\{\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{i,i-1}, \alpha_{i,i+1}, \ldots, \alpha_{i,q^n+1}\} = \Gamma_i \tag{1}$$

is an $(n-1)$-spread of β_i.

Now consider $\beta_i, \beta_j, \Gamma_i, \Gamma_j, \alpha_{ij}$, $j \neq i$. In Γ_j we next consider an $(n-1)$-regulus γ_j containing α_{ij}. The $(n-1)$-regulus γ_j is a set of maximal spaces of a Segre variety $\mathcal{S}_{1;n-1}$; see Section 4.5 in [Hirschfeld and Thas 2016]. The $(n-1)$-regulus γ_j and the $(n-1)$-spread Γ_i of β_i generate a regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$ of $\boldsymbol{PG}(3n-1, q)$. This can be seen as follows. The elements of Γ_i intersect n lines $U_1, U_2, \ldots, U_n$ which are conjugate in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$; that is, they form an orbit of the Galois group corresponding to this extension. Let $\alpha_{ij} \cap U_l = \{u_l\}$, with $l = 1, 2, \ldots, n$. Now consider the transversals $T_1, T_2, \ldots, T_n$ of the elements of γ_j, with T_l containing u_l. The n planes $T_l U_l = \theta_l$ intersect all elements of γ_j and Γ_i. The $(n-1)$-dimensional subspaces of $\boldsymbol{PG}(3n-1, q)$ intersecting $\theta_1, \theta_2, \ldots, \theta_n$ are the elements of the regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$. The elements of this spread correspond to the points of a plane $\boldsymbol{PG}(2, q^n)$, with its lines corresponding to the $(2n-1)$-dimensional spaces containing at least two (and then q^n+1) elements of the spread. Hence, the $q+2$ elements of $\widehat{\mathcal{O}}$ containing an element of γ_j, say $\beta_i = \beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_{q+1}}, \beta_{i_{q+2}} = \beta_j$, correspond to lines of $\boldsymbol{PG}(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+2}}$ correspond to points of $\boldsymbol{PG}(2, q^n)$.

Now consider β_{i_2} and γ_j, and repeat the argument. Then there arise n planes θ'_l intersecting all elements of γ_j and Γ_{i_2}. The $(n-1)$-dimensional subspaces of $\boldsymbol{PG}(3n-1, q)$ intersecting $\theta'_1, \theta'_2, \ldots, \theta'_n$ are the elements of the regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_{i_2})$. The elements of this spread correspond to the points of a plane $\boldsymbol{PG}'(2, q^n)$, and the lines of this plane correspond to the $(2n-1)$-dimensional spaces containing q^n+1 elements of the spread. Hence, $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_{q+2}}$ correspond to lines of $\boldsymbol{PG}'(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+2}}$ correspond to points of $\boldsymbol{PG}'(2, q^n)$.

First, assume that $\{\theta_1, \theta_2, \ldots, \theta_n\} \cap \{\theta'_1, \theta'_2, \ldots, \theta'_n\} = \varnothing$. Consider $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The planes of $\boldsymbol{PG}(3n-1, q^n)$ intersecting these four spaces constitute a set $\mathcal{M}$ of maximal spaces of a Segre variety $\mathcal{S}_{2;n-1}$ [Burau 1961]. The planes $\theta_1, \theta_2, \ldots, \theta_n, \theta'_1, \theta'_2, \ldots, \theta'_n$ are elements of $\mathcal{M}$. It follows that $(\theta_1 \cup \theta_2 \cup \cdots \cup \theta_n) \cap (\theta'_1 \cup \theta'_2 \cup \cdots \cup \theta'_n) = \varnothing$.

Now consider any $(n-1)$-dimensional subspace $\pi \in \{\pi_{i_5}, \pi_{i_6}, \ldots, \pi_{i_{q+2}}\}$ of $\boldsymbol{PG}(3n-1, q)$. We will show that π is a maximal subspace of $\mathcal{S}_{2;n-1}$. Let $\theta_i \cap \pi_j = \{t_{ij}\}$, $\theta'_i \cap \pi_j = \{t'_{ij}\}$, $i = 1, 2, \ldots, n$ and $j = i_1, i_2, \ldots, i_{q+2}$. If $t_{ij_1}t_{ij_2} \cap t_{ij_3}t_{ij_4} = \{v_i\}$ and $t'_{ij_1}t'_{ij_2} \cap t'_{ij_3}t'_{ij_4} = \{v'_i\}$, with j_1, j_2, j_3, j_4 distinct, then $v_1, v_2, \ldots, v_n$ are conjugate and similarly $v'_1, v'_2, \ldots, v'_n$ are conjugate. Hence, $\langle v_1, v_2, \ldots, v_n\rangle = \langle v'_1, v'_2, \ldots, v'_n\rangle$ defines an $(n-1)$-dimensional space over $\mathbb{F}_q$ which intersects $\theta_1, \theta_2, \ldots, \theta'_n$ (over $\mathbb{F}_{q^n}$). The points t_{ij}, with $j = i_1, i_2, \ldots, i_{q+2}$, generate a subplane of θ_i, and the points t'_{ij}, with $j = i_1, i_2, \ldots, i_{q+2}$, generate a subplane of θ'_i, with $i = 1, 2, \ldots, n$. Let $q = 2^h$, and let $\mathbb{F}_{2^v}$ be the subfield of $\mathbb{F}_{q^n} = \mathbb{F}_{2^{hn}}$ over which these subplanes are defined, so $v \mid hn$. Then $v < hn$ as otherwise the spreads of $\boldsymbol{PG}(3n-1, q)$ defined by $\theta_1, \theta_2, \ldots, \theta_n$ and $\theta'_1, \theta'_2, \ldots, \theta'_n$ coincide, which is clearly not possible. The $(n-1)$-regulus γ_j implies that the subplanes contain a line over $\mathbb{F}_q$, so $h \mid v$. As n is prime we have $v = h$, so $2^v = q$.

Hence, the $2n$ subplanes are defined over $\mathbb{F}_q$. It follows that the $q+2$ elements $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+2}}$ are maximal subspaces of the Segre variety $\mathcal{S}_{2;n-1}$. Hence, π is a maximal subspace of $\mathcal{S}_{2;n-1}$. It follows that $\pi_1, \pi_2, \ldots, \pi_{q+2}$ are maximal subspaces of $\mathcal{S}_{2;n-1}$.

Now consider a $\boldsymbol{PG}(2, q)$ intersecting $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The $(n-1)$-dimensional spaces $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+2}}$ are maximal spaces of $\mathcal{S}_{2;n-1}$ intersecting $\boldsymbol{PG}(2, q)$; they are maximal spaces of the Segre variety $\mathcal{S}_{2;n-1} \cap \boldsymbol{PG}(3n-1, q)$ of $\boldsymbol{PG}(3n-1, q)$.

Consider π_{i_1} and also a $\boldsymbol{PG}(2n-1, q)$ skew to π_{i_1}. If we project $\pi_{i_2}, \pi_{i_3}, \ldots, \pi_{i_{q+2}}$ from π_{i_1} onto $\boldsymbol{PG}(2n-1, q)$, then by the foregoing paragraph the $q+1$ projections constitute an $(n-1)$-regulus of $\boldsymbol{PG}(2n-1, q)$. We arrive at a similar conclusion if we project from π_{i_s}, s any element of $\{1, 2, \ldots, q+2\}$. Equivalently, if $s \in \{1, 2, \ldots, q+2\}$, then the spaces $\beta_{i_s} \cap \beta_{i_t}$, with $t = 1, 2, \ldots, s-1, s+1, \ldots, q+2$, form an $(n-1)$-regulus of β_{i_s}.

Now assume that the condition $\{\theta_1, \theta_2, \ldots, \theta_n\} \cap \{\theta'_1, \theta'_2, \ldots, \theta'_n\} = \varnothing$ is satisfied for any choice of $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$. In such a case every $(n-1)$-regulus contained in a spread Γ_s defines a Segre variety $\mathcal{S}_{2;n-1}$ over $\mathbb{F}_q$. Let us define the following design $\mathcal{D}$. Points of $\mathcal{D}$ are the elements of $\widehat{\mathcal{O}}$, a block of $\mathcal{D}$ is a set of $q+2$ elements of $\widehat{\mathcal{O}}$, containing at least one space of an $(n-1)$-regulus contained in some regular spread Γ_s, and incidence is containment. Then $\mathcal{D}$ is a $4-(q^n+2, q+2, 1)$ design. By Kantor [1974] this implies that $q = 2$, a contradiction.

Consequently, we may assume that for at least one quadruple $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$,

$$\{\theta_1, \theta_2, \ldots, \theta_n\} = \{\theta'_1, \theta'_2, \ldots, \theta'_n\}. \tag{2}$$

In such a case the q^n+2 elements of $\widehat{\mathcal{O}}$ correspond to lines of the plane $\boldsymbol{PG}(2, q^n)$. It follows that $\mathcal{O}$ is regular. □

Theorem 6.2. *Consider a pseudo-oval $\mathcal{O}$ in $\boldsymbol{PG}(3n-1, q)$, with $q = 2^h$, $h > 1$ and n prime. Then $\mathcal{O}$ is regular if and only if all $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}$ are regular.*

Proof. If $\mathcal{O}$ is regular, then clearly all $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}$ are regular.

Conversely, assume that the $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}$ are regular. Let $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n}\}$, let π_{q^n+1} be the nucleus of $\mathcal{O}$, let $\overline{\mathcal{O}} = \mathcal{O} \cup \{\pi_{q^n+1}\}$, let $\widehat{\mathcal{O}}$ be the dual of $\mathcal{O}$, let $\widehat{\overline{\mathcal{O}}}$ be the dual of $\overline{\mathcal{O}}$, and let β_i be the dual of π_i.

Choose β_i, $i \in \{0, 1, \ldots, q^n+1\}$, and let $\beta_i \cap \beta_j = \alpha_{ij}$, $j \neq i$. Then

$$\{\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{i,i-1}, \alpha_{i,i+1}, \ldots, \alpha_{i,q^n+1}\} = \Gamma_i \tag{3}$$

is an $(n-1)$-spread of β_i.

Now consider $\beta_i, \beta_j, \Gamma_i, \Gamma_j, \alpha_{ij}$, with $j \neq i$ and $i, j \in \{0, 1, \ldots, q^n\}$. In Γ_j we next consider an $(n-1)$-regulus γ_j containing α_{ij} and α_{j,q^n+1}. The $(n-1)$-regulus

γ_j is a set of maximal spaces of a Segre variety $\mathcal{S}_{1;n-1}$. The $(n-1)$-regulus γ_j and the $(n-1)$-spread Γ_i of β_i generate a regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$ of $\boldsymbol{PG}(3n-1, q)$. Such as in the proof of Theorem 6.1 we introduce the elements $\underset{\frown}{U}_l, u_l, T_l, \theta_l$, $l = 1, 2, \ldots, n$, and the plane $\boldsymbol{PG}(2, q^n)$. The $q+2$ elements of $\overline{\mathcal{O}}$ containing an element of γ_j, say $\beta_i = \beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_q}, \beta_j = \beta_{i_{q+1}}, \beta_{q^n+1}$, correspond to lines of $\boldsymbol{PG}(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+1}}, \pi_{q^n+1}$ correspond to points of $\boldsymbol{PG}(2, q^n)$.

Now consider β_{i_2} and γ_j, and repeat the argument. Then there arise n planes θ'_l of $\boldsymbol{PG}(3n-1, q^n)$ intersecting all elements of γ_j and Γ_{i_2}, and an $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_{i_2})$ of $\boldsymbol{PG}(3n-1, q)$. The elements of this spread correspond to the points of a plane $\boldsymbol{PG}'(2, q^n)$. The spaces $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_{q+1}}, \beta_{q^n+1}$ correspond to lines of $\boldsymbol{PG}'(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+1}}, \pi_{q^n+1}$ correspond to points of $\boldsymbol{PG}'(2, q^n)$.

First, assume $\{\theta_1, \theta_2, \ldots, \theta_n\} \cap \{\theta'_1, \theta'_2, \ldots, \theta'_n\} = \varnothing$. Consider $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The planes of $\boldsymbol{PG}(3n-1, q^n)$ intersecting these four spaces constitute a set $\mathcal{M}$ of maximal spaces of a Segre variety $\mathcal{S}_{2;n-1}$. The planes $\theta_1, \theta_2, \ldots, \theta_n, \theta'_1, \theta'_2, \ldots, \theta'_n$ are elements of $\mathcal{M}$. It follows that $(\theta_1 \cup \theta_2 \cup \cdots \cup \theta_n) \cap (\theta'_1 \cup \theta'_2 \cup \cdots \cup \theta'_n) = \varnothing$. Let $\pi \in \{\pi_{i_5}, \pi_{i_6}, \ldots, \pi_{i_{q+1}}, \pi_{q^n+1}\}$. As in the proof of Theorem 6.1 one shows that π is a maximal subspace of $\mathcal{S}_{2;n-1}$. It follows that $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+1}}, \pi_{q^n+1}$ are maximal subspaces of $\mathcal{S}_{2;n-1}$.

Next consider a $\boldsymbol{PG}(2, q)$ that intersects $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The $(n-1)$-dimensional spaces $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+1}}, \pi_{q^n+1}$ are maximal spaces of $\mathcal{S}_{2;n-1}$ which intersect the plane $\boldsymbol{PG}(2, q)$; they are maximal spaces of the Segre variety $\mathcal{S}_{2;n-1} \cap \boldsymbol{PG}(3n-1, q)$ of $\boldsymbol{PG}(3n-1, q)$. As in the proof of Theorem 6.1 it follows that the spaces $\beta_{q^n+1} \cap \beta_{i_t}$, with $t = 1, 2, \ldots, q+1$, form an $(n-1)$-regulus of β_{q^n+1}.

Now assume that the condition $\{\theta_1, \theta_2, \ldots, \theta_n\} \cap \{\theta'_1, \theta'_2, \ldots, \theta'_n\} = \varnothing$ is satisfied for any choice of $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$, $j \neq i$ and $i, j \in \{0, 1, \ldots, q^n\}$. Let $\alpha_1, \alpha_2, \alpha_3$ be distinct elements of Γ_{q^n+1}. Then $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$ can be chosen in such a way that $\alpha_1 \in \beta_i$, $\alpha_2 \in \beta_j$, $\alpha_2 \in \gamma_j$ and $\beta_{i_2} \cap \beta_j \in \gamma_j$ with $\alpha_3 \in \beta_{i_2}$. Hence, the $(n-1)$-regulus in β_{q^n+1} defined by $\alpha_1, \alpha_2, \alpha_3$ is a subset of Γ_{q^n+1}. From [Hirschfeld and Thas 2016, Theorem 4.123] now follows that the $(n-1)$-spread Γ_{q^n+1} of β_{q^n+1} is regular. By Theorem 6.1 the pseudo-hyperoval $\overline{\mathcal{O}}$ is regular, and so $\mathcal{O}$ is regular. But in such a case the condition $\{\theta_1, \theta_2, \ldots, \theta_n\} \cap \{\theta'_1, \theta'_2, \ldots, \theta'_n\} = \varnothing$ is never satisfied, a contradiction.

Consequently, we may assume that for at least one quadruple $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$ we have $\{\theta_1, \theta_2, \ldots, \theta_n\} = \{\theta'_1, \theta'_2, \ldots, \theta'_n\}$. In such a case the q^n+2 elements of $\overline{\mathcal{O}}$ correspond to lines of the plane $\boldsymbol{PG}(2, q^n)$. It follows that $\overline{\mathcal{O}}$, and hence also $\mathcal{O}$, is regular. □

Theorem 6.3. *A pseudo-hyperoval $\mathcal{O}$ in $\boldsymbol{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ and n prime, is regular if and only if at least $q^n - 1$ elements of $\{\Delta_0, \Delta_1, \ldots, \Delta_{q^n+1}\}$ are regular.*

Proof. If $\mathcal{O}$ is regular, then clearly all $(n-1)$-spreads Δ_i, with $i = 0, 1, \ldots, q^n+1$, are regular.

Conversely, assume that ρ, with $\rho \geq q^n - 1$, elements of $\{\Delta_0, \Delta_1, \ldots, \Delta_{q^n+1}\}$ are regular.

If $\rho = q^n+2$, then $\mathcal{O}$ is regular by Theorem 6.1; if $\rho = q^n+1$, then $\mathcal{O}$ is regular by Theorem 6.2.

Now assume that $\rho = q^n$ and that $\Delta_2, \Delta_3, \ldots, \Delta_{q^n+1}$ are regular. We have to prove that Δ_0 is regular. We use the arguments in the proof of Theorem 6.2. If one of the elements $\alpha_1, \alpha_2, \alpha_3$, say α_1, in the proof of Theorem 6.2 is $\beta_0 \cap \beta_1$, then let γ_j contain $\beta_j \cap \beta_i$, $\beta_j \cap \beta_0$, $\beta_j \cap \beta_1$ and let $\beta_{i_2} \neq \beta_1$, with $i, j \in \{2, 3, \ldots, q^n+1\}$. Now see the proof of the preceding theorem.

Finally, assume that $\rho = q^n - 1$ and that $\Delta_3, \Delta_4, \ldots, \Delta_{q^n+1}$ are regular. We have to prove that Δ_0 is regular. We use the arguments in the proof of Theorem 6.2. If exactly one of the elements $\alpha_1, \alpha_2, \alpha_3$, say α_1, in the proof of Theorem 6.2 is $\beta_0 \cap \beta_1$ or $\beta_0 \cap \beta_2$, then proceed as in the preceding paragraph with $\beta_{i_2} \neq \beta_1, \beta_2$. Now assume that two of the elements $\alpha_1, \alpha_2, \alpha_3$, say α_1 and α_2, are $\beta_0 \cap \beta_1$ and $\beta_0 \cap \beta_2$. Now consider all $(n-1)$-reguli in Δ_0 containing α_1 and α_3, and assume, by way of contradiction, that no one of these $(n-1)$-reguli contains α_2. The number of these $(n-1)$-reguli is $(q^n-2)/(q-1)$, and so $q = 2$, a contradiction. It follows that the $(n-1)$-regulus in β_0 defined by $\alpha_1, \alpha_2, \alpha_3$ is contained in Δ_0. Now we proceed as in the proof of Theorem 6.2. □

7. Final remarks

The cases q = 2 and n not prime. For $q = 2$ or n not prime other arguments have to be developed.

Improvement of Theorem 6.3. Let $\mathcal{D} = (P, B, \in)$ be an incidence structure satisfying the following conditions:

(i) $|P| = q^n + 1$, q even, $q \neq 2$,

(ii) the elements of B are subsets of size $q+1$ of P and every three distinct elements of P are contained in at most one element of B, and

(iii) Q is a subset of size δ of P such that any triple of elements in P with at most one element in Q is contained in exactly one element of B.

Assumption. Any such $\mathcal{D}$ is a $3-(q^n+1, q+1, 1)$ design whenever $\delta \leq \delta_0$ with $\delta_0 \leq q-2$.

Theorem 7.1. *Consider a pseudo-hyperoval $\mathcal{O}$ in $\boldsymbol{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ and n prime. Then $\mathcal{O}$ is regular if and only if at least $q^n + 1 - \delta_0$ elements of $\{\Delta_0, \Delta_1, \ldots, \Delta_{q^n+1}\}$ are regular.*

Proof. Similar to the proof of Theorem 6.3. □

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Conics in Baer subplanes

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This article studies conics and subconics of $PG(2, q^2)$ and their representation in the André/Bruck–Bose setting in $PG(4, q)$. In particular, we investigate their relationship with the transversal lines of the regular spread. The main result is to show that a conic in a tangent Baer subplane of $PG(2, q^2)$ corresponds in $PG(4, q)$ to a normal rational curve that meets the transversal lines of the regular spread. Conversely, every 3- and 4-dimensional normal rational curve in $PG(4, q)$ that meets the transversal lines of the regular spread corresponds to a conic in a tangent Baer subplane of $PG(2, q^2)$.

1. Introduction

This article investigates the representation of conics and subconics of $PG(2, q^2)$ in the Bruck–Bose representation in $PG(4, q)$. The Bruck–Bose representation of $PG(2, q^2)$ uses a regular spread $\mathcal{S}$ in the hyperplane at infinity of $PG(4, q)$. The regular spread $\mathcal{S}$ has two unique transversal lines g, g^q in the quadratic extension $PG(4, q^2)$. There are several known characterizations of objects of $PG(4, q)$ in terms of their relationship with these transversal lines. Firstly, a conic $\mathcal{C}$ in $PG(4, q)$ corresponds to a Baer subline of $PG(2, q^2)$ if and only if the extension of $\mathcal{C}$ to a conic of $PG(4, q^2)$ contains a point of g and a point of g^q [Casse and Quinn 2002]. A ruled cubic surface $\mathcal{V}$ in $PG(4, q)$ corresponds to a Baer subplane of $PG(2, q^2)$ if and only if the extension of $\mathcal{V}$ to $PG(4, q^2)$ contains g and g^q [Casse and Quinn 2002]. Further, an orthogonal cone $\mathcal{U}$ corresponds to a classical unital of $PG(2, q^2)$ if and only if the extension of $\mathcal{U}$ to $PG(4, q^2)$ contains g and g^q [Metsch 1997]. Hence the interaction of certain objects with the transversals of $\mathcal{S}$ is intrinsic to their characterization in $PG(2, q^2)$. In this article we study conics and subconics of $PG(2, q^2)$ and determine their relationship with the transversals of $\mathcal{S}$ in the Bruck–Bose setting in $PG(4, q)$. In particular, we characterize normal rational curves of $PG(4, q)$ whose extension meets the transversals as subconics of $PG(2, q^2)$.

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The article is set out as follows. Section 2 gives background and proves some preliminary results. In particular, in order to study how objects of the Bruck–Bose representation relate to the transversals of the regular spread $\mathcal{S}$, we formally define the notion of g-special sets, or special sets in $\mathrm{PG}(4, q)$ (page 90). In the last subsection (pages 93–95) we consider a Baer subplane $\mathcal{B}$ tangent to ℓ_∞, and give a geometric construction via $\mathrm{PG}(4, q)$ that partitions the affine points of $\mathcal{B}$ into q conics, one of which is degenerate.

In Section 3, we discuss how the notion of specialness relates to the known Bruck–Bose representation of Baer sublines and Baer subplanes.

In Section 4, we investigate nondegenerate conics of $\mathrm{PG}(2, q^2)$ in the $\mathrm{PG}(4, q)$ Bruck–Bose representation, and specifically the structure in the quadratic extension to $\mathrm{PG}(4, q^2)$. We show that in $\mathrm{PG}(4, q^2)$, the (extended) structure corresponding to a nondegenerate conic $\mathcal{O}$ is the intersection of two quadrics which meet g in the two points (possibly repeated or in an extension) corresponding to $\mathcal{O} \cap \ell_\infty$.

In Section 5 we characterize the Bruck–Bose representation of conics contained in Baer subplanes. In $\mathrm{PG}(2, q^2)$, let $\mathcal{B}$ be a Baer subplane tangent to ℓ_∞, and $\mathcal{C}$ a nondegenerate conic contained in $\mathcal{B}$. We show that in $\mathrm{PG}(4, q)$, $\mathcal{C}$ corresponds to a normal rational curve that meets the transversals of the regular spread. Conversely, we characterize every normal rational curve in $\mathrm{PG}(4, q)$ that meets the transversals of the regular spread as corresponding to a nondegenerate conic in a Baer subplane of $\mathrm{PG}(2, q^2)$.

While the proofs in Section 4 are largely coordinate-based, the proofs in Section 5 use geometrical arguments.

2. Background and preliminary results

In this section we give the necessary background, introduce the notation we use in this article, and prove a number of preliminary results.

Conjugate points. For q a prime power, we denote the unique finite field of order q by $\mathbb{F}_q$. We use the phrase conjugate points in different settings. Firstly, consider the automorphism $x \mapsto x^q$ for $x \in \mathbb{F}_{q^r}$ and the induced automorphic collineation of $\mathrm{PG}(n, q^r)$ given by $X = (x_0, \dots, x_n) \mapsto X^q = (x_0^q, \dots, x_n^q)$. The points $X, X^q, \dots, X^{q^{n-1}}$ are called *conjugate*. Secondly, let $\mathcal{B}$ be a Baer subplane of $\mathrm{PG}(2, q^2)$; there is a unique involutory collineation that fixes $\mathcal{B}$ pointwise, and we call this map *conjugacy with respect to* $\mathcal{B}$. Note that $P, Q \in \ell_\infty$ are conjugate with respect to the secant Baer subplane $\mathcal{B}$ if and only if P, Q are conjugate with respect to the Baer subline $\mathcal{B} \cap \ell_\infty$.

***Spreads in* $\mathbf{PG(3, q)}$.** The following construction of a regular spread of $\mathrm{PG}(3, q)$ will be needed, see [Hirschfeld and Thas 1991] for more information on spreads.

Embed PG$(3, q)$ in PG$(3, q^2)$ and let g be a line of PG$(3, q^2)$ disjoint from PG$(3, q)$. The line g has a conjugate line g^q with respect to the map $x \mapsto x^q$, $x \in \mathbb{F}_{q^2}$, and g^q is also disjoint from PG$(3, q)$. Let P_i be a point on g; then the line $\langle P_i, P_i^q \rangle$ meets PG$(3, q)$ in a line. As P_i ranges over all the points of g, we obtain $q^2 + 1$ lines of PG$(3, q)$ that partition PG$(3, q)$. These lines form a regular spread $\mathcal{S}$ of PG$(3, q)$. The lines g, g^q are called the (conjugate skew) *transversal lines* of the regular spread $\mathcal{S}$. Conversely, given a regular spread $\mathcal{S}$ in PG$(3, q)$, there is a unique pair of transversal lines in PG$(3, q^2)$ that generate $\mathcal{S}$ in this way.

The Bruck–Bose representation. We will use the linear representation of a finite translation plane of dimension at most two over its kernel, introduced independently in [André 1954] and [Bruck and Bose 1964; 1966]. Let Σ_∞ be a hyperplane of PG$(4, q)$ and let $\mathcal{S}$ be a spread of Σ_∞. The phrase *a subspace of* PG$(4, q) \setminus \Sigma_\infty$ will be used to mean a subspace of PG$(4, q)$ that is not contained in Σ_∞. Consider the following incidence structure: the *points* of $\mathcal{A}(\mathcal{S})$ are the points of PG$(4, q) \setminus \Sigma_\infty$; the *lines* of $\mathcal{A}(\mathcal{S})$ are the planes of PG$(4, q) \setminus \Sigma_\infty$ that contain an element of $\mathcal{S}$; and *incidence* in $\mathcal{A}(\mathcal{S})$ is induced by incidence in PG$(4, q)$. Then the incidence structure $\mathcal{A}(\mathcal{S})$ is an affine plane of order q^2. We can complete $\mathcal{A}(\mathcal{S})$ to a projective plane $\mathcal{P}(\mathcal{S})$; the points on the line at infinity ℓ_∞ have a natural correspondence to the elements of the spread $\mathcal{S}$. We call this the *Bruck–Bose representation* of $\mathcal{P}(\mathcal{S})$ in PG$(4, q)$. The projective plane $\mathcal{P}(\mathcal{S})$ is the Desarguesian plane PG$(2, q^2)$ if and only if $\mathcal{S}$ is a regular spread of $\Sigma_\infty \cong$ PG$(3, q)$ (see [Bruck 1969]). We use the following notation in the Bruck–Bose setting:

- $\mathcal{S}$ is a regular spread with transversal lines g, g^q.
- An affine point of PG$(2, q^2) \setminus \ell_\infty$ is denoted with a capital letter, A say, and $[A]$ denotes the corresponding point of PG$(4, q) \setminus \Sigma_\infty$.
- A point on ℓ_∞ in PG$(2, q^2)$ is denoted with an over-lined capital letter, $\bar{T}$ say, and the corresponding spread line is denoted $[T]$.
- The points of ℓ_∞ are in one-to-one correspondence with the points of g; for a point $\bar{T} \in \ell_\infty$, we denote the corresponding point of g by T.
- A set of points $\mathcal{X}$ in PG$(2, q^2)$ corresponds to a set of points denoted $[\mathcal{X}]$ in PG$(4, q)$.

We will work in the extension of PG$(4, q)$ to PG$(4, q^2)$ and to PG$(4, q^4)$. Let $\mathcal{K}$ be a primal of PG$(4, q)$, so $\mathcal{K}$ is the set of points of PG$(4, q)$ satisfying a homogeneous equation $f(x_0, \ldots, x_4) = 0$, with coefficients in $\mathbb{F}_q$. We define $\mathcal{K}^{\star}$ to be the (unique) primal of PG$(4, q^2)$ which is the set of points of PG$(4, q^2)$ satisfy the same homogeneous equation $f = 0$. Note that if $\mathcal{K} = \Pi$ is an r-dimensional subspace of PG$(4, q)$, then $\Pi^{\star}$ is the (unique) r-dimensional subspace of PG$(4, q^2)$ containing Π. Further, if $\mathcal{V}$ is a variety of PG$(4, q)$, the intersection of primals

$\mathcal{K}_1, \dots, \mathcal{K}_s$, then we define $\mathcal{V}^{☆} = \mathcal{K}_1^{☆} \cap \cdots \cap \mathcal{K}_s^{☆}$. Similarly, we can extend a primal $\mathcal{K}$ to PG(4, q^4), and we denote the resulting set by $\mathcal{K}^{\star}$. The transversals g, g^q of the regular spread $\mathcal{S}$ lie in PG(4, q^2), and we denote their extensions to lines of PG(4, q^4) by $g^{\star}, g^{q\star}$ respectively.

***Ruled cubic surfaces in* PG(4, *q*).** A ruled cubic surface $\mathcal{V}$ of PG(4, q) consists of a line directrix t, a conic directrix $\mathcal{C}$ lying in a plane disjoint from t, and a set of $q+1$ pairwise disjoint generator lines joining the points of t and $\mathcal{C}$ according to a projectivity $\omega \in \mathrm{PGL}(2, q)$. Specifically, if $\theta, \phi \in \mathbb{F}_q \cup \{\infty\}$ are the nonhomogeneous coordinates of t and $\mathcal{C}$, then ω maps $(1, \theta)$ to $(1, \phi)$. The generators of $\mathcal{V}$ are the lines joining points of t to the corresponding point of $\mathcal{C}$ under ω. We will need the following result, which shows how hyperplanes of PG(4, q) meet a ruled cubic surface.

Result 2.1 [Quinn 2002]. *A hyperplane of* PG(4, q) *meets a ruled cubic surface in one of the following*:

- *The line directrix*; $(q^2 - q)/2$ *hyperplanes do this.*
- *The line directrix and one generator line*; $q + 1$ *hyperplanes do this.*
- *The line directrix and two generator lines*; $(q^2 + q)/2$ *hyperplanes do this.*
- *A conic and a generator line*; $q^3 + q^2$ *hyperplanes do this.*
- *A twisted cubic curve* (*which meets the line directrix in a unique point*); $q^4 - q^2$ *hyperplanes do this.*

Corollary 2.2. *Let* Π *be a hyperplane of* PG(4, q) *that meets a ruled cubic surface* $\mathcal{V}$ *in a twisted cubic* $\mathcal{N}$. *Then* $\mathcal{N}$ *meets each generator line of* $\mathcal{V}$ *in a unique point.*

Proof. If $\mathcal{N}$ meets a generator line ℓ of $\mathcal{V}$ in two points, then the 3-space Π containing $\mathcal{N}$ also contains ℓ, which is not possible by Result 2.1. Hence $\mathcal{N}$ meets each generator line in at most one point. As $\mathcal{N}$ contains $q+1$ points, each generator of $\mathcal{V}$ contains a unique point of $\mathcal{N}$. □

There are two ways to extend the ruled cubic surface to PG(4, q^2), we show that they are equivalent. The ruled cubic surface $\mathcal{V}$ is a variety whose points are the exact intersection of three quadrics, $\mathcal{V} = \mathcal{Q}_0 \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$ (see for example [Barwick and Jackson 2012]). So extending this variety to PG(4, q^2) yields $\mathcal{V}^{☆} = \mathcal{Q}_0^{☆} \cap \mathcal{Q}_1^{☆} \cap \mathcal{Q}_2^{☆}$. Alternatively, we can extend $\mathcal{V}$ to PG(4, q^2) as in [Casse and Quinn 2002]: namely extending the line directrix t and conic directrix $\mathcal{C}$ to PG(4, q^2) by taking $\theta, \phi \in \mathbb{F}_{q^2} \cup \{\infty\}$, and extending the projectivity ω to act over $\mathbb{F}_{q^2}$. We denote this extension by $\mathcal{V}'$, thus $\mathcal{V}'$ is the ruled cubic surface with line directrix $t^{☆}$, conic directrix $\mathcal{C}^{☆}$, and ruled using the (extended) projectivity ω. We show that these two extensions $\mathcal{V}^{☆}, \mathcal{V}'$ are the same. The surface $\mathcal{V}$ contains exactly q^2 conics $\mathcal{C}_1, \dots, \mathcal{C}_{q^2}$, and these conics cover each point of $\mathcal{V} \setminus t$ q-times (see [Barwick and Ebert 2008] for more details). Hence both sets $\mathcal{V}^{\star}, \mathcal{V}'$ contain the extended conics $\mathcal{C}_1^{\star}, \dots, \mathcal{C}_{q^2}^{\star}$. Moreover, these conics together with $t^{\star}$ cover all the points of $\mathcal{V}'$, and so $\mathcal{V}^{\star}$ contains $\mathcal{V}'$. However, $\mathcal{V}^{\star}$ is the intersection of three quadrics over $\mathbb{F}_{q^2}$, whose intersection over $\mathbb{F}_q$ is a ruled cubic surface. By [Bernasconi and Vincenti 1981], all ruled cubic surfaces are projectively equivalent, hence $\mathcal{V}^{\star}$ and $\mathcal{V}'$ are the same ruled cubic surface of $\mathrm{PG}(4, q^2)$.

Coordinates in Bruck–Bose. We now show the relation between the coordinates of points in $\mathrm{PG}(2, q^2)$ and the coordinates of the corresponding points in the Bruck–Bose representation of $\mathrm{PG}(4, q)$. See [Barwick and Ebert 2008, Section 3.4] for more details. Let τ be a primitive element in $\mathbb{F}_{q^2}$ with primitive polynomial $x^2 - t_1 x - t_0$ over $\mathbb{F}_q$. Then every element $\alpha \in \mathbb{F}_{q^2}$ can be uniquely written as $\alpha = a_0 + a_1\tau$ with $a_0, a_1 \in \mathbb{F}_q$. That is, $\mathbb{F}_{q^2} = \{x_0 + x_1\tau \mid x_0, x_1 \in \mathbb{F}_q\}$. It is useful to keep in mind the relationships: $\tau\tau^q = -t_0$, $\tau + \tau^q = t_1$, $t_0/\tau = -\tau^q = \tau - t_1$ and $\tau^{q^2} = 1$. Points in $\mathrm{PG}(2, q^2)$ have homogeneous coordinates (x, y, z) with $x, y, z \in \mathbb{F}_{q^2}$, not all zero. We let the line at infinity ℓ_∞ have equation $z = 0$, so affine points of $\mathrm{PG}(2, q^2)$ have coordinates $(x, y, 1)$, with $x, y \in \mathbb{F}_{q^2}$. Points in $\mathrm{PG}(4, q)$ have homogeneous coordinates (x_0, x_1, y_0, y_1, z) with $x_0, x_1, y_0, y_1, z \in \mathbb{F}_q$, not all zero. We let the hyperplane at infinity Σ_∞ have equation $z = 0$, so the affine points of $\mathrm{PG}(4, q)$ have coordinates $(x_0, x_1, y_0, y_1, 1)$, with $x_0, x_1, y_0, y_1 \in \mathbb{F}_q$. Let A be an affine point in $\mathrm{PG}(2, q^2)$ with coordinates $A = (x_0 + x_1\tau, y_0 + y_1\tau, z)$, where $x_0, x_1, y_0, y_1, z \in \mathbb{F}_q$, $z \neq 0$. The map

$$\varphi : \mathrm{PG}(2, q^2) \setminus \ell_\infty \to \mathrm{PG}(4, q) \setminus \Sigma_\infty,$$
$$A = (x_0 + x_1\tau, y_0 + y_1\tau, z) \mapsto [A] = (x_0, x_1, y_0, y_1, z),$$

is a bijection from the affine points of $\mathrm{PG}(2, q^2)$ to the affine points of $\mathrm{PG}(4, q)$, called the *Bruck–Bose map*. We can extend this to a projective map: for a point $\bar{T} = (\delta, 1, 0) \in \ell_\infty$, write $\delta = d_0 + d_1\tau \in \mathbb{F}_{q^2}$, $d_0, d_1 \in \mathbb{F}_q$; then

$$\bar{T} = (\delta, 1, 0) \mapsto [T] = \langle (d_0, d_1, 1, 0, 0), (t_0 d_1, d_0 + t_1 d_1, 0, 1, 0) \rangle.$$

The transversal lines g, g^q of $\mathcal{S}$ have coordinates given by

$$g = \langle A_0 = (\tau^q, -1, 0, 0, 0), A_1 = (0, 0, \tau^q, -1, 0) \rangle,$$
$$g^q = \langle A_0^q = (\tau, -1, 0, 0, 0), A_1^q = (0, 0, \tau, -1, 0) \rangle.$$

Recall that each line of the regular spread $\mathcal{S}$ meets the transversal g of $\mathcal{S}$ in a point. The one-to-one correspondence between points of ℓ_∞ and points of g is

$$\bar{T} = (\delta, 1, 0) \in \ell_\infty \leftrightarrow T = \delta A_0 + A_1 \in g, \quad \delta \in \mathbb{F}_{q^2} \cup \{\infty\},$$

that is, $T = [T]^{\star} \cap g$ and $[T]^{\star} = TT^q$.

Coordinates and the quartic extension $\mathrm{PG}(4, q^4)$. We will be interested in nondegenerate conics of $\mathrm{PG}(2, q^2)$, and one of the cases to consider is when a conic $\mathcal{C}$ is exterior to ℓ_∞, and so meets ℓ_∞ in two points which lie in the quadratic extension of $\mathrm{PG}(2, q^2)$ to $\mathrm{PG}(2, q^4)$. That is, $\mathcal{C}$ meets ℓ_∞ in two points $\bar{P}$, $\bar{Q}$ over $\mathbb{F}_{q^4}$. Note that $\bar{P}$, $\bar{Q}$ are conjugate with respect to the map $x \mapsto x^{q^2}$, $x \in \mathbb{F}_{q^4}$, that is $\bar{Q} = \bar{P}^{q^2}$. There is no direct representation for the point $\bar{P}$ in the Bruck–Bose representation in $\mathrm{PG}(4, q)$. However, there is a related point in the quartic extension $\mathrm{PG}(4, q^4)$. We can extend the one-to-one correspondence between points ℓ_∞ and points of g to a one-to-one correspondence between points of the quadratic extension of ℓ_∞ and points of the extended transversal $g^\star$ in $\mathrm{PG}(4, q^4)$, so

$$\bar{P} = (\alpha, 1, 0) \leftrightarrow P = \alpha A_0 + A_1 \in g^\star, \quad \alpha \in \mathbb{F}_{q^4} \cup \{\infty\}.$$

If $\bar{P} = (\alpha, 1, 0)$ for some $\alpha \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, that is $\bar{P} \in \mathrm{PG}(2, q^4) \setminus \mathrm{PG}(2, q^2)$, then in $\mathrm{PG}(4, q^4)$ the corresponding point P lies in $g^\star \setminus g$, and the conjugate point $P^q = \alpha^q A_0^q + A_1^q$ lies on $g^{q\star} \setminus g^q$. As $\bar{P} \notin \mathrm{PG}(2, q^2)$, the line PP^q is not a line of the spread $\mathcal{S}$; PP^q is a line of $\mathrm{PG}(4, q^4)$ that does not meet Σ_∞.

g-special sets. When studying curves of $\mathrm{PG}(2, q^2)$ in the $\mathrm{PG}(4, q)$ Bruck–Bose setting, the transversals g, g^q of the regular spread $\mathcal{S}$ play an important role in characterizations. Let $\mathcal{V}$ be a variety or rational curve in $\mathrm{PG}(4, q)$, we are interested in how $\mathcal{V}^{\star}$ meets g, g^q in the extension to $\mathrm{PG}(4, q^2)$. Note that if $\mathcal{V}^{\star}$ meets g in a point P, then as $\mathcal{V}$ is defined over $\mathbb{F}_q$, $\mathcal{V}^{\star}$ also meets g^q in the point P^q. A nondegenerate conic $\mathcal{C}$ in $\mathrm{PG}(4, q)$ is called a *g-special conic* if in $\mathrm{PG}(4, q^2)$, $\mathcal{C}^{\star}$ contains one point of g, and one point of g^q. A twisted cubic $\mathcal{N}$ in $\mathrm{PG}(4, q)$ is called a *g-special twisted cubic* if in $\mathrm{PG}(4, q^2)$, $\mathcal{N}^{\star}$ contains one point of g, and one point of g^q. A 4-dimensional normal rational curve $\mathcal{N}$ in $\mathrm{PG}(4, q)$ is called a *g-special normal rational curve* if in $\mathrm{PG}(4, q^2)$, $\mathcal{N}^{\star}$ contains two points of g (possibly repeated) and two points of g^q. Further, $\mathcal{N}$ is called $g^\star$*-special* if in the quartic extension $\mathrm{PG}(4, q^4)$, $\mathcal{N}^\star$ contains two points of the extended transversal $g^\star \setminus g$. A ruled cubic surface $\mathcal{V}$ in $\mathrm{PG}(4, q)$ is called a *g-special ruled cubic surface* if in $\mathrm{PG}(4, q^2)$, $\mathcal{V}^{\star}$ contains g and g^q.

Representations of Baer sublines and subplanes. We use the following representations of Baer sublines and subplanes of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(4, q)$, see [Barwick and Ebert 2008] for more details.

Result 2.3. *Let $\mathcal{S}$ be a regular spread in a 3-space Σ_∞ in* $\mathrm{PG}(4, q)$ *and consider the representation of the Desarguesian plane $\mathcal{P}(\mathcal{S}) = \mathrm{PG}(2, q^2)$ defined by the Bruck–Bose construction.*

1. *A Baer subline of ℓ_∞ in* $\mathrm{PG}(2, q^2)$ *corresponds to a regulus of $\mathcal{S}$.*

2. *A Baer subline of* $\mathrm{PG}(2, q^2)$ *that meets* ℓ_∞ *in a point corresponds to a line of* $\mathrm{PG}(4, q) \setminus \Sigma_\infty$.
3. *A Baer subplane of* $\mathrm{PG}(2, q^2)$ *secant to* ℓ_∞ *corresponds to a plane, not containing a spread line, of* $\mathrm{PG}(4, q) \setminus \Sigma_\infty$.
4. *A Baer subline of* $\mathrm{PG}(2, q^2)$ *that is disjoint from* ℓ_∞ *corresponds in* $\mathrm{PG}(4, q)$ *to a* g*-special conic.*
5. *A Baer subplane tangent to* ℓ_∞ *at a point* $\bar{T}$ *corresponds in* $\mathrm{PG}(4, q)$ *to a* g*-special ruled cubic surface containing the corresponding spread line* $[T]$.

Moreover, the converse of each of these correspondences holds.

Remark 2.4. The correspondences in parts 2 and 3 are not exact at infinity. The exact at infinity representation of a Baer subline that meets ℓ_∞ in a point T is an affine line that meets the spread line $[T]$ *union* with the spread line $[T]$. Similarly, the exact at infinity representation of a secant Baer subplane is a plane α not containing a spread line, *union* the lines of $\mathcal{S}$ that α meets.

Representations of subconics. The representation of nondegenerate conics contained in a Baer subplane was considered in [Quinn 2002].

Result 2.5 [Quinn 2002]. *Let* $\mathcal{C}$ *be a nondegenerate conic contained in a Baer subplane* $\mathcal{B}$ *of* $\mathrm{PG}(2, q^2)$.

1. *Suppose* $\mathcal{B}$ *is secant to* ℓ_∞. *Then* $\mathcal{C}$ *corresponds to a nondegenerate conic in the plane* $[\mathcal{B}]$ *of* $\mathrm{PG}(4, q)$.
2. *Suppose* $\mathcal{B}$ *is tangent to* ℓ_∞, $\mathcal{B} \cap \ell_\infty \in \mathcal{C}$, *and* $q \geq 3$. *Then* $\mathcal{C}$ *corresponds to a twisted cubic on the ruled cubic surface* $[\mathcal{B}]$ *of* $\mathrm{PG}(4, q)$.
3. *Suppose* $\mathcal{B}$ *is tangent to* ℓ_∞, $\mathcal{B} \cap \ell_\infty \notin \mathcal{C}$, *and* $q \geq 4$. *Then* $\mathcal{C}$ *corresponds to a* 4*-dimensional normal rational curve on the ruled cubic surface* $[\mathcal{B}]$ *of* $\mathrm{PG}(4, q)$.

In Section 5, we show that the 3- and 4-dimensional normal rational curves of Result 2.5 are g-special. Conversely, we show that every g-special normal rational curve in $\mathrm{PG}(4, q)$ corresponds to a nondegenerate conic contained in a tangent Baer subplane.

Remark 2.6. The correspondence in Result 2.5 parts 1 and 2 is not exact at infinity (compare with Remark 2.4). For example, in part 2, the point $\bar{T} = \mathcal{B} \cap \ell_\infty$ is in $[\mathcal{C}]$, and the twisted cubic $[\mathcal{C}]$ meets Σ_∞ in a point of $[T]$. The exact-at-infinity representation is: the set $[\mathcal{C}]$ is a twisted cubic *union* the spread line $[T]$. We use the simpler, not exact-at-infinity correspondence given in Result 2.5 as it does not lead to any confusion.

***The circle geometry* CG(2, *q*).** Circle geometries $\mathrm{CG}(d, q)$, $d \geq 2$, were introduced in [Bruck 1973a; 1973b], and we summarize the results we need here. Note that $\mathrm{CG}(2, q)$ is an inversive plane. We can construct $\mathrm{CG}(2, q)$ from the line $\mathrm{PG}(1, q^2)$, in this case the circles are the Baer sublines of $\mathrm{PG}(1, q^2)$. Equivalently, we can construct $\mathrm{CG}(2, q)$ from the lines of a regular spread $\mathcal{S}$ of $\mathrm{PG}(3, q)$, in this case the circles are the reguli contained in $\mathcal{S}$. Using the representation of $\mathrm{CG}(2, q)$ as $\ell_\infty \cong \mathrm{PG}(1, q^2)$, we can use properties of the circle geometry to deduce several properties of the projective plane $\mathrm{PG}(2, q^2)$. If $\bar{P}$, $\bar{Q}$ are two distinct points on ℓ_∞ in $\mathrm{PG}(2, q^2)$, then there is a unique partition of ℓ_∞ into $\bar{P}$, $\bar{Q}$ and $q - 1$ Baer sublines $\ell_1, \ldots, \ell_{q-1}$, where the points $\bar{P}$, $\bar{Q}$ are conjugate with respect to each Baer subline ℓ_i. Further, if $\mathcal{B}$ is a Baer subplane secant to ℓ_∞, such that $\bar{P}$, $\bar{Q}$ are conjugate with respect to $\mathcal{B}$, then $\mathcal{B}$ meets ℓ_∞ in one of the Baer sublines ℓ_i. Of particular interest is an application to conics.

Result 2.7. *Let $\mathcal{O}$ be a nondegenerate conic of* $\mathrm{PG}(2, q^2)$ *that meets ℓ_∞ in $\{\bar{P}, \bar{Q}\}$. Then there is a unique partition of the $q^2 - 1$ affine points of $\mathcal{O}$ into $q - 1$ subconics $\mathcal{C}_1, \ldots, \mathcal{C}_{q-1}$, lying in Baer subplanes $\mathcal{B}_1, \ldots, \mathcal{B}_{q-1}$ which are secant to ℓ_∞. Further, the Baer sublines $\mathcal{B}_i \cap \ell_\infty$ are either equal or disjoint*

The properties of the circle geometry also lead to properties of a regular spread $\mathcal{S}$ in $\mathrm{PG}(3, q)$. Let g, g^q be the transversals of $\mathcal{S}$, so g and g^q lie in $\mathrm{PG}(3, q^2)$. Consider the set of lines of $\mathrm{PG}(3, q^2)$ that meet both g and g^q. This set is called the *hyperbolic congruence* of g and g^q in [Hirschfeld 1985]. Note that if two distinct lines in the hyperbolic congruence meet, then they meet on g or g^q. The hyperbolic congruence contains the extended spread lines $[P]^\star = PP^q$ for $P \in g$ and the lines PQ^q for distinct $P, Q \in g$. The lines PQ^q have an interesting relationship with the regular spread $\mathcal{S}$.

Result 2.8 [Bruck 1973b]. *Let $[P], [Q]$ be two lines of a regular spread $\mathcal{S}$ in* $\mathrm{PG}(3, q)$, *and denote their intersections with the transversal g of $\mathcal{S}$ by P, Q. There is a unique partition of $\mathcal{S}$ into $[P], [Q]$ and $q - 1$ reguli $\mathcal{R}_1, \ldots, \mathcal{R}_{q-1}$. Denote the opposite regulus of $\mathcal{R}_i$ by $\mathcal{R}'_i$. Then the set $\{[P], [Q], \mathcal{R}'_1, \ldots, \mathcal{R}'_{q-1}\}$ is a regular spread with transversals PQ^q, P^qQ.*

We will show that the lines in the hyperbolic congruence of g, g^q are related to the Bruck–Bose representation of nondegenerate conics of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(4, q)$.

Normal rational curves contained in quadrics. Next, we show that if a normal rational curve is contained in a quadric in $\mathrm{PG}(4, q)$, then the containment also holds in the quadratic extension $\mathrm{PG}(4, q^2)$, provided q is not small.

Lemma 2.9. *In* $\mathrm{PG}(4, q)$, $q > 7$, *let $\mathcal{N}$ be a 4-dimensional normal rational curve and $\mathcal{Q}$ a quadric, with $\mathcal{N} \subset \mathcal{Q}$. Then in the quadratic extension* $\mathrm{PG}(4, q^2)$, $\mathcal{N}^\star \subset \mathcal{Q}^\star$.

Proof. Without loss of generality, let $\mathcal{N} = \{P_\theta = (1, \theta, \theta^2, \theta^3, \theta^4) \mid \theta \in \mathbb{F}_q \cup \{\infty\}\}$. Let $\mathcal{Q}$ have equation $g(x_0, x_1, x_2, x_3, x_4) = 0$. Consider $g(P_\theta) = g(1, \theta, \theta^2, \theta^3, \theta^4) = f(\theta)$. As $\mathcal{Q}$ is a quadric, $f(\theta)$ is a polynomial in θ of degree at most 8. Now as $\mathcal{N} \subset \mathcal{Q}$, $f(P_\theta) = 0$ for all $\theta \in \mathbb{F}_q \cup \{\infty\}$. So if $q + 1 > 8$, f is identically 0, and so $f(P_\theta) = 0$ for all $\theta \in \mathbb{F}_{q^2}$. Using $\theta = \infty$, this implies that the coefficient of θ^8 is zero, thus the degree of f is at most 7. As $f(\theta) = 0$ for the q values of $\theta \in \mathbb{F}_q$, it follows that f has q roots, so if $q > 7$ then f is the zero polynomial, thus $f(\theta) = 0$ for all θ in any extension of $\mathbb{F}_q$, and so $g(P_\theta) = 0$ for all θ in any extension of $\mathbb{F}_q$. So if $q > 7$, the point P_θ, $\theta \in \mathbb{F}_{q^2} \cup \{\infty\}$, lies on $\mathcal{Q}^\star$, and so $\mathcal{N}^\star \subset \mathcal{Q}^\star$. □

The bound on q in Lemma 2.9 is tight as shown by the following example. In PG(4, 7), let $\mathcal{N}$ be the normal rational curve $\mathcal{N} = \{P_\theta = (1, \theta, \theta^2, \theta^3, \theta^4) \mid \theta \in \mathrm{GF}(7) \cup \{\infty\}\}$ and let $\mathcal{Q}$ be the quadric with equation $f(x_0, x_1, x_2, x_3, x_4) = -x_0x_1 - x_3^2 + x_2x_4 + x_3x_4$. First note that $f(P_\theta) = \theta^7 - \theta = 0$ for all $\theta \in \mathrm{GF}(7)$. Further, $P_\infty = (0, 0, 0, 0, 1)$, so $f(P_\infty) = 0$. Hence $\mathcal{N} \subset \mathcal{Q}$ in PG(4, 7). Now extend GF(7) to GF(7^2) using a primitive element τ. The point $P_\tau = (1, \tau, \tau^2, \tau^3, \tau^4)$ lies in the extended curve $\mathcal{N}^\star$. However $f(P_\tau) = \tau^7 - \tau \neq 0$ as $\tau \notin \mathrm{GF}(7)$, and so P_τ does not lie on the extended quadric $\mathcal{Q}^\star$, that is $\mathcal{N}^\star \not\subset \mathcal{Q}^\star$.

Baer pencils and partitions of Baer subplanes. In this section we investigate the representation in PG($2, q^2$) of a 3-space of PG($4, q$). We use this to partition tangent Baer subplanes into conics.

Definition 2.10. A *Baer pencil* $\mathcal{K}$ in PG($2, q^2$) is the cone of $q + 1$ lines joining a vertex point P to a Baer subline base b. An ℓ_∞*-Baer pencil* $\mathcal{K}$ is a Baer pencil with vertex in ℓ_∞ and base b meeting ℓ_∞ in a point.

Let $\mathcal{K}$ be a Baer pencil; then every line of PG($2, q^2$) not through the vertex of $\mathcal{K}$ meets $\mathcal{K}$ in a Baer subline. Also note that an ℓ_∞-Baer pencil $\mathcal{K}$ contains ℓ_∞ and a further q^3 affine points. It is straightforward to characterize the ℓ_∞-Baer pencils of PG($2, q^2$) in PG($4, q$).

Lemma 2.11. *Let* Π *be a* 3*-space in* PG($4, q$) *distinct from* Σ_∞*. Then* Π *corresponds in* PG($2, q^2$) *to an* ℓ_∞*-Baer pencil with vertex corresponding to the unique spread line in* Π*. Conversely, any* ℓ_∞*-Baer pencil in* PG($2, q^2$) *corresponds to a* 3*-space of* PG($4, q$).

We look at how ℓ_∞-Baer pencils meet a tangent Baer subplane.

Theorem 2.12. *Let* $\mathcal{B}$ *be a Baer subplane in* PG($2, q^2$) *tangent to* ℓ_∞ *at the point* $\bar{T} = \mathcal{B} \cap \ell_\infty$*. An* ℓ_∞*-Baer pencil with vertex* $\bar{P} \neq \bar{T}$ *meets* $\mathcal{B}$ *in either a nondegenerate conic through* $\bar{T}$ *or in two lines, namely the unique line of* $\mathcal{B}$ *whose extension contains* $\bar{P}$*, and one line through* $\bar{T}$*. Of the* ℓ_∞*-Baer pencils with vertex* $\bar{P}$*, there are* $q^2 - 1$ *of the first type, and* $q + 1$ *of the second type (each containing one of the* $q + 1$ *lines of* $\mathcal{B}$ *through* $\bar{T}$*).*

Proof. In PG(4, q), let X be a point on the spread line $[T]$ and let $\alpha = \langle X, [P]\rangle$. Label the 3-spaces of PG(4, q) (not equal to Σ_∞) that contain the plane α by $\mathcal{L} = \{\Pi_1, \ldots, \Pi_q\}$. By Lemma 2.11, each 3-space in $\mathcal{L}$ corresponds to an ℓ_∞-Baer pencil of PG(2, q^2) with vertex P. Result 2.1 describes how a 3-space meets the ruled cubic surface $[\mathcal{B}]$. As each 3-space in $\mathcal{L}$ meets $[T]$ in one point, and the 3-spaces in $\mathcal{L}$ partition the affine points, we deduce that one of the 3-spaces in $\mathcal{L}$, Π_1 say, meets $[\mathcal{B}]$ in a conic and the generator line of $[\mathcal{B}]$ through X, and the remaining 3-spaces in $\mathcal{L}$ meet $[\mathcal{B}]$ in a twisted cubic $\mathcal{N}_i = \Pi_i \cap [\mathcal{B}]$, $i = 2, \ldots, q$. By Result 2.5, the twisted cubics $\mathcal{N}_i$ each correspond in PG(2, q^2) to nondegenerate conics in $\mathcal{B}$ that contains $\bar{T}$. Note that there is a unique plane of PG(4, q) $\setminus \Sigma_\infty$ that contains the spread line $[P]$ and meets $[\mathcal{B}]$ in a conic; namely the plane that corresponds in PG(2, q^2) to the unique line m_P through $\bar{P}$ that meets $\mathcal{B}$ in a Baer subline. Hence $\Pi_1 \cap [\mathcal{B}]$ contains the generator line $[m]$ of $[\mathcal{B}]$ through the point X and a conic in the plane $[m_P]$. This corresponds in PG(2, q^2) to an ℓ_∞-Baer pencil with vertex $\bar{P}$ that meets $[\mathcal{B}]$ in the two Baer sublines $m_P \cap \mathcal{B}$ and m.

As there are $q+1$ choices for the point X on $[T]$, there are $(q+1)(q-1)$ 3-spaces about $[P]$ that meets $[\mathcal{B}]$ in a twisted cubic, and $q+1$ that meet $[\mathcal{B}]$ in a line and a conic, giving the required number of Baer pencils. □

The next result shows that a nondegenerate conic in $\mathcal{B}$ lies in a unique ℓ_∞-Baer pencil, and describes the relationship between the conic and the vertex of the pencil.

Theorem 2.13. *Let $\mathcal{B}$ be a Baer subplane in* PG(2, q^2) *tangent to ℓ_∞ at the point $\bar{T} = \mathcal{B} \cap \ell_\infty$ and let $\mathcal{C}$ be a nondegenerate conic in $\mathcal{B}$ with $\bar{T} \in \mathcal{C}$. Then $\mathcal{C}$ lies in a unique ℓ_∞-Baer pencil $\mathcal{K}$. Moreover, the vertex of $\mathcal{K}$ lies in the extension of $\mathcal{C}$ to* PG(2, q^2).

Proof. Let $\mathcal{C}$ be a nondegenerate conic in $\mathcal{B}$, with $\bar{T} = \mathcal{B} \cap \ell_\infty \in \mathcal{C}$. As ℓ_∞ is not a line of $\mathcal{B}$, it is not the tangent line of $\mathcal{C}$ at the point $\bar{T}$. Let $\mathcal{C}^+$ be the extension of $\mathcal{C}$ to PG(2, q^2); then ℓ_∞ is a secant to $\mathcal{C}^+$, so $\mathcal{C}^+ \cap \ell_\infty = \{\bar{T}, \bar{L}\}$. We will show that $\mathcal{C}$ lies in a unique ℓ_∞-Baer pencil $\mathcal{K}$ which has vertex $\bar{L}$.

We first show that any point $X \in \mathcal{C}^+$ projects $\mathcal{C}$ onto a Baer subline. Without loss of generality, let $\mathcal{C} = \{P_\theta = (1, \theta, \theta^2) \mid \theta \in \mathbb{F}_{q^2} \cup \{\infty\}\}$, so $\mathcal{C}^+ = \{(1, \theta, \theta^2) \mid \theta \in \mathbb{F}_q \cup \{\infty\}\}$. Let $\omega \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, so the point $X = (1, \omega, \omega^2)$ lies in $\mathcal{C}^+ \setminus \mathcal{C}$. The projection of the point P_θ, $\theta \in \mathbb{F}_q \cup \{\infty\}$ from X onto the line ℓ with equation $x = 0$ is $P'_\theta = (0, 1, \theta + \omega)$. That is, the projection of $\mathcal{C}$ from X onto ℓ is the set $\{(0, 1, \omega) + \theta(0, 0, 1) \mid \theta \in \mathbb{F}_q \cup \{\infty\}\}$, which is a Baer subline.

We next show that $\mathcal{C}$ lies in a unique ℓ_∞-Baer pencil. By Result 2.5, in PG(4, q), $[\mathcal{C}]$ is a twisted cubic meeting the spread line $[T]$ in one point and $[\mathcal{C}]$ lies in a 3-space Π that meets $[T]$ in exactly one point. Hence Π contains a unique spread line $[P]$, with $\bar{P} \neq \bar{T}$. By Lemma 2.11, Π corresponds to an ℓ_∞-Baer pencil $\mathcal{K}$

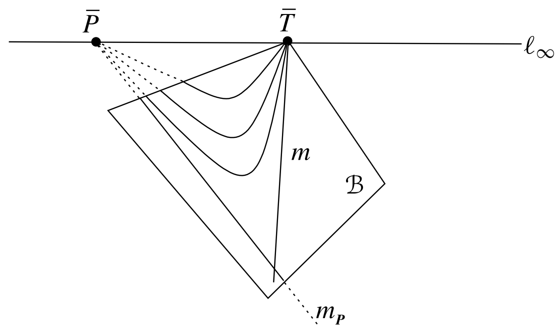

Figure 1. A partition of $\mathcal{B} \setminus \bar{T}$ into q conics through $\bar{T}$.

with vertex $\bar{P}$, so $\mathcal{C}$ lies in the pencil $\mathcal{K}$. If $\mathcal{C}$ were in two ℓ_∞-Baer pencils $\mathcal{K}, \mathcal{K}'$, then $[\mathcal{C}]$ would lie in two 3-spaces $\Pi_{\mathcal{K}}, \Pi_{\mathcal{K}'}$, which is not possible.

Hence $\mathcal{C}$ lies in a unique ℓ_∞-Baer pencil $\mathcal{K}$ with some vertex $\bar{P} \in \ell_\infty$. Further, as argued above, the point $\bar{L} \in \mathcal{C}^+ \cap \ell_\infty$ projects $\mathcal{C}$ onto a Baer subline, and so $\mathcal{C}$ lies in an ℓ_∞-Baer pencil with vertex $\bar{L}$. Thus $\bar{P} = \bar{L}$ as required. □

The ℓ_∞-Baer pencils give rise to partitions of the affine points of a tangent Baer subplane into q conics: one degenerate and $q-1$ nondegenerate.

Corollary 2.14. *Let $\mathcal{B}$ be a Baer subplane in* $\mathrm{PG}(2, q^2)$ *tangent to ℓ_∞ at the point $\bar{T} = \mathcal{B} \cap \ell_\infty$. For each line m of $\mathcal{B}$ through $\bar{T}$ and point $\bar{P} \in \ell_\infty$, $\bar{P} \neq \bar{T}$, there is a set of q ℓ_∞-Baer pencils with vertex $\bar{P}$ that partition the affine points of* $\mathrm{PG}(2, q^2)$ *and partition the affine points of $\mathcal{B}$ into q conics through $\bar{T}$, one being degenerate. Moreover, the extension of each of these conics to* $\mathrm{PG}(2, q^2)$ *contains the point $\bar{P}$* (see Figure 1).

Proof. The proof of Theorem 2.12 gives a construction for these partitions. The line m corresponds in $\mathrm{PG}(4, q)$ to a line $[m]$ that meets the spread line $[T]$ in a point X. Let $\mathcal{L}$ be the set of q 3-spaces of $\mathrm{PG}(4, q) \setminus \Sigma_\infty$ containing the plane $\alpha = \langle X, [P] \rangle$. These 3-spaces partition the affine points of $\mathrm{PG}(4, q)$ and hence partition the affine points of $[\mathcal{B}]$. As argued in the proof of Theorem 2.12, one of the 3-spaces in $\mathcal{L}$ gives rise in $\mathrm{PG}(2, q^2)$ to two lines in $\mathcal{B}$, and the remaining $q-1$ give rise to nondegenerate conics of $\mathcal{B}$ containing $\bar{T}$. By Theorem 2.13, the extension of these conics to $\mathrm{PG}(2, q^2)$ contains the point $\bar{P}$. □

3. Specialness and Baer sublines and subplanes

Parts 4 and 5 of Result 2.3 illustrate that the concept of g-specialness is important in the Bruck–Bose representation of Baer substructures. In this section we discuss how parts 1 and 3 of Result 2.3 relate to the notion of specialness.

Let b be a Baer subline of ℓ_∞; then by Result 2.3(1), in $\mathrm{PG}(4, q)$, $[b]$ is a regulus contained in the regular spread $\mathcal{S}$. Hence in $\mathrm{PG}(4, q^2)$, the transversals g, g^q of $\mathcal{S}$ are lines of the regulus opposite to $[b]^{\star}$. That is, the regulus $[b]$ is closely related to the transversals of $\mathcal{S}$. There is another way to express this relationship.

Theorem 3.1. 1. *Let b be a Baer subline of ℓ_∞ in* $\mathrm{PG}(2, q^2)$. *Then in the Bruck–Bose representation in* $\mathrm{PG}(4, q)$, *each nondegenerate conic contained in the regulus* $[b]$ *is a g-special conic.*

2. *Conversely, every g-special conic in Σ_∞ lies in a unique regulus of $\mathcal{S}$, and so corresponds to a Baer subline of ℓ_∞.*

Proof. Let b be a Baer subline of ℓ_∞ in $\mathrm{PG}(2, q^2)$. By Result 2.3(1), in $\mathrm{PG}(4, q)$, $[b]$ is a regulus contained in the regular spread $\mathcal{S}$. There are $q^3 - q$ planes of Σ_∞ that meet the regulus $[b]$ is a nondegenerate conic, namely the planes that do not contain a line of $[b]$. Let α be such a plane, so α contains a unique spread line $[L]$, and $\mathcal{C} = [b] \cap \alpha$ is a nondegenerate conic. In $\mathrm{PG}(4, q^2)$, the transversal g meets each extended spread line, and so g meets at least three lines of the extended regulus $[b]^{\star}$, hence g is a line of the opposite regulus. In particular, each point of g lies on one line of $[b]^{\star}$. Now $\mathcal{C}^{\star}$ is the exact intersection $[b]^{\star} \cap \alpha^{\star}$, and $\alpha^{\star}$ meets g in one point, hence $\mathcal{C}^{\star}$ contains the points $g \cap \alpha^{\star}$, $g^q \cap \alpha^{\star}$, and so $\mathcal{C}$ is a g-special conic.

Conversely, let $\mathcal{C}$ be a g-special conic in Σ_∞. So $\mathcal{C}$ lies in a plane α; moreover, α contains a spread line $[L]$, and in $\mathrm{PG}(4, q^2)$, $\mathcal{C}^{\star}$ contains the points $X = g \cap [L]^{\star}$ and $X^q = g^q \cap [L]^{\star}$. Let $\mathcal{K}$ be the set of lines of $\mathcal{S}$ that meet $\mathcal{C}$, we need to show that $\mathcal{K}$ is a regulus. Let $[P_1]$, $[P_2]$, $[P_3]$ be three lines of $\mathcal{K}$ and let $\mathcal{R}$ be the unique regulus containing the three lines. By the argument above, $\mathcal{D} = \mathcal{R} \cap \alpha$ is a g-special conic, and $\mathcal{D}^{\star}$ contains the points X and X^q. So $\mathcal{C}^{\star}$ and $\mathcal{D}^{\star}$ have five points in common, namely X, X^q, $[P_i] \cap \alpha$, $i = 1, 2, 3$. Hence $\mathcal{C}^{\star} = \mathcal{D}^{\star}$ and so $\mathcal{K} = \mathcal{R}$. That is, the points of $\mathcal{C}$ lie on lines of a regulus of $\mathcal{S}$, which by Result 2.3 corresponds to a Baer subline of ℓ_∞ in $\mathrm{PG}(2, q^2)$. □

Furthermore, the regulus $[b]$ has a relationship to the lines in the hyperbolic congruence of g, g^q.

Theorem 3.2. *Let b be a Baer subline of ℓ_∞, and let $\bar{P}, \bar{Q} \in \ell_\infty$ be conjugate with respect to b. Then in* $\mathrm{PG}(4, q^2)$, *the lines PQ^q, P^qQ are lines of the regulus* $[b]^{\star}$.

Proof. Let $\bar{P}$, $\bar{Q}$ be two points of ℓ_∞ that are conjugate with respect to a Baer subline $b \subset \ell_\infty$. By Result 2.3, in $\mathrm{PG}(4, q)$, $[b]$ is a regulus of $\mathcal{S}$. By Result 2.8, the unique partition of $\mathcal{S} \setminus \{[P], [Q]\}$ into reguli contains the regulus $[b]$; and in $\mathrm{PG}(4, q^2)$, the lines PQ^q, P^qQ meet each line of the regulus opposite to $[b]^{\star}$. Hence the lines PQ^q, P^qQ are lines of the regulus $[b]^{\star}$. □

Remark 3.3. Given a Baer subline b of ℓ_∞, the points of $\ell_\infty \setminus \{b\}$ can be partitioned into pairs of points $\{\bar{P}_i, \bar{Q}_i\}$ which are conjugate with respect to b. Hence the $q^2 - q$

lines $P_i Q_i^q$ and $(P_i Q_i^q)^q$ are exactly the lines of $\mathrm{PG}(4, q^2)$ in the regulus $[b]^{\star}$ that are not lines of $\mathrm{PG}(4, q)$.

We now consider a Baer subplane $\mathcal{B}$ of $\mathrm{PG}(2, q^2)$ secant to ℓ_∞. By Result 2.3, $[\mathcal{B}]$ is a plane of $\mathrm{PG}(4, q)$, and the line $[\mathcal{B}] \cap \Sigma_\infty$ meets $q+1$ lines of $\mathcal{S}$ which form a regulus denoted by $\mathcal{R}$. As noted above, in $\mathrm{PG}(4, q^2)$, the transversals g, g^q are lines of the regulus opposite to $\mathcal{R}$. Moreover, by Theorem 3.2 the extended regulus $\mathcal{R}^{\star}$ contains the line PQ^q where the corresponding points $\bar{P}, \bar{Q} \in \ell_\infty$ are conjugate with respect to $\mathcal{B}$.

Corollary 3.4. *Let $\mathcal{B}$ be a Baer subplane of* $\mathrm{PG}(2, q^2)$ *that is secant to ℓ_∞, and let $\bar{P}, \bar{Q} \in \ell_\infty$ be conjugate with respect to $\mathcal{B}$. Then in* $\mathrm{PG}(4, q^2)$, *the lines PQ^q, $P^q Q$ meet the plane* $[\mathcal{B}]^{\star}$.

4. Conics of $\mathrm{PG}(2, q^2)$

In [Barwick et al. 2011], it is shown that a nondegenerate conic $\mathcal{O}$ in $\mathrm{PG}(2, q^2)$ corresponds in $\mathrm{PG}(4, q)$ to the intersection of two quadrics. Moreover, this correspondence is exact-at-infinity: that is, an affine point $A \in \mathrm{PG}(2, q^2) \setminus \ell_\infty$ lies in $\mathcal{O}$ if and only if the affine point $[A] \in \mathrm{PG}(4, q) \setminus \Sigma_\infty$ lies in $[\mathcal{O}] = \mathcal{Q}_1 \cap \mathcal{Q}_2$ and a point $\bar{T} \in \ell_\infty$ lies in $\mathcal{O}$ if and only if the spread line $[T]$ is contained in $[\mathcal{O}] = \mathcal{Q}_1 \cap \mathcal{Q}_2$. So the set $[\mathcal{O}] = \mathcal{Q}_1 \cap \mathcal{Q}_2$ meets Σ_∞ either in the empty set, or in 1 or 2 spread lines. We determine the relationship of $[\mathcal{O}]$ with the transversals g, g^q of the regular spread $\mathcal{S}$.

The arguments used are coordinate-based. A conic $\mathcal{O}$ has equation $f(x, y, z) = 0$ where f is a homogeneous equation of degree two over $\mathbb{F}_{q^2}$. Using the Bruck–Bose map, this can be written as $f_\infty(x_0, x_1, y_0, y_1, z) + \tau f_0(x_0, x_1, y_0, y_1, z) = 0$, where $f_\infty = 0$ and $f_0 = 0$ are homogeneous quadratic equations over $\mathbb{F}_q$, which is to say equations of quadrics $\mathcal{Q}_\infty$, $\mathcal{Q}_0$ in $\mathrm{PG}(4, q)$; hence $[\mathcal{O}] = \mathcal{Q}_\infty \cap \mathcal{Q}_0$. Moreover, $[\mathcal{O}]$ is contained in the pencil of quadrics $\{\mathcal{Q}_t = t\mathcal{Q}_\infty + \mathcal{Q}_0, t \in \mathbb{F}_q \cup \{\infty\}\}$ where $\mathcal{Q}_t$ has equation $f_t = tf_\infty + f_0 = 0$. There is a natural extension to $\mathrm{PG}(4, q^2)$ and to $\mathrm{PG}(4, q^4)$, namely $[\mathcal{O}]^{\star} = \mathcal{Q}_\infty^{\star} \cap \mathcal{Q}_0^{\star}$ and $[\mathcal{O}]^{\bigstar} = \mathcal{Q}_\infty^{\bigstar} \cap \mathcal{Q}_0^{\bigstar}$. In order to study subconics in Baer subplanes, we will need a full analysis of how these sets meet the hyperplane at infinity, which we give in this section. We first show that none of the quadrics $\mathcal{Q}_t^{\star}$, $t \in \mathbb{F}_q \cup \{\infty\}$, contain g, and so $[\mathcal{O}]^{\star}$ does not contain g.

Theorem 4.1. *Let $\mathcal{O}$ be a nondegenerate conic in* $\mathrm{PG}(2, q^2)$, *so* $[\mathcal{O}] = \mathcal{Q}_\infty \cap \mathcal{Q}_0$. *In* $\mathrm{PG}(4, q^2)$, *the quadric* $\mathcal{Q}_t^{\star} = t\mathcal{Q}_\infty^{\star} + \mathcal{Q}_0^{\star}$, $t \in \mathbb{F}_q \cup \{\infty\}$, *meets g in* 0, 1 *or* 2 *points, according to whether $\mathcal{O}$ meets ℓ_∞ in* 0, 1 *or* 2 *points respectively.*

Proof. Consider first the case when $\mathcal{O}$ is tangent to ℓ_∞. The group $\mathrm{PGL}(3, q^2)$ is transitive on nondegenerate conics, and the subgroup fixing a nondegenerate conic $\mathcal{O}$ is transitive on the tangent lines of $\mathcal{O}$. Hence we can without loss of generality, prove the result for the conic $\mathcal{O}$ of equation $y^2 = xz$ in $\mathrm{PG}(2, q^2)$, which meets ℓ_∞ in one (repeated) point $\bar{T}=(1,0,0)$. The affine point $(x,y,1)=(x_0+x_1\tau, y_0+y_1\tau, 1)$ is on $\mathcal{O}$ if $(y_0+y_1\tau)^2=x_0+x_1\tau$, that is $(y_0^2+y_1^2t_0-x_0)+(y_1^2t_1+2y_0y_1-x_1)\tau=0$. The solutions $(x_0,x_1,y_0,y_1,1)\in \mathrm{PG}(4,q)$ to this are the affine points in $[\mathcal{O}]$. That is, $[\mathcal{O}]$ is the intersection of the two quadrics $\mathcal{Q}_\infty$, $\mathcal{Q}_0$ with homogeneous equations $f_\infty=0$, $f_0=0$ respectively, where

$$f_\infty = y_0^2+y_1^2t_0-x_0z \quad \text{and} \quad f_0=y_1^2t_1+2y_0y_1-x_1z. \tag{1}$$

Note that the intersection $[\mathcal{O}]=\mathcal{Q}_\infty\cap\mathcal{Q}_0$ is exact on Σ_∞; both $\mathcal{Q}_\infty$ and $\mathcal{Q}_0$ contain the spread line $[T]=\{(a,b,0,0,0)\mid a,b\in\mathbb{F}_q\}$, and these are the only points of Σ_∞ contained in both $\mathcal{Q}_\infty$ and $\mathcal{Q}_0$. Also note that in $\mathrm{PG}(4,q^2)$, $\mathcal{Q}_\infty^{\star}$ and $\mathcal{Q}_0^{\star}$ both contain the extended spread line $[T]^{\star}$, and so both contain at least one point of g, namely $[T]^{\star}\cap g=A_0$. Also, $[\mathcal{O}]$ lies in the pencil of quadrics $\{\mathcal{Q}_t=t\mathcal{Q}_\infty+\mathcal{Q}_0\mid t\in\mathbb{F}_q\cup\{\infty\}\}$ where $\mathcal{Q}_t$ has equation $f_t=tf_\infty+f_0=0$. Recall that the transversal g of $\mathcal{S}$ consists of the points $G_\beta=\beta A_0+A_1=(\beta\tau^q,-\beta,\tau^q,-1,0)$ for $\beta\in\mathbb{F}_{q^2}\cup\{\infty\}$. For $\beta\in\mathbb{F}_{q^2}$, we have $f_\infty(G_\beta)=\tau^q(\tau^q-\tau)$ and $f_0(G_\beta)=\tau-\tau^q$. Let $f_t=tf_\infty+f_0$; then $f_t(G_\beta)=(\tau^q-\tau)(t\tau^q-1)$ which is never zero when $t\in\mathbb{F}_q$. Hence $G_\infty=A_0$ is the only point of g contained in the quadric $\mathcal{Q}_t$. Similarly, A_0^q is the only point of the (other) transversal g^q contained in the quadric $\mathcal{Q}_t$. That is, when $\mathcal{O}$ is tangent to ℓ_∞, the quadrics $\mathcal{Q}_t^{\star}$ each meet g in one point, namely $[T]^{\star}\cap g=A_0$. A similar argument using the conic with equation $f(x,y,z)=x^2-\delta y^2+z^2$, $\delta\in\mathbb{F}_{q^2}\setminus\{0\}$ for q odd, and $\delta x^2+y^2+z^2+yx=0$, $\delta\in\mathbb{F}_{q^2}$ for q even completes the other cases. □

The proof of Theorem 4.1, and the one-to-one correspondence between points $\bar{P}$ of ℓ_∞ and points $P=[P]^{\star}\cap g$ of the transversal g, allow us to identify the points of the quadric $\mathcal{Q}_t^{\star}$ on g.

Corollary 4.2. *Let $\mathcal{O}$ be a nondegenerate conic in* $\mathrm{PG}(2,q^2)$; *then*

1. $\bar{P}\in\mathcal{O}\cap\ell_\infty$ *if and only if in* $\mathrm{PG}(4,q^2)$, $P\in g$;
2. $\bar{P}$ *is a point in the intersection of the extension of* $\mathcal{O}$ *and the extension of* ℓ_∞ *to* $\mathrm{PG}(2,q^4)$ *if and only if in* $\mathrm{PG}(4,q^4)$, $P\in g^{\bigstar}\setminus g$.

Next we consider the set $[\mathcal{O}]$ extended to $\mathrm{PG}(4,q^2)$ and $\mathrm{PG}(4,q^4)$, and determine the exact intersection with the hyperplane at infinity.

Theorem 4.3. *Let $\mathcal{O}$ be a nondegenerate conic in* $\mathrm{PG}(2,q^2)$.

1. *Suppose $\mathcal{O}$ is secant to ℓ_∞, so $\mathcal{O}\cap\ell_\infty=\{\bar{P},\bar{Q}\}$. Then*
   (a) *in* $\mathrm{PG}(4,q)$, $[\mathcal{O}]\cap\Sigma_\infty=\{[P],[Q]\}$;
   (b) *in* $\mathrm{PG}(4,q^2)$, $[\mathcal{O}]^{\star}\cap\Sigma_\infty^{\star}=\{[P]^{\star},[Q]^{\star},PQ^q,P^qQ\}$;
   (c) *in* $\mathrm{PG}(4,q^4)$, $[\mathcal{O}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=\{[P]^{\bigstar},[Q]^{\bigstar},(PQ^q)^{\bigstar},(P^qQ)^{\bigstar}\}$.
2. *Suppose $\mathcal{O}$ is tangent to ℓ_∞, so $\mathcal{O}\cap\ell_\infty=\{\bar{P}\}$. Then*
   (a) *in* $\mathrm{PG}(4,q)$, $[\mathcal{O}]\cap\Sigma_\infty=\{[P]\}$;

(b) *in* PG$(4, q^2)$, $[\mathcal{O}]^{\star} \cap \Sigma_\infty^{\star} = \{[P]^{\star}\}$,
(c) *in* PG$(4, q^4)$, $[\mathcal{O}]^{\bigstar} \cap \Sigma_\infty^{\bigstar} = \{[P]^{\bigstar}\}$.

3. *Suppose* $\mathcal{O}$ *is exterior to* ℓ_∞, *so in the extension to* PG$(2, q^4)$, *the extension of* $\mathcal{O}$ *meets the extension of* ℓ_∞ *in two points* $\{\bar{P}, \bar{P}^{q^2}\}$. *Then*
   (a) *in* PG$(4, q)$, $[\mathcal{O}] \cap \Sigma_\infty = \varnothing$;
   (b) *in* PG$(4, q^2)$, $[\mathcal{O}]^{\star} \cap \Sigma_\infty^{\star} = \varnothing$;
   (c) *in* PG$(4, q^4)$, $[\mathcal{O}]^{\bigstar} \cap \Sigma_\infty^{\bigstar} = \{\ell_P, \ell_P^q, \ell_P^{q^2}, \ell_P^{q^3}\}$, *where* $\ell_P = PP^q$.

Proof. As noted above, $[\mathcal{O}] = \mathcal{Q}_\infty \cap \mathcal{Q}_0$ for quadrics $\mathcal{Q}_\infty$, $\mathcal{Q}_0$, and this correspondence is exact, so $[\mathcal{O}]$ meets Σ_∞ in either the empty set, or in 1 or 2 spread lines (corresponding respectively to $\mathcal{O}$ meeting ℓ_∞ in 0, 1 or 2 points). The cases $\mathcal{O}$ tangent, secant and exterior to ℓ_∞, q odd and even, are proved separately using the same conic equations as in the proof of Theorem 4.1. We omit the calculations, noting that we rely on [Bruen and Hirschfeld 1988, Table 2] to show that the intersection of the two quadrics in the 3-space $\Sigma_\infty^{\star}$ is a set of four lines, possibly repeated. □

We have shown that in PG$(4, q^2)$, the set $[\mathcal{O}]^{\star}$ contains an extended spread line $[P]^{\star}$ if and only if in PG$(2, q^2)$, the point $\bar{P} \in \mathcal{O} \cap \ell_\infty$. We will need the next corollary which considers whether the set $[\mathcal{O}]^{\star}$ can contain a point of any other extended spread line.

Corollary 4.4. *Let* $\mathcal{O}$ *be a nondegenerate conic in* PG$(2, q^2)$. *Let* $\bar{L}$ *be a point of* ℓ_∞ *not in* $\mathcal{O}$. *In* PG$(4, q^2)$, *the corresponding extended spread line* $[L]^{\star}$ *is disjoint from* $[\mathcal{O}]^{\star}$.

Proof. If $\mathcal{O}$ is secant to ℓ_∞, so $\mathcal{O} \cap \ell_\infty = \{\bar{P}, \bar{Q}\}$, then by Theorem 4.3, $[\mathcal{O}]^{\star} \cap \Sigma_\infty^{\star}$ consists of the four lines $[P]^{\star}$, $[Q]^{\star}$, PQ^q, P^qQ. Let $[L]^{\star}$ be an extended spread line, $\bar{L} \neq \bar{P}, \bar{Q}$. Then $[L]^{\star}$, $[P]^{\star}$, $[Q]^{\star}$, PQ^q, P^qQ are all lines of the hyperbolic congruence of g and g^q, and so do not meet off g, g^q, and hence are mutually skew. So $[L]^{\star} \cap [\mathcal{O}]^{\star} = \varnothing$. If $\mathcal{O}$ is tangent to ℓ_∞, then by Theorem 4.3, $[\mathcal{O}]^{\star} \cap \Sigma_\infty^{\star} = [P]^{\star}$. Hence $[\mathcal{O}]^{\star}$ meets no other spread line. If $\mathcal{O}$ is exterior to ℓ_∞, then by Theorem 4.3, $[\mathcal{O}]^{\star} \cap \Sigma_\infty^{\star} = \varnothing$, so $[\mathcal{O}]^{\star}$ contains no point on any extended spread line, as required. □

5. Conics of Baer subplanes

In this section we improve Result 2.5 by characterizing the normal rational curves of PG$(4, q)$ that correspond to conics of a Baer subplane of PG$(2, q^2)$. In particular, we show that if $\mathcal{C}$ is a conic contained in a tangent Baer subplane $\mathcal{B}$ of PG$(2, q^2)$, then in PG$(4, q)$, the corresponding 3- or 4-dimensional normal rational curve $[\mathcal{C}]$ is g-special. Further, we show that any g-special 3- or 4-dimensional normal rational curve in PG$(4, q)$ corresponds to a conic in a Baer subplane of PG$(2, q^2)$.

$\mathbb{F}_{q^2}$-conics and $\mathbb{F}_q$-conics. In this section we show that the notion of specialness is also intrinsic to the Bruck–Bose representation of conics in Baer subplanes. First we introduce some notation to easily distinguish between conics in $\mathrm{PG}(2, q^2)$ and conics contained in a Baer subplane. An *$\mathbb{F}_{q^2}$-conic* in $\mathrm{PG}(2, q^2)$ is a nondegenerate conic of $\mathrm{PG}(2, q^2)$. Note that an $\mathbb{F}_{q^2}$-conic meets a Baer subplane $\mathcal{B}$ in either 0, 1, 2, 3 or 4 points, or in a nondegenerate conic of $\mathcal{B}$. We define an *$\mathbb{F}_q$-conic* of $\mathrm{PG}(2, q^2)$ to be a nondegenerate conic in a Baer subplane of $\mathrm{PG}(2, q^2)$. For the remainder of this article, $\mathcal{C}$ will denote an $\mathbb{F}_q$-conic. Further, we denote the *unique* $\mathbb{F}_{q^2}$-conic containing $\mathcal{C}$ by $\mathcal{C}^+$. An $\mathbb{F}_{q^2}$-conic contains many $\mathbb{F}_q$-conics.

Lemma 5.1. *Let $\mathcal{O}$ be an $\mathbb{F}_{q^2}$-conic in $\mathrm{PG}(2, q^2)$. Any three points of $\mathcal{O}$ lie in a unique $\mathbb{F}_q$-conic contained in $\mathcal{O}$, so there are $q(q^2+1)$ $\mathbb{F}_q$-conics contained in $\mathcal{O}$.*

Proof. The $\mathbb{F}_{q^2}$-conic $\mathcal{O}$ is equivalent to the line $\ell \cong \mathrm{PG}(1, q^2)$, and subconics of $\mathcal{O}$ are equivalent to Baer sublines of ℓ. Since three points of ℓ lie in a unique Baer subline of ℓ, three points of $\mathcal{O}$ lie in a unique subconic $\mathcal{C}$. As $\mathcal{C}$ is a normal rational curve over $\mathbb{F}_q$, by [Hirschfeld and Thas 1991, Theorem 21.1.1] there is a homography ϕ that maps $\mathcal{C}$ to $\mathcal{C}' = \phi(\mathcal{C}) = \{(1, \theta, \theta^2) \mid \theta \in \mathbb{F}_q \cup \{\infty\}\}$. As $\mathcal{C}'$ lies in the Baer subplane $\mathcal{B}' = \mathrm{PG}(2, q)$, $\mathcal{C}$ lies in the Baer subplane $\phi^{-1}(\mathcal{B}')$, that is, $\mathcal{C}$ is an $\mathbb{F}_q$-conic. Straightforward counting shows that the number of $\mathbb{F}_q$-conics in $\mathcal{O}$ is $(q^2+1)q^2(q^2-1)/(q+1)q(q-1) = q(q^2+1)$. □

Remark 5.2. Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in $\mathrm{PG}(2, q^2)$, $q > 4$, so there is a unique $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ with $\mathcal{C} \subset \mathcal{C}^+$. Then in $\mathrm{PG}(4, q)$, $[\mathcal{C}] \subset [\mathcal{C}^+]$. This is clearly true for the affine points. For the points at infinity, we recall Remark 2.6, if $\bar{T} \in \mathcal{C} \cap \ell_\infty \subseteq \mathcal{C}^+ \cap \ell_\infty$, then $[\mathcal{C}]$ meets the spread line $[T]$ in a point, while $[\mathcal{C}^+]$ contains the spread line $[T]$.

Conics in secant Baer subplanes. In this section we consider the Bruck–Bose representation of $\mathbb{F}_q$-conics in secant Baer subplanes of $\mathrm{PG}(2, q^2)$, in particular looking at the relationship with the lines of the hyperbolic congruence of g, g^q.

Theorem 5.3. *Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in a Baer subplane $\mathcal{B}$ secant to ℓ_∞. The $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ meets ℓ_∞ in two (possibly equal) points $\bar{P}$, $\bar{Q}$. In $\mathrm{PG}(4, q)$, $[\mathcal{C}]$ is a nondegenerate conic in the plane $[\mathcal{B}]$, and $[\mathcal{C}^+] \cap \Sigma_\infty$ is the two spread lines $[P], [Q]$.*

1. *If $\bar{P} = \bar{Q}$, then $\bar{P} \in \mathcal{B}$, and $[\mathcal{C}]$ meets Σ_∞ in one point $[P] \cap [\mathcal{B}]$.*
2. *If $\bar{P} \neq \bar{Q}$ and $\bar{P}$, $\bar{Q} \in \mathcal{B}$, then $[\mathcal{C}]$ meets Σ_∞ in two points $[P] \cap [\mathcal{B}]$ and $[Q] \cap [\mathcal{B}]$.*
3. *If $\bar{P} \neq \bar{Q}$ and $\bar{P}$, $\bar{Q} \notin \mathcal{B}$, then $[\mathcal{C}]$ is a (PQ^q)-special conic.*

Proof. By Results 2.3 and 2.5, in $\mathrm{PG}(4, q)$, $[\mathcal{B}]$ is a plane, and $[\mathcal{C}]$ is a conic in $[\mathcal{B}]$. Parts 1 and 2 follow immediately from the Bruck–Bose definition. For part 3, the set $[\mathcal{C}^+]$ contains the spread lines $[P]$ and $[Q]$. The set $[\mathcal{B}]$ is a plane, and the line $m = [\mathcal{B}] \cap \Sigma_\infty$ meets $q+1$ spread lines, but does not meet the spread lines $[P]$, $[Q]$.

The set $[\mathcal{C}]$ is a nondegenerate conic in $[\mathcal{B}]$ which does not meet m, and in the extension to $\mathrm{PG}(4, q^2)$, $[\mathcal{C}]^{\star}$ meets $\Sigma_\infty^{\star}$ in two points of the line $m^{\star} = [\mathcal{B}]^{\star} \cap \Sigma_\infty^{\star}$. In $\mathrm{PG}(2, q^2)$, we have $\mathcal{C} = \mathcal{B} \cap \mathcal{C}^+$, and in $\mathrm{PG}(4, q)$, $[\mathcal{C}] = [\mathcal{B}] \cap [\mathcal{C}^+]$. Moreover, in $\mathrm{PG}(4, q^2)$, $[\mathcal{C}]^{\star} = [\mathcal{B}]^{\star} \cap [\mathcal{C}^+]^{\star}$, hence $[\mathcal{C}]^{\star} \cap \Sigma_\infty^{\star} = \{[\mathcal{B}]^{\star} \cap \Sigma_\infty^{\star}\} \cap \{[\mathcal{C}^+]^{\star} \cap \Sigma_\infty^{\star}\}$. By Theorem 4.3, this is equal to $\{m^{\star}\} \cap \{g, g^q, PQ^q, P^qQ\}$. Now $m^{\star}$ does not meet g (or g^q) as the only lines of Σ_∞ whose extension meets g are the lines of $\mathcal{S}$. Hence the two points of $[\mathcal{C}]^{\star} \cap \Sigma_\infty^{\star}$ lie in PQ^q and P^qQ, that is, $[\mathcal{C}]$ is a (PQ^q)-special conic of $\mathrm{PG}(4, q)$. □

We now characterize $\mathbb{F}_q$-conics in secant Baer subplanes by showing that the converse is true.

Theorem 5.4. *In* $\mathrm{PG}(4, q)$, *let* α *be a plane not containing a spread line*, *and let* $\mathcal{N}$ *be a nondegenerate conic in* α.

1. *In* $\mathrm{PG}(2, q^2)$, *there is a secant Baer subplane* $\mathcal{B}$ *with* $[\mathcal{B}] = \alpha$, *and an* $\mathbb{F}_q$*-conic* $\mathcal{C}$ *in* $\mathcal{B}$ *with* $[\mathcal{C}] = \mathcal{N}$.
2. *If* $\mathcal{N}$ *meets* Σ_∞ *in a point of the spread line* $[T]$, *then* $\bar{T} \in \mathcal{C}$.
3. *If* $\mathcal{N}$ *is a* (PQ^q)*-special conic*, *then the* $\mathbb{F}_{q^2}$*-conic* $\mathcal{C}^+$ *containing* $\mathcal{C}$ *meets* ℓ_∞ *in the points* $\bar{P}$, $\bar{Q}$.

Proof. Parts 1 and 2 follow from Result 2.3. For part 3, in $\mathrm{PG}(4, q)$, let $\mathcal{N}$ be a (PQ^q)-special conic of $\mathrm{PG}(4, q)$ lying in a plane α that does not contain a spread line. By part 1, $[\mathcal{B}] = \alpha$ and $[\mathcal{C}] = \mathcal{N}$ where in $\mathrm{PG}(2, q^2)$, $\mathcal{B}$ is a secant Baer subplane containing the $\mathbb{F}_q$-conic $\mathcal{C}$. As $\mathcal{N}$ is a (PQ^q)-special conic, $\mathcal{N} \cap \Sigma_\infty = \varnothing$, and in $\mathrm{PG}(4, q^2)$, $\mathcal{N}^{\star}$ is a conic which meets the line $\alpha \cap \Sigma_\infty^{\star}$ is two points, one lying on each of PQ^q and P^qQ. As $\mathcal{N} \cap \Sigma_\infty = \varnothing$, in $\mathrm{PG}(2, q^2)$, the $\mathbb{F}_q$-conic $\mathcal{C}$ does not meet ℓ_∞, so the $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ meets ℓ_∞ in two points $\bar{A}, \bar{B} \notin \mathcal{B}$. By Theorem 5.3, $[\mathcal{C}] = \mathcal{N}$ is a (AB^q)-special conic. Hence $\{\bar{A}, \bar{B}\} = \{\bar{P}, \bar{Q}\}$, so $\mathcal{C}^+ \cap \ell_\infty = \{\bar{P}, \bar{Q}\}$ as required. □

Conics in tangent Baer subplanes. We now consider a Baer subplane $\mathcal{B}$ that is tangent to ℓ_∞ and look at $\mathbb{F}_q$-conics in $\mathcal{B}$. There are two cases to consider, namely whether the $\mathbb{F}_q$-conic contains the point $\mathcal{B} \cap \ell_\infty$ or not. In each case we generalize Result 2.5 by showing that the corresponding normal rational curve of $\mathrm{PG}(4, q)$ is g-special. Further, we characterize all g-special normal rational curves in $\mathrm{PG}(4, q)$ as corresponding to $\mathbb{F}_q$-conics in a tangent Baer subplane.

Conics in $\mathcal{B}$ *containing the point* $\bar{T} = \mathcal{B} \cap \ell_\infty$. We first look at an $\mathbb{F}_q$-conic $\mathcal{C}$ in a tangent Baer subplane $\mathcal{B}$, with $\mathcal{B} \cap \ell_\infty$ in $\mathcal{C}$.

Theorem 5.5. *In* $\mathrm{PG}(2, q^2)$, $q > 5$, *let* $\mathcal{B}$ *be a tangent Baer subplane and* $\mathcal{C}$ *an* $\mathbb{F}_q$*-conic in* $\mathcal{B}$ *containing the point* $\mathcal{B} \cap \ell_\infty$. *Then in* $\mathrm{PG}(4, q)$, $[\mathcal{C}]$ *is a* g*-special twisted cubic.*

Proof. Let $\mathcal{B}$ be a Baer subplane of $\mathrm{PG}(2, q^2)$ that is tangent to ℓ_∞ in the point $\bar{T} = \mathcal{B} \cap \ell_\infty$. Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic of $\mathcal{B}$ that contains $\bar{T}$. By Result 2.5, in $\mathrm{PG}(4, q)$, $\mathcal{C}$ corresponds to a twisted cubic $[\mathcal{C}]$ that lies in a 3-space denoted $\Pi_{\mathcal{C}}$. By Result 2.1, $\Pi_{\mathcal{C}}$ meets the ruled cubic surface $[\mathcal{B}]$ in exactly the twisted cubic $[\mathcal{C}]$. We show that $[\mathcal{C}]$ is g-special. By Lemma 2.11, the 3-space $\Pi_{\mathcal{C}}$ corresponds to an ℓ_∞-Baer pencil $\mathcal{K}$ of $\mathrm{PG}(2, q^2)$ that meets $\mathcal{B}$ in $\mathcal{C}$. By Theorem 2.13, $\mathcal{K}$ has vertex $\bar{P} \in \mathcal{C}^+$. Hence $\Pi_{\mathcal{C}}$ contains the spread line $[P]$ (and this is the only spread line in $\Pi_{\mathcal{C}}$). Consider the extension of $\mathrm{PG}(4, q)$ to $\mathrm{PG}(4, q^2)$. Note that Lemma 2.9 can be generalized to a 3-dimensional normal rational curve when $q > 5$. Hence as $[\mathcal{B}]$ is the intersection of three quadrics [Barwick and Jackson 2012], we have $[\mathcal{C}]^{☆} \subset [\mathcal{B}]^{☆}$ in $\mathrm{PG}(4, q^2)$. Thus by Corollary 2.2, the twisted cubic $[\mathcal{C}]^{☆}$ contains a unique point of each generator line of the ruled cubic surface $[\mathcal{B}]^{☆}$. By Result 2.3, $[\mathcal{B}]$ is g-special, so the transversal lines g, g^q of the regular spread $\mathcal{S}$ are generator lines of the extended ruled cubic surface $[\mathcal{B}]^{☆}$. Hence $[\mathcal{C}]^{☆}$ contains a point of g and g^q. Thus $[\mathcal{C}]^{☆}$ contains the points corresponding to the vertex of $\mathcal{K}$, that is, the point $g \cap {\Pi_{\mathcal{C}}}^{☆} = g \cap [P]^{☆} = P$ and P^q. That is, the twisted cubic $[\mathcal{C}]$ is g-special. □

The converse of Theorem 5.5 is also true.

Theorem 5.6. *A g-special twisted cubic in* $\mathrm{PG}(4, q)$ *corresponds to an $\mathbb{F}_q$-conic in some tangent Baer subplane of* $\mathrm{PG}(2, q^2)$.

Proof. Let $\mathcal{N}$ be a g-special twisted cubic in $\mathrm{PG}(4, q)$, so in $\mathrm{PG}(4, q^2)$, $\mathcal{N}^{☆}$ meets the transversal g of $\mathcal{S}$ in a point R, and meets g^q in the point R^q. The line RR^q meets Σ_∞ in a spread line denoted $[R]$, corresponding to the point $\bar{R} \in \ell_\infty$. Let $\Pi_{\mathcal{N}}$ be the 3-space containing $\mathcal{N}$, and recall that a twisted cubic meets a plane in three points, possibly repeated, or in an extension. As $\mathcal{N}$ meets the plane $\pi = \Pi_{\mathcal{N}} \cap \Sigma_\infty$ in two points R, R^q over $\mathbb{F}_{q^2}$, $\mathcal{N}$ meets π in one point X over $\mathbb{F}_q$. Let $[T]$ be the spread line containing the point X, so $[T] \notin \Pi_{\mathcal{N}}$. Let $[A]$, $[B]$, $[C]$ be three affine points of $\mathcal{N}$, and let $\alpha = \langle [A], [B], [C] \rangle$.

As α lies in the 3-space $\Pi_{\mathcal{N}}$, if α contained a spread line, it would contain $[R]$. However, if α contains $[R]$, then the plane $\langle [A], [B], [R] \rangle^{☆}$ would contain four points of $\mathcal{N}^{☆}$, namely $[A]$, $[B]$, R, R^q, a contradiction. If α contained the point X, then α would contain four points of $\mathcal{N}$, namely X, $[A]$, $[B]$, $[C]$, a contradiction. Hence α corresponds to a Baer subplane $\mathcal{B}_\alpha$ of $\mathrm{PG}(2, q^2)$ that is secant to ℓ_∞, with $\bar{T} \notin \mathcal{B}_\alpha$. Hence the points $\{\bar{T}, A, B, C\}$ form a quadrangle and so lie in a unique Baer subplane denoted $\mathcal{B}$. As $\mathcal{B}_\alpha$ is the unique Baer subplane containing A, B, C and secant to ℓ_∞, we have $\mathcal{B} \neq \mathcal{B}_\alpha$, and $\mathcal{B}$ is tangent to ℓ_∞ at the point $\bar{T}$.

In $\mathrm{PG}(4, q)$, $[\mathcal{B}]$ is a ruled cubic surface with line directrix $[T]$. As X, $[A]$, $[B]$, $[C]$ are points of $\mathcal{N}$, no three are collinear, so $[A]$, $[B]$, $[C]$ lie on distinct generators of $[\mathcal{B}]$. Recall that $\Pi_{\mathcal{N}}$ does not contain $[T]$, so by Result 2.1, $\Pi_{\mathcal{N}}$ meets $[\mathcal{B}]$ in a twisted cubic, denoted $\mathcal{N}_1$. The argument in the proof of Theorem 5.5 shows that in the quadratic extension, $\mathcal{N}_1^{\star}$ contains the points R and R^q. Hence $\mathcal{N}^{\star}$ and $\mathcal{N}_1^{\star}$ share six points, and so are equal. That is, $\mathcal{N}$ is a g-special twisted cubic contained in a g-special ruled cubic surface, and $\mathcal{N}$ meets Σ_∞ in one point.

Straightforward counting shows that in $\mathrm{PG}(2, q^2)$, the number of $\mathbb{F}_q$-conics in $\mathcal{B}$ that contain $\bar{T}$ is $q^4 - q^2$. By Result 2.1, the number of 3-spaces of $\mathrm{PG}(4, q)$ that meet the ruled cubic surface $[\mathcal{B}]$ in a twisted cubic is $q^4 - q^2$. Hence they are in one to one correspondence. That is, $\mathcal{N}$ corresponds to an $\mathbb{F}_q$-conic in the tangent Baer subplane $\mathcal{B}$ as required. □

The proofs of Theorems 5.5 and 5.6 show that the points of g on a g-special twisted cubic correspond to the points on ℓ_∞ contained in the associated $\mathbb{F}_{q^2}$-conic.

Corollary 5.7. *Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in a tangent Baer subplane $\mathcal{B}$ in* $\mathrm{PG}(2, q^2)$, $q > 5$, *with $\bar{T} = \mathcal{B} \cap \ell_\infty \in \mathcal{C}$. The $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ meets ℓ_∞ in a point $\bar{P} \neq \bar{T}$ if and only if in* $\mathrm{PG}(4, q^2)$ *the twisted cubic $[\mathcal{C}]^{\star}$ meets the transversals of $\mathcal{S}$ in the points P, P^q.*

Conics of $\mathcal{B}$ not containing the point $\bar{T} = \mathcal{B} \cap \ell_\infty$. We now look at an $\mathbb{F}_q$-conic $\mathcal{C}$ in a tangent Baer subplane $\mathcal{B}$, with $\mathcal{B} \cap \ell_\infty$ not in $\mathcal{C}$. The $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ meets ℓ_∞ in two distinct points (which may lie in $\mathrm{PG}(2, q^4)$). We show that if these two points lie in $\mathrm{PG}(2, q^2)$, then $[\mathcal{C}]$ is a g-special normal rational curve. Further, if the two points lie in the quadratic extension of $\mathrm{PG}(2, q^2)$ to $\mathrm{PG}(2, q^4)$, then $[\mathcal{C}]$ is an $g^\bigstar$-special normal rational curve.

Theorem 5.8. *In* $\mathrm{PG}(2, q^2)$, $q > 7$, *let $\mathcal{B}$ be a Baer subplane tangent to ℓ_∞ with $\bar{T} = \mathcal{B} \cap \ell_\infty$. Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in $\mathcal{B}$, $\bar{T} \notin \mathcal{C}$. In* $\mathrm{PG}(4, q)$, *$[\mathcal{C}]$ is a g-special or $g^\bigstar$-special* 4*-dimensional normal rational curve.*

Proof. Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in $\mathcal{B}$ not through $\bar{T} = \mathcal{B} \cap \ell_\infty$, and consider the $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ containing $\mathcal{C}$. Then either (i) $\mathcal{C}^+$ is secant to ℓ_∞ and $\mathcal{C}^+ \cap \ell_\infty$ consists of two distinct points $\bar{P}$, $\bar{Q}$, or (ii) $\mathcal{C}^+$ is tangent to ℓ_∞ and $\mathcal{C}^+ \cap \ell_\infty$ is a repeated point $\bar{P} = \bar{Q}$, or (iii) $\mathcal{C}^+$ is exterior to ℓ_∞ and in $\mathrm{PG}(2, q^4)$ the extension of $\mathcal{C}^+$ meets the extension of ℓ_∞ in two points $\bar{P}$, $\bar{Q}$ which are conjugate with respect to this extension from $\mathrm{PG}(2, q^2)$ to $\mathrm{PG}(2, q^4)$, that is, $\bar{Q} = \bar{P}^{q^2}$. By Result 2.5, as $\bar{T} \notin \mathcal{C}$, in $\mathrm{PG}(4, q)$, $[\mathcal{C}]$ is a 4-dimensional normal rational curve lying on the g-special ruled cubic surface $[\mathcal{B}]$, and $[\mathcal{C}]$ does not meet Σ_∞. Thus it remains to show that in $\mathrm{PG}(4, q)$ $[\mathcal{C}]$ is a g-special or $g^\bigstar$-special. We will show that in an appropriate extension of $\mathrm{PG}(4, q)$, the extension of the normal rational curve $[\mathcal{C}]$ contains the points P, Q of the (possibly extended) transversal g, giving the g-special property. Recall that a 4-dimensional normal rational curve of $\mathrm{PG}(4, q)$ meets the 3-space Σ_∞ in four points, possibly repeated or in an extension. As $[\mathcal{C}]$ is disjoint from Σ_∞, either (a) in $\mathrm{PG}(4, q^2)$, $[\mathcal{C}]^{\star}$ meets $\Sigma_\infty^{\star}$ in four points of the form X, X^q, Y, Y^q, possibly $X = Y$; or (b) in $\mathrm{PG}(4, q^4)$, $[\mathcal{C}]^\bigstar$ meets $\Sigma_\infty^\bigstar$ in four points of form X, X^q, X^{q^2}, X^{q^3}.

In $\mathrm{PG}(2,q^2)$, we have $\mathcal{C}\subset\mathcal{C}^+$, so as discussed in Remark 5.2, in $\mathrm{PG}(4,q)$, $[\mathcal{C}]\subset[\mathcal{C}^+]$. By [Barwick et al. 2011, Corollary 3.3], $[\mathcal{C}^+]$ is the exact intersection of two quadrics, so by Lemma 2.9 $[\mathcal{C}]^{\star}\subset[\mathcal{C}^+]^{\star}$ in $\mathrm{PG}(4,q^2)$ and $[\mathcal{C}]^{\bigstar}\subset[\mathcal{C}^+]^{\bigstar}$ in $\mathrm{PG}(4,q^4)$. Similarly, as $[\mathcal{C}]\subset[\mathcal{B}]$ and $[\mathcal{B}]$ is the intersection of three quadrics [Barwick and Jackson 2012], we have $[\mathcal{C}]^{\star}\subset[\mathcal{B}]^{\star}$ and $[\mathcal{C}]^{\bigstar}\subset[\mathcal{B}]^{\bigstar}$ by Lemma 2.9. In $\mathrm{PG}(2,q^2)$, we have $\mathcal{C}=\mathcal{B}\cap\mathcal{C}^+$. As $[\mathcal{C}]$ is disjoint from Σ_∞, in $\mathrm{PG}(4,q)$, we have $[\mathcal{C}]=[\mathcal{B}]\cap[\mathcal{C}^+]$. We need to determine $[\mathcal{C}]^{\star}\cap\Sigma_\infty^{\star}=[\mathcal{C}^+]^{\star}\cap[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}$ in $\mathrm{PG}(4,q^2)$ and $[\mathcal{C}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=[\mathcal{C}^+]^{\bigstar}\cap[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}$ in $\mathrm{PG}(4,q^4)$.

First we determine $[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}$ and $[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}$. In $\mathrm{PG}(2,q^2)$, $\bar{T}\in\mathcal{B}$, so in $\mathrm{PG}(4,q)$, $[T]\subset[\mathcal{B}]$, and $[\mathcal{B}]\cap\Sigma_\infty=[T]$. Hence in $\mathrm{PG}(4,q^2)$, $[T]^{\star}\subset[\mathcal{B}]^{\star}$, and in $\mathrm{PG}(4,q^4)$, $[T]^{\bigstar}\subset[\mathcal{B}]^{\bigstar}$. By Result 2.3, $[\mathcal{B}]$ is a g-special ruled cubic surface, so the transversal lines g, g^q lie in $[\mathcal{B}]^{\star}$. That is, $\{[T]^{\star}, g, g^q\}$ lie in $[\mathcal{B}]^{\star}$, and using Result 2.1 in $\mathrm{PG}(4,q^2)$, the 3-space $\Sigma_\infty^{\star}$ meets the ruled cubic surface $[\mathcal{B}]^{\star}$ in exactly these three lines, so $[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}=\{[T]^{\star}, g, g^q\}$. Similarly, in $\mathrm{PG}(4,q^4)$, the 3-space $\Sigma_\infty^{\bigstar}$ meets the ruled cubic surface $[\mathcal{B}]^{\bigstar}$ in the three lines $\{[T]^{\bigstar}, g^{\bigstar}, g^{q\bigstar}\}$.

Recall that Theorem 4.3 determines the intersection $[\mathcal{C}^+]^{\star}\cap\Sigma_\infty^{\star}$ and $[\mathcal{C}^+]^{\bigstar}\cap\Sigma_\infty^{\bigstar}$ for the three cases where $\mathcal{C}^+$ is (i) secant, (ii) tangent or (iii) exterior to ℓ_∞ in $\mathrm{PG}(2,q^2)$. For each case we determine $[\mathcal{C}^+]^{\star}\cap[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}$ in $\mathrm{PG}(4,q^2)$ and $[\mathcal{C}^+]^{\bigstar}\cap[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}$ in $\mathrm{PG}(4,q^2)$.

In case (i), $\mathcal{C}^+$ is secant to ℓ_∞, so $[\mathcal{C}^+]^{\star}\cap\Sigma_\infty^{\star}=\{[P]^{\star}, [Q]^{\star}, PQ^q, P^qQ\}$, by Theorem 4.3. Now $[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}=\{[T]^{\star}, g, g^q\}$, and by Corollary 4.4, $[\mathcal{C}^+]^{\star}\cap\Sigma_\infty^{\star}$ does not meet $[T]^{\star}$. Hence $[\mathcal{C}^+]^{\star}\cap[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}$ consists of the four points P, Q, P^q and Q^q. Similarly, $[\mathcal{C}^+]^{\bigstar}\cap[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=\{P, Q, P^q, Q^q\}$. As $[\mathcal{C}]^{\star}\cap\Sigma_\infty^{\star}=[\mathcal{C}^+]^{\star}\cap[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}$, $[\mathcal{C}]^{\star}$ meets g in two distinct points, namely P, Q and so $[\mathcal{C}]$ is a g-special normal rational curve.

In case (ii), $\mathcal{C}^+$ is tangent to ℓ_∞, so by Theorem 4.3, $\{[\mathcal{C}^+]^{\star}\cap\Sigma_\infty^{\star}\}\cap\{[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}\}=\{[P]^{\star}\}\cap\{[T]^{\star}, g, g^q\}=\{P, P^q\}$. Similarly, $[\mathcal{C}^+]^{\bigstar}\cap[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=\{P, P^q\}$. Hence $[\mathcal{C}]^{\star}$ meets g in the repeated point P, and so $[\mathcal{C}]$ is a g-special normal rational curve.

In case (iii), $\mathcal{C}^+$ is exterior to ℓ_∞, so in $\mathrm{PG}(2,q^4)$, the extension of $\mathcal{C}^+$ meets the extension of ℓ_∞ in two points $\bar{P}$, $\bar{Q}$, where $\bar{Q}=\bar{P}^{q^2}$. By Theorem 4.3, $[\mathcal{C}^+]^{\star}\cap\Sigma_\infty^{\star}=\varnothing$ and $[\mathcal{C}^+]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=\{\ell_P, \ell_P^q, \ell_P^{q^2}, \ell_P^{q^3}\}$, where $\ell_P=PP^q$. Hence $[\mathcal{C}^+]^{\star}\cap[\mathcal{B}]^{\star}\cap\Sigma_\infty^{\star}=\varnothing$, and $[\mathcal{C}^+]^{\bigstar}\cap[\mathcal{B}]^{\bigstar}\cap\Sigma_\infty^{\bigstar}=\{P, P^q, P^{q^2}, P^{q^3}\}$. So in this case the normal rational curve $[\mathcal{C}]$ meets Σ_∞ in four points over $\mathbb{F}_{q^4}$. As $[\mathcal{C}]^{\bigstar}$ meets $g^{\bigstar}$ in two points (namely P and $P^{q^2}=Q$) $[\mathcal{C}]$ is an $g^{\bigstar}$-special normal rational curve. □

Conversely, every g-special or $g^{\bigstar}$-special normal rational curve corresponds to an $\mathbb{F}_q$-conic:

Theorem 5.9. *Let $\mathcal{N}$ be a g-special or $g^{\bigstar}$-special* 4*-dimensional normal rational curve in* $\mathrm{PG}(4,q)$. *Then $\mathcal{N}=[\mathcal{C}]$ where $\mathcal{C}$ is an $\mathbb{F}_q$-conic in a tangent Baer subplane of* $\mathrm{PG}(2,q^2)$.

Proof. Let $\mathcal{N}$ be a g-special 4-dimensional normal rational curve in $\mathrm{PG}(4, q)$. So there are two (possibly equal) spread lines $[P]$, $[Q]$ such that $\mathcal{N}^{\star} \cap \Sigma_\infty^{\star}$ consists of the four points $P = g \cap [P]^{\star}$, $P^q = g^q \cap [P]^{\star}$, $Q = g \cap [Q]^{\star}$, $Q^q = g^q \cap [Q]^{\star}$. Note that as $\mathcal{N}^{\star}$ meets $\Sigma_\infty^{\star} \setminus \Sigma_\infty$ in four points, $\mathcal{N}$ is disjoint from Σ_∞. There are three cases to consider.

Case (i): Suppose first that $[P] \neq [Q]$. Let $[A], [B], [C]$ be three points of $\mathcal{N}$, so $[A], [B], [C] \notin \Sigma_\infty$. If the plane $\alpha = \langle [A], [B], [C] \rangle$ contained a point of the spread line $[P]$, then the 3-space $\langle \alpha, [P] \rangle^{\star}$ contains five points of $\mathcal{N}^{\star}$, namely $[A], [B], [C], P, P^q$, a contradiction. So α is disjoint from the spread lines $[P]$ and $[Q]$. If α contained a spread line $[X]$, then in $\mathrm{PG}(4, q^2)$, $\langle \alpha^{\star}, g \rangle$ is a 3-space that contains five points of $\mathcal{N}^{\star}$, namely $[A], [B], [C], P, Q$, a contradiction. So α corresponds to a Baer subplane $\mathcal{B}_\alpha$ of $\mathrm{PG}(2, q^2)$ that is secant to ℓ_∞, and does not contain $\bar{P}$ or $\bar{Q}$.

Consider the corresponding points $\bar{P}, \bar{Q}, A, B, C$ in $\mathrm{PG}(2, q^2)$. So $\bar{P}, \bar{Q} \in \ell_\infty$ and $A, B, C \in \mathrm{PG}(2, q^2) \setminus \ell_\infty$. Now A, B, C are not collinear as α does not contain a spread line. So $\mathcal{B}_\alpha$ is the unique Baer subplane that contains A, B, C and is secant to ℓ_∞. As $\bar{P}, \bar{Q} \in \ell_\infty \setminus \mathcal{B}$ and $A, B, C \in \mathcal{B} \setminus \ell_\infty$, no three of $\bar{P}, \bar{Q}, A, B, C$ are collinear, hence they lie on a unique $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$. By Lemma 5.1, A, B, C lie in a unique $\mathbb{F}_q$-conic $\mathcal{C}$ contained in $\mathcal{C}^+$, and $\mathcal{C}$ lies in a Baer subplane $\mathcal{B}$.

Suppose $\mathcal{B} = \mathcal{B}_\alpha$; then by Corollary 3.4, in $\mathrm{PG}(4, q^2)$, the plane $\alpha^{\star}$ meets PQ^q. Note that the line PQ^q contains two points of $\mathcal{N}^{\star}$, namely P, Q^q. Hence $\langle \alpha^{\star}, PQ^q \rangle$ is a 3-space of $\mathrm{PG}(4, q^2)$ that contains five points of $\mathcal{N}^{\star}$, namely $[A], [B], [C], P, Q^q$, a contradiction. Thus $\mathcal{B} \neq \mathcal{B}_\alpha$.

Hence the Baer subplane $\mathcal{B}$ is tangent to ℓ_∞. As $\mathcal{C}^+$ is secant to ℓ_∞, we are in case (i) of the proof of Theorem 5.8, hence in $\mathrm{PG}(4, q)$, $[\mathcal{C}]$ is a g-special 4-dimensional normal rational curve and $[\mathcal{C}]^{\star}$ contains the seven points A, B, C, P, P^q, Q, Q^q. As seven points lie on a unique 4-dimensional normal rational curve, we have $\mathcal{N}^{\star} = [\mathcal{C}]^{\star}$ and so $\mathcal{N} = [\mathcal{C}]$. That is, the normal rational curve $\mathcal{N}$ corresponds in $\mathrm{PG}(2, q^2)$ to an $\mathbb{F}_q$-conic $\mathcal{C}$ in the tangent Baer subplane $\mathcal{B}$ as required.

Case (ii): Suppose $[P] = [Q]$, the proof is very similar to case (i). Let $\mathcal{N}$ be a 4-dimensional normal rational curve of $\mathrm{PG}(4, q)$ such that $\mathcal{N} \cap \Sigma_\infty = \varnothing$, and $\mathcal{N}^{\star} \cap \Sigma_\infty^{\star}$ consists of two repeated points P, P^q. Let $[A], [B], [C] \in \mathcal{N}$ and $\alpha = \langle [A], [B], [C] \rangle$. Similarly to case (i), α corresponds to a Baer subplane $\mathcal{B}_\alpha$ of $\mathrm{PG}(2, q^2)$ that is secant to ℓ_∞, and does not contain $\bar{P}$. The points $\bar{P}, A, B, C$ lie in a unique $\mathbb{F}_{q^2}$-conic $\mathcal{C}^+$ that is tangent to ℓ_∞ at $\bar{P}$. By Lemma 5.1, A, B, C lie in a unique $\mathbb{F}_q$-conic $\mathcal{C}$ contained in $\mathcal{C}^+$, and $\mathcal{C}$ lies in a Baer subplane $\mathcal{B}$. If $\mathcal{B} = \mathcal{B}_\alpha$, then $\bar{P} \notin \mathcal{C}$, and so $\mathcal{C}^+$ meets ℓ_∞ in two points, a contradiction. Hence $\mathcal{B} \neq \mathcal{B}_\alpha$ and $\mathcal{B}$ is tangent to ℓ_∞. As $\mathcal{C}^+$ is tangent to ℓ_∞, we are in case (ii) of the proof of Theorem 5.8, hence in $\mathrm{PG}(4, q)$, $[\mathcal{C}]$ is a g-special 4-dimensional normal rational curve containing A, B, C, and $[\mathcal{C}]^{\star}$ meets $\Sigma_\infty^{\star}$ twice at P and twice at P^q. These conditions define a unique normal rational curve of PG(4, q^2), and so $\mathcal{N} = \mathcal{C}$ as required.

Case (iii): Suppose $\mathcal{N}$ is an $g^{\bigstar}$-special 4-dimensional normal rational curve. As $\mathcal{N}$ is a normal rational curve over $\mathbb{F}_q$, $\mathcal{N}$ meets $\Sigma_\infty^{\bigstar} \setminus \Sigma_\infty^{\star}$ in four points which are conjugate with respect to the map $x \mapsto x^q$, $x \in \mathbb{F}_q$. That is, points of form X, X^q, X^{q^2}, X^{q^3} with $X, X^{q^2} \in g^{\bigstar}$ and $X^q, X^{q^3} \in g^{q\bigstar}$. Recalling the one-to-one correspondence between points of $g^{\bigstar}$ and points of the quadratic extension of ℓ_∞ to PG(2, q^4), there are points $\bar{P}$, $\bar{Q}$ on the quadratic extension of ℓ_∞ such that $P = X$ and $Q = X^{q^2}$. The argument of case (i) now generalizes by working in the quadratic extension of PG(2, q^2) to PG(2, q^4) and the quartic extension of PG(4, q) to PG(4, q^4). □

Moreover, the proofs of Theorems 5.8 and 5.9 show that the normal rational curve corresponding to an $\mathbb{F}_q$-conic $\mathcal{C}$ meets the transversal g of the regular spread $\mathcal{S}$ in points corresponding to the points $\mathcal{C}^+ \cap \ell_\infty$. The three cases when $\mathcal{C}^+$ is tangent, secant or exterior to ℓ_∞ are summarized in the next result.

Theorem 5.10. *In* PG(2, q^2), $q > 7$, *let* $\mathcal{B}$ *be a Baer subplane tangent to* ℓ_∞. *Let* $\mathcal{C}$ *be an* $\mathbb{F}_q$*-conic in* $\mathcal{B}$ *with* $\mathcal{B} \cap \ell_\infty \notin \mathcal{C}$, *so* $[\mathcal{C}]$ *is a* 4*-dimensional normal rational curve. The* $\mathbb{F}_{q^2}$*-conic* $\mathcal{C}^+$ *meets* ℓ_∞ *in two points denoted* $\bar{P}$, $\bar{Q}$, *possibly equal or in an extension. The three possibilities when* $\mathcal{C}^+$ *is tangent, secant or exterior to* ℓ_∞ *are as follows.*

(1) $\bar{P} = \bar{Q}$ *if and only if, in* PG(4, q^2), $[\mathcal{C}]^{\star}$ *meets the transversal* g *of* $\mathcal{S}$ *in the point* P.

(2) $\bar{P}, \bar{Q} \in \ell_\infty$ *if and only if, in* PG(4, q^2), $[\mathcal{C}]^{\star}$ *meets the transversal* g *of* $\mathcal{S}$ *in the two points* P, Q.

(3) $\bar{P}, \bar{Q}$ *lie in the extension* PG(2, q^4) *if and only if, in* PG(4, q^4), $[\mathcal{C}]^{\bigstar}$ *meets the extended transversal* $g^{\bigstar}$ *in the two points* P *and* Q.

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On triples of ideal chambers in A_2-buildings

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We investigate the geometry in a real Euclidean building X of type A_2 of some simple configurations in the associated projective plane at infinity $\mathbb{P}$, seen as ideal configurations in X, and relate it with the projective invariants (from the cross ratio on $\mathbb{P}$). In particular we establish a geometric classification of generic triples of ideal chambers of X and relate it with the triple ratio of triples of flags.

Introduction

The triples of objects in the boundaries of geometric spaces X are basic tools, for example in the study of surface group representations. For instance, in the case where $X = \mathbb{H}^2$, the hyperbolic plane, ideal triples of points may be used to define the notion of Euler class [Goldman 1980], and Penner–Thurston shear coordinates on the Teichmüller space. In the case where $X = \mathbb{H}^2_{\mathbb{C}}$, the ideal triples are classified by Cartan's angular invariant, see for example [Goldman 1999, §7.1], and they may be for instance used to define Toledo's invariant and maximal representations [Toledo 1989]. See for instance [Clerc and Neeb 2006; Burger et al. 2010] for generalization to higher rank Hermitian symmetric spaces X, and triples in their Shilov boundary.

For higher rank symmetric spaces X of type A_{N-1}, corresponding to the group $\mathrm{PGL}_N(\mathbb{R})$, ideal configurations in X may be seen as configurations in the projective space $\mathbb{P} = \mathbb{P}(\mathbb{R}^N)$. In particular, ideal chambers of X correspond to complete flags in $\mathbb{P}$, and opposite pairs of flags (or generic N-tuples of points) in $\mathbb{P}$ correspond to maximal flats in X. This is still true in the non-Archimedean setting, i.e., replacing $\mathbb{R}$ by an ultrametric valued field $\mathbb{K}$, in which case X is a Euclidean building of type A_{N-1}.

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Configurations in projective spaces $\mathbb{P}(\mathbb{R}^N)$ have been widely studied and used. In particular, triples of flags in $\mathbb{P}(\mathbb{R}^N)$ and their classical invariants (the triple ratio for $N = 3$), are the basic building blocks used by Fock and Goncharov [2006] to define generalized shearing coordinates for higher Teichmüller space, parametrizing positive representations of punctured surface groups in $G = \mathrm{SL}_N(\mathbb{R})$. But the geometric properties in the symmetric space or Euclidean building X of these configurations remain mysterious.

In this article, we investigate the geometry of some simple ideal configurations in a (not necessarily discrete) Euclidean building X of type A_2, mainly the generic triples of ideal chambers, and the relationship with their projective geometry in the projective plane $\mathbb{P}$. Our first motivation is to use it to study actions of surface groups on Euclidean buildings of type A_2, and degenerations of Hitchin representations in $\mathrm{SL}_3(\mathbb{R})$ (see [Parreau 2015]).

The main result is a classification of ideal triples of chambers by the geometry of the five naturally associated flats in X, in relation with their triple ratio as triples of flags in $\mathbb{P}$. In the case where X is a real tree (e.g., a Euclidean building of type A_1), any generic ideal triple bounds a *tripod* in X, that is a convex subset consisting of union of three rays from a point $x \in X$ (the *center* of the tripod). This is no longer the case in general in higher rank buildings like A_2-buildings, and many types of configurations are possible. A special case was studied by A. Balser [2008], who established a characterization of triples of points in $\partial_\infty X$ bounding a tripod in X, and used it to study convex rank 1 subsets in A_2-buildings. We give here a complete and precise description.

We now get into more details. Let X be a real Euclidean building of (vectorial) type A_2, i.e., with model flat the Euclidean plane

$$\mathbb{A} = \left\{ \lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 / \sum_i \lambda_i = 0 \right\}$$

endowed with the finite reflection group $W = \mathfrak{S}_3$ acting by permutation of the coordinates. Note that X is not necessarily discrete (simplicial) nor locally compact, and possibly exotic.

The boundary at infinity of X may be identified with the incidence graph of an associated projective plane $\mathbb{P} = \mathbb{P}_\infty(X)$, equipped with an $\mathbb{R}$-valued additive cross ratio $\boldsymbol{\beta}$ (called a projective valuation in [Tits 1986]) defined on quadruples of pairwise distinct collinear points in $\mathbb{P}$ [Tits 1986]. In the algebraic case, i.e., when X is the Bruhat–Tits building $X(\mathbb{K}^3)$ associated with the group $\mathrm{PGL}(\mathbb{K}^3)$ for some ultrametric field $\mathbb{K}$, the projective plane $\mathbb{P}$ is $\mathbb{P}(\mathbb{K}^3)$ and $\boldsymbol{\beta}$ is the logarithm

$$\boldsymbol{\beta} = \log|\boldsymbol{b}|$$

of the absolute value of the usual $\mathbb{K}$-valued cross ratio $\boldsymbol{b}$ on $\mathbb{P}(\mathbb{K}^3)$, where conventions on cross ratios are taken such that

$$\boldsymbol{b}(\infty, -1, 0, Z) = Z$$

in $\mathbb{P}^1\mathbb{K} = \mathbb{K} \cup \{\infty\}$ (following [Fock and Goncharov 2006]). We will then call $\boldsymbol{\beta}$ the *geometric* cross ratio and $\boldsymbol{b}$ the *algebraic* cross ratio to distinguish between them.

We now turn to ideal triples of chambers. Let $T = (F_1, F_2, F_3)$ be a triple of chambers at infinity of X. We denote by $F_i = (p_i, D_i)$ the corresponding flag of $\mathbb{P}$, with p_i the point and D_i the line. The set $\{1, 2, 3\}$ of indices will be canonically identified with $\mathbb{Z}/3\mathbb{Z}$. A triple $T = (F_1, F_2, F_3)$ will be called *generic* if the flags $(F_i)_i$ are pairwise opposite, the points $(p_i)_i$ are not collinear and the lines $(D_i)_i$ are not concurrent.

In the algebraic case $\mathbb{P} = \mathbb{P}(\mathbb{K}^3)$ generic triples of flags $T = (F_1, F_2, F_3)$ are classified by one $\mathbb{K}$-valued invariant, the (*algebraic*) triple ratio (see for example [Fock and Goncharov 2006, §9.4]), that may be defined by:

$$\mathrm{Tri}(F_1, F_2, F_3) = \boldsymbol{b}(D_1, p_1p_2, p_1p_{23}, p_1p_3) \tag{0-1}$$

where $p_{ij} = D_i \cap D_j$. We recall that it is invariant under cyclic permutations of T, and that reversing the order inverses the algebraic triple ratio: $\mathrm{Tri}(\overline{T}) = \mathrm{Tri}(T)^{-1}$, where $\overline{T} = (F_3, F_2, F_1)$.

In the general case, we introduce an invariant for generic triples of flags in $\mathbb{P}$, analogous to the algebraic triple ratio: the *geometric triple ratio*, which still make sense when the building X is exotic (nonalgebraic), whereas the algebraic triple ratio is not defined anymore. We define it as the triple

$$\mathrm{tri}(T) = (\mathrm{tri}_m(T))_{m=1,2,3}$$

of the following cross ratios in $\mathbb{P}$, which are the cross ratios obtained from the four lines $D_1, p_1p_2, p_1p_{23}, p_1p_3$ by cyclic permutation of the three last one:

$$\mathrm{tri}_1(F_1, F_2, F_3) = \boldsymbol{\beta}(D_1, p_1p_2, p_1p_{23}, p_1p_3)$$
$$\mathrm{tri}_2(F_1, F_2, F_3) = \boldsymbol{\beta}(D_1, p_1p_3, p_1p_2, p_1p_{23})$$
$$\mathrm{tri}_3(F_1, F_2, F_3) = \boldsymbol{\beta}(D_1, p_1p_{23}, p_1p_3, p_1p_2).$$

To simplify notations, we denote from now on

$$z_m = \mathrm{tri}_m(T) \quad \text{and} \quad z = (z_1, z_2, z_3) = \mathrm{tri}(T)$$

In the algebraic case, we have $\mathbb{P} = \mathbb{P}(\mathbb{K}^3)$ and the geometric triple ratio is obtained from the algebraic cross ratio $Z = \mathrm{Tri}(T) \in \mathbb{K}$ by:

$$z_1 = \log|Z|, \quad z_2 = \log\left|\frac{1}{1+Z}\right| = -\log|1+Z|, \quad z_3 = \log|1+Z^{-1}|.$$

The geometric triple ratio z enjoys the following properties. It is invariant by cyclic permutations of the flags, and changed to $(-z_1, -z_3, -z_2)$ by permutations reversing the cyclic order. We also have $z_1 + z_2 + z_3 = 0$, and the stronger following property: for all $m \in \mathbb{Z}/3\mathbb{Z}$, if $z_m > 0$ then $z_{m-1} = 0$ and $z_{m+1} = -z_m < 0$. Note that the three natural cases: $z \in \mathbb{R}_+(0, 1, -1)$, $z \in \mathbb{R}_+(-1, 0, 1)$, and $z \in \mathbb{R}_+(1, -1, 0)$ subdivide in two types, as the case $z_1 = 0$ is invariant under reversing the order of T, whereas the two other cases are exchanged.

We now turn to the geometry inside the Euclidean building X. A generic triple $T = (F_1, F_2, F_3)$ of ideal chambers defines five natural flats in X: the three flats $A_{ij} = A(F_i, F_j)$ containing the opposite chambers F_i and F_j in their boundaries, the flat $A_p = A(p_1, p_2, p_3)$ containing the triple of ideal singular points (p_1, p_2, p_3) in its boundary, and the similarly defined flat $A_D = A(D_1, D_2, D_3)$. We will show that there are also six particular points in X naturally associated with the triple T, that may be defined as the orthogonal projections y_i and y_i^* (which happen to be unique) of p_i and D_i on the flat A_{jk} where $j = i + 1$ and $k = i + 2$.

We say that (F_1, F_2, F_3) is of type *"tripod"* if there exists a tripod in X joining the three (middle points of the) ideal chambers (F_1, F_2, F_3). The set of centers of such tripods is the intersection I of the three flats A_{ij}.

We show that either the three flats A_{ij} have a nonempty intersection, that is (F_1, F_2, F_3) is of type "tripod", or the two flats A_p and A_D have non empty intersection Δ, which is then a *flat singular triangle* (that is, a triangle in $\mathbb{A}$ with singular sides) (we then say that (F_1, F_2, F_3) is of type *"flat"*). The two following results describe more precisely the two possible types, and relate them with the points y_i, y_i^* and the geometric triple ratio z. We denote by $\mathfrak{C} = \{\lambda \in \mathbb{A} / \ \lambda_1 > \lambda_2 > \lambda_3\}$ the model Weyl chamber of $\mathbb{A}$ and we use the corresponding *simple roots coordinates* on $\mathbb{A}$, that is $\lambda = (\lambda_1 - \lambda_2,\ \lambda_2 - \lambda_3)$.

Theorem 0.1 (type "tripod"). *The intersection $I = A_{12} \cap A_{23} \cap A_{31}$ is nonempty if and only if $z_1 = 0$. Then $z_2 \geq 0$ and there exist a unique pair (x, x^*) in X such that*:

(i) $y_1 = y_2 = y_3 = x$ *and* $y_1^* = y_2^* = y_3^* = x^*$.

(ii) *I is the segment* $[x, x^*]$.

(iii) $[x, x^*]$ *is the unique shortest segment joining A_p to A_D.*

(iv) *Identifying A_{ij} with $\mathbb{A}$ by a marked flat $f : \mathbb{A} \mapsto A_{ij}$ sending $\mathfrak{C}$ to F_j, in simple roots coordinates, we have $\overrightarrow{xx^*} = (-z_2, z_2)$. In particular x^* is on the ray $[x, p_{ij})$ from x to p_{ij}.*

Theorem 0.2 (type "flat"). *The intersection $A_p \cap A_D$ is nonempty if and only if* $(z_2 = 0$ *or* $z_3 = 0)$, *or, equivalently, if and only if $z_2 \leq 0$. Then there exists a unique flat singular triangle $\Delta \subset X$ with vertices x_1, x_2, x_3 such that*:

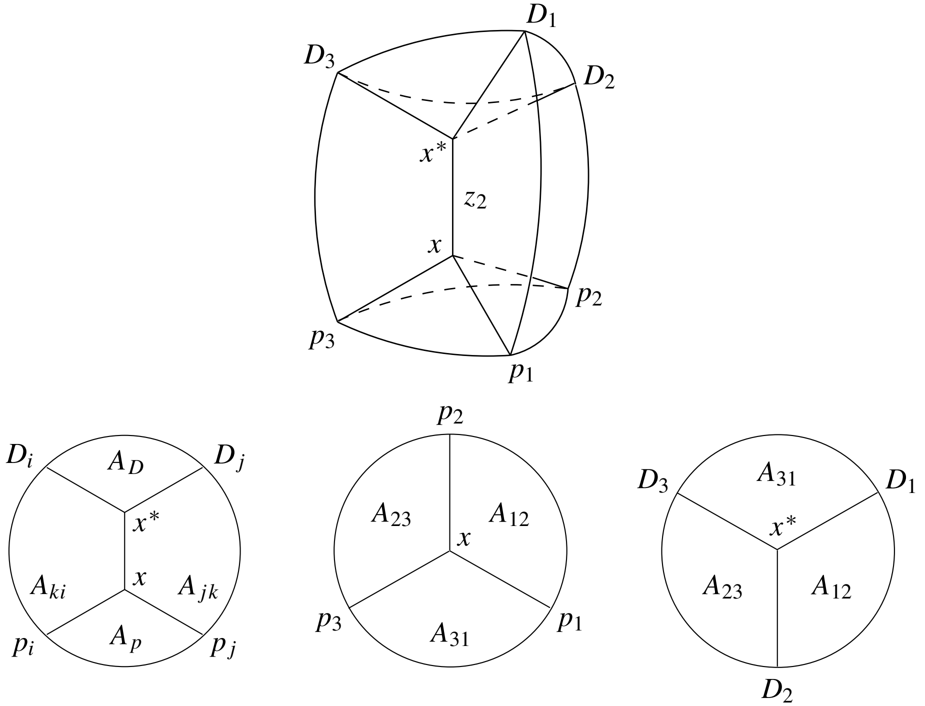


Figure 1. Type "tripod". Bottom row: projections in the flat A_{ij} (left), in the flat A_p (middle) and in the flat A_D.

(i) $A_p \cap A_D = \Delta$.

(ii) $A_{ij} \cap A_{ik}$ *is the Weyl chamber from* x_i *to* F_i.

(iii) *Let* $i \in \{1, 2, 3\}$ *and* $j = i+1$. *In a marked flat* $f : \mathbb{A} \mapsto A_{ij}$ *sending* $\mathfrak{C}$ *to* F_j, *in simple roots coordinates, we have* $\overrightarrow{x_i x_j} = ({z_1}^+, {z_1}^-)$ *where* ${z_1}^+ = \max(z_1, 0)$ *and* ${z_1}^- = \max(-z_1, 0)$. *In particular* x_j *is on the ray from* x_i *to* p_j (*if* $z_1 \geq 0$) *or* D_j (*if* $z_1 \geq 0$).

(iv) *The germs of Weyl chambers at* x_i *respectively defined by* Δ *and* F_i *are opposite* (*in the spherical building of directions at* x_i). *In particular there exists a flat containing* Δ, *and containing* F_i *in its boundary.*

Furthermore if $z_1 \geq 0$ *we have* $x_i = y_{i-1} = y^*_{i+1}$ *for all* i, *and if* $z_1 \leq 0$ *we have* $x_i = y_{i+1} = y^*_{i-1}$ *for all* i.

The intersections of each flat with the four other flats form a partition (i.e., a covering with disjoint interiors), which is described in Figure 1 for the type "tripod", and in Figure 2 for the type "flat" (see Proposition 4.2, Corollary 4.3 and Proposition 4.5).

The special case where the hypotheses of both Theorems 0.1 and 0.2 are satisfied corresponds to the case where $z_1 = z_2 = z_3 = 0$. Then the five flats intersect in

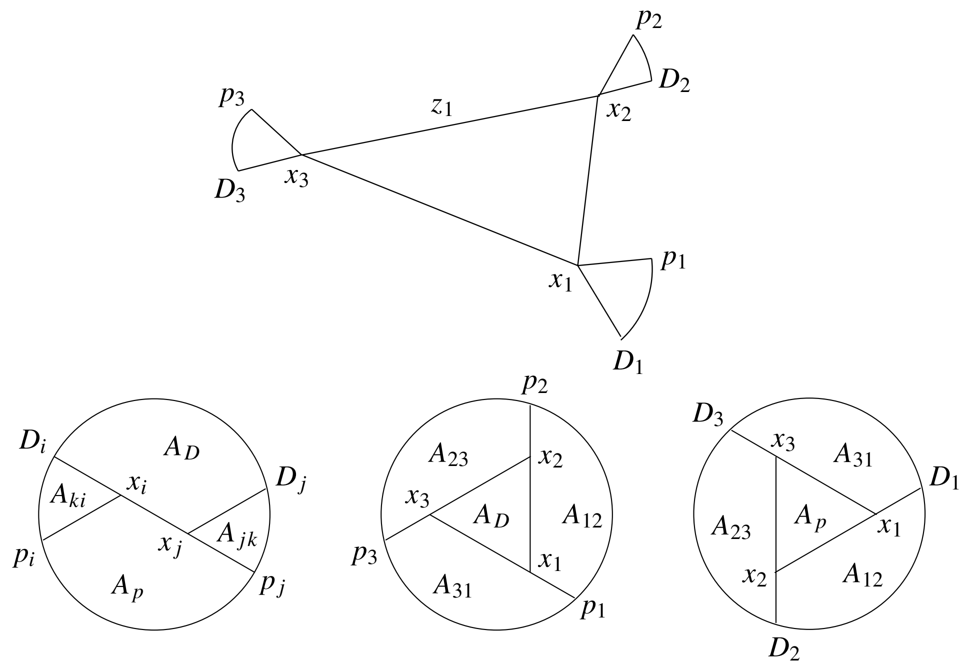


Figure 2. Type "flat", in the case where $z_1 \geq 0$. Bottom row: projections in A_{ij}, with $j = i+1$. (left), in A_p (middle) and in A_D. The case $z_1 \leq 0$ is obtained from the case $z_1 \geq 0$ by reversing the order of the flags F_i, that is, by exchanging 1 and 3 and i and j in the diagrams.

a unique point x, and, in the spherical building of directions at x, the triple of chambers induced by $T = (F_1, F_2, F_3)$ is generic.

In particular we recover the characterization of [Balser 2008] for triples of points in $\partial_\infty X$ bounding a tripod in X. Note that M. Talbi [2006] established some analogous geometric classification for interior triangles in discrete Euclidean buildings of type A_2.

Theorem 0.2 will be used in [Parreau 2015] to study actions of punctured surface groups on Euclidean buildings of type A_2. It allows us to give a metric interpretation, in the building, of Fock–Goncharov parameters associated with ideal triangulations. We are then able to construct in X an invariant weakly convex cocompact 2-complex for large families of actions. Theorem 0.2 enables us to associate to each triangle of the triangulation a flat singular triangle in X, the complex is then obtained by connecting them gluing flat strips. This allows to describe length spectra for large families of degenerations of convex projective structures on surfaces.

We also show that generic quadruples of points in $\mathbb{P}$ (which will be called *projective frames*) define a nice center in X, with various characterizations, see Proposition 2.4 (this result generalizes to higher rank $\mathbb{R}$-buildings of type A_{N-1}).

1. Preliminaries

1A. ***The model flat*** **$(\mathbb{A}, W)$** ***of type*** **A_{N-1}.** Let $N \geq 2$ be an integer. The *model flat* of type A_{N-1} is the vector space $\mathbb{A} = \mathbb{R}^N/\mathbb{R}(1, \dots, 1)$, endowed with the action of the *Weyl group* $W = \mathfrak{S}_N$ acting on $\mathbb{A}$ by permutation of coordinates (finite reflection group). We denote by $[\lambda]$ the projection in $\mathbb{A}$ of a vector λ in $\mathbb{R}^N$. The vector space $\mathbb{A}$ may be identified with the hyperplane $\{\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N \big/ \sum_i \lambda_i = 0\}$ of $\mathbb{R}^N$. Recall that a vector in $\mathbb{A}$ is called *singular* if it belongs to one the hyperplanes $\lambda_i = \lambda_j$, and *regular* otherwise. A *(open) (vectorial) Weyl chamber* of $\mathbb{A}$ is a connected component of regular vectors. We will call a *sector* a more general convex cone in $\mathbb{A}$, in particular the closed convex cone formed by the union of the closed Weyl chambers containing a given singular ray. The *model Weyl chamber* is the simplicial cone

$$\mathfrak{C} = \{\lambda \in \mathbb{A} / \ \lambda_1 > \cdots > \lambda_N\}.$$

Its closure $\overline{\mathfrak{C}}$ is a strict fundamental domain for the action of W on $\mathbb{A}$. Recall that two nonzero vectors λ and λ' of $\mathbb{A}$ are called *opposite* if $\lambda' = -\lambda$. Similarly, two Weyl chambers C and C' of $\mathbb{A}$ are *opposite* if $C' = -C$. The *type* of a vector $\lambda \in \mathbb{A}$ is its projection (modulo W) in $\overline{\mathfrak{C}}$.

We denote by $\partial\mathbb{A}$ the sphere of unitary vectors in $\mathbb{A}$, identified with the set $\mathbb{P}^+(\mathbb{A}) = (\mathbb{A} - \{0\})/\mathbb{R}_{>0}$ of rays issued from 0, and by $\partial : \mathbb{A} - \{0\} \to \partial\mathbb{A}$ the corresponding projection. The *type (of direction)* of a nonzero vector $\lambda \in \mathbb{A}$ is its canonical projection in $\partial\overline{\mathfrak{C}}$.

We denote by $(\varepsilon_1, \dots, \varepsilon_N)$ the canonical basis of $\mathbb{R}^N$. For $d = 1, \dots, N-1$, we will say that a nonzero vector in $\mathbb{A}$ (or a point in the sphere $\partial\mathbb{A}$) is *singular of type d* if its canonical projection in $\partial\overline{\mathfrak{C}}$ is $[\varepsilon_1 + \cdots + \varepsilon_d]$.

The *simple roots* (associated with $\mathfrak{C}$) are the following linear forms on $\mathbb{A}$

$$\varphi_i : \lambda \mapsto \lambda_i - \lambda_{i+1}$$

for $i = 1, \dots, N-1$. The set of simple roots is denoted by Λ. We will also use the root $\varphi_N : \lambda \mapsto \lambda_N - \lambda_1$ satisfying

$$\varphi_1 + \cdots + \varphi_N = 0.$$

The vector space $\mathbb{A}$ is endowed with the unique W-invariant Euclidean scalar product, which is well defined up to homothety (induced by the standard Euclidean scalar product of $\mathbb{R}^N$). We will normalize it by requiring that the simple roots have unit norm, i.e., the distance between the two hyperplanes with equation $\varphi_i = 0$ and $\varphi_i = 1$ is 1 for one (all) i. When $\dim \mathbb{A} = 1$, we will identify $\mathbb{A}$ with $\mathbb{R}$ by the basis $\{[\varepsilon_1]\}$, i.e., by the map from $s \mapsto [(s, 0)]$ from $\mathbb{R}$ to $\mathbb{A}$, which is an isometry in the above normalization.

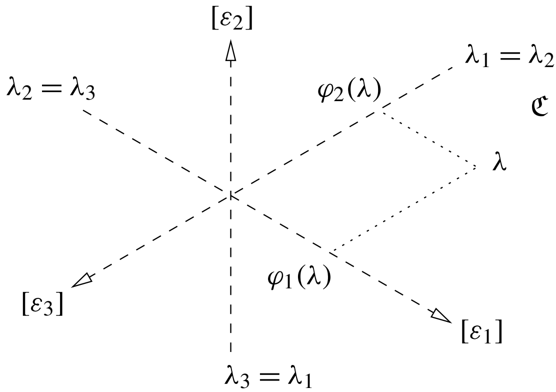


Figure 3. The model flat $\mathbb{A}$ of type A_2 (for $N = 3$), and simple roots coordinates. The arrows denote the singular directions of type 1.

1B. ***Projective spaces.*** We here collect the notations and vocabulary for projective spaces, which will be used throughout this article. We refer to [Tits 1974, §6.2]. Let $\mathbb{P}$ be a projective space of dimension $N-1$, with $N \geq 2$. We denote by Flags($\mathbb{P}$) the set of flags of $\mathbb{P}$, that is increasing sequences $(V_1, \ldots, V_M)$ of proper linear subspaces of $\mathbb{P}$. We denote by $\mathbb{P}^*$ the dual projective space, whose set of points is the set of hyperplanes of $\mathbb{P}$.

Two maximal flags $(V_1, \ldots, V_{N-1})$ and $(V'_1, \ldots, V'_{N-1})$ are *opposite* if they are in generic position, that is if $V_i \oplus V'_{n-i} = \mathbb{P}$ for all i. A finite subset $p_1, \ldots, p_M$ in $\mathbb{P}$, with $2 \leq M \leq N$, is called *independent* if it is not contained in any linear subspace of dimension $M-2$ of $\mathbb{P}$. Then it is contained in a unique $(M-1)$-dimensional linear subspace of $\mathbb{P}$, which will be denoted by $p_1 \oplus \cdots \oplus p_M$. When $M = 2$, we will also denote the line $p \oplus q$ by pq.

A *frame* of $\mathbb{P}$ is an independent N-tuple. A *projective frame* in $\mathbb{P}$ is a $(N+1)$-tuple $(p_0, p_1, \ldots, p_N)$ of points in $\mathbb{P}$ in generic position, i.e., such that the induced N-tuple $(p_0, \ldots, \hat{p}_i, \ldots, p_N)$ is a frame in $\mathbb{P}$ for all i.

If p is a point in $\mathbb{P}$, we denote by $\mathbb{P}/p$ the set of lines through p, which is a projective space of dimension $N-2$ whose linear subspaces are the linear subspaces of $\mathbb{P}$ containing p. The *projection at p* is the corresponding projection $\mathrm{proj}_p : q \mapsto pq$ from $\mathbb{P} - \{p\}$ to $\mathbb{P}/p$. If p is a point of $\mathbb{P}$ and $H \subset \mathbb{P}$ an hyperplane with $p \notin H$, then the projection proj_p induces a canonical isomorphism $\mathrm{proj}_{Hp} : H \xrightarrow{\sim} \mathbb{P}/p$ (called *perspectivity*).

Note that if $\mathcal{F} = (p_1, \ldots, p_M)$ is independent in $\mathbb{P}$, then its projection $\mathrm{proj}_{p_1}(\mathcal{F}) = (p_1 p_2, \ldots, p_1 p_M)$ at p_1 is independent in $\mathbb{P}/p_1$. In particular the projection of a (projective) frame at one of its points is a (projective) frame.

1C. ***Spherical buildings of type A_{N-1} and associated projective spaces.*** See [Tits 1974, §6]. A spherical building $\mathcal{B}$ of type A_{N-1} is the building of flags of an associated projective space $\mathbb{P} = \mathbb{P}(\mathcal{B})$ of dimension $N-1$. For $d = 0, 1, \dots, N-1$, the set of linear subspaces of dimension d of $\mathbb{P}$ identifies with the subset of vertices of type $d+1$ of $\mathcal{B}$. In particular, the projective space $\mathbb{P}$ itself is identified with the set of vertices of type 1 of $\mathcal{B}$, and the dual projective space $\mathbb{P}^*$ is identified with the set of vertices of type $N-1$.

In the algebraic case, that is when $\mathcal{B}$ is the spherical building of flags of some vector space V of dimension N over a field $\mathbb{K}$, then $\mathbb{P} = \mathbb{P}(V)$.

A basic fact is that frames in $\mathbb{P}$ correspond to apartments of $\mathcal{B}$.

Recall that, in (the geometric realization modeled on $(\partial\mathbb{A}, W)$ of) a spherical building, any two points (resp. chambers) are contained in a common apartment, and that they are *opposite* if they are opposite in that apartment, that is, for two points ξ and ξ', if and only if $\sphericalangle(\xi, \xi') = \pi$ for the canonical metric $\sphericalangle$ on $\mathcal{B}$. Note that $p \in \mathbb{P}$ and $H \in \mathbb{P}^*$ are opposite if and only if $\sphericalangle(p, H) = \pi$, if and only if $p \notin H$. Two chambers are opposite if and only if they are opposite as maximal flags in $\mathbb{P}$. In particular, in the type A_2 case, two chambers $F_1 = (p_1, D_1)$, $F_2 = (p_2, D_2)$ are opposite if and only if $p_1 \notin D_2$ and $p_2 \notin D_1$.

For any simplex σ of $\mathcal{B}$ the *residue* $\mathrm{St}(\sigma)$ of σ is the spherical building formed by the simplices of $\mathcal{B}$ containing σ. If H is a hyperplane of $\mathbb{P}$, the residue $\mathrm{St}(H)$ of H in $\mathcal{B}$ is the subset of flags of $\mathbb{P}$ containing H. It canonically identifies with the spherical building $\mathrm{Flags}(H)$ of flags of H by the map $(V_1, \dots, V_M, H) \mapsto (V_1, \dots, V_M)$. The residue $\mathrm{St}(p)$ of a point p in $\mathbb{P}$ identifies canonically with the flag building $\mathrm{Flags}(\mathbb{P}/p)$ of $\mathbb{P}/p$ by the map $(V_1 = p, \dots, V_M) \mapsto (V_2/p, \dots, V_M/p)$. If $p \notin H$ then the projection proj_p induces a canonical isomorphism $\mathrm{proj}_{Hp} : \mathrm{St}(H) \xrightarrow{\sim} \mathrm{St}(p)$ of spherical buildings (perspectivity).

1D. ***Euclidean buildings.*** Euclidean buildings considered in this article are (not necessarily discrete) Euclidean buildings of type A_{N-1}. We refer for example to [Parreau 2000] for the definition and properties of Euclidean buildings we use below (see also [Tits 1986; Kleiner and Leeb 1997; Rousseau 2009]). Recall that a *Euclidean building of type A_{N-1}* is a CAT(0) metric space X endowed with a (maximal) collection $\mathcal{A}$ of isometric embeddings $f : \mathbb{A} \to X$ called *marked apartments*, or *marked flats* by analogy with Riemannian symmetric spaces, satisfying the following properties:

- (**A1**) $\mathcal{A}$ is invariant by precomposition by W_{aff}.
- (**A2**) If f and f' are two marked flats, then the transition map $f^{-1} \circ f'$ is the restriction of an element of W_{aff}.
- (**A3′**) Any two rays of X are initially contained in a common marked flat.

Where W_{aff} denotes the subgroup of all affine isomorphisms of $\mathbb{A}$ with linear part in W. The *flats* and the *Weyl chambers* of X are the images of $\mathbb{A}$ and $\mathfrak{C}$ by the marked flats, respectively.

Algebraic case. Let $\mathbb{K}$ be an ultrametric field, i.e., a field endowed with an ultrametric absolute value $|\cdot|$ (not necessarily discrete). When V is a finite N-dimensional vector space over $\mathbb{K}$, we denote by $X = X(V)$ the Euclidean building associated with $G = \mathrm{PGL}(V)$. We refer for example to [Parreau 2000] for the model of norms for X (see [Goldman and Iwahori 1963; Bruhat and Tits 1984]). To each basis $\boldsymbol{v}$ of V is then associated a marked flat $f_{\boldsymbol{v}} : \mathbb{A} \to A_{\boldsymbol{v}} \subset X$, such that, if a is an element of G with diagonal matrix $\mathrm{diag}(a_1, \dots, a_N)$ in the basis $\boldsymbol{v}$, then a translates the flat $A_{\boldsymbol{v}}$ by the vector

$$\nu(a) = [(\log|a_i|)_i]$$

in $\mathbb{A}$ (identifying the flat $A_{\boldsymbol{v}}$ with the model flat $\mathbb{A}$ through the marking $f_{\boldsymbol{v}}$).

From now to Section 1H, X will denote a Euclidean building of type A_{N-1}.

1E. *Spherical building and projective space at infinity.* The CAT(0) boundary $\partial_\infty X$ of X is the geometric realization modeled on $(\partial\mathbb{A}, W)$ of a spherical building of type A_{N-1} whose chambers are the boundaries of the Weyl chambers of X, and whose apartments are the boundaries of the flats of X. It will be identified with the building of flags on the associated projective space $\mathbb{P} = \mathbb{P}_\infty(X)$, whose points are the vertices of type 1 of $\partial_\infty X$. If c_+ and c_- are opposite ideal chambers, then we denote by $A(c_-, c_+)$ the unique flat *joining* c_- to c_+ in X, that is, containing c_- and c_+ in its boundary. If $\mathcal{F}$ is a frame of $\mathbb{P}$ or $\mathbb{P}^*$, then there is a unique flat $A(\mathcal{F})$ of X containing $\mathcal{F}$ in its boundary.

1F. *Local spherical building and projective space at a point.* Recall that, in Euclidean buildings, two (unit speed) geodesic segments issued from a common point x have zero angle if and only if they have same germ at x (i.e., coincide in a neighborhood of x). A *direction at $x \in X$* is a germ of nontrivial geodesic segment from x. A direction, geodesic segment, ray or line has a well-defined *type (of direction)* in $\partial\overline{\mathfrak{C}}$, which is its canonical projection (through a marked flat) in $\partial\overline{\mathfrak{C}}$. It is called *singular* or *regular* accordingly.

The *space of directions* at x of X is the quotient space of non trivial geodesic segments from x for this relation, with the induced angular metric, and is denoted by $\Sigma_x X$. We denote by $\Sigma_x : X - \{x\} \to \Sigma_x X$, $y \to \Sigma_x y$, the associated projection. Its extension to the boundary at infinity will also be denoted by $\Sigma_x : \partial_\infty X \to \Sigma_x X$, $\xi \to \Sigma_x \xi$ and called the *canonical projection*.

The space of directions $\Sigma_x X$ inherits the structure of a spherical A_{N-1}-building, whose apartments are the germs $\Sigma_x A$ at x of the flats A of X passing through x, and whose chambers are the germs $\Sigma_x C$ at x of the Weyl chambers C of X with vertex

x (see for example [Parreau 2000]). The canonical projection $\Sigma_x : \partial_\infty X \to \Sigma_x X$ sends chambers to chambers (and, more generally, simplices to simplices) and preserves the type of points.

The *local projective space* $\mathbb{P}_x = \mathbb{P}_x(X)$ *at* x is the projective space of dimension $N-1$ associated with the spherical building $\Sigma_x X$ of type A_{N-1} (see Section 1C). Its underlying set is the set of vertices of type 1 of $\Sigma_x X$.

The canonical projection $\Sigma_x : \partial_\infty X \to \Sigma_x X$ induces (by restriction to vertices) a surjective morphism (of projective spaces) $\Sigma_x : \mathbb{P} \to \mathbb{P}_x$ from the projective space at infinity $\mathbb{P}$ to the local projective space $\mathbb{P}_x$ at x. Note that, in particular, if $\mathcal{F}$ is a frame of $\mathbb{P}$, then x belongs to the associated flat $A(\mathcal{F})$ if and only if $\Sigma_x(\mathcal{F})$ is a frame of $\mathbb{P}_x$.

1G. ***Transverse spaces at infinity.*** See for example [Tits 1986, §8; Leeb 2000, 1.2.3; Mühlherr et al. 2014, §4]. Let ξ be a vertex of $\partial_\infty X$ of type 1 or $N-1$, i.e., either a point p in the projective plane at infinity $\mathbb{P}$ or a hyperplane H of $\mathbb{P}$.

The *transverse space* X_ξ at ξ may be defined, from the metric viewpoint (as in [Leeb 2000, 1.2.3]), as the quotient space of the set of all rays to ξ by the pseudodistance d_ξ given by

$$d_\xi(r_1, r_2) = \inf_{t_1, t_2} d(r_1(t_1), r_2(t_2)).$$

We denote by $\pi_\xi : X \to X_\xi$ the canonical projection (which maps x to the class of the unique ray from x to ξ). The space X_ξ is a Euclidean building of type A_{N-2}, whose flats are the projections to X_ξ of the flats of X containing a ray to ξ. In particular, when X is of type A_2, the transverse space X_ξ is an $\mathbb{R}$-tree, and we will call it the *transverse tree* at ξ.

In the algebraic case, i.e., when $X = X(V)$, the transverse space X_H canonically identifies with the building $X(H)$ of H, where H is seen as an hyperplane of V, and X_p identifies with $X(V/p)$, where p is seen as a 1-dimensional subspace of V.

The spherical building $\partial_\infty X_\xi$ at infinity of X_ξ identifies canonically with the residue $\mathrm{St}(\xi)$ of ξ. In particular, if p is a point in $\mathbb{P}$, the projective space at infinity of X_p identifies with $\mathbb{P}/p$, and if H is an hyperplane of $\mathbb{P}$, the projective space at infinity of X_H identifies with H.

If $\mathcal{F} = (p_1, \dots, p_N)$ is a frame in $\mathbb{P} \subset \partial_\infty X$, then the projection on X_{p_1} of the flat $A(p_1, \dots, p_N)$ is the flat defined by the projection $\mathrm{proj}_{p_1}(\mathcal{F}) = (p_1 p_2, \dots, p_1 p_N)$ of the frame $\mathcal{F}$, i.e., $\pi_{p_1}(A(\mathcal{F})) = A(\mathrm{proj}_{p_1}(\mathcal{F}))$.

We now describe the canonical isomorphism $\pi_{\xi^-\xi^+} : X_{\xi^-} \xrightarrow{\sim} X_{\xi^+}$ for opposite points ξ^- and ξ^+ of $\partial_\infty X$. The union $F_{\xi^-\xi^+}$ of all geodesics joining ξ^- to ξ^- is a convex closed subspace and a subbuilding, whose flats are the flats of X containing a geodesic joining ξ^- to ξ^- (see [Kleiner and Leeb 1997, Proposition 4.8.1] and [Parreau 2012, 2.2.1]). We denote by $F_{\xi^-\xi^+} = X^{\xi^-\xi^+} \times \mathbb{R}$ the canonical decomposition (see [Parreau 2011, 1.2.10]). The restriction of the projection π_{ξ^+} to $F_{\xi^-\xi^+}$ is surjective and factorizes through the projection on the first factor, inducing a canonical isomorphism of Euclidean buildings $X^{\xi^-\xi^+} \xrightarrow{\sim} X_{\xi^+}$. We similarly have a isomorphism $X^{\xi^-\xi^+} \xrightarrow{\sim} X_{\xi^-}$, so it induces a canonical isomorphism $\pi_{\xi^-\xi^+} : X_{\xi^-} \xrightarrow{\sim} X_{\xi^+}$. It is easy to see that the map $\pi_{\xi^-\xi^+}$ extends to the boundaries at infinity of X_{ξ^-} and X_{ξ^+} by the canonical isomorphism of spherical buildings $\mathrm{proj}_{\xi^-\xi^+} : \mathrm{St}(\xi^-) \xrightarrow{\sim} \mathrm{St}(\xi^+)$ (perspectivity).

1H. ***The $\mathbb{A}$-valued Busemann cocycle.*** Let c be a chamber at infinity of X. We now define the $\mathbb{A}$-valued *Busemann cocycle*

$$B_c : X \times X \to \mathbb{A}$$

associated to c. It can be simply defined from canonical retractions as

$$B_c(x, y) := r(y) - r(x)$$

where $r : X \to \mathbb{A}$ is any canonical retraction centered at c, sending c to $\partial\mathfrak{C}$ (see [Parreau 2000, Proposition 1.19]). More precisely, the Buseman cocycle at c is characterized by the property:

$$B_c(f(\lambda), f'(\lambda')) = \lambda' - \lambda$$

for any two marked flats $f, f' : \mathbb{A} \to X$ sending $\partial\mathfrak{C}$ to c and such that $f = f'$ on some subchamber of $\mathfrak{C}$.

We clearly have

$$B_c(x, z) = B_c(x, y) + B_c(y, z).$$

When $\dim \mathbb{A} = 1$, it coincides with the usual Busemann cocycle, which is defined for $\xi \in \partial_\infty X$ by

$$B_\xi(x, y) = \lim_{z \to \xi} d(x, z) - d(y, z).$$

In the type A_2 case, the simple root coordinates of $\mathbb{A}$-valued Busemann cocycles may be determined by projecting in transverse trees at infinity, using the following relations (using the normalization of the metric).

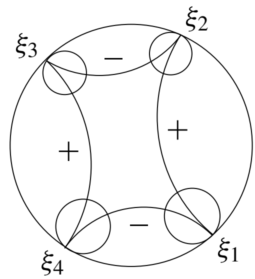
$$\begin{aligned} \varphi_1(B_{(p,D)}(x, y)) &= B_p(\pi_D(x), \pi_D(y)), \\ \varphi_2(B_{(p,D)}(x, y)) &= B_D(\pi_p(x), \pi_p(y)). \end{aligned} \tag{1-1}$$

We now turn to cross ratios.

1I. ***Cross ratio on the boundary of a tree.*** See [Tits 1986, §7], and in a more general setting [Otal 1992; Bourdon 1996]. In this section, we suppose that X is an $\mathbb{R}$-tree. Given three distinct ideal points ξ_1, ξ_2, ξ_3 in $\partial_\infty X$, we denote by $c(\xi_1, \xi_2, \xi_3)$ the *center* of the ideal triple (ξ_1, ξ_2, ξ_3), that is the unique common intersection point of the three geodesic lines joining two of the three points. Note that $c(\xi_1, \xi_2, \xi_3)$ is the (orthogonal) projection of ξ_3 on the geodesic joining ξ_1 to ξ_2. We denote by $B_\xi(x, y)$ the Busemann cocycle (see Section 1H).

Define the *cross ratio* of four pairwise distinct points $\xi_1, \xi_2, \xi_3, \xi_4$ in $\partial_\infty X$ by

$$\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) = \tfrac{1}{2}(\ell_{12} - \ell_{23} + \ell_{34} - \ell_{41})$$



where ℓ_{ij} is the length of the geodesic in X from ξ_i to ξ_j after removing disjoint fixed horoballs centered at each ξ_k. It does not depended on the choice of the horoballs since the horoballs centered at a given point are equidistant along the rays to that point.

The cross ratio naturally extends to *nondegenerate* quadruples, that are quadruples $(\xi_1, \xi_2, \xi_3, \xi_4)$ *without triple point* (i.e., any three of the points are not equal), which is equivalent to the following condition:

$$(\xi_1 \neq \xi_4 \text{ and } \xi_2 \neq \xi_3) \quad \text{or} \quad (\xi_1 \neq \xi_2 \text{ and } \xi_3 \neq \xi_4). \tag{1-2}$$

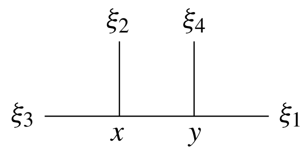
We then set

$$\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) = \begin{cases} 0 & \text{when } \xi_1 = \xi_3 \text{ or } \xi_2 = \xi_4, \\ -\infty & \text{when } \xi_1 = \xi_2 \text{ or } \xi_3 = \xi_4, \\ +\infty & \text{when } \xi_1 = \xi_4 \text{ or } \xi_2 = \xi_3. \end{cases}$$

We now recall some basic properties that we will use.

The cross ratio may be read inside the tree on the oriented geodesic from ξ_3 to ξ_1, as the oriented distance $\overrightarrow{xy}$ from the center x of the ideal triple (ξ_3, ξ_1, ξ_2) to the center y of the ideal triple (ξ_3, ξ_1, ξ_4):

$$\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) = \overrightarrow{xy} = B_{\xi_1}(x, y). \tag{1-3}$$



The cocycle identity is

$$\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) + \boldsymbol{\beta}(\xi_1, \xi_4, \xi_3, \xi_5) = \boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_5).$$

The cross ratio $\boldsymbol{\beta}$ is left unchanged by the double transpositions and changed to $-\boldsymbol{\beta}$ by the transpositions (13) and (24). We now consider the behavior under cyclic permutations of the three last terms. We have

$$\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) + \boldsymbol{\beta}(\xi_1, \xi_4, \xi_2, \xi_3) + \boldsymbol{\beta}(\xi_1, \xi_3, \xi_4, \xi_2) = 0. \tag{1-4}$$

Moreover, the following *ultrametricity* property (specific to the case of trees) is easy to prove using (1-3) (see [Tits 1986, §7, Proposition 3]):

If $\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) > 0$,

$$\text{then } \boldsymbol{\beta}(\xi_1, \xi_3, \xi_4, \xi_2) = 0 \text{ and } \boldsymbol{\beta}(\xi_1, \xi_4, \xi_2, \xi_3) = -\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4). \quad (1\text{-}5)$$

Note that (1-5) is equivalent (under (1-4)) to

$$\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) \le \max(0, -\boldsymbol{\beta}(\xi_1, \xi_4, \xi_2, \xi_3)). \quad (1\text{-}6)$$

which in the algebraic case follows from the symmetry properties of the cross ratio under 3-cyclic permutations (1-9).

1J. ***Algebraic case: link with usual cross ratio.*** Suppose that X is the tree $X(V)$ associated with a 2-dimensional vector space V over an ultrametric field $\mathbb{K}$ (see Section 1D). Then $\partial_\infty X$ identifies with the projective line $\mathbb{P}(V)$.

The usual cross ratio $\boldsymbol{b}$ on $\mathbb{P}(V)$ of a nondegenerate quadruple of points (see (1-2)) is defined by (following the convention of [Fock and Goncharov 2007], and taking values in $\mathbb{K} \cup \{\infty\}$)

$$\boldsymbol{b}(a_1, a_2, a_3, a_4) = \frac{(a_1 - a_2)(a_3 - a_4)}{(a_1 - a_4)(a_2 - a_3)} \quad (1\text{-}7)$$

in any affine chart $\mathbb{P}(V) \xrightarrow{\sim} \mathbb{K} \cup \{\infty\}$, so that $\boldsymbol{b}(\infty, -1, 0, a) = a$.

The cross ratio $\boldsymbol{\beta}$ defined in Section 1I will then be called the *geometric* cross ratio, to distinguish it from $\boldsymbol{b}$, which will be called the *algebraic* cross ratio. They are then related as follows:

$$\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) = \log|\boldsymbol{b}(\xi_1, \xi_2, \xi_3, \xi_4)|. \quad (1\text{-}8)$$

Proof. Let $x_4 = c(\xi_3, \xi_1, \xi_2)$ and $x_2 = c(\xi_3, \xi_1, \xi_4)$. In a suitable basis $\boldsymbol{v} = (v_1, v_2)$ of V, we have in homogeneous coordinates $\xi_1 = [1:0]$, $\xi_3 = [0:1]$, $\xi_2 = [-1:1]$ and $\xi_4 = [b:1]$, where $b = \boldsymbol{b}(\xi_1, \xi_2, \xi_3, \xi_4)$. Then $g = \left(\begin{smallmatrix} -b & 0 \\ 0 & 1 \end{smallmatrix}\right)$ fixes ξ_1 and ξ_3 and sends ξ_2 to ξ_4. Hence $g(x_4) = x_2$. In the flat $A(\xi_3, \xi_1)$ identified with $\mathbb{A} = \mathbb{R}^2/\mathbb{R}(1, 1)$ by the marked flat $f_{\boldsymbol{v}}$, we have $\overrightarrow{x_4 x_2} = \nu(g) = [(\log|b|, 0)]$, hence $\overrightarrow{x_4 x_2} = \log|b|$ as needed. □

We recall that the algebraic cross ratio $\boldsymbol{b}$ satisfies the following symmetry properties: It is left unchanged by the double transpositions and changed to $\boldsymbol{b}^{-1}$ by the transpositions (13) and (24). Furthermore we have an additional symmetry under 3-cycles not satisfied by the geometric cross ratio:

$$\begin{aligned} \boldsymbol{b}(a_1, a_3, a_4, a_2) &= -1 - \boldsymbol{b}(a_1, a_2, a_3, a_4)^{-1}, \\ \boldsymbol{b}(a_1, a_4, a_2, a_3) &= -(1 + \boldsymbol{b}(a_1, a_2, a_3, a_4))^{-1}. \end{aligned} \quad (1\text{-}9)$$

1K. ***Cross ratio on the boundary of an A_2-Euclidean building.*** See [Tits 1986]. Let X be a Euclidean building of type A_2, and $\mathbb{P}$ the associated projective plane at infinity.

Let (p_1, p_2, p_3, p_4) be a nondegenerate quadruple of points of $\mathbb{P}$ on a common line D. Then their *cross ratio* $\boldsymbol{\beta}(p_1, p_2, p_3, p_4)$ (i.e., *projective valuation* in [Tits 1986]) is by definition their cross ratio as ideal points of the transverse tree X_D. The cross ratio of a nondegenerate quadruple of lines in $\mathbb{P}$ passing through a common point p is similarly defined as their cross ratio as ideal points of the transverse tree X_p.

The main additional property is that perspectivities preserve cross ratio, which follows from the fact that perspectivities extend isometries between the transverse trees (see Section 1G):

Proposition 1.1. *Let p be a point of $\mathbb{P}$ and D a line of $\mathbb{P}$ with $p \notin D$. The canonical isomorphisms (perspectivities) $\operatorname{proj}_{pD} : \operatorname{St}(D) \xrightarrow{\sim} \operatorname{St}(p)$, $q \mapsto pq$ and $\operatorname{proj}_{Dp} : \operatorname{St}(p) \xrightarrow{\sim} \operatorname{St}(D)$, $L \mapsto D \cap L$, preserve the cross ratio $\boldsymbol{\beta}$, i.e.,*

$$\boldsymbol{\beta}(p_1, p_2, p_3, p_4) = \boldsymbol{\beta}(pp_1, pp_2, pp_3, pp_4),$$
$$\boldsymbol{\beta}(D_1, D_2, D_3, D_4) = \boldsymbol{\beta}(D \cap D_1, D \cap D_2, D \cap D_3, D \cap D_4) \qquad \square$$

2. Some basic ideal configurations

2A. ***Extension of orthogonal projection to the boundary in* CAT(0) *spaces.*** In this section X is a general CAT(0) metric space, and we prove the following basic property: the usual orthogonal projection onto a proper convex subset $Y \subset X$ extends to the boundary outside the closed $\frac{\pi}{2}$-neighborhood of $\partial_\infty Y$ for the Tits metric (note that the projection is no longer unique). This property is quite elementary but we did not see it in the classical literature, so we include the proof. We refer to the book [Bridson and Haefliger 1999] for CAT(0) spaces.

We denote by $\partial_\infty X$ the CAT(0) boundary of X, and by $\sphericalangle_{Tits}(\xi, \eta)$ the Tits angle between two ideal points $\xi, \eta \in \partial_\infty X$. For a subset A of $\partial_\infty X$, we define $\sphericalangle_{Tits}(\xi, A) = \inf_{\eta \in A} \sphericalangle_{Tits}(\xi, \eta)$.

Definition 2.1. Let Y be a subspace of X and $\xi \in \partial_\infty X$ an ideal point. We say that a point $x \in Y$ is an *orthogonal projection of ξ on Y* if $\sphericalangle_x(\xi, y) \geq \frac{\pi}{2}$ for all $y \in Y - \{x\}$.

Proposition 2.2. *Let Y be a convex subspace of a* CAT(0) *space X which is proper for the induced metric, and ξ in $\partial_\infty X$. Suppose that $\sphericalangle_{Tits}(\xi, \partial_\infty Y) > \frac{\pi}{2}$. Then there exists an orthogonal projection x of ξ on Y.*

Proof. Consider a sequence (x_n) converging to ξ in X, and let y_n be the orthogonal projection of x_n on Y. If $(y_n)_{n \in \mathbb{N}}$ is not bounded then, up to passing to a subsequence, y_n converges to η in $\partial_\infty Y$. Then for any fixed y in Y we have $\sphericalangle_y(\xi, y_n) \leq \frac{\pi}{2}$

for all n, hence $\sphericalangle_y(\xi, \eta) \leq \frac{\pi}{2}$. Therefore $\sphericalangle_{Tits}(\xi, \eta) \leq \frac{\pi}{2}$. Thus $(y_n)_{n\in\mathbb{N}}$ is bounded, hence, since Y is proper, it has a converging subsequence, and the limit point x is then an orthogonal projection of ξ on Y. □

2B. *Centers of generic* $(N+1)$*-tuples.* In this section, we show that the notion of center of ideal triples in trees extends to Euclidean buildings of type A_{N-1}, for generic $(N+1)$-tuples of points (or hyperplanes) in the associated projective space at infinity (Proposition 2.4).

Let X be a Euclidean building of type A_{N-1}, and $\mathbb{P}$ be its projective space at infinity (i.e., the set of singular points of type 1 in $\partial_\infty X$, see Section 1). Recall from Section 1B that a *projective frame* in a projective space of dimension $N-1$ is a generic $(N+1)$-tuple of points.

We first observe that the orthogonal projection of a point of $\mathbb{P}$ on a flat of X exists under a simple necessary and sufficient condition.

Proposition 2.3. *Let A be a flat of X and $p \in \mathbb{P}$. Let $(p_1, \dots, p_N) = (\partial_\infty A) \cap \mathbb{P}$ be the points of type* 1 *in $\partial_\infty A$. Then p admits an orthogonal projection on A if and only if $(p, p_1, \dots, p_N)$ is a projective frame.*

The analogous property is also valid for points $H \in \mathbb{P}^*$. Note that these properties also hold in symmetric spaces of type A_{N-1}.

Proof. If $p \in H$ for some hyperplane H in $\mathbb{P}^* \cap \partial_\infty A$, then p and H are in a common chamber of the spherical building $\partial_\infty X$, and, as the diameter d of the model spherical Weyl chamber $\partial\overline{\mathfrak{C}}$ is strictly less that $\pi/2$ (for the angle metric), we have $\sphericalangle_{Tits}(p, H) < \pi/2$, hence the orthogonal projection does not exist. Else, for every hyperplane H in $\mathbb{P}^* \cap \partial_\infty A$, we have $p \notin H$, hence $\sphericalangle_{Tits}(p, H) = \pi$, which implies that since $\sphericalangle_{Tits}(p, \eta) \geq \pi - d > \pi/2$ for all $\eta \in \partial_\infty A$, and the orthogonal projection exists by Proposition 2.2. □

We now turn to the main result of this section.

Proposition 2.4. *Let $\mathcal{F} = (p_0, p_1, \dots, p_N)$ be a projective frame in $\mathbb{P} \subset \partial_\infty X$. For each $i \in \{0, \dots, N\}$ let A_i be the unique flat of X through $(p_0, \dots, \hat{p}_i, \dots, p_N)$. There exists a unique point $x \in X$ satisfying the following equivalent conditions*:

(i) $x \in \cap_i A_i$.

(ii) *For all i and for all H in $\partial_\infty A_i \cap \mathbb{P}^*$ the angle $\sphericalangle_x(p_i, H)$ is π.*

(iii) *The $(N+1)$-tuple $\Sigma_x \mathcal{F} = (\Sigma_x p_i)_{i=0,\dots,N}$ of directions at x form a projective frame in $\mathbb{P}_x$.*

(iv) *For all i, the point x is an orthogonal projection of p_i on the flat A_i.*

(v) *There exists i such that x is an orthogonal projection of p_i on A_i.*

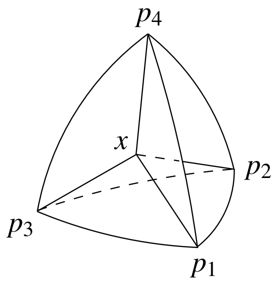


Figure 4. The center $x \in X$ of a projective frame (p_1, p_2, p_3, p_4) (for $N = 3$).

We will call x the center *of the projective frame $\mathcal{F} = (p_0, p_1, \dots, p_N)$ and denote it by $c(p_0, p_1, \dots, p_N)$ or $c(\mathcal{F})$.*

Proof. The existence of x, as an orthogonal projection of p_0 on A_0, is ensured by Proposition 2.3.

For $i \neq j$, denote by H_{ij} the hyperplane $\oplus_{k \neq i,j} p_k$ in the projective space $\mathbb{P}$. Let $x \in X$. Conditions (iii) and (i) are equivalent (see Section 1F).

We first show (i) $\Rightarrow$ (ii): Fix i and $H \in \mathbb{P}^*$ in $\partial_\infty A_i$. The opposite of H in $\partial_\infty A_i$ is some p_j. Then $H = H_{ij}$, so H is also the opposite of p_i in the apartment $\partial_\infty A_j$. As $x \in A_j$, we then have $\sphericalangle_x(p_i, H) = \pi$. We now prove (ii) $\Rightarrow$ (iii): First recall that for $p \in \mathbb{P}$ and $H \in \mathbb{P}^*$, we have $\sphericalangle_x(p_i, H) = \pi$ if and only if $\Sigma_x\, p \notin \Sigma_x\, H$ in the projective space $\mathbb{P}_x$. So (ii) means that $\Sigma_x\, p_i \notin \Sigma_x\, H_{ij}$ for all $i \neq j$. Let U_i be the minimal linear subspace of the projective space $\mathbb{P}_x$ containing $\Sigma_x\, p_0, \dots, \Sigma_x\, p_i$. Then, for $i \leq N-1$, we have that $\Sigma_x\, p_i$ is not in U_{i-1}, else $\Sigma_x\, p_i$ would belong to $\Sigma_x\, H_{i,i+1}$. Hence $(\Sigma_x\, p_0, \dots, \Sigma_x\, p_i)$ is independent in $\mathbb{P}_x$ by induction on i. Therefore $(\Sigma_x\, p_0, \dots, \Sigma_x\, p_{N-1})$ is a frame, and (iii) follows by permuting the p_i.

We now prove (ii) $\Rightarrow$ (iv). Let $i \in \{0, \dots, N\}$. Let $v \in \Sigma_x\, A_i$. Let $C \subset A_i$ be a closed Weyl chamber with vertex x containing v. Let $H \in \mathbb{P}^*$ be the singular point of type $N-1$ in $\partial_\infty C$. Then $\sphericalangle_x(p_i, H) = \pi$, hence $\sphericalangle_x(p_i, v) \geq \pi - d > \frac{\pi}{2}$, as the diameter d of $\partial\overline{\mathfrak{C}}$ is strictly less that $\pi/2$.

(iv) $\Rightarrow$ (v) is clear. Assume now that (v) holds. For $j \neq i$ in $\{0, \dots, N\}$, as $\sphericalangle_x(p_i, H_{ij}) \geq \frac{\pi}{2}$, the direction $\Sigma_x\, p_i$ is not in a closed chamber of $\Sigma_x\, X$ containing $\Sigma_x\, H_{ij}$. Hence by type considerations we must have $\sphericalangle_x(p_i, H_{ij}) = \pi$. So (ii) holds.

So the equivalence of all assertions is proven. We now prove the uniqueness of x. Suppose that x' is another point of X with the same properties, and $x' \neq x$. We proved above that we have then $\sphericalangle_x(p_i, x') > \frac{\pi}{2}$ and $\sphericalangle_{x'}(p_i, x) > \frac{\pi}{2}$, which is impossible. □

We now state some properties of centers of projective frames. Consider a projective frame $\mathcal{F} = (p_0, p_1, \dots, p_N)$ in $\mathbb{P}$, and let $x \in X$ be its center. Consider the $N+1$ associated flats $A_i = A(p_0, \dots, \hat{p}_i, \dots, p_N)$ in X. We first describe the intersections of the flats A_i with A_0.

Proposition 2.5. *For $i = 1, \dots, N$, let S_i be the convex hull of the rays from x to the points $p_1, \dots, \hat{p}_i, \dots, p_N$ — in other words the sector on those points, with basepoint x. Let $H_i = p_1 \oplus \cdots \oplus \hat{p}_i \oplus \cdots \oplus p_N$ denote the point in $\partial_\infty A_0$ opposite to p_i. For $i \in \{1, \dots, N\}$, we have:*

(i) *Let y be an interior point of S_i. Then $\Sigma_y p_0 = \Sigma_y p_i$.*

(ii) *For $y \in A_0$, we have $y \in A_0 \cap A_i$ if and only if $\Sigma_y p_0$ is opposite to $\Sigma_y H_i$.*

(iii) *$A_0 \cap A_i = S_i$.*

In particular, the intersections $A_0 \cap A_i$, $i = 1, \dots, N$, form a partition (i.e., a covering with disjoint interiors) of A_i.

Note that the sector S_i is the union of the Weyl chambers of the flat A_0 based at x and containing the singular ray to H_i.

Proof. The inclusion $S_i \subset A_0 \cap A_i$ is clear since $x \in A_0 \cap A_i$ and p_j is in both $\partial_\infty A_0$ and $\partial_\infty A_i$ for $j \neq i$ in $\{1, \dots, N\}$.

If y is an interior point of S_i, then in the local spherical building $\Sigma_y X$ at y, we have that $\Sigma_y p_0 \in \Sigma_y A_0$. Moreover, $y \in A_i$ as previously observed, so $\Sigma_y p_0$ is opposite to $\Sigma_y H_i$ (in $\Sigma_y A_i$). Hence $\Sigma_y p_0$ is equal to the opposite of $\Sigma_y H_i$ in $\Sigma_y A_0$, which is $\Sigma_y p_i$, proving (i).

We now prove (ii): In $\mathbb{P}_y$, the points $(\Sigma_y p_1, \dots, \Sigma_y p_N)$ form a frame (since $y \in A_0$). Hence the $N-1$ points $(\Sigma_y p_1, \dots, \widehat{\Sigma_y p_i}, \dots, \Sigma_y p_N)$ are independent Therefore $(\Sigma_y p_0, \dots, \widehat{\Sigma_y p_i}, \dots, \Sigma_y p_N)$ is a frame in $\mathbb{P}_y$ (i.e., $y \in A_i$) if and only if $\Sigma_y p_0 \notin \Sigma_y H_i$.

We finish by proving the remaining inclusion $A_0 \cap A_i \subset S_i$: The S_i clearly form a partition of A_0. So it is enough to prove that $A_0 \cap A_i$ does not meet the interior of S_j for $j \neq i$. Else, at such a point y, by (i), we would have $\Sigma_y p_0 = \Sigma_y p_j$, which is not opposite to $\Sigma_y H_i$, providing a contradiction. □

The following proposition shows that the notion of center of projective frames behaves well with respect to projections to transverse spaces at infinity.

Proposition 2.6. *For each i, the projection of x in the transverse building at infinity X_{p_i} is the center of the projective frame of $\partial_\infty X_{p_i}$ formed by the projections $\mathrm{proj}_{p_i}(p_j) = p_i p_j$ of the p_j, $j \neq i$, that is:*

$$\pi_{p_i}(c(p_0, p_1, \dots, p_N)) = c\big(p_i p_0, p_i p_1, \dots, \widehat{p_i p_i}, \dots, p_i p_N\big).$$

Proof. For all $j \neq i$, the ray from x to p_i is in the flat A_j hence its projection $\pi_{p_i}(x)$ in the transverse building X_{p_i} is in $\pi_{p_i}(A_j)$, which is the flat defined by the frame $\mathrm{proj}_{p_i}(p_k) = p_i p_k$, $k \neq i, j$. □

In the algebraic case, i.e., when X is the Euclidean building $X(V)$ associated with some vector space V of dimension N over an ultrametric field $\mathbb{K}$, we have the following characterization of the center as a norm on V.

Proposition 2.7. *Let $\mathcal{F} = (p_0, p_1, \ldots, p_N)$ be a projective frame in $\mathbb{P} = \mathbb{P}(V)$. The center of $\mathcal{F}$ is the norm η on V canonically associated to any basis $\boldsymbol{v} = (v_i)_{i=1,\ldots N}$ of V such that $p_i = [v_i]$ for $1 \le i \le N$ and $p_0 = [v_1 + \cdots + v_N]$ in $\mathbb{P}(V)$, i.e., the norm defined by*

$$\eta\left(\sum_{i=1}^{N} a_i v_i\right) = \max_{1\le i\le N} |a_i|.$$

Proof. Let $\boldsymbol{v} = (v_1, \ldots, v_N)$ be a basis of V such that $p_0 = [v_1 + \cdots + v_N]$ in $\mathbb{P}(V)$ and $p_i = [v_i]$. Let η be the associated canonical norm on V. We clearly have $\eta \in A_0$ by the definition of marked flats in the model of norms. Let g be the element of $\mathrm{GL}(V)$ sending the basis $\boldsymbol{v}$ to the basis $(v_1, \ldots, v_{N-1}, v_1 + \cdots + v_N)$. Then g preserves the norm η and sends A_0 to A_N and hence η is in the flat A_N. Permuting the basis v, we similarly get that η is in the flat A_i for all $i \neq 0$. □

Remark 2.8. By duality, the similar properties hold for generic $(N+1)$-tuples (projective frames) in $\mathbb{P}^* \subset \partial_\infty X$.

2C. *Projecting two ideal points onto a flat.* From now on we return to the case where $N = 3$ (type A_2).

Proposition 2.9. *Let (p_1, p_2, p_3) be a independent triple in $\mathbb{P}$. Let p, q be two points in $\mathbb{P}$, in generic position relatively to the p_i (i.e., not on any of the lines $p_i p_j$). Denote by x and y the respective orthogonal projections of p and q on the flat $A = A(p_1, p_2, p_3)$. Identify A with $\mathbb{A}$ by a marked flat sending $\partial\mathfrak{C}$ to (p_1, p_1p_2). Then the roots coordinates of $\overrightarrow{xy}$ are given by the three natural cross ratios at the vertices of the triangle*:

$$\varphi_1(\overrightarrow{xy}) = \boldsymbol{\beta}(p_3p_1, p_3p, p_3p_2, p_3q),$$
$$\varphi_2(\overrightarrow{xy}) = \boldsymbol{\beta}(p_1p_2, p_1p, p_1p_3, p_1q),$$
$$\varphi_3(\overrightarrow{xy}) = \boldsymbol{\beta}(p_2p_3, p_2p, p_2p_1, p_2q).$$

The analogous dual result holds for projections of two lines of $\mathbb{P}$ on a flat (exchanging the roles of points and lines in $\mathbb{P}$).

Proof. Projecting on the transverse tree X_{p_1} in direction p_1, we have

$$\varphi_2(\overrightarrow{xy}) = \varphi_2(B_{(p_1,p_1p_2)}(x, y)) = B_{p_1p_2}(\pi_{p_1}(x), \pi_{p_1}(y))$$

by (1-1). Since the projections of x and y on the tree X_{p_1} are the respective centers of the ideal triples (p_1p_2, p_1p_3, p_1p) and (p_1p_2, p_1p_3, p_1q) (Proposition 2.6), we have

$$B_{p_1p_2}(\pi_{p_1}(x), \pi_{p_1}(y)) = \boldsymbol{\beta}(p_1p_2, p_1p, p_1p_3, p_1q)$$

by (1-3), hence $\varphi_2(\overrightarrow{xy}) = \boldsymbol{\beta}(p_1p_2, p_1p, p_1p_3, p_1q)$. The remaining assertions follow by applying cyclic permutation, since

$$\varphi_1(B_{(p_1,p_1p_2)}(x, y)) = \varphi_2(B_{(p_3,p_3p_1)}(x, y)),$$
$$\varphi_3(B_{(p_1,p_1p_2)}(x, y)) = \varphi_2(B_{(p_2,p_2p_3)}(x, y)). \qquad \square$$

For the projections of a point and a line, we have the following result.

Proposition 2.10. *Let $F_- = (p_-, D_-)$ and $F_+ = (p_+, D_+)$ be two opposite flags in $\mathbb{P}$ and A the flat in X joining them, identified with $\mathbb{A}$ by a marked flat sending $\partial\mathfrak{C}$ to F_+. Let p be a point and D a line in $\mathbb{P}$ in generic position with respect to F_- and F_+, (i.e., p does not belong to any of the lines p_-p_+, D_-, D_+, and D does not contain any of the points $D_- \cap D_+$, p_-, p_+).*

Denote by x and x^* the respective orthogonal projections of p and D on A. Then in simple roots coordinates we have

$$\overrightarrow{xx}^* = (z_-, z_+),$$

with

$$z_- = \boldsymbol{\beta}(p_+, D_+ \cap (p_-p), D_+ \cap D_-, D_+ \cap D)$$
$$= \boldsymbol{\beta}(D_-, p_- \oplus (D_+ \cap D), p_-p_+, p_-p),$$
$$z_+ = \boldsymbol{\beta}(p_-, D_- \cap D, D_- \cap D_+, D_- \cap (p_+p))$$
$$= \boldsymbol{\beta}(D_+, p_+p, p_+p_-, p_+ \oplus (D_- \cap D))$$

Proof. See Figure 5. The projection of x on the transverse tree X_{p_-} is the center of the ideal triple $(p_-p_+, p_-(D_- \cap D_+), p_-p)$, and the projection of x^* on the tree X_{D_+} is the center of the ideal triple $(p_+, D_+ \cap D_-, D_+ \cap D)$ (Proposition 2.6). As x lies on a geodesic from p_- to D_+, we have

$$\begin{aligned}\pi_{D_+}(x) &= \pi_{D_+,p_-}(\pi_{p_-}(x))\\ &= \pi_{D_+,p_-}(c(p_-p_+, p_-(D_- \cap D_+), p_-p))\\ &= c(p_+, D_- \cap D_+, D_+ \cap (p_-p)).\end{aligned}$$

Then projecting on the transverse tree X_{D_+} we have

$$\varphi_1(\overrightarrow{xx}^*) = B_{p_+}(\pi_{D_+}(x), \pi_{D_+}(x^*)) = \boldsymbol{\beta}(p_+, D_+ \cap (p_-p), D_+ \cap D_-, D_+ \cap D)$$

as needed. The remaining assertions have identical proofs. $\square$

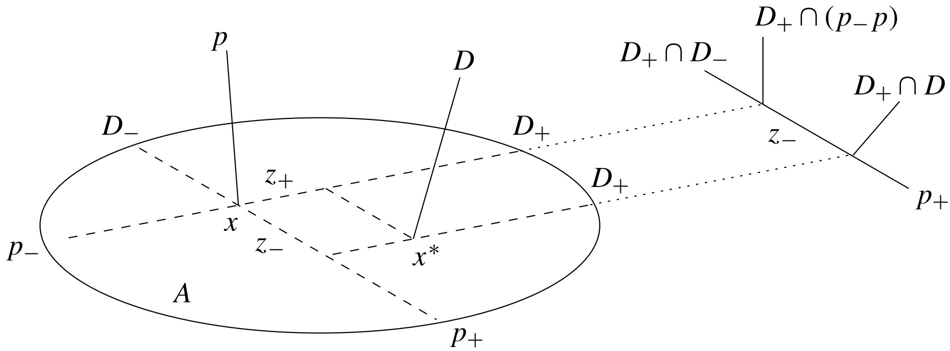


Figure 5. Projecting a point and a line on a flat. The left part of the diagram represents the situation in X, and the right part the situation in X_{D_+}.

3. Triple ratio of a triple of ideal chambers

In this section, we introduce the *(geometric) triple ratio* of a nondegenerate triple of ideal chambers in a real Euclidean building X of type A_2, establish its basic properties, and the links with the usual $\mathbb{K}$-valued (algebraic) triple ratio of triples of flags (see e.g., [Fock and Goncharov 2007]) in the algebraic case $\mathbb{P} = \mathbb{P}(\mathbb{K}^3)$.

We first give a precise definition of *nondegenerate* and *generic* triples of flags in an arbitrary projective plane $\mathbb{P}$.

3A. ***Nondegenerate and generic triples of flags.*** Let $\mathbb{P}$ be a projective plane and $T = (F_1, F_2, F_3)$ be a triple of flags $F_i = (p_i, D_i)$ in $\mathbb{P}$. We will denote by p_{ij} the point $D_i \cap D_j$ (resp. D_{ij} the line $p_i p_j$), when defined.

The natural nondegeneracy condition on the triple (F_1, F_2, F_3) for the triple ratios to be well defined is the following:

$$\text{either} \quad \forall i,\ p_i \notin D_{i+1} \quad \text{or} \quad \forall i,\ p_i \notin D_{i-1}. \tag{ND}$$

This condition is clearly equivalent to: the points are pairwise distinct, the lines are pairwise distinct, none of the points is on the three lines (i.e., $D_i \cap D_j \neq p_k$ for all $\{i, j, k\} = \{1, 2, 3\}$) and none of the lines contains the three points (i.e., $p_i p_j \neq D_k$ for all i, j, k). We will then say that the triple (F_1, F_2, F_3) is *nondegenerate*.

It is easy to check that the triple T defines then a nondegenerate quadruple $(D_i, p_i p_j, p_i p_{jk}, p_i p_k)$ of lines through each point p_i, and a nondegenerate quadruple $(p_i, D_i \cap D_j, D_i \cap D_{jk}, D_i \cap D_k)$ of points on each line D_i.

The triple of flags $T = (F_1, F_2, F_3)$ is *generic* if the flags $F_i = (p_i, D_i)$ are pairwise opposite, the points $(p_i)_i$ are not collinear and the lines $(D_i)_i$ are not concurrent. In particular, T is then nondegenerate, and the induced quadruples of points on each line (resp. of lines through each point) are generic (i.e., pairwise distinct).

3B. ***Algebraic triple ratio.*** When $\mathbb{P} = \mathbb{P}(\mathbb{K}^3)$ is the projective plane associated with an arbitrary field $\mathbb{K}$, the algebraic triple ratio of a nondegenerate triple of flags $T = (F_1, F_2, F_3)$ (see Section 3A), with values in $\mathbb{K} \cup \{\infty\}$, is defined by (see [Fock and Goncharov 2006, §9.4])

$$\mathrm{Tri}(F_1, F_2, F_3) = \frac{\tilde{D}_1(\tilde{p}_2)\tilde{D}_2(\tilde{p}_3)\tilde{D}_3(\tilde{p}_1)}{\tilde{D}_1(\tilde{p}_3)\tilde{D}_2(\tilde{p}_1)\tilde{D}_3(\tilde{p}_2)},$$

where $\tilde{p}_i$ is any vector in $\mathbb{K}^3$ representing p_i and $\tilde{D}_i$ is any linear form in $(\mathbb{K}^3)^*$ representing D_i, and $F_i = (p_i, D_i)$. It is invariant under cyclic permutation of the flags and inverted by reversing the order

$$\mathrm{Tri}(F_3, F_2, F_1) = \mathrm{Tri}(F_1, F_2, F_3)^{-1}.$$

It may be expressed as the following cross ratio:

$$\mathrm{Tri}(F_1, F_2, F_3) = \boldsymbol{b}(D_1, p_1p_2, p_1p_{23}, p_1p_3). \tag{3-1}$$

3C. ***Geometric triple ratio.*** We suppose now that the projective plane $\mathbb{P}$ is the projective plane at infinity of some a real Euclidean building X of type A_2, possibly exotic. Let $\boldsymbol{\beta}$ be the associated geometric cross ratio on $\mathbb{P}$ (see Section 1K). Let $T = (F_1, F_2, F_3)$ be a nondegenerate triple of ideal chambers of X, i.e., a nondegenerate triple of flags $F_i = (p_i, D_i)$ in $\mathbb{P}$.

The idea is to define the geometric triple ratio of T by analogy with the expression of the algebraic triple ratio as a cross ratio (3-1), replacing $\boldsymbol{b}$ by $\boldsymbol{\beta}$, in such a way that, in the algebraic case, the geometric triple ratio of a triple T with algebraic triple ratio Z should be $\log|Z|$. But for the purpose of geometric classification, this geometric cross ratio $\boldsymbol{\beta}(D_1, p_1p_2, p_1p_{23}, p_1p_3)$ alone will not retain enough information. In particular, in contrast to the algebraic cross ratio, it does not determine the geometric cross ratios obtained from the original 4-tuple by cyclic permutations of the three last arguments, which in the algebraic case are $\log|1 + Z^{-1}|$ and $-\log|1 + Z|$, see (1-9), and have geometric significance. For example, in the algebraic case, it will not distinguish between two triples T and T' with respective algebraic triple ratios $Z = -1$ and $Z' = -1 + a$ with $|a| < 1$.

In order to retain this information we define the *geometric triple ratio* of T as the triple

$$\mathrm{tri}(T) = (\mathrm{tri}_m(T))_{m=1,2,3}$$

where

$$\mathrm{tri}_1(F_1, F_2, F_3) = \boldsymbol{\beta}(D_1, p_1p_2, p_1p_{23}, p_1p_3),$$
$$\mathrm{tri}_2(F_1, F_2, F_3) = \boldsymbol{\beta}(D_1, p_1p_3, p_1p_2, p_1p_{23}),$$
$$\mathrm{tri}_3(F_1, F_2, F_3) = \boldsymbol{\beta}(D_1, p_1p_{23}, p_1p_3, p_1p_2),$$

are the geometric cross ratios obtained from $(D_1, p_1p_2, p_1p_{23}, p_1p_3)$ by cyclic permutation of the three last lines. Note these cross ratios are well defined, since the four lines D_1, p_1p_2, p_1p_{23}, p_1p_3 are well defined and form a nondegenerate quadruple of lines through p_1 (see Section 3A above).

The next proposition gathers the properties of the geometric triple ratio, and show in particular that this invariant is in fact 1-dimensional, as it takes values in one of the three rays $\mathbb{R}_+(0, 1, -1)$, $\mathbb{R}_+(-1, 0, 1)$, and $\mathbb{R}_+(1, -1, 0)$.

Proposition 3.1. (i) *The geometric triple ratio is invariant by cyclic permutations of the flags*; *i.e., for* $m = 1, 2, 3$,

$$\mathrm{tri}_m(F_2, F_3, F_1) = \mathrm{tri}_m(F_1, F_2, F_3).$$

(ii) *Exchanging two flags, we have*

$$\mathrm{tri}_1(F_1, F_3, F_2) = -\,\mathrm{tri}_1(F_1, F_2, F_3), \quad \mathrm{tri}_2(F_1, F_3, F_2) = -\,\mathrm{tri}_3(F_1, F_2, F_3).$$

(iii) *We have* $\mathrm{tri}_1(T) + \mathrm{tri}_2(T) + \mathrm{tri}_3(T) = 0$.

(iv) *For all* $m \in \mathbb{Z}/3\mathbb{Z}$, *if* $\mathrm{tri}_m(T) > 0$, *then we have* $\mathrm{tri}_{m-1}(T) = 0$ *and* $\mathrm{tri}_{m+1}(T) = -\,\mathrm{tri}_m(T) < 0$.

In order to prove this proposition, in particular, the invariance of the triple ratio by cyclic permutation of the flags, we first introduce the natural dual invariants given by the cross ratios of the natural induced quadruple of points on the line D_1 (that is, exchanging the role of points and lines):

$$\mathrm{tri}_1^*(F_1, F_2, F_3) = \boldsymbol{\beta}(p_1, D_2 \cap D_1, D_{23} \cap D_1, D_3 \cap D_1),$$
$$\mathrm{tri}_2^*(F_1, F_2, F_3) = \boldsymbol{\beta}(p_1, D_3 \cap D_1, D_2 \cap D_1, D_{23} \cap D_1),$$
$$\mathrm{tri}_3^*(F_1, F_2, F_3) = \boldsymbol{\beta}(p_1, D_{23} \cap D_1, D_3 \cap D_1, D_2 \cap D_1).$$

The following property is straightforward.

$$\begin{aligned} \mathrm{tri}_1^*(F_1, F_3, F_2) &= -\,\mathrm{tri}_1^*(F_1, F_2, F_3), \\ \mathrm{tri}_2^*(F_1, F_3, F_2) &= -\,\mathrm{tri}_3^*(F_1, F_2, F_3). \end{aligned} \tag{3-2}$$

We will need the following property showing that the invariants behave nicely under duality.

Lemma 3.2. *For* $m = 1, 2, 3$, *we have* $\mathrm{tri}_m^*(F_1, F_2, F_3) = \mathrm{tri}_m(F_3, F_2, F_1)$.

Proof of Lemma 3.2. By invariance under perspectivities and double transpositions, we have

$$\begin{aligned}\mathrm{tri}_1^*(F_1, F_2, F_3) &= \boldsymbol{\beta}(p_1, D_2 \cap D_1, D_{23} \cap D_1, D_3 \cap D_1)\\ &= \boldsymbol{\beta}(p_1 p_3, p_{12} p_3, D_{23}, D_3)\\ &= \boldsymbol{\beta}(D_3, p_2 p_3, p_{12} p_3, p_1 p_3)\\ &= \mathrm{tri}_1(F_3, F_2, F_1).\end{aligned}$$

The proof of $\mathrm{tri}_m^*(F_1, F_2, F_3) = \mathrm{tri}_m(F_3, F_2, F_1)$ for $m = 2, 3$ is similar. □

We now turn to the proof of Proposition 3.1.

Proof of Proposition 3.1. Assertions (iii) and (iv) follow immediately from the properties of the cross ratio $\boldsymbol{\beta}$ under cyclic permutation of the three last points (see (1-4) and (1-5)).

Assertion (ii) follows immediately from the definition and from the symmetries of the cross ratio.

We finally prove Proposition 3.1(i). Using (ii), Lemma 3.2 and (3-2), we have

$$\begin{aligned}\mathrm{tri}_1(F_2, F_3, F_1) &= -\,\mathrm{tri}_1(F_2, F_1, F_3)\\ &= -\,\mathrm{tri}_1^*(F_3, F_1, F_2) = \mathrm{tri}_1^*(F_3, F_2, F_1) = \mathrm{tri}_1(F_1, F_2, F_3),\\ \mathrm{tri}_2(F_2, F_3, F_1) &= -\,\mathrm{tri}_3(F_2, F_1, F_3)\\ &= -\,\mathrm{tri}_3^*(F_3, F_1, F_2) = \mathrm{tri}_2^*(F_3, F_2, F_1) = \mathrm{tri}_2(F_1, F_2, F_3).\end{aligned}$$

The case where $m = 3$ is similar to the case $m = 2$. □

3D. ***Geometric triple ratio from algebraic triple ratio.*** When $\mathbb{P}$ is the projective plane on some field $\mathbb{K}$ endowed with some ultrametric absolute value, and $\boldsymbol{\beta} = \log|\boldsymbol{b}|$ where $\boldsymbol{b}$ is the usual $\mathbb{K}$-valued cross ratio on $\mathbb{P}$, the three geometric triple ratios $\mathrm{tri}_m(T)$, $m = 1, 2, 3$ of T are obtained from the single algebraic triple ratio $Z = \mathrm{Tri}(T)$ of T by the relations

$$\begin{aligned}\mathrm{tri}_1(T) &= \log|Z|,\\ \mathrm{tri}_2(T) &= \log\left|\frac{1}{1+Z}\right| = -\log|1+Z|,\\ \mathrm{tri}_3(T) &= \log|1+Z^{-1}|,\end{aligned} \tag{3-3}$$

which are easily derived from the expression of algebraic triple ratio as a cross ratio (3-1) and from the symmetry properties of the algebraic cross ratio (1-9).

Remark 3.3. Note that the geometric invariants do not determine the triple of flags up to automorphisms of $\mathbb{P}$ (unlike the usual (algebraic) triple ratio): for example in the algebraic case $\mathbb{P} = \mathbb{P}(\mathbb{K}^3)$, take T with triple ratio $Z \in \mathbb{K}$ with $|Z| > 1$ and T' with triple ratio $Z' = Za$ where $a \in \mathbb{K}$ with $|a| = 1$ and $a \neq 1$. Then T and T' are not in the same $\mathrm{PGL}(\mathbb{K}^3)$-orbit, but have the same three geometric invariants, as $\mathrm{tri}_1(T) = \log|Z| = \mathrm{tri}_1(T')$, $\mathrm{tri}_2(T) = -\log|Z| = \mathrm{tri}_2(T')$, $\mathrm{tri}_3(T) = 0 = \mathrm{tri}_3(T')$.

4. Proof of the main result

In this section we prove Theorems 0.1 and 0.2. Let X be a Euclidean building of type A_2 and $T = (F_1, F_2, F_3)$ be a generic triple of flags in the projective plane $\mathbb{P}$ at infinity of X. We denote by $z_m = \mathrm{tri}_m(F_1, F_2, F_3)$, $m = 1, 2, 3$, its geometric triple ratio, and by $A_{ij} = A(F_i, F_j)$, $A_p = A(p_1, p_2, p_3)$ and $A_D = A(D_1, D_2, D_3)$ the five associated flats.

We first define the six associated points in X.

4A. ***Associated points in the building.*** For $\{i, j, k\} = \{1, 2, 3\}$, denote by y_k the center in X of the projective frame (p_1, p_2, p_3, p_{ij}), where $p_{ij} = D_i \cap D_j$, and by y_k^* the center of the projective frame (D_1, D_2, D_3, D_{ij}), where $D_{ij} = p_i p_j$, as defined in Proposition 2.4. In particular the point y_k is the orthogonal projection of p_{ij} on A_p, the point y_k^* is the orthogonal projection of D_{ij} on A_D, the point y_k is the orthogonal projection of p_k on $A_{ij} = A(p_i, p_j, p_{ij})$, and the point y_k^* is the orthogonal projection of D_k on $A_{ij} = A(D_i, D_j, D_{ij})$.

4B. ***In the flat A_{ij}.*** We now link the respective position of the points y_k and y_k^* in the flat A_{ij} to the geometric triple ratio of T. Suppose that the indices i, j, k respect the cyclic order, i.e., that $(i, j, k) = (123)$ as cyclic permutations. We identify A_{ij} with the model flat $\mathbb{A}$ by a marked flat $f_{ij} : \mathbb{A} \to A_{ij}$ sending $\partial\mathfrak{C}$ to F_j. For x, y in $A_{ij} \simeq \mathbb{A}$, we define then $\overrightarrow{xy} = y - x = B_{F_j}(x, y)$. Recall that $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ denotes the canonical basis of $\mathbb{R}^3$. In particular, the directions of p_i, p_{ij} and p_j are respectively identified with the directions of $[\varepsilon_1]$, $[\varepsilon_2]$, and $[\varepsilon_3]$ in $\mathbb{A}$.

Proposition 4.1. (i) *In simple roots coordinates, we have* $\overrightarrow{y_k^* y_k} = (z_2, z_3)$.

(ii) *For* $m = 1, 2, 3$, *if* $z_m > 0$ *then* $\overrightarrow{y_k y_k^*} = z_m[\varepsilon_m]$. *In particular* y_k^* *is on one of the three singular rays of type* 1 *issued from* y_k (*i.e the rays to* p_i, p_j *and* p_{ij}).

Proof. As y_k and y_k^* are the respective orthogonal projections on the flat A_{ij} of p_k and D_k, by Proposition 2.10 and cyclic invariance of the geometric triple ratio, we have

$$\varphi_1(\overrightarrow{y_k^* y_k}) = \boldsymbol{\beta}(D_i, p_i p_k, p_i p_j, p_i p_{jk}) = \mathrm{tri}_2(F_i, F_j, F_k) = z_2 \quad \text{and}$$
$$\varphi_2(\overrightarrow{y_k^* y_k}) = \boldsymbol{\beta}(D_j, p_j p_{ki}, p_j p_i, p_j p_k) = \mathrm{tri}_3(F_j, F_k, F_i) = z_3.$$

Assertion (ii) follows, since we have then $z_{m-1} = 0$ and $z_{m+1} = -z_m$ by ultrametricity of the geometric triple ratio (Proposition 3.1(iv)). □

We now describe the intersections of A_{ij} with the four other flats (see Figures 1 and 2 in the introduction). These intersections happen to be sectors in $\mathbb{A}$ bounded by two singular rays of same type, equivalently the union of two adjacent Weyl chambers.

Proposition 4.2. *Let* $x \in A_{ij}$. *Then*:

(i) *The intersection* $A_{ij} \cap A_p$ *is the sector at* y_k *bounded by the rays to* p_i *and* p_j. *That is,*

$$x \in A_p \text{ if and only if } \begin{cases} \varphi_1(x) \geq \varphi_1(y_k), \\ \varphi_2(x) \leq \varphi_2(y_k). \end{cases}$$

(ii) *The intersection* $A_{ij} \cap A_D$ *is the sector at* y_k^* *bounded by the rays to* D_i *and* D_j. *That is,*

$$x \in A_D \text{ if and only if } \begin{cases} \varphi_1(x) \leq \varphi_1(y_k^*), \\ \varphi_2(x) \geq \varphi_2(y_k^*). \end{cases}$$

(iii) *The intersection* $A_{ij} \cap A_{jk}$ *is the intersection of the sector at* y_k *bounded by the rays to* p_j *and* $D_i \cap D_j$, *and the sector at* y_k^* *bounded by the rays to* D_j *and* $p_i p_j$. *That is,*

$$x \in A_{jk} \text{ if and only if } \begin{cases} \varphi_1(x) \geq \varphi_1(y_k^*), \\ \varphi_2(x) \geq \varphi_2(y_k), \\ \varphi_3(x) \leq \min(\varphi_3(y_k), \varphi_3(y_k^*)). \end{cases}$$

(iv) *The intersection* $A_{ij} \cap A_{ki}$ *is the intersection of the sector at* y_k *bounded by the rays to* p_i *and* $D_i \cap D_j$, *and the sector at* y_k^* *bounded by the rays to* D_i *and* $p_i p_j$. *That is,*

$$x \in A_{ki} \text{ if and only if } \begin{cases} \varphi_1(x) \leq \varphi_1(y_k), \\ \varphi_2(x) \leq \varphi_2(y_k^*), \\ \varphi_3(x) \geq \max(\varphi_3(y_k), \varphi_3(y_k^*)). \end{cases}$$

Proof. Since y_k is the center of the projective frame (p_i, p_j, p_{ij}, p_k), assertion (i) comes from Proposition 2.5, as $A_{ij} = A(p_i, p_j, p_{ij})$ and $A_p = A(p_i, p_j, p_k)$. Assertion (ii) is similar. Assertion (iii): A point $x \in A_{ij}$ lies in A_{jk} if and only if, in the spherical building of directions at $\Sigma_x X$, the direction $\Sigma_x D_j$ is opposite to $\Sigma_x p_k$ and $\Sigma_x p_j$ is opposite to $\Sigma_x D_k$. Moreover, $\Sigma_x D_j$ is opposite to $\Sigma_x p_k$ if and only if $x \in A(p_k, p_j, p_{ij})$. As y_k is the center of the projective frame (p_i, p_j, p_{ij}, p_k) and $A_{ij} = A(p_i, p_j, p_{ij})$, the set of such x is the sector at y_k bounded by the rays to p_j and $D_i \cap D_j$ (by Proposition 2.5). This is the subset of $x \in A_{ij}$ satisfying: $\varphi_2(x) \geq \varphi_2(y_k)$ and $\varphi_3(x) \leq \varphi_3(y_k)$. Similarly, as y_k^* is the center of the projective frame (D_i, D_j, D_{ij}, D_k) and $A_{ij} = A(D_i, D_j, D_{ij})$, the direction $\Sigma_x p_j$ is opposite to $\Sigma_x D_k$ if and only if x is in the sector at y_k^* bounded by the rays to D_j and $D_{ij} = p_i p_j$. That is, if and only if $\varphi_1(x) \geq \varphi_1(y_k^*)$ and $\varphi_3(x) \leq \varphi_3(y_k^*)$, and we are done. Assertion (iv) is similar. □

In particular, as y_k^* is on one of the three singular rays of type 1 issued from y_k by Propositions 4.1, from Proposition 4.2 we easily get the following result.

Corollary 4.3. *The intersections with A_{ij} of A_{jk},A_{ki}, A_p and A_D form a partition of A_{ij}.* □

4C. ***In the flat A_p.*** We now consider the flat $A_p = A(p_1, p_2, p_3)$. The following proposition describes the respective positions in A_p of the points y_1, y_2, y_3. We identify A_p with $\mathbb{A}$ by a marked flat $f_p : \mathbb{A} \to A_p$ sending $\partial\mathfrak{C}$ to $(p_1, p_1 p_2)$ (hence direction $[\varepsilon_i]$ to p_i for $i = 1, 2, 3$). Recall that we then have $\overrightarrow{xx'} = x' - x = B_{(p_1, p_1 p_2)}(x, x')$ for $x, x' \in A_p$.

Proposition 4.4. *In the flat A_p we have*:

(i) *In simple roots coordinates, we have $\overrightarrow{y_2 y_3} = (z_1, 0)$.*

(ii) *If $z_1 \geq 0$, the point y_{i+1} is in the ray $[y_i, p_{i+2})$ (for all i), and if $z_1 \leq 0$, the point y_i is in the ray $[y_{i+1}, p_{i+2})$ for all i.*

In particular the triangle $\Delta \subset A_p$ with vertices y_1, y_2, y_3 is singular, *i.e., the sides have singular type in $\overline{\mathfrak{C}}$.*

Proof. Recall that the point y_k is the orthogonal projection on the flat A_p of the singular boundary point $p_{ij} = D_i \cap D_j$. Then, by Proposition 2.6 the points y_2 and y_3 have the same projection in the transverse tree X_{p_1}, that is the center of the ideal triple $(p_1 p_{13}, p_1 p_2, p_1 p_3) = (D_1, p_1 p_2, p_1 p_3) = (p_1 p_{23}, p_1 p_2, p_1 p_3)$, proving that $\varphi_2(\overrightarrow{y_2 y_3}) = 0$. Furthermore, by Proposition 2.9 we have

$$\begin{aligned}\varphi_2(\overrightarrow{y_3 y_1}) &= \boldsymbol{\beta}(p_1 p_2, p_1 p_{12}, p_1 p_3, p_1 p_{23})\\ &= \boldsymbol{\beta}(p_1 p_2, D_1, p_1 p_3, p_1 p_{23})\\ &= \boldsymbol{\beta}(D_1, p_1 p_2, p_1 p_{23}, p_1 p_3)\\ &= z_1,\end{aligned}$$

proving that $\varphi_2(\overrightarrow{y_3 y_1}) = z_1$. Applying this to the permuted triple (F_3, F_1, F_2), we obtain $\varphi_1(\overrightarrow{y_2 y_3}) = z_1$ (by invariance of the geometric triple ratio z_1 by cyclic permutation). Assertion (ii) follows from (ii), applying cyclic permutations. □

We now describe the intersections of A_p with the other flats; see Figure 6.

Proposition 4.5. *Let $S_i = A_p \cap A_{i,i+1}$ and let Δ be the triangle with vertices y_1, y_2, y_3. Then*:

(i) *S_i is the sector of A_p bounded by the rays from y_{i+2} to p_i and p_{i+1}.*

(ii) *S_1, S_2, S_3 and Δ form a partition of A_p.*

Proof. Assertion (i) follows from Proposition 4.2(i). In the case where $z_1 \geq 0$, assertion (ii) then comes from the fact that for all i, y_{i+1} is in the ray $[y_i, p_{i+2})$ (Proposition 4.4). The case where $z_1 \leq 0$ is similar. □

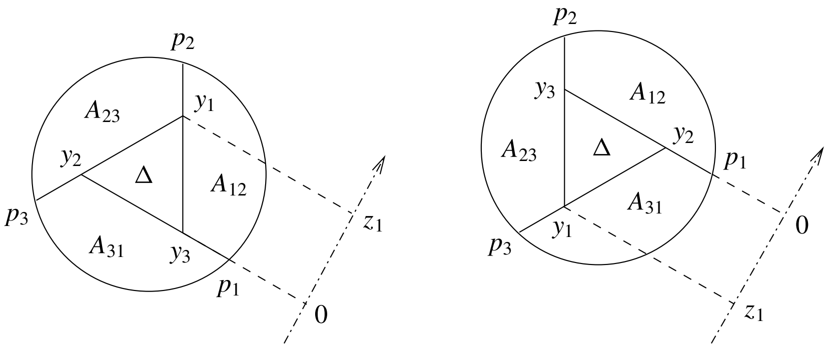


Figure 6. Situation in the flat A_p: when $z_1 \geq 0$ (left) and when $z_1 \leq 0$ (right).

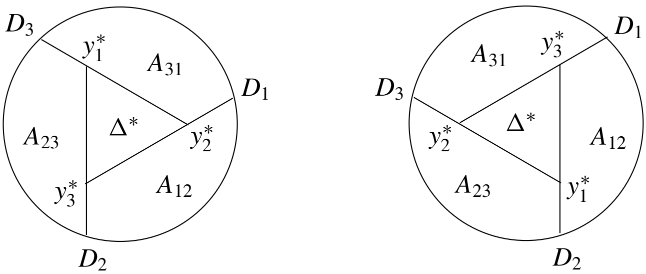


Figure 7. Situation in the flat A_D: when $z_1 \geq 0$ (left) and when $z_1 \leq 0$ (right).

4D. ***In the flat A_D.*** We now state the similar properties in the dual flat $A_D = A(D_1, D_2, D_3)$, which have same proofs, exchanging the role of points and lines.

Proposition 4.6. *In the flat A_D identified with $\mathbb{A}$ by a marked flat sending $\partial\mathfrak{C}$ to $(D_1 \cap D_2, D_1)$, we have*:

(i) $\overrightarrow{y_2^* y_3^*} = (0, -z_1)$ *in simple roots coordinates. In particular y_2^* and y_3^* are on a common singular geodesic to D_1.*

(ii) *The points y_1^*, y_2^*, y_3^* form a singular triangle Δ^* in A_D.*

(iii) *For all $i \in \mathbb{Z}/3\mathbb{Z}$, $S_i^* = A_D \cap A_{i,i+1}$ is the sector of A_D bounded by the rays from y_{i+2}^* to D_i and D_{i+1}.*

(iv) S_1^*, S_2^*, S_3^* *and Δ^* form a partition of A_D.* □

4E. ***The classification.*** We now combine the previous results to establish the classification in two geometric types, finishing to prove Theorems 0.1 and 0.2.

Proof of Theorem 0.1. Let $x = y_3$ and $x^* = y_3^*$. We identify the flat A_{12} with the model flat $\mathbb{A}$ by a marked flat sending $\partial\mathfrak{C}$ to F_2, and 0 to y_3^*. By Proposition 4.2

applied to the flat A_{12}, we have $\varphi_1(y_3) = z_2$, $\varphi_2(y_3) = z_3$, and $\varphi_3(y_3) = z_1$. By Proposition 4.2 applied to the flat A_{12}, the intersection $I = A_{12} \cap A_{23} \cap A_{31}$ is the subset of $y \in A_{12}$ such that

$$\begin{aligned} 0 \leq \varphi_1(y) \leq \varphi_1(y_3) = z_2, \\ 0 \geq \varphi_2(y) \geq \varphi_2(y_3) = z_3, \\ \max(\varphi_3(y_3), 0) \leq \varphi_3(y) \leq \min(\varphi_3(y_3), 0). \end{aligned}$$

In particular, if I is not empty, then $z_1 = \varphi_3(y_3) = 0$.

Suppose from now on that $z_1 = 0$. Then $z_2 \geq 0$ and $z_3 = -z_2$ by the ultrametricity of the geometric triple ratio (Proposition 3.1(iv)). By the description above, I is then the subset of the line $\varphi_3 = 0$ (which contains $y_3^* = 0$ and y_3) consisting of the y such that $0 \leq \varphi_1(y) \leq \varphi_1(y_3)$ (since $\varphi_2(y) = -\varphi_1(y)$ when $\varphi_3(y) = 0$). Hence I is not empty and is the segment from $0 = y_3^*$ to y_3 i.e., $[x, x^*]$. Furthermore, as $z_1 = 0$, Proposition 4.4 implies that $y_1 = y_2 = y_3$. Similarly, we have $y_1^* = y_2^* = y_3^*$ by Proposition 4.6. Suppose now $x \neq x^*$. Since the segment $[x, x^*]$ lies in the ray $[x, p_{ij})$, and $x = y_k$ is the orthogonal projection of p_{ij} on A_p, we have $\sphericalangle_x(x^*, D) = \pi$ for all lines D in $\partial_\infty A_p$ (Proposition 2.4). Therefore we have $\sphericalangle_x(x^*, y) \geq \frac{2\pi}{3}$ for all $y \neq x$ in A_p. Similarly, we have that $\sphericalangle_{x^*}(x, y) \geq \frac{2\pi}{3}$ for all $y \neq x$ in A_p. Hence $[x, x^*]$ is the unique segment of minimal length joining A_p to A_D. Assertion (iv) follows from Proposition 4.1. □

Proof of Theorem 0.2. If $z_2 > 0$, then $z_1 = 0$ by the ultrametricity of the geometric triple ratio (Proposition 3.1(iv)), and $A_p \cap A_D$ is empty by Theorem 0.1. Suppose now that $z_2 \leq 0$. Since the case $z_1 \leq 0$ reduces to the case $z_1 \geq 0$ by exchanging F_2 and F_3, it is enough to handle the case $z_1 \geq 0$. Then $z_3 = 0$ and $z_2 = -z_1$. Let $x_i = y_{i+2}$ for $i \in \mathbb{Z}/3\mathbb{Z}$. In A_{ij} identified with $\mathbb{A}$ in such a way that $y_k^* = 0$, by Proposition 4.1 we have $\varphi_1(y_k) = z_2 = -z_1 \leq 0$, $\varphi_2(y_k) = z_3 = 0$, hence $\varphi_3(y_k) = z_1 \geq 0$. By Proposition 4.2(iv), $A_{ij} \cap A_{ik}$ is the set of $x \in A_{ij} \simeq \mathbb{A}$ such that $\varphi_1(x) \leq \varphi_1(y_k)$, $\varphi_2(x) \leq 0 = \varphi_2(y_k)$ and $\varphi_3(x) \geq \max(\varphi_3(y_k), 0) = \varphi_3(y_k)$. This is the Weyl chamber $y_k - \overline{\mathfrak{C}}$, i.e., the Weyl chamber from $y_k = x_i$ to F_i. Similarly, $A_{ij} \cap A_{jk}$ is the Weyl chamber from y_k^* to F_j. Applying a cyclic permutation (ijk), i.e., working in the flat A_{jk}, we also similarly get that $A_{ij} \cap A_{jk}$ is the Weyl chamber from y_i to F_j. Therefore $y_k^* = y_i$.

By Proposition 4.2 $A_p \cap A_D \cap A_{ij}$ is the intersection of the sector at y_k^* bounded by the rays to D_i and D_j, with the sector at y_k bounded by the rays to p_i and p_j. As the point y_k is on the ray from y_k to D_i, this is equal to the segment $[y_k, y_k^*]$. In particular $A_p \cap A_D$ contains y_k. Then $A_p \cap A_D$ contains y_1, y_2 and y_3, hence the triangle Δ with vertices y_1, y_2 and y_3, and since $A_p \cap A_D \cap A_{ij} = [y_k, y_i] \subset \Delta$, Proposition 4.5(ii) provides the reverse inclusion. Assertion (iii) comes from Proposition 4.1.

We finally prove (iv). Let $(i, j, k) = (123)$. Looking in the flat A_p, we see that the singular triangle Δ is contained in the Weyl chamber of X with tip x_i and that at x_i, we have $\Sigma_{x_i} x_j = \Sigma_{x_i} p_j$. Looking in the flat A_D we get $\Sigma_{x_i} x_k = \Sigma_{x_i} D_k$. Hence $\Sigma_{x_i} \Delta = (\Sigma_{x_i} p_j, \Sigma_{x_i} D_k)$. Since x_i belongs to the flats $A(F_i, F_j)$ and $A(F_i, F_k)$, we have that $\Sigma_{x_i} p_j$ is opposite to $\Sigma_{x_i} D_i$ and that $\Sigma_{x_i} D_k$ is opposite to $\Sigma_{x_i} p_i$. Therefore the Weyl chambers $\Sigma_{x_i} \Delta$ and $\Sigma_{x_i} F_i$ are opposite. It implies that Δ and the Weyl chamber from x_i to F_i are contained in a common flat of X by basic properties of real Euclidean buildings (see property (CO) of [Parreau 2000]). □

In the algebraic case the following remark provides an alternative proof of some of the assertions of Theorem 0.2.

Remark 4.7. Let $\tilde{p}_i$ in $V = \mathbb{K}^3$ be a vector representing p_i and $\tilde{D}_i$ in V^* be a linear form representing D_i. Let $\boldsymbol{v} = (v_1, v_2, v_3)$ be the basis of V dual to the basis $(\tilde{D}_1, \tilde{D}_2, \tilde{D}_3)$ of V^*. Then in the projective plane $[v_i] = D_j \cap D_k$. We may suppose that $\tilde{p}_1 = (0, 1, 1)$, $\tilde{p}_2 = (Z, 0, 1)$, $\tilde{p}_3 = (1, 1, 0)$ in the basis $\boldsymbol{v}$, with $Z = \mathrm{Tri}(F_1, F_2, F_3)$. Then the element $g \in \mathrm{GL}(V)$ whose matrix in the basis $\boldsymbol{v}$ is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1/Z & 0 & 1 \end{pmatrix}$$

sends $[v_i]$ to p_{i+1}, hence A_D to A_p. If $|1+Z| \geq 1$ and $z = \log|Z| \geq 0$, then the fixed point set of g in A_D is the image by the marked flat $f_{\boldsymbol{v}}$ of the singular triangle $\{\lambda \in \overline{\mathfrak{C}} \mid \lambda_1 - \lambda_3 \leq \log|Z|\}$ (that is, Δ).

4F. *Complements.* We add here for future use a simple description of the vertices x_i, x_j, x_k of the singular triangle Δ in Theorem 0.2 by the projections on transverse trees at infinity.

Lemma 4.8. *We keep the hypotheses and notation of Theorem 0.2.*

(i) *The projection $\pi_{p_i}(x_i)$ of x_i on the tree X_{p_i} is the center of the ideal tripod $(D_i, p_i p_j, p_i p_k)$.*

(ii) *The projection $\pi_{D_i}(x_i)$ of x_i on the tree X_{D_i} is the center of the ideal tripod $(p_i, D_i \cap D_j, D_i \cap D_k)$.*

(iii) *The projection $\pi_{p_i}(x_j)$ is the center of the ideal tripod $(D_i, p_i p_j, p_i p_{jk})$.*

(iv) *The projection $\pi_{D_i}(x_j)$ is the center of the ideal tripod $(p_i, D_i \cap D_j, D_i \cap D_{jk})$.*

Proof. As the point x_i belongs to the three flats $A(F_k, F_i)$ and $A(F_j, F_i)$ and $A(p_i, p_j, p_k)$, its projection in the tree X_{p_i} belongs to the projection of $A(F_j, F_i)$, which is the line from D_i to $p_i p_j$, to the projection of $A(F_k, F_i)$, which is the line from D_i to $p_i p_k$, and to the projection of $A(p_i, p_j, p_k)$, which is the line from $p_i p_j$ to $p_i p_k$. Hence (i) is proven. Assertion (ii) is proven in the same way.

We now prove (iii). By (ii) applied to x_j, we have that $\pi_{D_j}(x_j)$ is the center of the ideal tripod p_j, $p_{jk} = D_j \cap D_k$, $D_j \cap D_i$. As x_j is on a geodesic from D_j to p_i, we may deduce that $\pi_{p_i}(x_j)$ is the center of the ideal tripod $p_i p_j$, $p_i p_{jk}$, D_i (using the canonical isomorphism $X_{D_j} \xrightarrow{\sim} X_{p_i}$). The last assertion (iv) has identical proof. □

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Opposition diagrams for automorphisms of small spherical buildings

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An automorphism θ of a spherical building Δ is called *capped* if it satisfies the following property: if there exist both type J_1 and J_2 simplices of Δ mapped onto opposite simplices by θ then there exists a type $J_1 \cup J_2$ simplex of Δ mapped onto an opposite simplex by θ. In previous work we showed that if Δ is a thick irreducible spherical building of rank at least 3 with no Fano plane residues then every automorphism of Δ is capped. In the present work we consider the spherical buildings with Fano plane residues (the *small buildings*). We show that uncapped automorphisms exist in these buildings and develop an enhanced notion of "opposition diagrams" to capture the structure of these automorphisms. Moreover we provide applications to the theory of "domesticity" in spherical buildings, including the complete classification of domestic automorphisms of small buildings of types F_4 and E_6.

Introduction

Let θ be an automorphism of a thick irreducible spherical building Δ of type (W, S). The *opposite geometry* of θ is the set $\mathrm{Opp}(\theta)$ of all simplices σ of Δ such that σ and σ^θ are opposite in Δ. This geometry forms a natural counterpart to the more familiar fixed element geometry $\mathrm{Fix}(\theta)$, however by comparison very little is known about $\mathrm{Opp}(\theta)$.

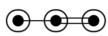
This paper is the continuation of [Parkinson and Van Maldeghem 2019], where we initiated a systematic study of $\mathrm{Opp}(\theta)$ for automorphisms of spherical buildings. In particular in [Parkinson and Van Maldeghem 2019] we showed that if Δ is a thick irreducible spherical building of rank at least 3 containing no Fano plane residues then $\mathrm{Opp}(\theta)$ has the following weak closure property: if there exist both type J_1 and J_2 simplices in $\mathrm{Opp}(\theta)$ then there exists a type $J_1 \cup J_2$ simplex in $\mathrm{Opp}(\theta)$.

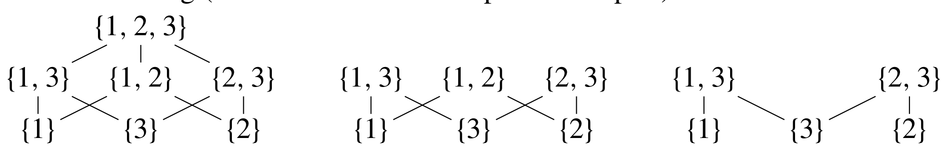
MSC2010: 20E42, 51E24.

Keywords: spherical building, opposition diagram, capped automorphism, domestic automorphism, displacement.

Automorphisms with this property are called *capped*, and the thick irreducible spherical buildings of rank at least 3 with no Fano plane residues are called *large buildings*. Thus every automorphism of a large building is capped.

In the present paper we investigate $\mathrm{Opp}(\theta)$ for the thick irreducible spherical buildings of rank at least 3 containing a Fano plane residue. These are called the *small buildings*. In particular we show that, in contrast to the case of large buildings, uncapped automorphisms exist for all small buildings (with the possible exception of $\mathsf{E}_8(2)$ where we provide conjectural examples).

A key tool in [Parkinson and Van Maldeghem 2019] was the notion of the *opposition diagram* of an automorphism θ, consisting of the triple (Γ, J, π), where Γ is the Coxeter graph of (W, S), J is the union of all $J' \subseteq S$ such that there exists a type J' simplex in $\mathrm{Opp}(\theta)$, and π is the automorphism of Γ induced by θ (less formally, the opposition diagram is drawn by encircling the nodes J of Γ). If θ is capped then this diagram turns out to encode a lot of information about the automorphism, essentially because it completely determines the partially ordered set $\mathcal{T}(\theta)$ of all types of simplices mapped onto opposite simplices by θ. However for an uncapped automorphism the opposition diagram does not necessarily determine $\mathcal{T}(\theta)$. For example in the polar space $\Delta = \mathsf{B}_3(2)$ there are collineations θ_1, θ_2 and θ_3 each with opposition diagram  (that is, each θ_i maps a vertex of each type to an opposite vertex) whose partially ordered sets $\mathcal{T}(\theta_i)$, for $i = 1, 2, 3$, are the following (see Theorem 3.7 for explicit examples):



Note that only θ_1 is capped (hence, in particular, analogues of θ_2 and θ_3 cannot exist for polar spaces $\mathsf{B}_3(\mathbb{F})$ with $|\mathbb{F}| > 2$ by the main result of [Parkinson and Van Maldeghem 2019]).

Thus the opposition diagram of an uncapped automorphism needs to be enhanced to properly understand these automorphisms. We achieve this by defining the *decorated opposition diagram* of an uncapped automorphism.

The full definition is given in Section 1, however for the purpose of this introduction consider the following simplified situation. Suppose that θ is an automorphism with the property that the induced automorphism π of the Coxeter graph Γ is the opposition automorphism w_0. Then the *decorated opposition diagram* of θ is the quadruple (Γ, J, K, π) where (Γ, J, π) is the opposition diagram, and

$$K = \{ j \in J \mid \text{there exists a type } J \backslash \{j\} \text{ simplex mapped onto an opposite simplex by } \theta\}.$$

Less formally, the decorated opposition diagram is drawn by encircling the nodes of J, and then shading those nodes of K. Thus, for example, the decorated

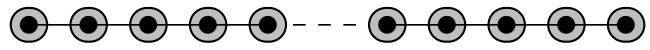
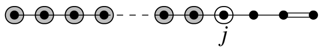
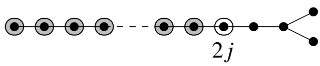
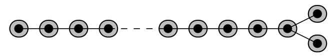
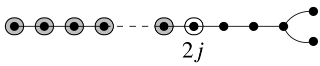
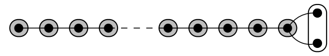
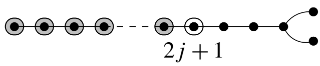
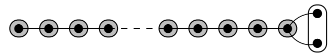
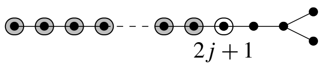
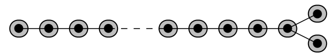
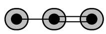
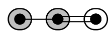
Δ	*diagrams*	
$\mathsf{A}_n(2)$		
$\mathsf{B}_n(2)$ *or* $\mathsf{B}_n(2,4)$, $(3 \le j \le n)$		
$\mathsf{D}_n(2)$, $n \ge 4$ *even* $(4 \le 2j \le n-2)$		
$\mathsf{D}_n(2)$, $n \ge 4$ *odd* $(4 \le 2j \le n-3)$		
$\mathsf{D}_n(2)$, $n \ge 4$ *even* $(3 \le 2j+1 \le n-3)$		
$\mathsf{D}_n(2)$, $n \ge 4$ *odd* $(3 \le 2j+1 \le n-2)$		

Table 1. Decorated opposition diagrams of uncapped automorphisms (classical types).

opposition diagrams of the two uncapped automorphisms of $\mathsf{B}_3(2)$ given above are

 and .

At an intuitive level, the more encircled nodes that are shaded on the decorated opposition diagram of an uncapped automorphism, the "closer" the automorphism is to being capped.

The main theorem of this paper is Theorem 1 below. Part (a) of the theorem shows that the decorated opposition diagram of an uncapped automorphism lies in a small list of diagrams, hence severely restricting the structure of uncapped automorphisms. Part (b) deals with the existence of uncapped automorphisms, showing that the list provided in part (a) has no redundancies, with only the $\mathsf{E}_8(2)$ case remaining open due to the size of the building rendering our computational techniques inadequate. We strongly believe that the two $\mathsf{E}_8(2)$ diagrams are indeed realised as opposition diagrams; see Conjecture 4.8 for details.

Theorem 1.

(a) *Let θ be an uncapped automorphism of a thick irreducible spherical building Δ of rank at least* 3. *Then the decorated opposition diagram of θ appears in Table 1 or Table 2.*

(b) *Let Δ be a small building. Each diagram appearing in the respective row of Table 1 or Table 2 can be realised as the decorated opposition diagram of some uncapped automorphism of Δ, with the exception perhaps of the two* $\mathsf{E}_8(2)$ *diagrams.*

Let us briefly describe corollaries to Theorem 1(a) (see Section 2B for details and precise statements). Recall that the *displacement* $\mathrm{disp}(\theta)$ of an automorphism θ is the maximum length of $\delta(C, C^\theta)$, with C a chamber.

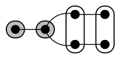
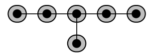
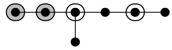
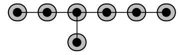
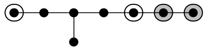
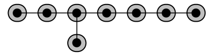
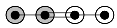
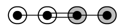
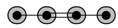
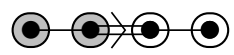
Δ	*diagrams*
$\mathsf{E}_6(2)$	 
$\mathsf{E}_7(2)$	 
$\mathsf{E}_8(2)$	 
$\mathsf{F}_4(2)$	  
$\mathsf{F}_4(2,4)$	

Table 2. Decorated opposition diagrams of uncapped automorphisms (exceptional types). The arrow in the $\mathsf{F}_4(2,4)$ diagram indicates that the residues of type $\{1,2\}$ are projective planes of order 2.

Corollary 2. *Let θ be an automorphism of a thick irreducible spherical building Δ.*

(a) *If θ is an involution, θ is capped.*

(b) *If θ is uncapped, $\mathcal{T}(\theta)$ is determined by the decorated opposition diagram of θ.*

(c) *If θ is uncapped,* $\mathrm{disp}(\theta)$ *is determined by the decorated opposition diagram of θ.*

In particular, if Δ has type (W, S) and $J = \mathrm{Typ}(\theta)$ then Corollary 2(c) implies that (see Corollary 2.29)

$$\mathrm{disp}(\theta) = \begin{cases} \mathrm{diam}(W) - \mathrm{diam}(W_{S\backslash J}) & \text{if } \theta \text{ is capped,} \\ \mathrm{diam}(W) - \mathrm{diam}(W_{S\backslash J}) - 1 & \text{if } \theta \text{ is uncapped.} \end{cases}$$

To illustrate this in an example, it follows that if θ is a nontrivial automorphism of a thick E_8 building then $\mathrm{disp}(\theta) \in \{57, 90, 107, 108, 119, 120\}$, which is a surprisingly restricted list of possibilities (see Remark 2.30). Moreover, displacements of 107 or 119 can only occur for uncapped automorphisms of the small building $\mathsf{E}_8(2)$.

We also provide applications of Theorem 1(a) to the study of *domesticity* in spherical buildings (recall that an automorphism is called *domestic* if it maps no chamber to an opposite chamber). These automorphisms have recently enjoyed extensive investigation, including the series [Temmermans et al. 2011; 2012a; 2012b] where domesticity in projective spaces, polar spaces, and generalised quadrangles is studied, [Van Maldeghem 2012] where symplectic polarities of large E_6 buildings are classified in terms of domesticity, [Van Maldeghem 2014] where domestic trialities of D_4 buildings are classified, and [Parkinson et al. 2015] where domesticity in generalised polygons is studied.

To give one example of our applications to domesticity, suppose that Δ is a simply laced spherical building, and that θ is a domestic automorphism inducing opposition on the type set with the property that θ maps at least one vertex of each type onto an opposite vertex (such automorphisms are called “exceptional domestic"). Then we show that in fact θ maps simplices of each type $J \subsetneq S$ onto opposite simplices (such automorphisms are called "strongly exceptional domestic"). In particular, this implies that $\mathrm{disp}(\theta) = \mathrm{diam}(\Delta) - 1$ for exceptional domestic automorphisms.

Theorem 1(b) provides the first known examples of exceptional domestic automorphisms of spherical buildings of rank at least 3 (examples were previously only known for generalised polygons; see [Parkinson et al. 2015]). In fact Theorem 1(b) shows that, with the possible exception of $\mathsf{E}_8(2)$, every small building admits a strongly exceptional domestic automorphism.

The proof of Theorem 1(b) for the small buildings of exceptional type involves computations using [Magma], and in particular the groups of Lie type package [Cohen et al. 2004]. In fact for the small buildings of type F_4 and E_6 we are able to prove a much stronger result and completely classify the domestic automorphisms of these buildings. To perform these calculations we implemented the minimal faithful permutation representations of the ATLAS groups $\mathsf{F}_4(2)$, $\mathsf{F}_4(2).2$, $\mathsf{E}_6(2)$, $\mathsf{E}_6(2).2$, ${}^2\mathsf{E}_6(2^2)$, and ${}^2\mathsf{E}_6(2^2).2$ (respective permutation degrees 69615, 139230, 139503, 279006, 3968055 and 3968055) into the Magma system. At the time of writing these representations were not readily available in either Magma or GAP, and therefore they are provided on Parkinson's webpage.

We conclude this introduction with an outline of the structure of the paper. In Section 1 we provide definitions and background. The proofs of Theorem 1(a) and its corollaries are contained in Section 2. The proof of Theorem 1(b) is divided across Section 3 for the classical types and Section 4 for the exceptional types. Moreover, Section 4 contains the complete classification of domestic automorphisms of the small buildings of types F_4 and E_6.

1. Definitions and background

We refer to [Abramenko and Brown 2008] for the general theory of buildings. In this section we will briefly recall some notation, mainly from [Parkinson and Van Maldeghem 2019, Section 1]. Let Δ be a spherical building of type (W, S), typically considered as a simplicial complex with type map $\tau : \Delta \to 2^S$. Let $\mathcal{C}$ be the set of chambers (maximal simplices) of Δ, and let $\delta : \mathcal{C} \times \mathcal{C} \to W$ be the Weyl distance function.

Chambers C and D of Δ are *opposite* if and only if they are at maximal distance in the chamber graph (with adjacency given by the union of the s-adjacency relations: $C \sim_s D$ if and only $\delta(C, D) = s$). Equivalently, chambers $C, D \in \mathcal{C}$ are opposite if and only if $\delta(C, D) = w_0$ where w_0 is the longest element of W.

If $J \subseteq S$ we write $J^{\mathrm{op}} = J^{w_0} = w_0^{-1} J w_0$ (the "opposite type" to J). The definition of opposition for chambers extends naturally to arbitrary simplices as follows (see [Abramenko and Brown 2008, Lemma 5.107]).

Definition 1.1. Simplices α, β of Δ are *opposite* if $\tau(\beta) = \tau(\alpha)^{\mathrm{op}}$ and there exists a chamber A containing α and a chamber B containing β such that A and B are opposite.

An *automorphism* of Δ is a simplicial complex automorphism $\theta : \Delta \to \Delta$. Note that θ does not necessarily preserve types. Indeed each automorphism $\theta : \Delta \to \Delta$ induces a permutation π_θ of the type set S, given by $\delta(C, D) = s$ if and only if $\delta(C^\theta, D^\theta) = s^{\pi_\theta}$, and this permutation is a diagram automorphism of the Coxeter graph Γ of (W, S). If Δ is irreducible, then from the classification of irreducible spherical Coxeter systems we see that $\pi_\theta : S \to S$ either

(1) is the identity, in which case θ is called a *collineation* (or *type-preserving*),

(2) has order 2, in which case θ is called a *duality*, or

(3) has order 3, in which case θ is called a *triality*; this only occurs in type D_4.

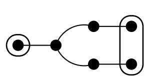
Automorphisms $\theta : \Delta \to \Delta$ that induce opposition on the type set (that is, $\pi_\theta = w_0$, where w_0 is the diagram automorphism given by $s^{w_0} = w_0^{-1} s w_0$) are called *oppomorphisms*. For example, oppomorphisms of an E_6 building are dualities, and oppomorphisms of an E_7 building are collineations (see, for example, [Abramenko and Brown 2008, Section 5.7.4]).

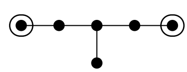
Let θ be an automorphism of Δ. The *opposite geometry* of θ is

$$\mathrm{Opp}(\theta) = \{\sigma \in \Delta \mid \sigma \text{ is opposite } \sigma^\theta\}.$$

A fundamental result of Leeb [2000, Section 5] and Abramenko and Brown [2009, Proposition 4.2] states that if θ is a nontrivial automorphism of a thick spherical building then $\mathrm{Opp}(\theta)$ is necessarily nonempty (this result has been generalised to the setting of twin buildings; see [Devillers et al. 2013]).

The *type* $\mathrm{Typ}(\theta)$ of an automorphism θ is the union of all subsets $J \subseteq S$ such that there exists a type J simplex in $\mathrm{Opp}(\theta)$. The *opposition diagram* of θ is the triple $(\Gamma, \mathrm{Typ}(\theta), \pi_\theta)$. Less formally, the opposition diagram of θ is depicted by drawing Γ and encircling the nodes of $\mathrm{Typ}(\theta)$, where we encircle nodes in minimal subsets invariant under $w_0 \circ \pi_\theta$. We draw the diagram "bent" (in the standard way) if $w_0 \circ \pi_\theta \neq 1$. For example, consider the following diagrams:





The diagram on the left represents a collineation θ of an E_6 building with $\mathrm{Typ}(\theta) = \{1, 2, 6\}$, and the diagram on the right represents a duality θ of an E_6 building with $\mathrm{Typ}(\theta) = \{1, 6\}$.

We call an opposition diagram *empty* if no nodes are encircled (i.e., $\mathrm{Typ}(\theta) = \varnothing$), and *full* if all nodes are encircled (i.e., $\mathrm{Typ}(\theta) = S$).

Definition 1.2. Let Δ be a spherical building of type (W, S). Let θ be a nontrivial automorphism of Δ, and let $J \subseteq S$. Then θ is called:

(a) *Capped* if there exists a type $\mathrm{Typ}(\theta)$ simplex in $\mathrm{Opp}(\theta)$, and *uncapped* otherwise.

(b) *Domestic* if $\mathrm{Opp}(\theta)$ contains no chamber.

(c) *J-domestic* if $\mathrm{Opp}(\theta)$ contains no type J simplex (this terminology is reserved for subsets J which are stable under $w_0 \circ \pi_\theta$).

(d) *Exceptional domestic* if θ is domestic with full opposition diagram.

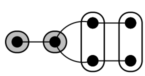
(e) *Strongly exceptional domestic* if θ is domestic, but not J-domestic for any strict subset J of S invariant under $w_0 \circ \pi_\theta$.

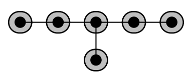
Note that if θ is a domestic automorphism with $w_0 \circ \pi_\theta = 1$ then θ is exceptional domestic if and only if there exists a vertex of each type mapped to an opposite vertex, and θ is strongly exceptional domestic if and only if there exists a panel of each cotype mapped to an opposite panel (recall that a *panel* is a codimension 1 simplex).

To study uncapped automorphisms θ we introduce the decorated opposition diagram. Let $\mathcal{J}_\theta$ denote the set of subsets $I \subseteq S$ which are minimal with respect to the condition $I^{\pi_\theta w_0} = I$. For example, if θ induces opposition on Γ then $\mathcal{J}_\theta = \{\{s\} \mid s \in S\}$ is the set of all singleton subsets of S.

Definition 1.3. The *decorated opposition diagram* of an uncapped automorphism θ is the quadruple $(\Gamma, J, K_\theta, \pi_\theta)$ where $J = \mathrm{Typ}(\theta)$ and $K_\theta \subseteq J$ is the union of all $J' \in \mathcal{J}_\theta$ such that there exists a type $J \backslash J'$ simplex mapped onto an opposite simplex.

Less formally, the decorated opposition diagram is drawn by shading the nodes of K_θ on the opposition diagram. For example, consider the following.





The decorated opposition diagram on the left represents an uncapped collineation of $\mathsf{E}_6(2)$ with the property that there are simplices of types $S\backslash\{2\}$ and $S\backslash\{4\}$ mapped onto opposite simplices, and no simplices of types $S\backslash\{3, 5\}$ nor $S\backslash\{1, 6\}$ mapped onto opposite simplices — this automorphism is exceptional domestic, but it is not strongly exceptional domestic. The diagram on the right represents an uncapped duality of $\mathsf{E}_6(2)$ with the property that there are panels of each cotype mapped onto opposite panels — this automorphism is strongly exceptional domestic.

Residue arguments are used extensively in the proof of Theorem 1(a), and so we conclude this section with a summary of the techniques. We first briefly define residues and projections (see [Abramenko and Brown 2008] for details). The *residue* $\mathrm{Res}(\alpha)$ of a simplex $\alpha \in \Delta$ is the set of all simplices of Δ which contain α, together with the order relation induced by that of Δ. Then $\mathrm{Res}(\alpha)$ is a building whose diagram is obtained from the diagram of Δ by removing all nodes which belong to $\tau(\alpha)$. The *projection onto* α is the map $\mathrm{proj}_\alpha : \Delta \to \mathrm{Res}(\alpha)$ defined as follows. Firstly, if B is a chamber of Δ then there is a unique chamber $A \in \mathrm{Res}(\alpha)$ such that $\ell(\delta(A, B)) < \ell(\delta(A', B))$ for all chambers $A' \in \mathrm{Res}(\alpha)$ with $A' \neq A$, and we define $\mathrm{proj}_\alpha(B) = A$. In other words, $\mathrm{proj}_\alpha(B)$ is the unique chamber A of $\mathrm{Res}(\alpha)$ with the property that every minimal length gallery from B to $\mathrm{Res}(\alpha)$ ends with the chamber A. Now, if β is an arbitrary simplex we define

$$\mathrm{proj}_\alpha(\beta) = \bigcap_B \mathrm{proj}_\alpha(B),$$

where the intersection is over all chambers B in $\mathrm{Res}(\beta)$. In other words, $\mathrm{proj}_\alpha(\beta)$ is the unique simplex γ of $\mathrm{Res}(\alpha)$ which is maximal subject to the property that every minimal length gallery from a chamber of $\mathrm{Res}(\beta)$ to $\mathrm{Res}(\alpha)$ ends in a chamber containing γ.

Let θ be an automorphism of Δ, and suppose that $\sigma \in \mathrm{Opp}(\theta)$. It follows from [Tits 1974, Theorem 3.28] that the projection map $\mathrm{proj}_\sigma : \mathrm{Res}(\sigma^\theta) \to \mathrm{Res}(\sigma)$ is an isomorphism. Define

$$\theta_\sigma : \mathrm{Res}(\sigma) \xrightarrow{\sim} \mathrm{Res}(\sigma) \quad \text{by} \quad \theta_\sigma = \mathrm{proj}_\sigma \circ \theta.$$

The type map induced by θ_σ is as follows.

Proposition 1.4. *Let θ be an automorphism of a spherical building Δ of type (W, S). Suppose that $\sigma \in \mathrm{Opp}(\theta)$ and let $J = \tau(\sigma)$. Then the type map on $S\backslash J$ induced by θ_σ is $w_{S\backslash J} \circ w_0 \circ \pi_\theta$.*

Proof. This follows easily from [Abramenko and Brown 2008, Corollary 5.116]. □

Example 1.5. We will use Proposition 1.4 many times in our residue arguments. For example, consider a duality θ of an D_n building, and suppose that $v \in \mathrm{Opp}(\theta)$ is a type i vertex, with $i \le n-2$. The residue of v is a building of type $\mathsf{A}_{i-1} \times \mathsf{D}_{n-i}$, and the induced automorphism θ_v of $\mathrm{Res}(v)$ is a duality on the A_{i-1} component, and a duality (respectively, collineation) on the D_{n-i} component if i is even (respectively, odd).

It is useful to note that if θ is an oppomorphism, and if $\sigma \in \mathrm{Opp}(\theta)$, then θ_σ is a oppomorphism of $\mathrm{Res}(\sigma)$ (this follows immediately from Proposition 1.4).

From [Tits 1974, Proposition 3.29] we have:

Proposition 1.6. *Let θ be an automorphism of a spherical building Δ and let $\alpha \in \mathrm{Opp}(\theta)$. If $\beta \in \mathrm{Res}(\alpha)$ then β is opposite β^θ in the building Δ if and only if β is opposite β^{θ_α} in the building* $\mathrm{Res}(\alpha)$.

The following corollary facilitates inductive residue arguments.

Corollary 1.7. *Let $\theta : \Delta \to \Delta$ be a domestic automorphism and let $\sigma \in \mathrm{Opp}(\theta)$. Then $\theta_\sigma : \mathrm{Res}(\sigma) \to \mathrm{Res}(\sigma)$ is a domestic automorphism of the building $\mathrm{Res}(\sigma)$.*

Proof. Let $J = \tau(\sigma)$. If θ_σ is not domestic then there is a chamber σ' of $\mathrm{Res}(\sigma)$ mapped onto an opposite chamber by θ_σ. Then $\sigma \cup \sigma'$ is a chamber of Δ, and from Proposition 1.6 this chamber is mapped onto an opposite chamber, which is a contradiction. □

2. Theorem 1(a) and its corollaries

In this section we prove Theorem 1(a) and give applications to determining the partially ordered set $\mathcal{T}(\theta)$, domesticity, cappedness of involutions, and calculating displacement.

2A. *Proof of Theorem 1(a).* By [Parkinson and Van Maldeghem 2019, Theorem 1] if θ is an uncapped automorphism of a thick irreducible spherical building Δ of rank at least 3 then Δ is a small building. These are precisely the buildings listed in the first column of Tables 1 and 2. Moreover, the following proposition from [Parkinson and Van Maldeghem 2019] explains why collineations of A_n, trialities of D_4, and dualities of F_4 do not appear in Tables 1 and 2.

Proposition 2.1. *Every collineation of a thick A_n building is capped, every triality of a thick D_4 building is capped, and every duality of a thick F_4 building is capped.*

Proof. See [Parkinson and Van Maldeghem 2019, Corollary 3.9, Theorem 3.17, Lemma 4.1]. □

Buildings of type A_n play an important role in our proof techniques owing to their prevalence as residues of spherical buildings of arbitrary type. Every thick building of type A_n with $n > 2$ is a projective space $\mathsf{PG}(n, \mathbb{K})$ over a division ring $\mathbb{K}$, where the type i vertices of the building are the $(i-1)$-spaces of the projective space. Thus points have type 1, lines have type 2, and so on.

Definition 2.2. Let $\mathbb{F}$ be a field. A duality of $\mathsf{A}_{2n-1}(\mathbb{F})$ with $U^\theta = \{v \mid (u, v) = 0$ for all $u \in U\}$ for some nondegenerate symplectic form $(\cdot\,,\cdot)$ on $\mathbb{F}^{2n}$ is called a *symplectic polarity*.

Let us recall some useful facts concerning dualities of type A buildings.

Lemma 2.3 [Temmermans et al. 2011, Lemma 3.2]. *If the projective space $\Delta = \mathsf{PG}(n, \mathbb{K})$ admits a duality θ for which all points are absolute (equivalently no type 1 vertex is mapped to an opposite), then n is odd, $\mathbb{K}$ is a field, and θ is a symplectic polarity.*

Lemma 2.4 [Parkinson and Van Maldeghem 2019, Lemma 3.4]. *If θ is a symplectic polarity of an A_{2n-1} building then θ is $\{i\}$-domestic for each odd i, and* *each vertex mapped to an opposite vertex is contained in a type* $\{2, 4, \ldots, 2n-2\}$ *simplex mapped to an opposite simplex. Hence symplectic polarities are capped.*

Theorem 2.5 [Parkinson and Van Maldeghem 2019, Theorems 3.10 and 3.11]. *Let* θ *be a domestic duality of the small building* $\Delta = \mathsf{A}_n(2)$ *with* $n \geq 2$. *Then either* θ *is a strongly exceptional domestic duality or* n *is odd and* θ *is a symplectic polarity.*

The following proposition shows that the diagrams for uncapped dualities of A_n buildings are as claimed in the first row of Table 1.

Proposition 2.6. *Every uncapped duality of* $\mathsf{A}_n(2)$ *is a strongly exceptional domestic duality.*

Proof. If θ is uncapped then necessarily θ is domestic, and so by Theorem 2.5 θ is either a symplectic polarity or is strongly exceptional domestic. The first case is eliminated by Lemma 2.4. □

We now consider the small buildings of types B_n and D_n. We first require some preliminary results. It is convenient at times to use terminology like "x is domestic for θ" and "x is nondomestic for θ" as short hand for "θ does not map x to an opposite" and "x is mapped to an opposite by θ". If the automorphism θ is clear from context we will simply say "x is domestic" or "x is nondomestic".

Lemma 2.7. *Let* $n \geq 4$ *and let* Δ *be a building of type* B_n *or* D_{n+2} *with thick projective plane residues. Let* θ *be an automorphism and let* $J = \mathrm{Typ}(\theta)$. *If there exists* $j \in J$ *odd with* $j \leq n$, *then* $\{1, 2, \ldots, j\} \subseteq J$.

Proof. Let v be a nondomestic type j vertex. Then θ_v acts as a duality on the A_{j-1} component of the residue of v (by Proposition 1.4). Since j is odd, this duality is either nondomestic or is exceptional domestic (see Theorem 2.5), and in either case $1, 2, \ldots, j-1 \in J$, and hence the result. □

Lemma 2.8. *Let* Δ *be a building of type* B_n *or* D_{n+2} *with* $n \geq 4$ *and thick projective plane residues, and let* θ *be a collineation. Let* $J = \mathrm{Typ}(\theta)$. *Suppose that* $3 \leq j < n$, *and that* $\{j-1, j\} \subseteq J$ *and* $j+1 \notin J$. *Then there exists a type* $\{1, j\}$*-simplex mapped onto an opposite simplex by* θ.

Proof. We first show that θ is not $\{j-1, j\}$-domestic. For if θ is $\{j-1, j\}$-domestic, then since θ is also $\{j-1, j+1\}$-domestic it follows from [Parkinson and Van Maldeghem 2019, Lemma 3.25] that either θ is $\{j-1\}$-domestic or $\{j\}$-domestic, a contradiction. Thus there exists a type $\{j-1, j\}$ simplex σ mapped onto an opposite. If v is the type j vertex of this simplex then θ_v acts as a duality on the A_{j-1} component (Proposition 1.4) mapping a hyperplane to an opposite (by Proposition 1.6). Thus θ_v is either nondomestic or strongly exceptional domestic on the A_{j-1} component, and in either case there exists a nondomestic type $\{1, j\}$ simplex (note that $j-1 \geq 2$). □

Lemma 2.9. *Let Δ be a small building of type B_n or D_{n+1}, and let $j < n$. Suppose that θ is an uncapped collineation of type $J = \{1, 2, 3, \dots, j\}$. Then θ is $\{1, 2, 3, \dots, j-1\}$-domestic.*

Proof. Suppose that there is a nondomestic type $\{1, 2, \dots, j-1\}$ simplex, and let v be the type $j-1$ vertex of this simplex. If θ is uncapped then necessarily θ_v acts as the identity on the "upper" residue of type B_{n-j+1} or D_{n-j+2} (by Proposition 1.6). Thus [Parkinson and Van Maldeghem 2019, Lemma 3.28] with $i = j-2$ and $\ell = j-3$ (note the index shift due to the fact that we used projective dimension in [Parkinson and Van Maldeghem 2019]) implies that every $(j-1)$-space in the polar space of Δ has a fixed point. Thus no type j vertex of Δ is mapped onto an opposite vertex, contradicting the fact that $j \in J$. □

We can now complete the proof of Theorem 1(a) for buildings of type B_n. We allow the additional generality of thin cotype n panels to facilitate our later arguments for type D_n.

Proposition 2.10. *Let Δ be a (possibly nonthick) building of type B_n with Fano plane residues and $n \geq 3$, and let θ be a collineation of Δ. If θ is uncapped, then the decorated opposition diagram of θ is one of the diagrams in Table 1.*

Proof. Suppose that θ is uncapped. Let $J = \mathrm{Typ}(\theta)$, and let $j = \max J$. Then $j \geq 3$, for if $j = 1$ then θ is capped, and if $j = 2$ then either $J = \{2\}$ and θ is capped, or $J = \{1, 2\}$ in which case [Parkinson and Van Maldeghem 2019, Fact 3.21] implies that θ is capped.

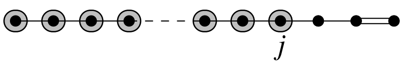
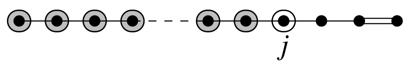
We claim that J contains an odd element. For if every element of J is even then for each nondomestic type j-vertex v the induced automorphism θ_v is a point domestic duality of an A_{j-1} building (by Propositions 1.4 and 1.6). Thus θ_v is a symplectic polarity (Lemma 2.3), and so there exists a type $\{2, 4, \dots, j-2\}$ simplex of the residue mapped to an opposite (Lemma 2.4). Hence by Proposition 1.6 there is a type $\{2, 4, \dots, j-2, j\} = J$ simplex of Δ mapped onto an opposite and so θ is capped, a contradiction.

Let $k \in J$ be the maximal odd node. By Lemma 2.7 we have $\{1, 2, \dots, k\} \subseteq J$. Consider the following cases.

(1) If $j = n$ then by [Parkinson and Van Maldeghem 2019, Proposition 3.12(2)] there is a nondomestic type $\{1, n\}$ simplex. In the A_{n-1} residue of the type n vertex of this simplex we have a strongly exceptional domestic duality of A_{n-1} (since it is domestic and maps a point to an opposite), and hence there are panels of each cotype $1, 2, \dots, n-1$ mapped onto opposites in Δ. Thus θ has either the first diagram listed in Table 1 (with $j = n$) or the second diagram listed in Table 1 (strongly exceptional domestic).

(2) If $k = j < n$ then $J = \{1, 2, \dots, j\}$, and by Lemma 2.8 there exists a nondomestic

type $\{1, j\}$ simplex. Considering the type A_{j-1} residue of the type j vertex of this simplex, and noting that $j-1$ is even, we see that in Δ there are nondomestic simplices of each type $J\backslash\{j'\}$ with $j' = 1, 2, \ldots, j-1$ (using Theorem 2.5), and hence the diagram of θ is either

 or  (2-1)

The first digram is eliminated by Lemma 2.9.

(3) If $k < j < n$ then j is even, and as above we have $\{2, 4, \ldots, j-2, j\} \subseteq J$. In particular $\{k, k+1\} \subseteq J$ and $k+2 \notin J$ (as k is the maximum odd node of J, and note that $k+2 \leq n$). Lemma 2.8 implies that there is a nondomestic type $\{1, k+1\}$ simplex. If $k+1 = j$ then as above we have the diagrams (2-1) and Lemma 2.9 eliminates the first of the diagrams. If $k+1 < j$ then $k+3 \leq j < n$. If θ is $\{1, k+3\}$-domestic, then since θ is not $\{k+3\}$-domestic, [Parkinson and Van Maldeghem 2019, Lemma 3.29] implies that θ is $\{1, k+1\}$-domestic, a contradiction. Hence there exists a type $\{1, k+3\}$ simplex mapped onto an opposite. However, considering the A_{k+2} residue of the type $k+3$ vertex of this simplex we see that θ is not $\{k+2\}$-domestic, contradicting the maximality of k.

Hence the result. □

Corollary 2.11. *Let Δ be a building of type B_n with thick projective spaces, and let θ be a collineation and $n \geq i \geq 3$. If θ is $\{1, i\}$-domestic then θ is either $\{1\}$-domestic or $\{i\}$-domestic.*

Proof. If θ is capped then the result is true by definition. If θ is uncapped then the result follows directly from the classification of uncapped diagrams given above. □

Remark 2.12. The assumption $i \geq 3$ cannot be removed from Corollary 2.11. For example, consider the exceptional domestic collineation of the generalised quadrangle $\mathsf{B}_2(2)$ (see [Temmermans et al. 2012b, Section 4]). More generally, for each $n \geq 2$ there exists an uncapped collineation of $\mathsf{B}_n(2)$ with $\mathrm{Typ}(\theta) = \{1, 2\}$ (see Theorem 3.7).

We now continue with the analysis of buildings of type D_n. Recall that each building of type D_n can be realised as the oriflamme geometry of the space $\mathbb{F}^{2n}$ equipped with an orthogonal form of Witt index n, for some field $\mathbb{F}$. The vertices of type j for $j \in \{1, \ldots, n-2\}$ are the totally isotropic spaces of dimension j, and the vertices of type $n-1$ and n are the totally isotropic subspaces of dimension n (corresponding to the orbits of the action of the associated simple orthogonal group). To each such building Δ of type D_n there is an associated (nonthick) building Δ' of type B_n. The type j vertices of Δ', for $1 \leq j \leq n$, are the totally isotropic subspaces of dimension j. Each type $n-1$ vertex of Δ' determines a type $\{n-1, n\}$

simplex of Δ, and vice versa, as follows. A type $n-1$ vertex of Δ' is an $(n-1)$-dimensional totally isotropic space W, and there are precisely two totally isotropic n-dimensional subspaces U, V containing W and (U, V) is an $\{n-1, n\}$-simplex of Δ. Conversely, if (U, V) is a type $\{n-1, n\}$ simplex of Δ then $W = U \cap V$ is a type $n-1$ vertex of Δ'.

We first recall two facts from [Parkinson and Van Maldeghem 2019].

Lemma 2.13 [Parkinson and Van Maldeghem 2019, Lemma 3.32]. *Let Δ be a thick building of type D_n with n odd, and let Δ' be the associated nonthick B_n building. A collineation θ maps a type $\{n-1, n\}$ simplex of Δ to an opposite simplex if and only if it maps the associated type $n-1$ vertex of Δ' to an opposite vertex.*

Lemma 2.14 [Parkinson and Van Maldeghem 2019, Proposition 3.16]. *No duality of a thick building of type D_n is $\{1\}$-domestic.*

Lemma 2.15. *Let Δ be a thick building of type D_n with $n \geq 5$ odd, and let θ be a collineation. If θ is $\{1, n-1, n\}$-domestic then θ is either $\{1\}$-domestic or $\{n-1, n\}$-domestic.*

Proof. Suppose that θ is neither $\{1\}$-domestic nor $\{n-1, n\}$-domestic. Since θ maps a type $\{n-1, n\}$-simplex to an opposite, by familiar residue arguments there are vertices of types $2, 4, \ldots, n-3$ mapped onto opposite vertices. These vertex types are therefore also mapped onto opposites in the associated nonthick B_n building Δ'. If there are no type $n-2$ or $n-1$ vertices of Δ' mapped onto opposite vertices, then θ is $\{n-3, n-2\}$-domestic and $\{n-3, n-1\}$-domestic (on Δ') and thus since θ is not $\{n-3\}$-domestic it follows from [Parkinson and Van Maldeghem 2019, Lemma 3.25] that every space of vector space dimension at least $n-2$ contains a fixed point. However by Lemma 2.13 there are $n-1$ dimensional spaces mapped onto opposites, a contradiction. Thus either (i) θ is not $\{n-3, n-2\}$-domestic, or (ii) θ is not $\{n-3, n-1\}$-domestic (on Δ').

Consider case (i). Let v be the type $n-2$ vertex of a nondomestic type $\{n-3, n-2\}$ simplex. Then θ_v acts on the upper type $\mathsf{A}_1 \times \mathsf{A}_1$ residue by permuting the components, and thus θ_v is nondomestic on this upper residue (see [Parkinson and Van Maldeghem 2019, Lemma 3.7]). Moreover θ_v is a duality on the lower type A_{n-3} residue mapping a hyperplane (a type $n-3$ vertex) of this residue onto an opposite, and thus θ_v also maps a point (a type 1 vertex) to an opposite. Thus θ maps a type $\{1, n-1, n\}$ simplex to an opposite, a contradiction.

Consider case (ii). Since θ is neither $\{1\}$-domestic nor $\{n-1\}$-domestic on Δ', and since $n-1 \leq 4$, Corollary 2.11 implies that there exists a type $\{1, n-1\}$ simplex of Δ' mapped to an opposite. Now Lemma 2.13 implies that θ is not $\{1, n-1, n\}$-domestic on Δ. This contradiction establishes the result. □

Proposition 2.16. *Let Δ be the building $\mathsf{D}_n(2)$, $n \geq 4$, and let θ be a collineation of Δ. If θ is uncapped then the decorated opposition diagram of θ is contained in Table 1.*

Proof. Let θ be an uncapped collineation of $\mathsf{D}_n(2)$, and let $J = \mathrm{Typ}(\theta)$. Let $j = \max J$.

Case 1: $j \in \{n-1, n\}$ with n odd. Then necessarily $\{n-1, n\} \subseteq J$. If $J \backslash \{n-1, n\}$ contains no odd types, then the induced automorphism in every residue of a non-domestic $\{n-1, n\}$-simplex is a symplectic polarity, and hence θ is capped, a contradiction. Thus $J \backslash \{n-1, n\}$ contains an odd node, and so by Lemma 2.7 we have $1 \in J$. Thus by Lemma 2.15 there exists a type $\{1, n-1, n\}$ simplex mapped onto an opposite simplex, and it easily follows that θ maps simplices of each type $S \backslash \{i\}$ with $i = 1, 2, \ldots, n-2$ to opposite. Hence the claimed diagram.

Case 2: $j \in \{n-1, n\}$ with n even. By duality symmetry we may assume that $j = n$. If $n-1 \in J$, then by [Parkinson and Van Maldeghem 2019, Proposition 3.12(3)(b)] there is a type $\{n-1, n\}$-simplex mapped onto an opposite, and then considering the type A_{n-2} residue we easily deduce that there are simplices of each cotype $S \backslash \{i\}$ with $i = 1, 2, \ldots, n-2$ mapped onto opposites. It then easily follows that there are also simplices of each type $S \backslash \{n-1\}$ and $S \backslash \{n\}$ mapped onto opposite. So suppose that $n-1 \notin J$. If $J \backslash \{n-1, n\}$ contains no odd indices, then as above we deduce that θ is capped. Thus $J \backslash \{n-1, n\}$ contains an odd node, and so $1 \in J$ by Lemma 2.7, and by [Parkinson and Van Maldeghem 2019, Proposition 3.12(3)(a)] there is a type $\{1, n\}$ simplex mapped onto an opposite. It now easily follows that θ is strongly exceptional domestic.

Case 3: $j \notin \{n-1, n\}$. If j is odd, then considering the upper residue of a type j nondomestic we obtain a duality of a D_{n-j}, and since every duality of a D_{n-j} maps a point to an opposite point (Lemma 2.14) we have $j+1 \in J$, a contradiction. Thus j is even. If $j = 2$ then θ is capped (see [Parkinson and Van Maldeghem 2019, Fact 3.22]). So $j \geq 4$ (and hence $n \geq 6$). If J has only even types then clearly θ is capped. Thus J contains an odd node, and hence by Lemma 2.7 we have $1 \in J$. Applying Corollary 2.11 in the nonthick B_n building it follows that there is a type $\{1, j\}$-simplex mapped onto an opposite, and the result easily follows, using Lemma 2.9 to show that the last node is not shaded. □

Proposition 2.17. *Let θ be a duality of the $\mathsf{D}_n(2)$ building. If θ is uncapped then the decorated opposition diagram of θ is contained in Table 1.*

Proof. Let θ be an uncapped duality of $\mathsf{D}_n(2)$, and let $J = \mathrm{Typ}(\theta)$. Let $j = \max J$.

Case 1: $j \in \{n-1, n\}$ with n even. Then necessarily $\{n-1, n\} \subseteq J$. In the residue of such a simplex we have an exceptional domestic duality of $\mathsf{A}_{n-2}(2)$, and the result easily follows.

Case 2: $j \in \{n-1, n\}$ with n odd. In the residue of a nondomestic type j vertex we obtain an exceptional domestic duality of $\mathsf{A}_{n-1}(2)$, and again the result easily follows.

Case 3: $j \notin \{n-1, n\}$. If j is even, then considering the upper residue of a nondomestic type j vertex we obtain a duality of $\mathsf{D}_{n-j}(2)$, and since every duality of $\mathsf{D}_{n-j}(2)$ maps a point to an opposite point we have $j+1 \in J$, a contradiction. Thus j is odd. If $j = 1$ then θ is obviously capped. So $j \geq 3$ (and hence $n \geq 5$). In the lower residue of a nondomestic type j vertex we obtain an exceptional domestic duality of $\mathsf{A}_{j-1}(2)$, and hence the result, using Lemma 2.9 to see that the last node is not shaded. □

Propositions 2.16 and 2.17 establish Theorem 1(a) for buildings of type D_n. We now consider the exceptional types.

Lemma 2.18. *Let Δ be the building $\mathsf{F}_4(2)$, and let θ be a collineation. If* $\mathrm{Typ}(\theta) = \{1, 2, 3, 4\}$ *then there exists either a nondomestic type* $\{1, 2\}$ *simplex, or a nondomestic type* $\{3, 4\}$ *simplex.*

Proof. This follows from the classification given in Theorem 4.3. We note that no circular logic is introduced by postponing the proof until Section 4. □

We are now ready to prove Theorem 1(a) for the small exceptional buildings. Before doing so we would like to correct [Van Maldeghem 2012, Main result 2.2], where it is asserted that every domestic duality of an E_6 building is a symplectic polarity. In fact this result only holds for large E_6 buildings. The oversight in the proof of [Van Maldeghem 2012, Main result 2.2] is in the proof of [Van Maldeghem 2012, Lemma 5.2], where the existence of exceptional domestic automorphisms of $\mathsf{A}_4(2)$ is overlooked.

Proposition 2.19. *If θ is an uncapped automorphism of a building of exceptional type then the decorated opposition diagram of θ is contained in Table 2.*

Proof. (1) Let θ be an uncapped collineation of $\mathsf{E}_6(2)$ and let $J = \mathrm{Typ}(\theta)$. Suppose that $J = S$, and so the opposition diagram has the subsets $\{2\}$, $\{4\}$, $\{3, 5\}$ and $\{1, 6\}$ encircled. Let σ be a nondomestic type $\{3, 5\}$ simplex. Then θ_σ is an automorphism of an $\mathsf{A}_2 \times \mathsf{A}_1 \times \mathsf{A}_1$ building acting as a duality on the A_2 component and interchanging the two A_1 components (by Proposition 1.4). Thus θ_σ is not domestic on the $\mathsf{A}_1 \times \mathsf{A}_1$ component (see [Parkinson and Van Maldeghem 2019, Lemma 3.7]) and must be exceptional domestic on the A_2 component (for otherwise θ is capped). Hence there are nondomestic simplices of types $S\backslash\{2\}$ and $S\backslash\{4\}$, and so the encircled nodes 2 and 4 are shaded. Suppose that there is a nondomestic simplex σ' either of type $S\backslash\{3, 5\}$ or $S\backslash\{1, 6\}$. Then $\theta_{\sigma'}$ is an automorphism of an $\mathsf{A}_1 \times \mathsf{A}_1$ building interchanging the two components (again by Proposition 1.4), and hence is not domestic, and hence θ is capped, a contradiction. Thus the encircled subsets $\{3, 5\}$ and $\{1, 6\}$ are not shaded.

Suppose that $J \neq S$. Then the first argument of the previous paragraph shows that $\{3, 5\} \cap J = \varnothing$. A similar argument shows that $4 \notin J$. Thus if $J \neq S$ we have $\{3, 4, 5\} \cap J = \varnothing$. If $\{1, 6\} \subseteq J$ then $2 \in J$ (for in the residue of a nondomestic type $\{1, 6\}$ simplex we obtain a duality of D_4, and no duality of D_n is point domestic; see [Parkinson and Van Maldeghem 2019, Proposition 3.16]), and θ is capped. If $J = \{2\}$ then θ is obviously capped. Thus there are no uncapped collineations of E_6 with $\mathrm{Typ}(\theta) \neq S$.

(2) Let θ be an uncapped duality of an E_6 building and let $J = \mathrm{Typ}(\theta)$. We claim that $J = S$. If $1 \in J$ then $6 \in J$, and vice versa (since no duality of D_n is point domestic), and this argument shows that if $J = \{1, 6\}$ then θ is capped, a contradiction. So $\{2, 3, 4, 5\} \cap J \neq \varnothing$. If $3 \in J$ then $\{2, 3, 4, 5, 6\} \subseteq J$ (considering the A_4 component of the residue of a nondomestic type 3 vertex) and similarly if $5 \in J$ then $\{1, 2, 3, 4, 5\} \subseteq J$. Thus if either $3 \in J$ or $5 \in J$ then $J = S$. If $2 \in J$ then $\{2, 3, 5\} \subseteq J$ (considering the A_5 residue of a nondomestic type 2 vertex), and thus again $J = S$. If $4 \in J$ then $\{1, 3, 4, 5, 6\} \subseteq J$ (considering the $\mathsf{A}_2 \times \mathsf{A}_2$ component of the residue of a nondomestic type 4 vertex), and so once more $J = S$.

Thus all nodes are encircled. We claim that θ is strongly exceptional domestic, and so all nodes are shaded. To prove that there exist cotype j panels mapped onto opposite panels for each $j \in \{1, 3, 4, 5, 6\}$, note first that there exists a nondomestic type $\{2, 4\}$ simplex (by considering the A_4 component of the residue of a nondomestic type 3 vertex). If v is the type 2 vertex of such a simplex, then θ_v is a domestic duality of A_5 mapping a plane of this projective space onto an opposite, and thus θ_v is strongly exceptional domestic, and hence the result. Finally, to see that there is a nondomestic cotype 2 panel, let v be the type 1 vertex of a nondomestic cotype 4 panel. Using the classification of uncapped D_5 diagrams we see that θ_v is strongly exceptional domestic, and it follows that there exists a cotype 2 panel of E_6 mapped onto an opposite.

(3) Let θ be an uncapped collineation of an E_7 building and let $J = \mathrm{Typ}(\theta)$. If $J = S$ then θ is strongly exceptional domestic (considering the A_6 residue of a nondomestic type 2 vertex shows that θ maps simplices of each type $S \backslash \{j\}$ onto opposites for $j = 1, 3, 4, 5, 6, 7$, and considering the E_6 residue of the type 7 vertex of a nondomestic type $\{2, 7\}$ simplex, and using (2), shows that there is a simplex of type $S \backslash \{2\}$ mapped onto an opposite).

Suppose that $J \neq S$. Then $2 \notin J$ (for otherwise the induced duality of the A_6 residue is strongly exceptional domestic) and $5 \notin J$ (for otherwise the induced dualities of the A_4 and A_2 residues are both strongly exceptional domestic). We note the following: if $3 \in J$ then $\{3, 4, 6\} \subseteq J$ (considering the A_5 component of the residue) and if $4 \in J$ then $\{1,3,4,6\} \subseteq J$ (considering the A_2 and A_3 components of the residue). Thus if either $3 \in J$ or $4 \in J$ then $\{1,3,4,6\} \subseteq J$. If $6 \in J$ then $\{1,6\} \subseteq J$ (since no duality of the D_5 component of the residue is point domestic). If $7 \in J$ then $\{1,6,7\} \subseteq J$ (since every duality of E_6 maps both type 1 and type 6 vertices to opposites). It follows that either $J = \{1\}$, $J = \{1,6\}$, $J = \{1,6,7\}$, $J = \{1,3,4,6\}$, or $J = \{1,3,4,6,7\}$. In the first, second, and third cases it is clear using the above arguments that θ is capped, a contradiction. We claim that $J = \{1,3,4,6,7\}$ is impossible (for any collineation, capped or uncapped), for if so, then by [Parkinson and Van Maldeghem 2019, Proposition 4.3(2)] there exists a type $\{3,7\}$ simplex σ mapped to an opposite simplex, and if v is the type 7 vertex of σ then θ_v is a duality of an E_6 building mapping a type 3 vertex to an opposite, thus forcing $2, 5 \in J$, a contradiction.

The previous paragraph shows that if θ is uncapped and $J \neq S$ then $J = \{1,3,4,6\}$. Considering the $\mathsf{A}_2 \times \mathsf{A}_3$ component of the residue of a nondomestic type 4 vertex shows that there are simplices of types $\{3,4,6\}$ and $\{1,4,6\}$ mapped onto opposites, thus the nodes 1 and 3 are shaded. If there exist either type $\{1,3,6\}$ or $\{1,3,4\}$ simplices mapped onto opposite simplices then considering the residue of the type 1 vertex of such a simplex we deduce that θ is capped, a contradiction. Thus the nodes 4 and 6 are not shaded.

(4) Let θ be an uncapped (hence nontrivial) collineation of an E_8 building and let $J = \mathrm{Typ}(\theta)$. If $J = S$ then easy residue arguments show that θ is strongly exceptional domestic.

We claim that if $J \neq S$ then $J \subseteq \{1,6,7,8\}$. To see this, note that if $2 \in J$ then $\{3,5,7\} \in J$ (considering an A_7 residue), if $3 \in J$ then $\{2,4,5,6,7,8\} \subseteq J$ (considering the A_6 component of the residue), if $4 \in J$ then $\{1,3,5,6,7,8\} \subseteq J$ (considering the $\mathsf{A}_2 \times \mathsf{A}_4$ component of the residue), and if $5 \in J$ then $\{1,2,3,4,7\} \subseteq J$ (considering the $\mathsf{A}_4 \times \mathsf{A}_3$ residue). Combining these statements it follows that if $\{2,3,4,5\} \cap J \neq \varnothing$ then $J = S$, and hence the claim.

Suppose $J \neq S$, and so $J \subseteq \{1,6,7,8\}$. We claim $J = \{1,6,7,8\}$. For if $1 \in J$ then $8 \in J$ (since no duality of D_7 is point domestic), if $6 \in J$ then $J = \{1,6,7,8\}$ (considering the $\mathsf{D}_5 \times \mathsf{A}_2$ residue and recalling that no duality of D_5 is point domestic), and if $7 \in J$ then $6 \in J$ (considering the duality of E_6 and using (2) above) and so again $J = \{1,6,7,8\}$. Thus $J = \{8\}, \{1,8\}$ or $\{1,6,7,8\}$. The first two cases are clearly capped, hence the claim. Now considering the residue of a type 6 nondomestic vertex we see that there are simplices of types $\{1,6,7\}$ and $\{1,6,8\}$ mapped onto opposite simplices (hence the nodes 7 and 8 are shaded). If there exists a simplex of type $\{6,7,8\}$ or $\{1,7,8\}$ mapped onto an opposite then considering the D_5 residue we deduce that θ is capped, and so the nodes 1 and 6 are not shaded.

(5) Let θ be an uncapped collineation of an F_4 building and let $J = \mathrm{Typ}(\theta)$. If $2 \in J$

then $3, 4 \in J$ (by the duality in the A_2 component of the residue) and similarly if $3 \in J$ then $1, 2 \in J$. Thus either $J = \{1\}$, $J = \{4\}$, $J = \{1, 4\}$, or $J = \{1, 2, 3, 4\}$. The first and second cases are trivially capped. The third case is capped by [Parkinson and Van Maldeghem 2019, Lemma 4.5]. Thus $J = \{1, 2, 3, 4\}$.

If $\Delta = \mathsf{F}_4(2)$ then by Lemma 2.18 there is either a type $\{1, 2\}$ or $\{3, 4\}$ simplex mapped onto an opposite simplex. In the first case, by considering the residue of the type 2 vertex, we see that there are panels of cotype 3 and 4 mapped onto opposites, and hence the nodes 3 and 4 are shaded. The second case is symmetric, with the nodes 1 and 2 shaded. Of course both cases may occur simultaneously, and then all nodes are shaded. Finally, note that if either nodes 1 or 2 are shaded then both are shaded (if the i node is shaded and $i \in \{1, 2\}$ then consider the residue of the type 3 vertex of a nondomestic cotype i panel). Similarly, if either nodes 3 or 4 are shaded then both are shaded. Hence the result for $\mathsf{F}_4(2)$.

If $\Delta = \mathsf{F}_4(2, 4)$ then considering the $\mathsf{A}_2(4)$ component of a type 2 nondomestic vertex we deduce that there are simplices of type $\{2, 3, 4\}$ mapped onto opposites. Then considering the $\mathsf{A}_2(2)$ residue of a type $\{3, 4\}$ nondomestic simplex we deduce that there are also simplices of type $\{1, 3, 4\}$ mapped onto opposites. Thus the nodes $1, 2$ are shaded. If there exists a simplex of type $\{1, 2, 4\}$ or $\{1, 2, 3\}$ mapped onto an opposite, then considering the type $\mathsf{A}_2(4)$ residue of the $\{1, 2\}$ subsimplex we deduce that θ is nondomestic, and hence capped, a contradiction. Thus the nodes 3 and 4 are not shaded. □

Theorem 1(a) now follows from Propositions 2.1, 2.6, 2.10, 2.16, 2.17, and 2.19.

2B. *Applications.* This section contains applications and corollaries of Theorem 1(a).

Corollary 2.20. *Let θ be a an exceptional domestic automorphism of a thick irreducible spherical building Δ.*

(a) *If θ is an oppomorphism and Δ is simply laced, then θ is strongly exceptional domestic.*

(b) *If θ is not an oppomorphism then θ is not strongly exceptional domestic.*

Proof. The first statement follows by noting that in Tables 1 and 2, if θ is an oppomorphism and Δ is simply laced, then whenever all nodes are encircled they are all shaded (see the first, third, sixth rows of Table 1 and the first, second, and third rows of Table 2). The second statement follows by inspecting the third and fourth rows of Table 1 and the first row of Table 2. □

The following lemma is in preparation for our next corollary to Theorem 1(a).

Lemma 2.21. *Let θ be an involution of a thick spherical building, and suppose that the simplex σ is mapped onto an opposite simplex. Then the induced automorphism θ_σ of* $\mathrm{Res}(\sigma)$ *is either the identity or it is an involution.*

Proof. Let α be a simplex of $\mathrm{Res}(\sigma)$. If $\alpha^\theta = \mathrm{proj}_{\sigma^\theta}(\alpha)$ then $\alpha^{\theta_\sigma} = \alpha$ (because the projection maps $\mathrm{proj}_\sigma : \mathrm{Res}(\sigma^\theta) \to \mathrm{Res}(\sigma)$ and $\mathrm{proj}_{\sigma^\theta} : \mathrm{Res}(\sigma) \to \mathrm{Res}(\sigma^\theta)$ are mutually inverse bijections). If $\alpha^\theta = \mathrm{proj}_{\sigma^\theta}(\alpha)$ then $\alpha^{\theta_\sigma} = \alpha$. If $\alpha^\theta \neq \mathrm{proj}_{\sigma^\theta}(\alpha)$ then, since θ maps α^θ onto α, the projection $\mathrm{proj}_\sigma(\alpha^\theta)$ is mapped onto $\mathrm{proj}_{\sigma^\theta}(\alpha)$. Thus $\theta_\sigma^2 = 1$. □

Corollary 2.22. *Every involution of a thick irreducible spherical building is capped.*

Proof. The result is of course true for large buildings of rank at least 3 (where all automorphisms are capped, by [Parkinson and Van Maldeghem 2019]), and thus it remains to show that involutions of small buildings and of arbitrary generalised polygons are capped. Let us begin with the former. We use the decorated opposition diagrams in Tables 1 and 2 to show that every uncapped automorphism has order strictly greater than 2. Consider type A_n, and let θ be uncapped. By Theorem 1(a) there exists a nondomestic type $\{3, 4, \dots, n\}$ simplex σ. Then θ_σ is a domestic duality of the Fano plane. However by [Parkinson et al. 2015] the only domestic duality of the Fano plane is the unique exceptional domestic duality, and this has order 8. Thus, by Lemma 2.21 θ has order strictly greater than 2.

The arguments are similar for all other uncapped diagrams. The key fact is that in some residue one finds a domestic duality of the Fano plane. For example, in the first $\mathsf{E}_6(2)$ diagram in Table 2 we have a nondomestic type $\{1, 3, 5, 6\}$ simplex σ (because, for example, the node 2 is shaded), and θ_σ is a domestic duality of the Fano plane residue.

We now show that every involution of an arbitrary generalised m-gon, $m \geq 2$, is capped. Recall that a generalised m-gon Δ is a bipartite graph with diameter m and girth $2m$. A chamber is a pair of vertices connected by an edge. If $\{x, y\}$ is a chamber we write $x \sim y$ and call x and y adjacent. In particular, if $x \sim y$ then the vertices x and y have different types. Vertices x and y of Δ are opposite if and only if the distance between them is m, and this in turn is equivalent to the existence of a path $x = x_0 \sim x_1 \sim \cdots \sim x_m = y$ with $x_j \neq x_{j+2}$ for all $j = 0, \dots, m-2$. If the distance between vertices x, y is $k < m$ then there is a unique geodesic from x to y. In this case, writing $x = z_0 \sim z_1 \sim \cdots \sim z_k = y$ the vertex z_1 (resp. z_{k-1}) is the projection of y onto x (resp. x onto y).

Claim 1: Every involutory collineation of a thick generalised $2n$-gon Δ, $n \geq 1$, is capped.

Proof of Claim 1. The case $n = 1$ is trivial, and so suppose that θ is an uncapped involutory collineation of a generalised $2n$-gon with $n \geq 2$. Thus θ is domestic (on chambers), and maps at least one vertex of each type onto an opposite vertex. Let Δ' denote the fixed elements of θ. Let x_0 be a type 1 vertex mapped onto an opposite vertex $x_{2n} = x_0^\theta$, and consider any geodesic path $x_0 \sim x_1 \sim \cdots \sim x_{2n-1} \sim x_{2n}$. If $x_1^\theta \neq x_{2n-1}$ then the chamber $\{x_0, x_1\}$ is mapped onto an opposite chamber and θ is capped. Hence $x_1^\theta = x_{2n-1}$, and it follows that $x_i^\theta = x_{2n-i}$, for all $i \in \{0, 1, 2, \ldots, 2n\}$. In particular $x_n^\theta = x_n$ is fixed. Consider another geodesic $x_0 \sim y_1 \sim \cdots \sim y_{2n-1} \sim x_{2n}$ with $y_1 \neq x_1$. Then $y_n^\theta = y_n$. By considering the path from x_n to x_0 to y_n we see that x_n and y_n are opposite, and thus there is a pair of opposite vertices $x_n, y_n \in \Delta'$.

Similarly, by considering a type 2 vertex x_0' that is mapped onto an opposite vertex we deduce the existence of a pair of opposite vertices $x_n', y_n' \in \Delta'$. Since the vertices x_n', y_n' have different type to the vertices x_n, y_n we conclude that for each type $j \in \{1, 2\}$ there are pairs of opposite vertices of type j in Δ'. It follows that Δ' is a sub-$2n$-gon (because the fixed structure of a collineation of a $2n$-gon is either empty, consists of pairwise opposite elements, is a tree of diameter at most $2n$, or is a sub-$2n$-gon, and the first three options are impossible from the above considerations).

Now, the distance from x_n' to x_n is at most $2n-1$ (by types and diameter) and hence the unique geodesic from x_n' to x_n is fixed by θ. In particular the chamber $\{z, x_n\}$ is fixed, where $z \sim x_n$ is the projection of x_n' onto x_n. Note that $z \neq x_{n-1}, x_{n+1}$ because $x_{n-1}^\theta = x_{n+1}$ is not fixed. We claim that every vertex $z_1 \sim z$ is fixed. With y_j as above, note that z and y_{n-1} are opposite (consider the path from z to x_0 to y_{n-1}). Hence the distance from z_1 to y_{n-1} is $2n-1$, and so there is a unique geodesic $z_1 \sim z_2 \sim \cdots z_{2n-1} = y_{n-1}$. If $z_1^\theta \neq z_1$ then z_n and z_n^θ are opposite (consider the path from z_n to z_0 to z_n^θ). Similarly, since $y_{n-1}^\theta = y_{n+1}$ we have $y_{n-1} \neq y_{n-1}^\theta$ and so z_{n+1} and z_{n+1}^θ are opposite. Hence the chamber $\{z_n, z_{n+1}\}$ is mapped onto an opposite chamber, a contradiction.

It now follows from [Van Maldeghem 1998, Proposition 1.8.1] that the sub-$2n$-gon Δ' has the property that whenever $x \in \Delta'$ has the same type as z, then all neighbours of x are fixed (and hence are in Δ'). But x_n' has the same type as z, contradicting the fact that the projection of x_0' onto x_n' is mapped onto the projection of $x_0'^\theta$ onto x_n' and that these projections are distinct. This contradiction completes the proof of Claim 1. □

Claim 2: Every involutary duality of a thick generalised $(2n-1)$-gon Δ, $n \geq 2$, is capped.

Proof of Claim 2. Let θ be a polarity of a generalised $(2n-1)$-gon and suppose that θ maps some element x_0 to an opposite element x_{2n-1}. Suppose that θ is not capped, i.e., θ does not map any chamber to an opposite chamber. Let $x_1 \sim x_0$ be arbitrary. Consider the path $x_0 \sim x_1 \sim \cdots \sim x_{2n-1}$. Using a similar approach to the one in the previous proof, we deduce that $x_i^\theta = x_{2n-1-i}$ for all $i \in \{0, 1, 2, \ldots, 2n-1\}$. Hence $x_n^\theta = x_{n-1}$. Consider a second path $x_0 \sim y_1 \sim \cdots \sim y_{2n-2} \sim x_{2n-1}$ with $y_1 \neq x_1$. Then also $y_{n-1}^\theta = y_n$. Let $z_0 \sim x_n$ be arbitrary but distinct from x_{n-1} and x_{n+1} (using thickness). There is a unique path $z_0 \sim z_1 \sim \cdots \sim z_{2n-2} = y_{n-1}$ from

z_0 to y_{n-1}. By considering the path $z_{n-2} \sim \cdots \sim z_0 \sim x_n \sim x_n^\theta \sim z_0^\theta \sim \cdots \sim z_{n-2}^\theta$ we see that z_{n-2} is mapped onto an opposite vertex. Similarly, since $y_{n-1}^\theta = y_n$ we see that z_{n-1} is mapped onto an opposite vertex (consider the path $z_{n-1} \sim \cdots \sim y_{n-1} \sim y_{n-1}^\theta \sim \cdots \sim z_{n-1}^\theta$). Hence the chamber $\{z_{n-2}, z_{n-1}\}$ is mapped onto an opposite chamber, a contradiction. This completes the proof of Claim 2. □

Finally, we note that no duality of a thick generalised $2n$-gon is domestic and no collineation of a thick generalised $(2n-1)$-gon is domestic (see [Parkinson et al. 2015, Lemmas 3.1 and 3.2]), completing the proof of the corollary. □

Corollary 2.22 shows that every uncapped automorphism has order at least 3. Since every known example of an uncapped automorphism has order at least 4 (see the examples in Sections 3 and 4, and also the rank 2 classification in [Parkinson et al. 2015]) we are led to make the following conjecture.

Conjecture 2.23. *If θ is an automorphism of a thick irreducible spherical building, and if θ has order* 3, *then θ is capped.*

Note that if we remove the shading from the diagrams in Tables 1 and 2 then the diagrams we obtain are contained in [Parkinson and Van Maldeghem 2019, Tables 1–5]. Thus Theorem 1(a) has the following immediate corollary.

Corollary 2.24. *The* (*undecorated*) *opposition diagram of any automorphism of a thick irreducible spherical building is contained in* [*Parkinson and Van Maldeghem 2019, Tables* 1–5].

We now use Theorem 1(a) to determine the partially ordered set $\mathcal{T}(\theta)$ for all automorphisms θ. We first note that, by the proposition below, it is sufficient to determine the maximal elements of $\mathcal{T}(\theta)$.

Proposition 2.25. *Let $\mathcal{M}(\theta)$ be the set of maximal elements of $\mathcal{T}(\theta)$. Then*

$$\mathcal{T}(\theta) = \{J \subseteq S \mid J^{\pi_\theta w_0} = J \text{ and } J \subseteq M \text{ for some } M \in \mathcal{M}(\theta)\}.$$

Proof. This follows immediately from the facts that if σ is a nondomestic type K simplex then (i) K is preserved by $w_0 \circ \pi_\theta$, and (ii) if $J \subseteq K$ is preserved under $w_0 \circ \pi_\theta$ then the type J subsimplex of σ is also nondomestic (see [Parkinson and Van Maldeghem 2019, Lemma 1.3]). □

Thus it remains to compute the set $\mathcal{M}(\theta)$ of maximal elements of $\mathcal{T}(\theta)$. We do this in the corollary below. Recall that if θ is uncapped then the decorated opposition diagram of θ is $(\Gamma, \mathrm{Typ}(\theta), K_\theta, \pi_\theta)$ where, in particular, K_θ is the set of shaded nodes.

Corollary 2.26. *Let θ be an automorphism of a spherical building Δ.*

(a) *If θ is capped then $\mathcal{M}(\theta) = \{\mathrm{Typ}(\theta)\}$.*

(b) *If θ is uncapped then $\mathcal{M}(\theta) = \{\mathrm{Typ}(\theta)\backslash\{k\} \mid k \in K_\theta\}$.*

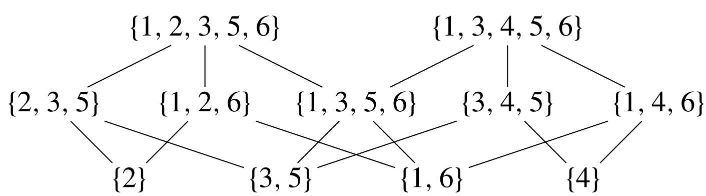
Proof. The first statement is obvious, so consider the second statement. Let (Γ, J, K, π) be the decorated opposition diagram, and so $J = \mathrm{Typ}(\theta)$. If $J = K$ then there are nondomestic simplices of each type $\mathrm{Typ}(\theta)\backslash\{k\}$ with $k \in J$, and these are clearly the maximal types mapped to opposite (otherwise θ is capped). Suppose now that $J\backslash K$ consists of a single minimal $w_0 \circ \pi$ invariant subset J' (thus J' is either a singleton, or J' consists of a pair, as in the second $\mathsf{D}_{2n}(2)$ diagram in Table 1). In this case the only $w_0 \circ \pi$ stable strict subset of J that is not contained in an element of $\{J\backslash\{k\} \mid k \in K\}$ is $J\backslash J'$, and since J' is not shaded all simplicies of this type are domestic. Hence the result in this case.

By Theorem 1(a) the only remaining cases are the six diagrams where $J\backslash K$ consists of precisely two minimal $w_0 \circ \pi$ invariant sets. Specifically, these examples are the $\mathsf{E}_6(2)$ collineation diagram, the first $\mathsf{E}_7(2)$ and $\mathsf{E}_8(2)$ diagrams, the first two $\mathsf{F}_4(2)$ diagrams (these are dual to one another), and the $\mathsf{F}_4(2,4)$ diagram. In these cases the result is implied by the following claim.

Claim: Suppose that the decorated opposition diagram of θ is one of the six diagrams listed above. Then θ is $\{i, j\}$-domestic where i and j are the two shaded nodes.

Proof of Claim. Consider the E_6 diagram. If there is a nondomestic type $\{2,4\}$ simplex then with v the type 4 vertex of this simplex the map θ_v acts on the $\mathsf{A}_2 \times \mathsf{A}_2$ component of the residue swapping the components (by Proposition 1.4). It follows that θ is not domestic, a contradiction. Similar arguments apply for E_7 and E_8, using an A_5 and E_6 residue, respectively. For the first $\mathsf{F}_4(2)$ diagram, suppose there is a nondomestic type $\{1,2\}$ simplex σ. Then θ_σ is a domestic duality of $\mathsf{A}_2(2)$, and hence is the exceptional domestic duality of the Fano plane. It follows that there is a nondomestic type $\{1,2,3\}$ simplex, contradicting the node 4 being unshaded. A dual argument applies to the second $\mathsf{F}_4(2)$ diagram. The $\mathsf{F}_4(2,4)$ diagram is similar. Hence the proof of the claim is complete, and the corollary follows. □

Example 2.27. Suppose that θ has the $\mathsf{E}_6(2)$ collineation diagram in Table 2. Then the partially ordered set $\mathcal{T}(\theta)$ is (using Proposition 2.25 and Corollary 2.26):



As a final application we will compute the displacement of an arbitrary automorphism θ in Corollary 2.29. Recall that $\mathrm{disp}(\theta) = \max\{d(C, C^\theta) \mid C \in \mathcal{C}\}$, where $\mathcal{C}$ is the set of chambers of Δ, and $d(C, D) = \ell(\delta(C, D))$ for chambers $C, D \in \mathcal{C}$.

Proposition 2.28. *Let θ be any automorphism of a thick irreducible spherical building of type (W, S). Then*

$$\mathrm{disp}(\theta) = \mathrm{diam}(W) - \min\{\mathrm{diam}(W_{S\backslash J}) \mid J \in \mathcal{M}(\theta)\}.$$

Proof. Let $N = \min\{\mathrm{diam}(W_{S\backslash J}) \mid J \in \mathcal{M}(\theta)\}$. We note first that

$$N = \min\{\mathrm{diam}(W_{S\backslash J}) \mid \text{there exists a type } J \text{ simplex in } \mathrm{Opp}(\theta)\} \tag{2-2}$$

because the minimum is obviously attained at a maximal element of $\mathcal{T}(\theta)$.

Let $J \subseteq \mathrm{Typ}(\theta)$ be any subset for which there exists a nondomestic type J simplex. Then for all chambers C containing this simplex we have $\delta(C, C^\theta) \in W_{S\backslash J} w_0$ (see [Parkinson and Van Maldeghem 2019, Lemma 2.5]) and thus

$$\mathrm{disp}(\theta) \geq \ell(\delta(C, C^\theta)) \geq \ell(w_0) - \ell(w_{S\backslash J}) = \mathrm{diam}(W) - \mathrm{diam}(W_{S\backslash J}).$$

Since this inequality holds for all J such that there exists a type J simplex in $\mathrm{Opp}(\theta)$ the formula (2-2) gives

$$\mathrm{disp}(\theta) \geq \mathrm{diam}(W) - N.$$

On the other hand, let C be any chamber with $\ell(\delta(C, C^\theta))$ maximal. By the arguments of [Abramenko and Brown 2009, Lemma 2.4 and Theorem 4.2] we have $\delta(C, C^\theta) = w_I w_0$ for some $I \subseteq S$ with $I^{\pi_\theta} = I^{w_0}$. Hence the type $J = S\backslash I$ simplex of C is mapped onto an opposite simplex. Thus

$$\mathrm{disp}(\theta) = \ell(\delta(C, C^\theta)) = \ell(w_0) - \ell(w_{S\backslash J}) = \mathrm{diam}(W) - \mathrm{diam}(W_{S\backslash J}) \leq \mathrm{diam}(W) - N,$$

and hence the result. □

Corollary 2.29. *Let θ be an automorphism of a thick irreducible spherical building and let $J = \mathrm{Typ}(\theta)$. Then*

$$\mathrm{disp}(\theta) = \begin{cases} \mathrm{diam}(W) - \mathrm{diam}(W_{S\backslash J}) & \text{if } \theta \text{ is capped}, \\ \mathrm{diam}(W) - \mathrm{diam}(W_{S\backslash J}) - 1 & \text{if } \theta \text{ is uncapped}. \end{cases}$$

In particular, if θ is exceptional domestic then $\mathrm{disp}(\theta) = \mathrm{diam}(\Delta) - 1$.

Proof. The case of capped automorphisms is [Parkinson and Van Maldeghem 2019, Theorem 5]. In the case of an uncapped automorphism we note that by Corollary 2.26 the maximal elements of $\mathcal{T}(\theta)$ are of the form $\mathrm{Typ}(\theta)\backslash\{j\}$ for some $j \in \mathrm{Typ}(\theta)$, and then the result follows from Proposition 2.28. □

Remark 2.30. Corollary 2.29 shows that the set of possible displacements is extremely restricted. For example, consider an E_8 building Δ, where a priori there are $\ell(w_0) = 120$ potential displacements. However, by Corollary 2.29, [Parkinson and Van Maldeghem 2019, Theorem 3], and Theorem 1(a) the only possible displacements of an automorphism θ are:

$$\begin{aligned}
0 &= \mathrm{diam}(\mathsf{E}_8) - \mathrm{diam}(\mathsf{E}_8) && \text{if } \theta \text{ is the identity,}\\
57 &= \mathrm{diam}(\mathsf{E}_8) - \mathrm{diam}(\mathsf{E}_7) && \text{if } \theta \text{ is capped with type } \{8\},\\
90 &= \mathrm{diam}(\mathsf{E}_8) - \mathrm{diam}(\mathsf{D}_6) && \text{if } \theta \text{ is capped with type } \{1,8\},\\
107 &= \mathrm{diam}(\mathsf{E}_8) - \mathrm{diam}(\mathsf{D}_4) - 1 && \text{if } \theta \text{ is uncapped with type } \{1,6,7,8\},\\
108 &= \mathrm{diam}(\mathsf{E}_8) - \mathrm{diam}(\mathsf{D}_4) && \text{if } \theta \text{ is capped with type } \{1,6,7,8\},\\
119 &= \mathrm{diam}(\mathsf{E}_8) - 1 && \text{if } \theta \text{ is uncapped with type } S,\\
120 &= \mathrm{diam}(\mathsf{E}_8) && \text{if } \theta \text{ is nondomestic.}
\end{aligned}$$

In particular, note that for E_8 buildings the displacement determines the (decorated) opposition diagram of the automorphism. This phenomenon is not true for all types; for example in $\mathsf{B}_7(\mathbb{F})$ displacement 45 is obtained by both capped automorphisms with $\mathrm{Typ}(\theta) = \{1, 2, 3, 4, 5\}$ and capped automorphisms with $\mathrm{Typ}(\theta) = \{2, 4, 6\}$.

3. Uncapped automorphisms for classical types

In this section we prove Theorem 1(b) for classical types. Thus our aim is to construct uncapped automorphisms with each of the diagrams listed in Tables 1 and 2 for the buildings $\mathsf{A}_n(2)$, $\mathsf{B}_n(2)$, $\mathsf{B}_n(2,4)$, and $\mathsf{D}_n(2)$.

3A. *The buildings* $\mathsf{A}_n(2)$. In this section we work with the concrete model $\mathsf{A}_n(2) = \mathsf{PG}(n, \mathbb{F}_2)$ for the small building of type A_n. Thus an i-space of $\mathsf{A}_n(2)$ means a subspace of $\mathbb{F}_2^n$ of (projective) dimension i, and this corresponds to a type $i+1$ vertex of the building. Let θ be a duality of $\mathsf{A}_n(2)$. Recall that a point p of $\mathsf{A}_n(2)$ is called *absolute* with respect to θ if $p \in p^\theta$ (that is, p is not mapped to an opposite hyperplane). Dually, a hyperplane π is absolute if $\pi^\theta \in \pi$ (that is, π is not mapped to an opposite point).

Lemma 3.1. *Let θ be a duality of a projective space. Suppose that U is an m-space consisting of absolute points of θ, and let $k = \dim(U \cap U^\theta)$. Then $m-k$ is even.*

Proof. The hyperplanes through $\langle U^\theta, U\rangle$ form a dual space of (projective) dimension k, and the inverse image is a k-space contained in U. Choose a complementary $(m-k-1)$-space H in U, and so H intersects neither U^θ nor $U^{\theta^{-1}}$. Then for each $x \in H$ we have that $x^\theta \cap H$ is a hyperplane of H through x, and hence is absolute. Thus θ is a symplectic polarity on H, and so $m-k$ is even (see Lemma 2.3). □

Theorem 3.2. *For each $n \geq 2$ there exists a unique duality θ of $\mathsf{A}_n(2)$ (up to conjugation) with the property that the set of absolute points of θ is the union of two distinct hyperplanes. This duality is strongly exceptional domestic, with order 8 if n is even and 4 if n is odd.*

Proof. We first demonstrate the existence of a duality whose absolute points form the union of two hyperplanes. Let J_1, J_2, and J_3 be the matrices

$$J_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad J_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

and let A be the $(n+1) \times (n+1)$ matrix in block diagonal form,

$$A = \operatorname{diag}(J, J_1, J_1, \dots, J_1) \quad \text{with } J = J_2 \text{ for even } n \text{ and } J = J_3 \text{ for odd } n.$$

Let θ be the duality of $\mathsf{A}_n(2)$ with matrix A. That is, $X^\theta = (AX)^\perp$ where X is written as a column vector. Then X is absolute if and only if $X \in (AX)^\perp$, and hence by direct calculation X is absolute if and only if $X_0 X_1 = 0$. The matrix for the collineation θ^2 is given by $A^{-t}A$, and it follows by calculation that θ has order 8 if n is even, and order 4 if n is odd.

We now prove that there is at most one duality θ up to conjugation with the given property, and that such a duality is necessarily strongly exceptional domestic. We proceed by induction on n, the case $n = 2$ being contained in [Parkinson et al. 2015].

So let θ be a duality of $\mathsf{A}_n(2)$ such that $\alpha_1 \cup \alpha_2$ is the set of absolute points for θ with $\alpha_1 \neq \alpha_2$ two hyperplanes of $\mathsf{A}_n(2)$. Let β be the hyperplane containing $\alpha_1 \cap \alpha_2$ and different from both α_1 and α_2. Note that $\alpha_1 \cup \alpha_2 \cup \beta$ is the entire point set. Let $p_i = \alpha_i^\theta$, $i = 1, 2$ and $q = \beta^\theta$; then $L = \{p_1, p_2, q\}$ is a line.

Note that q is absolute (for if $q \in \beta$ we have $q^\theta \ni \beta^\theta = q$). Thus $q \in \alpha_1 \cup \alpha_2$. In fact we claim that $q \in \alpha_1 \cap \alpha_2$. For if not we have $\beta^\theta = q \notin \beta$ and so β is not absolute, contradicting the fact that $\beta = q^{\theta^{-1}}$ is absolute (since q is absolute).

Since $L = \{p_1, p_2, q\}$ is a line and $q \in \alpha_1 \cap \alpha_2$ we either have $p_1, p_2 \in \beta \backslash (\alpha_1 \cup \alpha_2)$ or $p_1, p_2 \in \alpha_1 \cup \alpha_2$. We treat these two cases below. Before doing this, we observe that in the first case n is necessarily even, and in the second case n is necessarily odd. To see this, note that if $p_1, p_2 \in \beta \backslash (\alpha_1 \cup \alpha_2)$ then the point p_1 is nonabsolute and the mapping $\rho_1 : z \mapsto z^\theta \cap \alpha_1$, $z \in \alpha_1$, is a duality on α_1 every point of which is absolute, forcing n to be even (see Lemma 2.3). On the other hand, if $p_1, p_2 \in \alpha_1 \cup \alpha_2$ then we have $(\alpha_1 \cap \alpha_2)^\theta = \langle p_1, p_2 \rangle \subseteq \alpha_1 \cap \alpha_2$ and so Lemma 3.1 implies $(n-2) - 1 = n - 3$ is even, and so n is odd. We also observe that since α_1 and α_2 are the only two hyperplanes all of whose points are absolute, every even power of θ preserves the set $\{\alpha_1, \alpha_2\}$, and hence also the set $\{p_1, p_2\}$. It follows that $p_i^\theta \in \{\alpha_1, \alpha_2\}$ for $i = 1, 2$.

Case 1: $p_1, p_2 \in \beta \backslash (\alpha_1 \cup \alpha_2)$. As noted above n is even, and so we may assume $n \geq 4$. Let $\sigma = \{x, \xi\}$ be any nondomestic (point-hyperplane)-flag for θ (that is, a nondomestic type $\{1, n\}$-simplex of the building). We note that such simplices exist, and indeed they obviously all arise as follows: Since the absolute hyperplanes for θ are precisely the hyperplanes through one of the points p_1 or p_2, if we select any point $x \in \beta \setminus (\alpha_1 \cup \alpha_2)$ and any hyperplane ξ through x not containing p_1 or p_2, then $\sigma = \{x, \xi\}$ is nondomestic.

We claim that the mapping $\theta_\sigma : z \mapsto z^\theta \cap \xi \cap x^\theta$ for $z \in \xi \cap x^\theta$ has exactly two hyperplanes consisting entirely of absolute points. Note that $q \in \xi$ and also $q \in x^\theta$. Note also that, since p_i^θ contains the absolute point $q_i := \langle p_i, x\rangle \cap (\alpha_1 \cap \alpha_2)$, also x^θ contains q_i, $i = 1, 2$. Since ξ does not contain p_i, but it does contain x, it does not contain q_i, $i = 1, 2$. Consequently $x^\theta \cap \alpha_1 \cap \alpha_2$ is not contained in ξ and the claim follows.

Thus for every nondomestic (point-hyperplane)-pair $\sigma = \{x, \xi\}$ the induced duality θ_σ on the $\mathsf{A}_{n-2}(2)$ residue has precisely two hyperplanes of absolute points. Since $n-2$ is even this duality again satisfies the condition of Case 1, and so by induction θ is domestic. Since θ has nondomestic points necessarily θ is strongly exceptional domestic by Theorem 2.5.

We now show that θ is unique, up to a projectivity (and under the assumptions of Case 1). Let ρ_1 be the symplectic polarity on α_1 introduced in the paragraph before Case 1. Noting that $q^{\rho_1} = \alpha_1 \cap \alpha_2$, we see that the data α_1, α_2 and ρ_1 are projectively unique. This determines q. All choices of p_1 outside $\alpha_1 \cup \alpha_2$ are projectively equivalent, and then p_2 is the third point on the line determined by p_1 and q. We then know the image of an arbitrary point x_1 of $\alpha_1 \setminus \alpha_2$, as $x_1^\theta = \langle x^{\rho_1}, p_1\rangle$. This determines the images of all points of α_1. Since $p_1^\theta = \alpha_1$, we know the images of a basis, which suffices to determine the whole duality.

<u>Case 2</u>: $p_1, p_2 \in \alpha_1 \cup \alpha_2$. As noted above, n is odd. Take an arbitrary point $z \in \beta \setminus (\alpha_1 \cup \alpha_2)$ and set $H := z^\theta$. Then $\varphi : x \mapsto x^\theta \cap H$ is a duality in the $(n-1)$-dimensional projective space H such that its absolute points form two hyperplanes $H \cap \alpha_i$, $i = 1, 2$. Hence by the previous case φ is domestic, and since z was arbitrary amongst the nondomestic points for θ we conclude that θ is domestic. Thus by Theorem 2.5 θ is strongly exceptional domestic.

It remains to show that θ is unique up to conjugation with a projectivity. Let $D_i = H \cap \alpha_i$, $i = 1, 2$. Set $\{i, j\} = \{1, 2\}$ and $D_i^{\varphi^{-1}} = p_i'$. Then $\{q, p_1', p_2'\}$ is a line in $H \cap \beta$ (since $p_i'^{\varphi} = D_i$ it suffices to see that $q^\varphi = \beta \cap H$, and this follows from the definition of φ as $\beta = q^\theta$). It also follows that $D_i^{\theta^{-1}} = \langle p_i', z\rangle$. Since $D_i \subseteq \alpha_i$, we conclude $\alpha_i^{\theta^{-1}} \in \langle p_i', z\rangle$. But $\alpha_i^{\theta^{-1}} \in \{p_1, p_2\}$. We claim that $\alpha_i^{\theta^{-1}} = p_i$. Suppose not. Then $\alpha_i^{\theta^{-1}} = p_j$. Now from $z^\theta = H$ and $p_i^\theta = \alpha_j$ follows that $t_i^\theta = \langle D_j, z\rangle$, with $\{t_i, p_i, z\}$ a line. But $p_j'^\theta$ is a hyperplane through D_j distinct from α_j and H (as $p_j \in H$ and is not absolute); hence $p_j'^\theta = \langle D_j, z\rangle$ and so $t_i = p_j'$. Now $p_j'^{\theta^{-1}} = \langle D_i, z\rangle$ and $p_i^{\theta^{-1}} = \alpha_i$. It follows that $z^{\theta^{-1}} = H$. Hence $z^{\theta^2} = z$, for all $z \in \beta \setminus (\alpha_1 \cup \alpha_2)$. It follows that $p_i^{\theta^2} = p_i$, contradicting $p_i^{\theta^2} = \alpha_j^\theta = p_j$. Our claim follows.

But now, just like in the proof of our previous claim, we have that $\{p_i, p_i', z\}$ is a line and $p_i'^{\theta} = \langle D_j, z\rangle$. It follows that $p_i^{\theta^2} = p_j$ and so $z^{\theta^2} = z'$, with $\{z, z', q\}$ a line.

Now, α_1, α_2, H, z and φ are unique up to conjugation with a projectivity. But then, given $z^\theta = H$, the duality θ is completely determined, since q is determined and hence also z' (with the above notation). This determines the image x^θ of an arbitrary point in H as $x^\theta = \langle x^\varphi, z'\rangle$. Furthermore, we also have $z^\theta = H$, and so θ is determined. □

3B. *The buildings* $\mathsf{B}_n(2)$*,* $\mathsf{B}_n(2,4)$*, and* $\mathsf{D}_n(2)$. It will be more convenient for us to regard $\mathsf{B}_n(2) \cong \mathsf{C}_n(2)$ as a symplectic polar space. We begin by recalling the standard models of the $\mathsf{C}_n(2)$, $\mathsf{D}_n(2)$, and $\mathsf{B}_{n-1}(2,4)$ buildings in the ambient projective space $\mathsf{PG}(2n-1,2)$. Let $V = \mathbb{F}_2^{2n}$, and let $(\cdot,\cdot)$ be the (symplectic and symmetric) bilinear form on $V = \mathbb{F}_2^{2n}$ given by

$$(X, Y) = X_1Y_{2n} + X_2Y_{2n-1} + \cdots + X_{2n}Y_1. \tag{3-1}$$

The points of the polar space $\mathsf{C}_n(2)$ are the 0-spaces of $\mathsf{PG}(2n-1,2)$, and points $p = \langle X\rangle$ and $q = \langle Y\rangle$ are collinear (including the case $p = q$) if and only if $(X, Y) = 0$. A subspace U of V is *totally isotropic* if $(X, Y) = 0$ for all $X, Y \in U$. The totally isotropic subspaces of maximal dimension have projective dimension $n-1$, and for each $0 \le k \le n-1$ the k-spaces of the polar space $\mathsf{C}_n(2)$ are the totally isotropic subspaces of V with projective dimension k. To obtain the building of $\mathsf{C}_n(2)$ as a labelled simplicial complex one takes the totally isotropic $(k-1)$-spaces to be the type k vertices of the building for $1 \le k \le n$, with incidence of vertices given by symmetrised containment of the corresponding spaces. The full collineation group of $\mathsf{C}_n(2)$ is the symplectic group $\mathsf{Sp}_{2n}(2)$ consisting of all matrices $g \in \mathsf{GL}_{2n}(2)$ satisfying $g^TJg = J$, where J is the matrix of the symplectic form $(\cdot,\cdot)$ (see [Tits 1974, Corollary 5.9]).

Let F^+ and F^- be quadratic forms on V with Witt indices n and $n-1$, respectively. We will fix the specific choices

$$F^+(X) = X_1X_{2n} + X_2X_{2n-1} + \cdots + X_nX_{n+1},$$
$$F^-(X) = X_1X_{2n} + X_2X_{2n-1} + \cdots + X_nX_{n+1} + X_n^2 + X_{n+1}^2.$$

For $\epsilon \in \{-,+\}$, a subspace $U \subseteq V$ is *singular* with respect to F^ϵ if $F^\epsilon(X) = 0$ for all $X \in U$. The maximal dimensional singular subspaces of V with respect to F^ϵ have vector space dimension equal to the Witt index of F^ϵ. The points of $\mathsf{D}_n(2)$, (respectively, the polar space $\mathsf{B}_{n-1}(2,4)$), are those points of $\mathsf{PG}(2n-1,2)$ that are singular with respect to F^+, (respectively, F^-). In both cases points $p = \langle X\rangle$ and $q = \langle Y\rangle$ are collinear (including the case $p = q$) if and only if $(X, Y) = 0$, where $(\cdot,\cdot)$ is as in (3-1).

Let $\mathsf{GO}^{\epsilon}_{2n}(2)$ be the group of all matrices of $\mathsf{GL}_{2n}(2)$ preserving the quadratic form F^{ϵ}, and let $\mathsf{O}^{\epsilon}_{2n}(2)$ be the corresponding index 2 simple subgroup of $\mathsf{GO}^{\epsilon}_{2n}(2)$ (see [Conway et al. 1985, §2.4]). Since $\mathsf{GO}^{\epsilon}_{2n}(2)$ preserves collinearity, the group $\mathsf{GO}^{+}_{2n}(2)$ acts on $\mathsf{D}_n(2)$ and the group $\mathsf{GO}^{-}_{2n}(2)$ acts on $\mathsf{B}_{n-1}(2,4)$. In fact the group $\mathsf{GO}^{-}_{2n}(2)$ is the full automorphism group of $\mathsf{B}_{n-1}(2,4)$ (see [Tits 1974]). In the case of $\mathsf{D}_n(2)$ the maximal singular subspaces are partitioned into two sets of equal cardinality by the action of $\mathsf{O}^{+}_{2n}(2)$, and an automorphism θ of $\mathsf{D}_n(2)$ mapping points to points is called a *collineation* if this partition of maximal singular subspaces is preserved by θ, and a *duality* otherwise. Then $\mathsf{O}^{+}_{2n}(2)$ is the group of all collineations of $\mathsf{D}_n(2)$, and $\mathsf{GO}^{+}_{2n}(2)\backslash\mathsf{O}^{+}_{2n}(2)$ is the set of all dualities of $\mathsf{D}_n(2)$ (see [Tits 1974]).

To obtain the building of $\mathsf{B}_{n-1}(2,4)$ as a labelled simplicial complex one takes the singular $(k-1)$-spaces to be the type k vertices of the building for $1 \le k \le n-1$, with incidence of vertices given by symmetrised containment of the corresponding spaces. The situation for $\mathsf{D}_n(2)$ is slightly different: For $1 \le k \le n-2$ the singular $(k-1)$-spaces are taken to be the type k vertices of the building, and the singular $(n-1)$-spaces in one part of the partition mentioned above are taken to be the type $n-1$ vertices of the building, and those in the other part of the partition are taken to be the type n vertices of the building. A type $n-1$ vertex is declared to be incident with a type n vertex if the corresponding $(n-1)$-spaces meet in an $(n-2)$-space. For all other types incidence is given by symmetrised containment of the corresponding spaces.

Note the index shifts that occur (for example an $\{n\}$-domestic collineation of a $\mathsf{C}_n(2)$ building is a collineation that is domestic on the totally isotropic $(n-1)$-spaces). A point p of a polar space is an *absolute point* of an automorphism θ if p^{θ} is collinear with p (including $p^{\theta} = p$).

Lemma 3.3. *Let θ be a collineation of $\mathsf{C}_n(2)$.*

(a) *If θ fixes a subspace of $\mathsf{PG}(2n-1,2)$ of projective dimension $k \ge n$ then θ is $\{j\}$-domestic for each $2n-k \le j \le n$.*

(b) *If the set of absolute points of θ strictly contains the union of two distinct hyperplanes of $\mathsf{PG}(2n-1,2)$ then θ is $\{1\}$-domestic.*

Proof. (a) By considering dimensions, each $(j-1)$-space of $\mathsf{PG}(2n-1,2)$ with $j \ge 2n-k$ intersects the subspace of fixed points. In particular, no totally isotropic $(j-1)$-space is mapped onto an opposite and so θ is $\{j\}$-domestic for all $2n-k \le j \le n$.

(b) A point X is an absolute point of $\theta \in \mathsf{Sp}_4(2)$ if and only if $(X, \theta X) = X^T J\theta X = 0$, where J is the matrix of the symplectic form $(\cdot\,,\cdot)$. Thus the set of absolute points of θ is a quadric, and so if it strictly contains the union of two distinct hyperplanes then all points are absolute. □

In the following proofs we use the standard notations $p \perp q$ if points p and q are collinear (including the case $p = q$), and $p^\perp$ for the set of all points collinear to p.

Lemma 3.4. *Let $\Delta = \mathsf{C}_n(2)$ with $n \geq 2$ and let θ be a collineation.*

(a) *If the fixed points of θ form a $(2n-3)$-space W, then the absolute points form a subspace containing W.*

(b) *If the fixed points of θ form a $(2n-2)$-space W, then every absolute point is fixed.*

Proof. (a) Let p be a point not contained in W and suppose p is absolute. Let $q \in \langle W, p\rangle \setminus W$. We claim that q is absolute. Indeed, let $r := \langle p, q\rangle \cap W$. If $p \perp q$, then the plane $\pi = \langle p, q, p^\theta\rangle$ contains the triangle $\{p, p^\theta, r\}$ of points collinear in $\mathsf{C}_n(2)$ and so $q \perp q^\theta$, as both points belong to π. If $p \notin q^\perp$, then π contains the line $\langle p, p^\perp\rangle$, which belongs to $\mathsf{C}_n(2)$, but also contains the line $\langle p, r\rangle$, which does not belong to $\mathsf{C}_n(2)$. Also $\langle p^\theta, r\rangle$ does not belong to $\mathsf{C}_n(2)$, and it follows that the line $\langle r, s\rangle$, where $\{p, p^\theta, s\}$ is the line of $\mathsf{C}_n(2)$ through p and p^θ, belongs to $\mathsf{C}_n(2)$. Hence also the line $\{s, q, q^\theta\}$ belongs to $\mathsf{C}_n(2)$, which proves our claim.

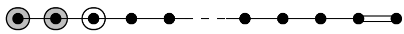
So, if there are no absolute points besides those in W, then (a) holds. If some absolute point $p \notin W$ exists, then there are three possibilities. Either exactly one hyperplane through W consists of absolute points (and then (a) holds), or all three hyperplanes through W consist of absolute points (and then, again, (a) holds), or exactly two hyperplanes H_1 and H_2 through W consist of absolute points. In this final case, let H be the third hyperplane through W. Let t, t_1, t_2 be points such that $t^\perp = H$ and $t_i^\perp = H_i$, $i = 1, 2$. Then, since θ fixes H, we have $t \in W$. Since $t_i \in t_i^\perp = H_i$, $i = 1, 2$, we deduce $t_i \in W$, $i = 1, 2$. Hence θ induces collineations in H, H_1, H_2 having a hyperplane W as fixed points. Consequently, these collineations are central involutions. Since all points of W are fixed, all subspaces through $\{t, t_1, t_2\}$ are fixed. Hence the centres of the above collineations are t, t_1, t_2. Since the collineations in H_i, $i = 1, 2$, map points to a collinear point, the centers are t_i. But then the centre of the collineation in H is t and hence it also maps points to collinear points, a contradiction. This shows (a).

(b) If the fixed points of θ form a $(2n-2)$-space W, then θ is a central elation in $\mathsf{PG}(2n-1, 2)$, and the centre is necessarily $W^\perp$ since every point of W is fixed, and hence every hyperplane through $W^\perp$ is fixed. No line through $W^\perp$ not contained in W is a line of $\mathsf{C}_n(2)$, whence (b). □

Lemma 3.5. *A collineation θ of the generalised quadrangle $\mathsf{C}_2(2)$ is exceptional domestic if and only if the set of absolute points of θ equals the union of two distinct hyperplanes in $\mathsf{PG}(3, 2)$.*

Proof. It is known that $\mathsf{C}_2(2)$ admits a unique exceptional domestic collineation (see [Temmermans et al. 2012b]), and direct inspection shows that the set of absolute points of this collineation forms the union of two distinct hyperplanes in $\mathsf{PG}(3,2)$. It remains to show that no other collineation of $\mathsf{C}_2(2)$ has such a structure of absolute points. This can be done, for example, using the character tables in the ATLAS; see [Conway et al. 1985, page 5]. We omit the details. □

Lemma 3.6. *Let $\Delta = \mathsf{C}_n(2)$ with $n \geq 3$ and let θ be a collineation. If the absolute points of θ lie on a union of two hyperplanes, and if the fixed points of θ form a $(2n-4)$-space W, then θ has the following decorated opposition diagram*:



Proof. The hypothesis implies that every 3-space contains a fixed point, and thus θ is $\{i\}$-domestic for all $4 \leq i \leq n$.

By the hypothesis on the structure of the absolute points of θ there exist points in $\mathrm{Opp}(\theta)$. Let p be an arbitrary point in $\mathrm{Opp}(\theta)$. We will show below that the induced collineation θ_p of $\mathsf{C}_{n-1}(2)$ is $\{2\}$-domestic (in the inherited labelling). Hence θ is $\{1,2\}$-domestic. So if θ is capped then θ is $\{2\}$-domestic, however by [Temmermans et al. 2012a, Theorem 5.1] every such collineation fixes a geometric hyperplane pointwise, contrary to our hypothesis that the fixed points form a $(2n-4)$-space. Thus θ is uncapped, and then by Theorem 1(a) the decorated opposition diagram of θ is forced to be as claimed.

Therefore it only remains to show that θ_p is $\{2\}$-domestic (that is, point-domestic on $\mathsf{C}_{n-1}(2)$). We fix some notation. Let H_i, $i = 1, 2$, be the two hyperplanes all points of which are absolute. Set $S = H_1 \cap H_2$ and let H be the hyperplane distinct from H_i, $i = 1, 2$, and containing S. Note that all points of $\mathrm{Opp}(\theta)$ are contained in H (more precisely they form the set $H \setminus S$).

First we claim that any line in $\mathrm{Opp}(\theta)$ incident to p must necessarily be contained in the hyperplane H. Suppose the such a line L is not contained in H. Then $L = \{p, q_1, q_2\}$, with $q_i \in H_i$ and hence $q_i^\theta \perp q_i$, $i = 1, 2$. Since p is not collinear to p^θ, it must be collinear to q_i^θ for some $i \in \{1, 2\}$. But then q_i^θ is collinear to all points of L, and so the line $L^\theta \ni q_i^\theta$ is not opposite the line L. Hence the claim.

Consider the subspace $\xi := p^\perp \cap (p^\theta)^\perp$ of dimension $2n-3$. Then clearly ξ contains the subspace $p^\perp \cap W$. We claim that $\dim(p^\perp \cap W) = 2n-5$. Indeed, if not, then W is a hyperplane of ξ. By Lemma 3.4(b) and our previous claim, all lines of $\mathsf{C}_n(2)$ through p are contained in H, implying $p^\perp = H$. But since H is fixed by θ we deduce that $p \in W$, a contradiction. Our claim follows.

Hence $\dim(p^\perp \cap W) = 2n-5$. It follows that $\dim(\xi \cap W) = 2n-5$ as well, since $p^\perp \cap W = (p^\theta)^\perp \cap W$. Now let $q \in \xi \setminus W$. Suppose $q \notin H$. Then the line $\langle p, q\rangle$ is not mapped to an opposite, as we showed above. Suppose $q \in S \setminus W$. Then $q^\theta \perp q$, and since $p^\theta \perp q$, we deduce that q is collinear to $\langle p, q\rangle^\theta$, implying that

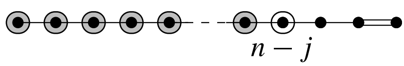
$\langle p, q\rangle \notin \mathrm{Opp}(\theta)$. Hence, if θ_p is not $\{2\}$-domestic, then $\xi \cap (H \setminus S) \neq \varnothing$. Under that conditon, if ξ is not contained in H, then $\xi \cap H_i$ is a hyperplane of ξ, $i = 1, 2$, and this contradicts Lemma 3.4(a).

Hence we deduce that if θ_p is not $\{2\}$-domestic, then $\xi \subseteq H$. In this case, since both p and p^θ are in H, we have $p^\perp = \langle p, \xi\rangle = H$ and $(p^\theta)^\perp = \langle p^\theta, \xi\rangle = H$. However $\perp$ is a symplectic polarity and so $p^\perp = H = (p^\theta)^\perp$ forces $p = p^\theta$, a contradiction. The lemma is proved. □

Theorem 3.7. *Let θ be a collineation of $\mathsf{C}_n(2)$. Suppose that the set of absolute points of θ equals the union of two distinct hyperplanes of $\mathsf{PG}(2n-1, 2)$. Then θ is domestic. Moreover, if k is the projective dimension of the subspace of points of $\mathsf{PG}(2n-1, 2)$ fixed by θ, then*

(a) *if $k = n-2$ then θ is strongly exceptional domestic, and*

(b) *if $k = n-1+j$ for some $0 \leq j \leq n-3$ then θ is uncapped with the following decorated opposition diagram:*



Moreover examples exist for each $n-2 \leq k \leq 2n-4$.

Proof. Suppose that θ is a collineation of $\mathsf{C}_n(2)$ such that the set of absolute points of θ is the union of two distinct hyperplanes H_1 and H_2 of $\mathsf{PG}(2n-1, 2)$. We show by induction on $n-j$ that θ is domestic, with Lemma 3.5 providing the base case $n-j = 3$.

Let p be any point not in $H_1 \cup H_2$. Thus p is mapped to an opposite point by θ. Let $\mathrm{Res}(p)$ be the set of totally isotropic subspaces containing p. Thus $\mathrm{Res}(p)$ is a $\mathsf{C}_{n-1}(2)$ building, whose points are the lines through p, whose lines are the planes through p, and so forth. Let $\theta_p = \mathrm{proj}_{\mathrm{Res}(p)} \circ \theta$, regarded as a collineation of $\mathsf{C}_{n-1}(2)$. Since $p^\perp$ and $(p^\theta)^\perp$ are hyperplanes of $\mathsf{PG}(2n-1, 2)$ the spaces $H_i' = p^\perp \cap (p^\theta)^\perp \cap H_i$ are $(2n-4)$-spaces for $i = 1, 2$ (as in the proof of Lemma 3.5). Let $q \in p^\perp \cap (p^\theta)^\perp \cap (H_1 \cup H_1)$, and let $L = \langle p, q\rangle$. Similar arguments as those in Lemma 3.5 show that

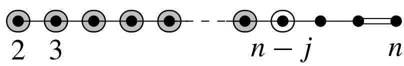
(i) if q is fixed by θ, then L is fixed by θ_p, and

(ii) if q is mapped to a distinct collinear point by θ then L is either fixed by θ_p, or is mapped to a distinct coplanar line by θ_p.

Thus for all nondomestic points p the induced collineation θ_p of the $\mathsf{C}_{n-1}(2)$ building $\mathrm{Res}(p)$ has the property that the set of points mapped to collinear points (including fixed points) contains the union of two distinct hyperplanes in $\mathsf{PG}(2n-3, 2)$. Thus by Lemma 3.3 and the induction hypothesis the collineation θ_p is domestic, and hence θ is domestic.

Now suppose that the absolute points of θ form a union of two hyperplanes, and that the fixed point set F of θ is an $(n-2)$-space of $\mathsf{PG}(2n-1,2)$. We prove by induction on n that θ is strongly exceptional domestic, with Lemma 3.5 providing the base case. The above argument shows that θ is necessarily domestic, and so it remains to show that there are nondomestic panels of each cotype $1,2,\ldots,n$. We claim that for $n\geq 3$ there exists a nondomestic point p such that the hyperplane $p^{\perp}$ intersects F in an $(n-3)$-space F'. To see this it suffices to show that there is a point p with $p\notin H_1\cup H_2$ and $p\notin F^{\perp}$. The number of points in $H_1\cup H_2$ is $3\cdot 2^{2n-2}-1$ and the number of points in $F^{\perp}$ is $2^{n+1}-1$. Thus for $n\geq 3$ there is a point $p\notin H_1\cup H_2$ and $p\notin F^{\perp}$. By the induction hypothesis, there are panels of cotypes $2,3,\ldots,n$ of $\mathrm{Res}(p)$ mapped to an opposite panels by θ_p, and thus there are panels of each cotype $2,3,\ldots,n$ of $\mathsf{C}_n(2)$ mapped to an opposite by θ. It is then easy to see that there is also a nondomestic cotype 1 panel (by a residue argument) and hence θ is strongly exceptional domestic.

Now suppose the absolute points of θ form a union of two hyperplanes, and that the fixed point set F of θ is a k-space with $k=n-1+j$ for some $0\leq j\leq n-3$. An argument as in the previous paragraph shows there is a nondomestic point p such that $p^{\perp}$ intersects F in an $(n-2+j)$-space. By induction, with Lemma 3.6 as the base case, the collineation θ_p of the $\mathsf{C}_{n-1}(2)$ building $\mathrm{Res}(p)$ has the following diagram:



Moreover, for any other nondomestic point p we have that either θ_p has the above diagram, or θ_p is domestic on type $n-1-j$ vertices. Thus no simplex $\mathsf{C}_n(2)$ of type

$$\{1,2,\ldots,n-j-1\}$$

is mapped to an opposite by θ, hence the result.

To conclude we prove existence of collineations with each diagram. Recursively define elements $g_n\in\mathsf{Sp}_{2n}(2)$, for $n\geq 2$, by

$$g_2=\begin{bmatrix}0&1&0&0\\1&0&0&0\\1&0&0&1\\0&0&1&0\end{bmatrix},\quad g_3=\begin{bmatrix}0&0&1&0&0&0\\0&1&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&1\\1&0&0&0&1&0\\0&0&0&1&0&0\end{bmatrix},\quad g_n=\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\0&0&g_{n-2}&0&0\\0&0&0&1&0\\0&0&0&1&1\end{bmatrix}.$$

Moreover, for each $j\geq 0$ define $g_n^{(j)}\in\mathsf{Sp}_{2n}(2)$ in block diagonal form by

$$g_n^{(j)}=\mathrm{diag}(I_j,g_{n-j},I_j)\quad\text{where } I_j \text{ is the } j\times j \text{ identity matrix.}$$

By direct calculation, the absolute points of g_{2n} and $g_{2n}^{(j)}$ are given by

$$X_{2n-1}X_{2n} = 0$$

and the collinear points of g_{2n+1} and $g_{2n+1}^{(j)}$ are given by

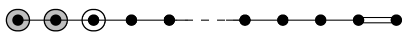
$$X_{n-1}(X_{n-2} + X_n) = 0.$$

Moreover, the fixed points of g_n form an $(n-2)$-space of $\mathsf{PG}(2n-1,2)$, and the fixed points of $g_n^{(j)}$ form an $(n-2+j)$-space of $\mathsf{PG}(2n-1,2)$. Thus, by the arguments above, g_n is a strongly exceptional domestic collineation of $\mathsf{C}_n(2)$ for each $n \geq 2$, and $g_n^{(j+1)}$ diagram as in (b). □

Similar theorems hold, with similar proofs, for the $\mathsf{B}_n(2,4)$ and $\mathsf{D}_n(2)$ buildings. We will only sketch the details below. Consider first the case $\mathsf{B}_n(2,4)$. The following lemmas are similar to the $\mathsf{C}_n(2)$ case.

Lemma 3.8. *A collineation θ of the generalised quadrangle $\mathsf{B}_2(2,4)$ is exceptional domestic if and only if the set of absolute points of θ is the set of points of $\mathsf{B}_2(2,4)$ lying on the union of two distinct hyperplanes in $\mathsf{PG}(5,2)$.*

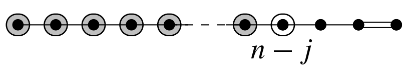
Lemma 3.9. *Let $\Delta = \mathsf{B}_n(2,4)$ with $n \geq 3$ and let θ be a collineation. If the absolute points of θ lie on a union of two hyperplanes, and if the fixed points of θ are the isotropic points of a $(2n-3)$-space in $\mathsf{PG}(2n+1,2)$, then θ has the following decorated opposition diagram:*



Theorem 3.10. *Let θ be a collineation of $\mathsf{B}_n(2,4)$. Suppose that the set of absolute points of θ is the set of isotropic points lying on the union of two hyperplanes of $\mathsf{PG}(2n+1,2)$. Let k be the projective dimension of the subspace of points of $\mathsf{PG}(2n+1,2)$ fixed by θ. Then θ is domestic, and*

(a) *if $k = n$ then θ is strongly exceptional domestic, and*

(b) *if $k = n+1+j$ for some $0 \leq j \leq n-3$ then θ is uncapped with the following decorated diagram:*



Moreover examples exist for each $n \leq k \leq 2n-2$.

Proof. The proofs are very similar to Theorem 3.7, with the base cases given by Lemma 3.8 and 3.9, and we omit the details. Thus it only remains to exhibit the existence of collineations of $\mathsf{B}_n(2,4)$ with the desired properties. To this end, define matrices g_n, $n \geq 3$ by

$$g_2 = \begin{bmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&1&0 \end{bmatrix}, \quad g_n = \begin{bmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 0&0&g_{n-2}&0&0\\ 0&0&0&1&0\\ 0&0&0&1&1 \end{bmatrix}.$$

Moreover, for each $j \geq 1$ define $g_n^{(j)}$ in block diagonal form by

$$g_n^{(j)} = \mathrm{diag}(I_j, g_{n-j}, I_j).$$

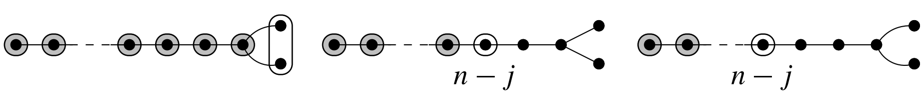
Since $g_n, g_n^{(j)} \in \mathsf{GO}^-_{2n+2}(2)$ these matrices induce collineations of $\mathsf{B}_n(2,4)$. It is straightforward to check that g_n satisfies the condition (a) and $g_n^{(j+1)}$ satisfies the condition (b). □

Consider now the case $\mathsf{D}_n(2)$.

Theorem 3.11. *Let θ be an automorphism of $\mathsf{D}_n(2)$. Suppose that the set of absolute points of θ is the set of points of $\mathsf{D}_n(2)$ lying on the union of two hyperplanes of $\mathsf{PG}(2n-1,2)$. Let k be the projective dimension of the subspace of points of $\mathsf{PG}(2n-1,2)$ fixed by θ. Then θ is domestic, and*

(a) *if $k = n-1$ and θ is an oppomorphism then θ is strongly exceptional domestic, and*

(b) *if $k = n-1+j$ for some $1 \leq j \leq n-3$ and θ is a nonoppomorphism (for odd j) and an oppomorphism (for even j) then θ has the following diagram:*



(*if $j = 1$*) (*if j is even*) (*if $j > 1$ is odd*)

Moreover examples exist for all $n-1 \leq k \leq 2n-4$.

Proof. The proofs of the statements (a) and (b) are again analogous to those in Theorem 3.7, with an appropriate start to the induction. We omit the details.

To prove existence, note that the matrices g_{n-1}, $n \geq 3$, from the proof of Theorem 3.10 are also elements of $\mathsf{GO}^+_{2n}(2)$. Let $h_3 = g_2$ and $h_4 = g_3$. Then h_3 induces a duality of $\mathsf{D}_3(2)$ and h_4 induces a collineation of $\mathsf{D}_4(2)$. Let $h_n = g_{n-1}$, and for each $1 \leq j \leq n-3$ let $h_n^{(j)} = g_{n-1}^{(j)}$. It is easy to check that h_n satisfies conditions (a), and $h_n^{(j)}$ satisfies conditions (b). □

4. Uncapped automorphisms for exceptional types

In this section we prove Theorem 1(b) for the small buildings of exceptional type. Moreover we completely classify the domestic automorphisms of the buildings $\mathsf{F}_4(2)$, $\mathsf{F}_4(2,4)$, and $\mathsf{E}_6(2)$. We begin, in Section 4A, by developing a (computationally feasible) method of detecting whether a given automorphism is domestic. In Section 4B we briefly describe the implementation of the minimal faithful permutation representations of the relevant ATLAS groups into Magma, and then in Section 4C we give the classification of domestic automorphisms of the buildings $\mathsf{F}_4(2)$, $\mathsf{F}_4(2,4)$, and $\mathsf{E}_6(2)$ making use of these permutation representations. We provide examples of uncapped automorphisms in $\mathsf{E}_7(2)$, and give conjectures for $\mathsf{E}_8(2)$ in Section 4D.

Throughout this section we use standard notation for Chevalley and twisted Chevalley groups G, and we refer to Carter [1989] for details. In particular, the symbols B, H, N, U, W, S, R, $x_\alpha(a)$, $n_\alpha(a)$, etc., have their usual meanings. However we note that in the twisted case we use these symbols for the objects in the twisted group (rather than the untwisted group). Then the quadruple (B, N, W, S) forms a Tits system in G, and thus (Δ, δ) is a building of type (W, S) where $\Delta = G/B$ and $\delta(gB, hB) = w$ if and only if $g^{-1}h \in BwB$. In the case of graph automorphisms of a simply laced Dynkin diagram we assume the Chevalley generators are chosen so that [Carter 1989, Proposition 12.2.3] holds (in particular $x_\alpha(a)^\sigma = x_{\sigma(\alpha)}(\pm a)$).

4A. ***Detecting domesticity.*** The following lemma shows that under certain hypotheses, to verify domesticity it is sufficient to show that no chamber opposite a given chamber is mapped onto an opposite.

Lemma 4.1. *Let θ be an automorphism of a thick spherical building Δ, and let $L = \mathrm{disp}(\theta)$. Let C be any chamber. Suppose that either*

(i) *each panel of Δ has at least 4 chambers, or*

(ii) *θ is an involution, or*

(iii) *θ induces opposition on the type set and $L = \ell(w_0)$.*

Then there exists a chamber D with $\delta(C, D) = w_0$ and $\ell(\delta(D, D^\theta)) = L$.

Proof. Let E be a chamber with $\ell(\delta(E, E^\theta)) = L$, and write $v = \delta(E, E^\theta)$. Let $w = \delta(C, E)$, and suppose that $w \neq w_0$. Then there exists $s \in S$ with $\ell(ws) > \ell(w)$. We show that there is a chamber D with $\delta(E, D) = s$ such that $\ell(\delta(D, D^\theta)) = L$. Consider each case.

(1) $\ell(sv) < \ell(v)$. Then either:

 (a) $\ell(svs^\theta) = \ell(v)$, in which case we choose the unique D with $\delta(E, D) = s$ such that $\delta(D, E^\theta) = sv$. Since $\delta(E^\theta, D^\theta) = s^\theta$ and $\ell(svs^\theta) > \ell(sv)$ we have $\delta(D, D^\theta) = svs^\theta$ and so $\ell(\delta(D, D^\theta)) = L$.

(b) $\ell(svs^\theta) < \ell(v)$, in which case necessarily $\ell(vs^\theta) < \ell(v)$, and it follows that there exists a reduced expression for v starting with s and ending with s^θ. Thus there exists a minimal length gallery

$$E = E_0 \sim_{s_1} E_1 \sim_{s_2} \cdots \sim_{s_{\ell-1}} E_{\ell-1} \sim_{s_\ell} E_\ell = E^\theta$$

with $s_1 = s$ and $s_\ell = s^\theta$.

(i) If every panel of Δ has at least 4 chambers then there exists a chamber D with $\delta(E, D) = s$ such that $D \notin \{E_1, E_{\ell-1}^{\theta^{-1}}\}$. Then there is a gallery $D \sim_{s_1} E_1 \sim_{s_2} \cdots \sim_{s_{\ell-1}} E_{\ell-1} \sim_{s_\ell} D^\theta$, and hence $\delta(D, D^\theta) = v$ has length L.

(ii) If θ is an involution then θ maps every minimal length gallery from E to E^θ to a minimal length gallery from E^θ to E, and it follows by considering types of first and last steps that $E_1^\theta = E_{\ell-1}$. Thus for any D with $\delta(E, D) = s$ and $D \neq E_1$ we again have $\delta(D, D^\theta) = v$.

(iii) If θ induces opposition and $L = \ell(w_0)$ then $v = w_0$, and $svs^\theta = sw_0s^\theta = w_0s^\theta s^\theta = w_0$, and so case (1)(b) cannot occur.

(2) $\ell(sv) > \ell(v)$. Then either:

(a) $\ell(svs^\theta) > \ell(v)$, in which case every chamber D with $\delta(E, D) = s$ has $\delta(D, D^\theta) = svs^\theta$, contradicting $\ell(v) = \mathrm{disp}(\theta)$. Thus this case cannot occur.

(b) $\ell(svs^\theta) = \ell(v)$, in which case we can choose D to be any chamber with $\delta(E, D) = s$. Then $\delta(D, E^\theta) = sv$ (since $\ell(sv) > \ell(v)$), and thus $\delta(D, D^\theta) = sv$ or $\delta(D, D^\theta) = svs^\theta$. The first case is impossible by the definition of displacement, and thus $\delta(D, D^\theta) = svs^\theta$ has length L.

Hence the result. □

Remark 4.2. The following examples illustrate that the conclusion of Lemma 4.1 may fail if the hypotheses of the lemma are not satisfied.

(1) The collineation θ of the Fano plane $\mathsf{PG}(2, \mathbb{F}_2)$ given by the upper triangular 3×3 matrix with all upper triangular entries equal to 1 maps no chamber opposite the base chamber $C = (\langle e_1 \rangle, \langle e_1 + e_2 \rangle)$ to an opposite chamber. However this collineation has displacement $\ell(w_0) = 3$, since no nontrivial collineation of a projective plane is domestic. Note that this collineation has order 4, and so none of the conditions of Lemma 4.1 are satisfied.

(2) The exceptional domestic collineation of the generalised quadrangle $\mathsf{GQ}(2, 2) = \mathsf{C}_2(2)$ (see [Temmermans et al. 2012b, Section 4]) is given by $\theta = x_1(1)x_2(1)$ in Chevalley generators. The chambers opposite the base chamber B of G/B are mapped to distances s_1s_2 or s_2s_1, however θ has displacement 3 (by both $s_1s_2s_1$ and $s_2s_1s_2$). Note that this collineation has order 4, and so again none of the conditions of Lemma 4.1 are satisfied.

4B. ***Minimal faithful permutation representations.*** Let $\mathcal{G}$ be the following set of ATLAS groups:

$$\mathcal{G} = \{\mathsf{F}_4(2), \mathsf{F}_4(2).2, {}^2\mathsf{E}_6(2^2), {}^2\mathsf{E}_6(2^2).2, \mathsf{E}_6(2), \mathsf{E}_6(2).2\}.$$

These groups are, respectively, the collineation group of $\mathsf{F}_4(2)$, the full automorphism group of $\mathsf{F}_4(2)$ (including dualities), the "inner" automorphism group of $\mathsf{F}_4(2,4)$, the full automorphism group of $\mathsf{F}_4(2,4)$, the collineation group of $\mathsf{E}_6(2)$, and the full automorphism group of $\mathsf{E}_6(2)$. In the following section we will need an explicit set of conjugacy class representatives for the groups in $\mathcal{G}$. With the exception of perhaps $\mathsf{F}_4(2)$, these groups appear to be too large for the conjugacy class algorithms in Magma (or GAP) when input as matrix groups using the adjoint representation. For example $\mathsf{E}_6(2).2$ has order 429683151044011150540800, and in any case it is not an entirely trivial task to construct such extensions as matrix groups. However the available algorithms in both Magma and GAP for permutation groups turn out to be considerably more efficient, and therefore we require faithful permutation representations of the groups in $\mathcal{G}$.

The degrees $\deg(G)$ of the minimal faithful permutation representations of the groups in $\mathcal{G}$ are well known (see for example [Vasilev 1996; 1997; 1998]):

$$\deg(\mathsf{F}_4(2)) = 69615, \quad \deg(\mathsf{F}_4(2).2) = 139230,$$
$$\deg({}^2\mathsf{E}_6(2^2)) = \deg({}^2\mathsf{E}_6(2^2).2) = 3968055,$$
$$\deg(\mathsf{E}_6(2)) = 139503, \quad \deg(\mathsf{E}_6(2).2) = 279006.$$

In each case the permutation representation can naturally be realised by the action of G on certain maximal parabolic coset spaces (equivalently, on certain vertices of the building). For example, for $G = \mathsf{E}_6(2).2$ we consider the action on $G/P_1 \cup G/P_6$ (the set of type 1 and type 6 vertices of the $\mathsf{E}_6(2)$ building), and for $G = {}^2\mathsf{E}_6(2^2).2$ we consider the action on ${}^2\mathsf{E}_6(2^2)/P_1$ (the set of type 1 vertices of the $\mathsf{F}_4(2,4)$ building), where P_i denotes the maximal parabolic subgroup of type $S\backslash\{s_i\}$.

To our knowledge, at the time of writing these minimal faithful permutation representations were not available in either GAP or Magma. Therefore we have implemented these permutation representations in Magma, using the above action on vertices of the building, and making use of the "Groups of Lie Type" package [Cohen et al. 2004]. The resulting permutation representations are available on Parkinson's webpage, where we also provide lists of conjugacy class representatives and code relevant to the computations in the following sections. We would like to thank Bill Unger from the Magma team at Sydney University for helping us generate the conjugacy class representatives from the permutation representations.

4C. ***Domestic automorphism of small buildings of types*** F_4 ***and*** E_6**.** In this section we classify domestic automorphisms of the buildings $\mathsf{F}_4(2)$, $\mathsf{F}_4(2,4)$, and $\mathsf{E}_6(2)$.

This requires two main steps. We first exhibit a list of n examples of pairwise nonconjugate domestic automorphisms for each building (for some integer n). Next, using an explicit set of conjugacy class representatives, we show that all but n of these representatives map some chamber to an opposite and are hence nondomestic. Thus we conclude that our list of n examples is complete.

We make frequent use of both commutator relations, and the formula

$$n_\alpha(a) = x_\alpha(a)x_{-\alpha}(-a^{-1})x_\alpha(a). \tag{4-1}$$

We will also use the following observation: for the buildings $\mathsf{E}_n(2)$, $n = 6, 7, 8$, the displacement of an automorphism θ determines the (decorated) opposition diagram of θ (see Remark 2.30). For the buildings $\mathsf{F}_4(2)$ and $\mathsf{F}_4(2,4)$ the (capped) automorphisms with types $\{1\}$ and $\{4\}$ are not distinguished by displacement, and furthermore in $\mathsf{F}_4(2)$ the three uncapped diagrams all have displacement 23.

Before beginning we outline a useful technique. Suppose that $\theta \in G$ induces an automorphism of $\Delta = G/B$ such that the hypothesis of Lemma 4.1 holds. Then there exists $gB \in Bw_0B/B$ such that $\mathrm{disp}(\theta) = \ell(\delta(gB, \theta gB))$. Each $gB \in Bw_0B/B$ can be written as $gB = uw_0B$ with $u \in U$, and $\delta(gB, \theta gB)$ is the unique $w \in W$ such that

$$w_0^{-1}u^{-1}\theta u w_0 \in BwB. \tag{4-2}$$

Thus to determine $\mathrm{disp}(\theta)$ it is sufficient to analyse the terms $w_0^{-1}u^{-1}\theta u w_0$ with $u \in U$. However $|U| = |\mathbb{F}|^{\ell(w_0)}$, and so even for relatively small buildings it is not computationally feasible to check each $u \in U$ (for example, in $\mathsf{E}_6(2)$ we have $|U| = 2^{36}$).

The following idea often provides a considerable computational efficiency. Note that each $u \in U$ can be written as $\prod_{\alpha \in R^+} x_\alpha(a_\alpha)$ with $a_\alpha \in \mathbb{F}$ and the product taken in any order (see [Steinberg 2016, Lemma 17]; of course the a_α depend on the order chosen). Writing

$$A = \{\alpha \in R^+ \mid x_\alpha(a)\theta \neq \theta x_\alpha(a) \text{ for all } a \in \mathbb{F}\}$$

we can write $u = u_A' u_A$ where u_A is a product over terms $\alpha \in A$, and u_A' is a product over the remaining positive roots. Then u_A' commutes with θ, and so

$$w_0^{-1}u^{-1}\theta u w_0 = w_0^{-1}u_A^{-1}\theta u_A w_0. \tag{4-3}$$

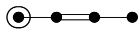
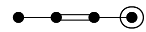
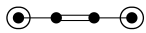
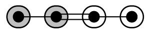
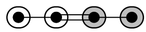
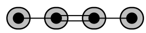
There are $|\mathbb{F}|^{|A|}$ such elements, and so the technique works best if a conjugacy class representative for θ is chosen with the property that it commutes with as many elements $x_\alpha(a)$, $\alpha \in R^+$, as possible.

The residue of the type J simplex of the chamber gB is the coset $gP_{S\backslash J}$, and this residue is nondomestic for θ if and only if $g^{-1}\theta g \in P_{S\backslash J} w_0 P_{S\backslash J}$, and thus if and only if

$$g^{-1}\theta g \in BwB \quad \text{for some } w \in w_0 W_{S\backslash J} \tag{4-4}$$

In the following we write $g_1 \sim g_2$ to mean that g_1 and g_2 are conjugate in G.

Theorem 4.3. *Let $G = \mathsf{F}_4(2)$, and let $\Delta = G/B$ be the associated building. Let $\varphi = (2342)$ and $\varphi' = (1232)$ be the highest root and highest short root (respectively) of the F_4 root system. There are precisely six conjugacy classes of domestic collineations of Δ, as follows*:

θ	capped	diagram	fixed type 1/4 vertices	ATLAS
$\theta_1 = x_\varphi(1)$	yes		2287/5103	$2B$
$\theta_2 = x_{\varphi'}(1)$	yes		5103/2287	$2A$
$\theta_3 = x_\varphi(1)x_{\varphi'}(1)$	yes		1263/1263	$2C$
$\theta_4 = x_1(1)x_2(1)$	no		127/399	$4D$
$\theta_5 = x_4(1)x_3(1)$	no		399/127	$4C$
$\theta_6 = x_2(1)x_3(1)$	no		151/151	$4E$

Moreover, $\theta_{3+i}^2 \sim \theta_i$ for $i = 1, 2, 3$, and $\theta_2 = \sigma(\theta_1)$, $\theta_3 = \sigma(\theta_3)$, $\theta_5 = \sigma(\theta_4)$, and $\theta_6 = \sigma(\theta_6)$.

Proof. We first show that the automorphisms have the claimed diagrams. Note that θ_1, θ_2, and θ_3 are involutions, and hence the hypothesis of Lemma 4.1 applies. Consider θ_1. Following the strategy of (4-2) we notice that $\theta_1 = x_\varphi(1)$ is central in U (by the commutator formulae), and hence, for all $u \in U$, using (4-1) we have

$$w_0^{-1}u^{-1}\theta_1 u w_0 = w_0^{-1}x_\varphi(1)w_0 = x_{-\varphi}(1) = x_\varphi(1)n_\varphi(1)x_\varphi(1) \in Bs_\varphi B.$$

Thus $\delta(gB, \theta_1 gB) = s_\varphi$ for all $gB \in Bw_0B/B$, and so $\mathrm{disp}(\theta) = \ell(s_\varphi) = 15$ (using Lemma 4.1). Moreover, note that $s_\varphi = w_0 w_{\{2,3,4\}}$ (for example, by computing inversion sets), and so there exists a nondomestic type 1 vertex. All type 2 or 3 vertices are domestic, for if, for example, there is a nondomestic type 2 vertex then there is $g \in G$ with $\delta(gB, \theta gB) \in w_0 W_{\{1,3,4\}}$ and hence $\mathrm{disp}(\theta) \geq 24 - 4 > 15$. If there exists a nondomestic type 4 vertex then by [Parkinson and Van Maldeghem 2019, Lemma 4.5] there exists a nondomestic type $\{1, 4\}$ simplex, which again contradicts the displacement calculation. Thus the diagram for θ_1 is as claimed, and since $\theta_2 = \sigma(\theta_1)$ (with σ the graph automorphism) the result for θ_2 also follows.

Consider θ_3. Since $x_{\varphi'}(1)$ is also central in U (this special feature of characteristic 2 follows from the commutator relations) we see that θ_3 is central in U. Thus, using commutator relations and (4-1) we have

$$\begin{aligned} w_0^{-1}u^{-1}\theta_3 u w_0 &= x_{-\varphi'}(1)x_{-\varphi}(1) = x_{-\varphi'}(1)x_\varphi(1)n_\varphi(1)x_\varphi(1) \\ &= x_\varphi(1)x_{(1110)}(1)x_{-(0122)}(1)x_{-\varphi'}(1)n_\varphi(1)x_\varphi(1) \in Bx_{-(0122)}(1)x_{-\varphi'}(1)s_\varphi B \\ &= Bs_\varphi x_{-(0122)}(1)x_{(1110)}(1)B = Bs_\varphi s_{(0122)}B. \end{aligned}$$

We have $s_\varphi s_{(0122)} = w_0 w_{\{2,3\}}$ (for example, by computing the inversion sets), and hence there exists a nondomestic type $\{1,4\}$ simplex; see (4-4). By Lemma 4.1 the above calculation also shows that $\mathrm{disp}(\theta) = \ell(w_0 w_{\{2,3\}}) = 20$, and the diagram of θ_3 follows.

Consider θ_4. We first show that θ_4 is domestic. We will work with the conjugate

$$\theta_4' = x_{(1220)}(1) x_{1122}(1) = w^{-1}\theta_4 w \quad \text{where } w = s_{(0110)} s_{(1242)}$$

because this representative commutes with more elements $x_\alpha(1)$ with $\alpha \in R^+$, making (4-2) more effective. Indeed θ_4' commutes with all $x_\alpha(1)$ with $\alpha \in R^+ \backslash A$, where

$$A = \{(0100), (0001), (0110), (0011), (0120), (1220), (0122), (1122)\}.$$

Then, as in (4-3), we have $w_0^{-1} u^{-1} \theta_4' u w_0 = w_0^{-1} u_A^{-1} \theta_4' u_A w_0$. There are 2^8 distinct elements u_A, and using the groups of Lie type package in Magma we can easily verify that $w_0^{-1} u_A^{-1} \theta_4' u_A w_0 \notin B w_0 B$ for all u_A (see Parkinson's webpage for the code). This implies that θ_4' is domestic, for if θ_4' were not domestic then the third hypothesis of Lemma 4.1 would hold and hence there would exist an element u_A with $w_0^{-1} u_A^{-1} \theta_4' u_A w_0 \in B w_0 B$.

One may see that θ_4' maps panels of cotypes 1 and 2 to opposites by simply exhibiting such panels (the groups of Lie type package is helpful here). Checking that there are no cotype 3 or 4 panels mapped to opposite panels is more complicated, and we have resorted to exhaustively verifying this by computation. However some efficiencies must be found to make the search feasible. Firstly, it is sufficient to check that there are no nondomestic type $\{1,2\}$ simplices (by a simple residue argument). Writing $P = P_{\{3,4\}}$, the (residues of the) type $\{1,2\}$ simplices of Δ are the cosets gP, $g \in G$. Let $T \subseteq W$ denote a transversal of minimal length representatives for cosets in $W/W_{\{3,4\}}$. A complete set of representatives for P cosets in G (and hence type $\{1,2\}$ simplices in Δ) is

$$\{u_w(a) w \mid w \in T,\ a \in \mathbb{F}_2^{\ell(w)}\} \quad \text{where } u_w(a) = x_{\beta_1}(a_1) \cdots x_{\beta_k}(a_k),$$

where $R(w) = \{\beta_1, \dots, \beta_k\}$ is the inversion set of w. Thus, using (4-4), it is sufficient to check that $\delta(g, \theta_4' g) \notin w_0 W_{\{3,4\}}$ for all $g = u_w(a) w$ with $w \in T$. However there are 4385745 such elements g (the cardinality of G/P) and this would be computationally expensive. Considerable efficiency can be gained by using the fact that the product $u_w(a)$ can be taken in any order (again, see [Steinberg 2016, Lemma 17]). Thus, applying the technique (4-3), we only need to consider terms $u_w'(a) = x_{\gamma_1}(a_1) \cdots x_{\gamma_\ell}(a_\ell)$ with $\{\gamma_1, \dots, \gamma_\ell\} = R(w) \cap A$. This drastically reduces the number of cases needing checking. In fact it turns out that there are only 3885 elements to check, and these are very quickly checked by the computer.

Since $\theta_5 = \sigma(\theta_4)$ the result for θ_5 follows.

Consider θ_6. Again we use a different conjugate $\theta_6 \sim \theta_6' = x_{(1110)}(1)x_{(0122)}(1)$. This element commutes with all $x_\alpha(1)$ with $\alpha \in R^+ \backslash A$, where

$$A = \{(0001),(0011),(0122),(0111),(0121),(1120),(1220),(1110),(1100),(1000)\}.$$

A similar argument to before, this time checking 2^{10} cases, verifies that θ_6' (and hence θ_6) is domestic. It is then straightforward to provide panels of each cotype mapped onto opposites, and hence θ_6 has the claimed diagram.

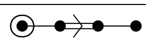
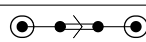
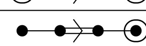
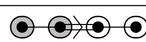
There are 95 conjugacy classes in the group $\mathsf{F}_4(2)$ (computed using the permutation representation), and for 88 of these classes a quick search finds nondomestic chambers. The seven remaining classes must therefore be domestic, because the six examples given above are clearly nonconjugate (they have distinct decorated opposition diagrams), and the identity is also trivially domestic.

The number of fixed type 1 vertices for each example is easily computed using the permutation representation, and the number of fixed type 4 vertices is obtained by considering the dual. Finally the ATLAS classes can be determined by the orders and fixed structures. □

Since no duality of a thick F_4 building is domestic the classification of domestic automorphisms of $\mathsf{F}_4(2)$ is complete (see [Parkinson and Van Maldeghem 2019, Lemma 4.1]). We also note that Lemma 2.18 follows from the above classification.

We now consider the building $\mathsf{F}_4(2,4)$. The full automorphism group of this building is ${}^2\mathsf{E}_6(2^2).2$ (that is, ${}^2\mathsf{E}_6(2^2)$ extended by the diagram automorphism σ of E_6; see [Tits 1974, Section 10.4] and [Conway et al. 1985, page 191]). We write $x_\alpha(a)$ for the Chevalley generators in the twisted group ${}^2\mathsf{E}_6(2^2)$. Thus $a \in \mathbb{F}_2$ (respectively, $a \in \mathbb{F}_4$) if α is a long root (respectively, short root) of the twisted root system.

Theorem 4.4. *Let $G = {}^2\mathsf{E}_6(2^2)$, and let $\Delta = G/B$ be the associated building of type $\mathsf{F}_4(2,4)$. Let φ (respectively, φ') be the highest root (respectively, highest short root) of the F_4 root system. There are precisely four classes of nontrivial domestic collineations, as follows (where σ is the graph automorphism of E_6):*

θ	capped	diagram	fixed points	ATLAS
$\theta_1 = x_\varphi(1)$	yes		46135	$2A$
$\theta_2 = x_{\varphi'}(1)$	yes		20279	$2B$
$\theta_3 = \sigma$	yes		69615	$2E$
$\theta_4 = x_1(1)x_2(1)$	no		855	$4A$

Here $x_\alpha(a)$ denote the Chevalley generators in the twisted group. Further, $\theta_4^2 \sim \theta_1$.

Proof. The analysis for θ_1 is similar to the analysis of θ_1 for $\mathsf{F}_4(2)$. Specifically, this element commutes with all terms $x_\alpha(a)$, and the result easily follows.

Consider θ_2. This element commutes with all terms $x_\alpha(a)$ with $\alpha \in R^+$ except for $x_{(0010)}(a)$, $x_{(0110)}(a)$ and $x_{(1110)}(a)$ with $a \in \{\xi, \xi^2\}$ (where ξ is a generator of $\mathbb{F}_4^*$). By commutator relations, if $a \in \{\xi, \xi^2\}$ we have

$$x_{(0010)}(-a)\theta_2 x_{(0010)}(a) = \theta_2 x_{\varphi-\alpha_1-\alpha_2}(1)$$
$$x_{(0110)}(-a)\theta_2 x_{(0110)}(a) = \theta_2 x_{\varphi-\alpha_1}(1)$$
$$x_{(1110)}(-a)\theta_2 x_{(1110)}(a) = \theta_2 x_{\varphi}(1),$$

and it follows that for all $u \in U$ we have

$$w_0^{-1}u^{-1}\theta_2 u w_0 = x_{-\varphi'}(1)x_{-\varphi+\alpha_1+\alpha_2}(a_1)x_{-\varphi+\alpha_1}(a_2)x_{-\varphi}(a_3) \text{ with } a_1, a_2, a_3 \in \{0, 1\}.$$

Considering each of the eight possibilities for the triple $(a_1, a_2, a_3) \in \mathbb{F}_2^3$ we see that the maximum length of $w = \delta(uw_0B, \theta_2 uw_0B)$ is 20 with $w = s_\varphi s_{(0122)}$, and the result follows.

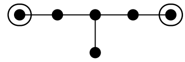
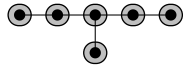
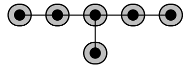
Consider θ_4. This element is conjugate to $\theta_4' = x_{(1220)}(1)x_{(1122)}(1)$, and then an analysis very similar to the case of θ_4 for $\mathsf{F}_4(2)$ applies. In particular, with A as in the $\mathsf{F}_4(2)$ case, we need to check each of the elements $\delta(u_A w_0 B, \theta_4' u_A w_0 B)$. This time there are $2048 = 4^3 \times 2^5$ elements u_A to check (since there are three roots in A whose root subgroup is isomorphic to $\mathbb{F}_4$ and the remaining five root subgroups are isomorphic to $\mathbb{F}_2$). A quick check with the computer shows that the maximum length of $\delta(u_A w_0 B, \theta_4' u_A w_0 B)$ is 23, and hence $\theta_4' \sim \theta_4$ is domestic. Then necessarily θ_4 maps no panels of cotypes 3 or 4 to opposite (by a simple residue argument), and then since $\mathrm{disp}(\theta_4) = 23$ it is forced that there are panels of cotypes both 1 and 2 mapped onto opposites.

Consider $\theta_3 = \sigma$. This element acts on the untwisted group $\mathsf{E}_6(4)$ as a symplectic polarity, and thus is $\{i\}$-domestic for $i \in \{2, 3, 4, 5\}$ (see [Van Maldeghem 2012]). It follows that σ is $\{i\}$-domestic for $i \in \{1, 2, 3\}$ on the building $\mathsf{F}_4(2, 4)$, hence the result.

Thus the diagrams of the four automorphisms are as claimed. Next, as in the $\mathsf{F}_4(2)$ example, we use the permutation representation of ${}^2\mathsf{E}_6(2^2).2$ to compute a complete list of conjugacy class representatives of this group. It turns out that there are 189 conjugacy classes, and for 184 of these classes one can exhibit a chamber mapped onto an opposite chamber. Thus there are at most 4 classes of nontrivial domestic collineations, and since the examples exhibited above are pairwise nonconjugate (by decorated opposition diagrams) the list is complete.

Finally, the calculation of the numbers of fixed points is immediate from the permutation representation, and the ATLAS classes can be determined by the orders and fixed structures. □

Theorem 4.5. *Let $G = \mathsf{E}_6(2).2$, and let $\Delta = \mathsf{E}_6(2)/B$ be the associated building of type $\mathsf{E}_6(2)$. There are precisely three classes of domestic dualities* (*up to* *conjugation in the full automorphism group), as follows:*

θ	capped	diagram	order
$\theta_1 = \sigma$	yes		2
$\theta_2 = x_1(1)\sigma$	no		4
$\theta_3 = x_1(1)x_3(1)\sigma$	no		8

Proof. As noted in Theorem 4.4, the element $\theta_1 = \sigma$ acts as a symplectic polarity on $\mathsf{E}_6(2)$, and thus has the diagram claimed (see [Van Maldeghem 2012]). For the remaining cases θ_2 and θ_3 we note that it is easy to find vertices of each type mapped onto opposite vertices. Thus it remains to show that these dualities are domestic. The working here is slightly more complicated than the case of collineations of the F_4 buildings. Writing $\theta = \tilde{\theta}\sigma$ with $\tilde{\theta} \in G$, we must show $w_0^{-1}u^{-1}\tilde{\theta}u^{\sigma}w_0 \notin Bw_0B$ for all $u \in U$ (here we are applying Lemma 4.1).

Consider θ_2. We use the conjugate $\theta_2' = x_\beta(1)\sigma$ with $\beta = (111221)$. It turns out, by commutator relations, that if $u \in U$ is arbitrary then $u^{-1}x_\beta(1)u^\sigma$ can be written in the following form (where we use Magma's built-in lexicographic order on the positive roots $\alpha_1, \ldots, \alpha_{32}$):

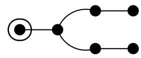
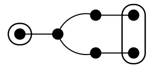
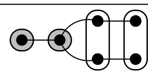
$$\begin{aligned} &x_1(a_1)x_7(a_2)x_{12}(a_3)x_{18}(a_4)x_{23}(0)x_{17}(a_5)x_{22}(a_6)x_{27}(0)x_{26}(a_7)x_{30}(0)\\ &\quad x_{29}(a_8)x_{32}(a_9)x_{33}(a_9+1)x_{34}(a_{10})x_{35}(a_{11})x_{36}(a_{12})x_3(a_{13})x_9(a_{14})\\ &\qquad x_{13}(a_{15})x_{15}(0)x_{19}(0)x_{21}(a_4)x_{25}(a_6)x_{24}(0)x_{28}(a_7)x_{31}(a_{16})x_4(0)\\ &\qquad\quad x_{10}(a_{14})x_8(0)x_{14}(a_{15})x_{16}(a_3)x_{20}(a_5)x_5(a_{13})x_{11}(a_2)x_2(0)x_6(a_1), \end{aligned}$$

where $a_1, \ldots, a_{16} \in \mathbb{F}_2$. The point is that there are only 2^{16} such terms, rather than $2^{36} = |U|$ terms. It is then a quick check on the computer to verify that θ_2 is domestic (and hence strongly exceptional domestic by Corollary 2.20).

The analysis of θ_3 is slightly more challenging. Using the conjugate $\theta_3' = x_\beta(1)x_{\beta'}(1)\sigma$ with $\beta = (010111)$ and $\beta' = (001111)$ we see that $u^{-1}x_\beta(1)x_{\beta'}(1)u^\sigma$ can be written in a similar way to the θ_2 case above, this time with 2^{22} degrees of freedom. The verification that θ_3 is domestic is then a long search with the computer. The details are on Parkinson's webpage.

To verify that our list of domestic examples is complete we again use explicit conjugacy class representatives computed from the minimal faithful permutation representation, as in the previous theorems. See Parkinson's webpage for the relevant code. Note that the character table of $\mathsf{E}_6(2)$ is not printed in ATLAS, and therefore it is not possible to provide the ATLAS conjugacy class names. □

Theorem 4.6. *Let $G = \mathsf{E}_6(2)$, and let $\Delta = G/B$ be the associated building of type* $\mathsf{E}_6(2)$. *There are precisely* 3 *classes of domestic collineations, as follows*:

θ	capped	diagram	fixed points	order
$\theta_1 = x_1(1)$	yes		10479	2
$\theta_2 = x_1(1)x_2(1)$	yes		2543	2
$\theta_3 = x_1(1)x_3(1)$	no		847	4

Proof. To analyse θ_1 we work with the conjugate $\theta_1 \sim x_\varphi(1)$, where φ is the highest root. Then an analysis very similar to the $\mathsf{F}_4(2)$ case shows that θ_1 has the diagram claimed.

The analysis for θ_2 can be done by hand. We work with the conjugate $\theta_2' = x_\varphi(1)x_{\varphi'}(1)$ where φ is the highest root and $\varphi' = (101111)$ is the highest root of the A_5 subsystem. Let $u \in U$. By commutator relations and a simple induction we see that $u^{-1}\theta_2' u$ is a product of terms $x_\alpha(a)$ with $\alpha \geq \varphi'$ (with $\geq$ being the natural dominance order). In particular, each such α is in $R^+\backslash\mathsf{D}_5$, where D_5 is the subsystem generated by $\alpha_2, \ldots, \alpha_6$. Let $v = w_0 w_{\mathsf{D}_5}$, where w_{D_5} is the longest element of the parabolic subgroup $\langle s_2, \ldots, s_6\rangle$. Then $R^+\backslash\mathsf{D}_5 = \{\alpha \in R^+ \mid v^{-1}\alpha \in -R^+\}$. It follows that $v^{-1}(w_0^{-1}u^{-1}\theta_2' u w_0)v \in B$ for all $u \in U$, and therefore

$$w_0^{-1}u^{-1}\theta_2' u w_0 \in vBv^{-1} \subseteq BvB \cdot Bv^{-1}B.$$

Hence $w_0^{-1}u^{-1}\theta_2' u w_0 \in BwB$ for some w with $\ell(w) \leq 2\ell(v) = 2(\ell(w_0) - \ell(w_{\mathsf{D}_5})) = 32$ (in fact we necessarily have strict inequality here by double coset combinatorics). Thus $\mathrm{disp}(\theta) \leq 32$, and it then follows from the classification of diagrams (and hence of possible displacements) that $\mathrm{disp}(\theta) \leq 30$. On the other hand, a quick calculation shows that $w_0^{-1}\theta_2' w_0 \in Bs_\varphi s_{\varphi'}B$, and by computing inversion sets we have $s_\varphi s_{\varphi'} = w_0 w_{\mathsf{A}_3}$ (where A_3 is the subsystem generated by $\alpha_3, \alpha_4, \alpha_5$). Thus θ_2' maps the type $\{1, 2, 6\}$ simplex of the chamber $w_0 B$ to an opposite simplex, hence the result.

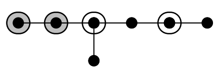
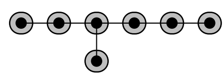
The working for θ_3 is more involved. Here Lemma 4.1 cannot be applied, and it is not practical to directly check every chamber for domesticity (there are 3126356394525 of them). Instead we argue in a similar fashion as we did for the collineation θ_4 in Theorem 4.3. First replace θ_3 by the conjugate $\theta_3 \sim \theta_3' = x_{(111210)}(1)x_{(011111)}(1)$. Then θ_3' commutes with all $x_\alpha(a)$ with $\alpha \in R^+\backslash A$ where

$$A = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6, (000110), (000011), (101100), (101110), (001111), (0111111), (111210)\}.$$

By a residue argument it is sufficient to show that there are no nondomestic type $\{2, 4\}$ simplices (see the claim in the proof of Corollary 2.26). Again one cannot feasibly check all type $\{2, 4\}$ simplices (there are 7089243525 of them). However, as in Theorem 4.3, with T a transversal of minimal length representatives for the cosets in $W/W_{\{1,3,5,6\}}$, it is sufficient to check that $\delta(g, \theta_3' g) \notin w_0 W_{\{1,3,5,6\}}$ for all $g = u_w'(a)w$ with $w \in T$ and $u_w'(a) = x_{\gamma_1}(a_1) \cdots x_{\gamma_\ell}(a_\ell)$ with $\{\gamma_1, \dots, \gamma_\ell\} = R(w) \cap A$. It turns out that there are only 64158 such elements g, and they are readily checked by computer in under an hour. □

4D. ***Automorphisms of small buildings of types E_7 and E_8.*** Consider the E_7 root system R. Fix the ordering $\alpha_1, \dots, \alpha_{63}$ of the positive roots according to increasing height, using the natural lexicographic order for roots of the same height (for example, $(1122100) < (1112110)$). Note that this is the inbuilt order in Magma. With this order, the roots $\alpha_{44} = (1112111)$, $\alpha_{45} = (0112211)$, and $\alpha_{46} = (1122210)$ play a special role below.

Theorem 4.7. *Let $\theta_1 = x_{44}(1)x_{46}(1)$ and $\theta_2 = x_{44}(1)x_{45}(1)x_{46}(1)$ in $\mathsf{E}_7(2)$. Then θ_1 and θ_2 are uncapped with the following respective decorated opposition diagrams*:

 *and* 

Moreover $\theta_1^2 = \theta_2^2 = x_\varphi(1)$ where φ is the highest root, and hence θ_1 and θ_2 have order 4.

Proof. Consider θ_2 first. We show that θ_2 is domestic using Lemma 4.1. Applying (4-3) verbatim requires us to check 2^{26} elements. The following modification of the theme is more efficient. It follows from commutator relations that

$$w_0^{-1} u^{-1} \theta_2 u w_0 = \prod_{\beta \in B} x_{-\beta}(a_\beta),$$

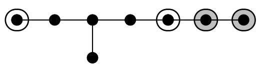
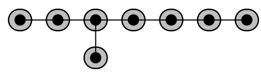
where $B = \{\beta \in R^+ \mid \beta \geq \alpha_{44}$ or $\beta \geq \alpha_{45}$ or $\beta \geq \alpha_{46}\}$ (where here $\alpha \geq \beta$ if and only if $\alpha - \beta$ is a nonnegative combination of simple roots). There are 20 roots in B. Moreover $a_{44} = a_{45} = a_{46} = 1$ (by commutator relations), and so there remain only 2^{17} elements to consider. It is then readily checked by computer that θ_2 is domestic, and we easily find vertices of each type mapped onto opposite vertices. Finally, commutator relations show that $\theta_2^2 = x_\varphi(1)$.

For θ_1 we do a similar search to the above to show that θ_1 is domestic. The remaining difficultly is showing that θ_1 is $\{1, 3\}$-domestic. Arguing as we did for θ_4 in Theorem 4.3 it turns out that there are 1141419 elements to check, and this can be done in an overnight run on the computer. □

Thus the proof of Theorem 1(b) is complete. Our computational techniques are not efficient enough to handle the two diagrams for $\mathsf{E}_8(2)$ due to the formidable size of the group. Thus for these diagrams we provide conjectural examples. For each of these conjectures we have randomly selected 10^5 chambers and verified that restricted to this subset of the chamber set the structure of the automorphism is as claimed.

Fix the ordering $\alpha_1, \ldots, \alpha_{120}$ of the positive roots of E_8 according to increasing height, using the natural lexicographic order for roots of the same height. Then the roots $\alpha_{88} = (11232221)$, $\alpha_{89} = (12243210)$ and $\alpha_{90} = (12233211)$ play a special role below.

Conjecture 4.8. *Let* $\theta_1 = x_{88}(1)x_{90}(1)$ *and* $\theta_2 = x_{88}(1)x_{89}(1)x_{90}(1)$ *in* $\mathsf{E}_8(2)$. *Then* θ_1 *and* θ_2 *are uncapped with the following respective decorated opposition diagrams*:

 *and* 

We note that $\theta_1^2 = \theta_2^2 = x_\varphi(1)$ where φ is the highest root, and hence θ_1 and θ_2 have order 4. It is not difficult to verify that $\mathrm{Typ}(\theta_1) = \{1, 6, 7, 8\}$ and $\mathrm{Typ}(\theta_2) = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Thus the difficulty in the above conjecture is to show that θ_1 is $\{7, 8\}$-domestic, and that θ_2 is domestic. In principle the approach taken for $\mathsf{E}_7(2)$ is applicable, however in practice the enormous size of the group $\mathsf{E}_8(2)$ makes the search impractical. For example, applying the technique of Theorem 4.7 to θ_2 amounts to checking $2^{30} = 1073741824$ elements. Each of these checks requires a sequence of commutator relations in the group $\mathsf{E}_8(2)$, and while Magma has remarkably efficient algorithms implemented for this, the number of cases renders this computational approach unfeasible.

Remark 4.9. The examples of uncapped automorphisms that we have constructed thus far fix a chamber of the building. This is clear for the examples in exceptional types because the representatives are either in the Borel subgroup B, or are a composition of an element of B with a standard graph automorphism. For the examples constructed in classical types we note that all examples have either order 4 or 8. It follows that they lie in a Sylow 2-group of the automorphism group, and hence are conjugate to an element of B (or $\langle B, \sigma\rangle$ in the case of an order 2 graph automorphism). However there do exist uncapped automorphisms that do not fix a chamber. For example, in $\mathsf{C}_3(2) = \mathsf{Sp}_6(2)$ the element

$$\theta = x_2(1)x_3(1)n_2 = E_{11} + E_{23} + E_{24} + E_{25} + E_{32} + E_{33} + E_{45} + E_{54} + E_{55} + E_{66}$$

is exceptional domestic (in fact strongly exceptional domestic), with order 6. Thus θ does not lie in any conjugate of B, and hence θ fixes no chamber of $\mathsf{C}_3(2)$. In fact the fixed structure of θ consists of three points p_1, p_2, p_3, a line L, and three planes π_1, π_2, and π_3 such that π_1, π_2 and π_3 intersect in L, $p_i \in \pi_i$ for $i = 1, 2, 3$, and $p_i \notin L$ for $i = 1, 2, 3$.

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