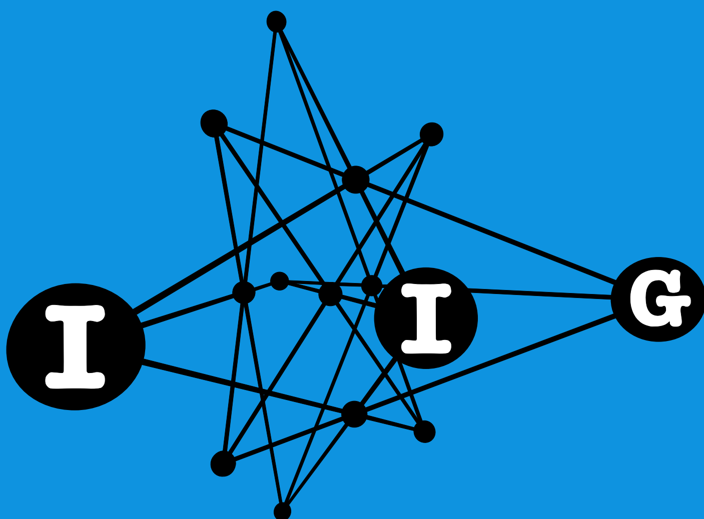
Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial



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The global structure of the chamber graph of certain rank 3 geometries that are almost buildings is determined. Computer files containing extensive details of these graphs accompany this paper.

1. Introduction

The study of geometries that are almost buildings was instigated by Tits [1981]. The acronym "GAB" was bestowed upon them in [Kantor 1981], and they also go under the names of "geometries of type M" or "Tits geometries of type M". These geometries are Buekenhout–Tits geometries [Buekenhout 1979a] all of whose rank-2 residue geometries are generalized polygons (though they are not required to satisfy the intersection property). That is, they are incidence geometries satisfying axioms (1) and (2) but not necessarily (3) of [Buekenhout 1979a].

We recall that an incidence geometry over a set I is a triple $(\Gamma, *, \tau)$ where Γ is a set, τ an onto map from Γ to I and $*$ is an incidence relation on Γ such that if $x, y \in \Gamma$ and $x * y$ then $\tau(x) \neq \tau(y)$. The map τ is called the type map and $|I|$ the rank of Γ. As is customary, we shall abbreviate $(\Gamma, *, \tau)$ to Γ. A flag F of Γ is a subset of Γ such that $x * y$ for all $x, y \in F$, $x \neq y$ and the type of F is $\{\tau(x) \mid x \in F\}$. The residue of F in Γ, Γ_F, is the (subgeometry) given by $\{x \in \Gamma \mid y * x \text{ for all } y \in F\}$. If $F = \{x\}$, then we write Γ_x instead of $\Gamma_{\{x\}}$. We shall call a maximal flag of Γ a chamber of Γ. Note that, by axiom (1) of [Buekenhout 1979a], the type of a chamber of a GAB is I. The chamber graph $\mathcal{C}(\Gamma)$ is defined as follows. The vertices are the chambers of Γ with distinct chambers γ and γ' deemed adjacent in $\mathcal{C}(\Gamma)$ if $|\gamma \cap \gamma'| = |I| - 1$. We sometimes also say that γ and γ' are i-adjacent if $I = \{i\} \cup \{\tau(x) \mid x \in \gamma \cap \gamma'\}$. Let γ be a chamber of Γ. The i-th disc of γ, denoted by $\Delta_i(\gamma)$, consists of all the chambers which are distance i from γ in the graph $\mathcal{C}(\Gamma)$. We shall use $d(\,,\,)$ for

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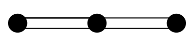
the distance metric on $\mathcal{C}(\Gamma)$ and $\mathrm{Diam}(\mathcal{C}(\Gamma))$ for the diameter of $\mathcal{C}(\Gamma)$. For more on incidence geometries, consult [Buekenhout 1979b; 1995], while for GAB's the survey paper [Kantor 1986] contains much interesting material.

The chamber graph of a building contains all the important geometric information about the building. For example, the (chambers of the) apartments of the building can be detected in the chamber graph. The sets $\Delta_i(\gamma)$, for γ a chamber, encode data relating to the Weyl group of the building. Further, if d is the diameter of the chamber graph and G is the automorphism group of the building, then G_γ, a Borel subgroup of G, acts transitively on $\Delta_d(\gamma)$. See [Ronan 2009; Tits 1974; 1981] for more on buildings. It is natural to wonder about chamber graphs of other geometries associated with groups which are, in some sense, close to buildings. This has prompted a number of papers which have focussed on analyzing the disc structure of such chamber graphs. See [Carr and Rowley 2018; Rowley 1998; 2009; 2010]. Most of the geometries of interest have a large number of chambers and so these investigations have necessarily involved extensive computation using packages such as MAGMA [Cannon and Playoust 1997]. Here we continue this line of work, examining the chamber graphs of rank 3 GAB's. The examples we look at have been drawn from [Aschbacher and Smith 1983; Cooperstein 1989; Kantor 1981; Ronan and Smith 1980] (see also [Connor 2011; Kantor 1985; Yoshiara 1988]). We now state our main results on the disc structure of these GAB's.

Theorem 1.1. *Let G denote one of the five groups $P\Omega_6^-(3)$, $G_2(3)$, $U_6(2)$, $\Omega_8^+(2)$ and Suz, and let Γ denote a GAB associated to one of these groups. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let γ_0 be a fixed chamber of $\mathcal{C}$. Put $B = \mathrm{Stab}_G(\gamma_0)$.*

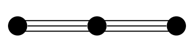
(i) *If $G \cong P\Omega_6^-(3)$ and Γ has diagram*



then $\mathcal{C}$ has 25515 *chambers*, 196 *B-orbits*, *diameter* 10 *and disc structure*

i-th disc	1	2	3	4	5	6	7	8	9	10
$\lvert\Delta_i(\gamma_0)\rvert$	6	20	64	176	416	1024	2432	5120	9088	7168
# of B-orbits	3	5	8	12	15	19	27	35	43	28

(ii) *If $G \cong G_2(3)$ and Γ has diagram*



then $\mathcal{C}$ has 66339 *chambers*, 1144 *B-orbits*, *diameter* 12 *and disc structure*

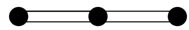
i-th disc	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert\Delta_i(\gamma_0)\rvert$	6	20	64	208	600	1728	4640	10368	17920	20416	9472	896
# of B-orbits	3	6	10	18	27	42	90	176	288	321	148	14

(iii) *If* $G \cong G_2(3)$ *and* Γ *has diagram*

then $\mathcal{C}$ *has* 66339 *chambers*, 1144 *B-orbits*, *diameter* 13 *and disc structure*

i-th disc	1	2	3	4	5	6	7	8	9	10	11	12	13
$\|\Delta_i(\gamma_0)\|$	6	20	56	144	384	960	2176	4864	10368	19072	21248	6976	64
# of B-orbits	3	6	9	14	21	31	51	92	172	302	332	109	1

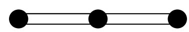
(iv) *If* $G \cong U_6(2)$ *and* Γ *has diagram*



then $\mathcal{C}$ *has* 1576960 *chambers*, 505 *B-orbits*, *diameter* 8 *and disc structure*

i-th disc	1	2	3	4	5	6	7	8
$\|\Delta_i(\gamma_0)\|$	15	117	972	6075	35721	203391	875043	455625
# of B-orbits	3	6	10	17	35	98	246	89

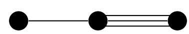
(v) *If* $G \cong \Omega_8^+(2)$ *and* Γ *has diagram*



then $\mathcal{C}$ *has* 179200 *chambers*, 317 *B-orbits*, *diameter* 9 *and disc structure*

i-th disc	1	2	3	4	5	6	7	8	9
$\|\Delta_i(\gamma_0)\|$	9	45	216	891	3159	11421	37098	80676	45684
# of B-orbits	3	6	10	16	26	43	68	95	49

(vi) *If* $G \cong Suz$ *and* Γ *has diagram*



then $\mathcal{C}$ *has* 18243225 *chambers*, 1276 *B-orbits*, *diameter* 16 *and disc structure*

i-th disc	1	2	3	4	5	6	7	8
$\|\Delta_i(\gamma_0)\|$	8	32	128	432	1216	3712	11008	29184
# of B-orbits	3	5	8	12	15	19	26	33
i-th disc	9	10	11	12	13	14	15	16
$\|\Delta_i(\gamma_0)\|$	81920	229376	598016	1576960	3595264	5410816	5304320	1400832
# of B-orbits	44	66	99	155	241	270	222	57

The GAB associated with the Lyons sporadic simple group is beyond our computational reach having 207060716016 chambers. However, we can give bounds on the diameter of the chamber graph.

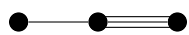
Theorem 1.2. *Let* Γ *be the GAB for Ly. Then* $10 \leq \mathrm{Diam}(\mathcal{C}(\Gamma)) \leq 16$.

2. Properties of $\mathcal{C}(\Gamma)$

The information collated in Theorem 1.1 was obtained using the code available with [Carr and Rowley 2018] and employing MAGMA. In fact, much more intricate details about $\mathcal{C}(\Gamma)$ were obtained, and these are available in the files in the online supplement (see article web page, doi 10.2140/iig.2019.17.189). We give a brief summary of such things.

The chambers of Γ are viewed as the right cosets of B. The panel stabilizers will be denoted by P_1, P_2 and P_3 (recall we are only looking at rank 3 geometries). The data obtained and program code is underpinned by DB, a sequence containing the (B, B) double coset representatives. So for $g = DB[j]$, the Bg coset is a representative for the B-orbits on the chambers of Γ. To minimise storage, we record j rather than $DB[j]$ whenever possible.The important output files are BorbitsDiscs and Neighbours. The first is a sequence where BorbitsDiscs[i] tells us the B-orbits making up $\Delta_i(\gamma_0)$ (where γ_0 is identified with the coset B). Here we give B-orbit representatives Bg, where $g = DB[k]$, by recording k. Neighbours is also a sequence where Neighbours[j] is giving information on the neighbours of Bg (where $g = DB[j]$). Suppose we have $[P_i : B] = 3$ for $i = 1, 2, 3$ (as happens for the GAB associated with $P\Omega_6^-(3)$, for example), so $\mathcal{C}(\Gamma)$ has valency 6. Returning to Neighbours[j], in this case this would be a 6-tuple $[k_1, k_2, k_3, k_4, k_5, k_6]$. This is saying that the six neighbours of Bg are in the B-orbits of $B * DB[k_i]$ $(i = 1, \ldots, 6)$. More than this we are also keeping track of the kind of adjacency. So the neighbours in the B-orbits of $B * DB[k_1]$ and $B * DB[k_2]$ are 1-adjacent to Bg, those in the B orbits of $B * DB[k_3]$ and $B * DB[k_4]$ are 2-adjacent to Bg, and those in the B-orbits of $B * DB[k_5]$ and $B * DB[k_6]$ are 3-adjacent to Bg.

Proof of Theorem 1.2. Let $G = Ly$ and let γ_0 be a chamber of $\mathcal{C}(\Gamma)$, and put $B = \mathrm{Stab}_G(\gamma_0)$. Recall that the diagram for Γ is



Let x be a point of Γ. Then by Section 6 of [Kantor 1981], Γ_x is a generalized hexagon dual to the usual $G_2(5)$ generalized hexagon. In particular, for any two chambers γ, γ' of Γ containing x we have $d(\gamma, \gamma') \leq 6$. Let the point, line and plane of γ_0 be respectively x_0, l_0, p_0 and γ_1 a chamber whose point, line and plane are respectively x_1, l_0, p_1 where $x_0 \neq x_1$. So x_0 and x_1 are collinear in Γ. Now $\gamma_0 = \{x_0, l_0, p_0\}, \{x_0, l_0, p_1\}, \{x_1, l_0, p_1\} = \gamma_1$ is a path in $\mathcal{C}(\Gamma)$, whence $d(\gamma_0, \gamma_1) \leq 2$. Since the point-line collinearity graph of Γ has diameter 2 (see Section 6 of [Kantor 1981] again), we infer that $\mathrm{Diam}(\mathcal{C}(\Gamma)) \leq 2 + 6 + 2 + 6 = 16$.

The number of chambers in the GAB associated with the Lyons group is

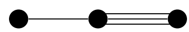
$$\frac{|G|}{N_G(S)} = \frac{|G|}{5^6 \cdot 2^4} = 207060716016,$$

where $S \in Syl_5(G)$. We find a lower bound for the diameter of the $\mathcal{C}(\Gamma)$ by working out the maximum number of chambers that can be in each disc. We have $[P_i : B] = 6$, $i = 1, 2, 3$, and so the valency of $\mathcal{C}(\Gamma)$ is 15. Therefore each chamber γ in $\Delta_1(\gamma_0)$ is joined to 5 chambers in $\Delta_1(\Gamma_0) \cup \{\gamma_0\}$. Hence $|\Delta_1(\gamma) \cap \Delta_2(\gamma)| = 10$. Of course for $i \geq 2$, a chamber in $\Delta_i(\gamma_0)$ can have at most 14 neighbours in $\Delta_{i+1}(\gamma_0)$. Thus, letting $d = \mathrm{Diam}(\mathcal{C}(\Gamma))$,

$$207060716016 \leq 1 + 15 + 150 + 150 \cdot 14 + \cdots + 150 \cdot 14^{d-2} = 16 + 150\left(\tfrac{14^{d-1}-1}{14-1}\right).$$

This gives $d - 1 \geq \log_{14}\left(\frac{13}{150}(207060716001) + 1\right)$, whence $d - 1 \geq 8.947$. Consequently, $\mathrm{Diam}(\mathcal{C}(\Gamma)) \geq 10$, which completes the proof of Theorem 1.2. □

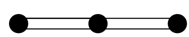
Collapsed adjacency graphs. For a GAB with diameter of say d, we call $\Delta_d(\gamma_0)$ the last disc (of γ_0) of the chamber graph. When examining the number of B-orbits which comprise the last disc we see, from the point of the chamber graph, the appellation of "almost building" is something of a misnomer. Of the GAB's investigated here only the GAB associated with $G_2(3)$, diagram



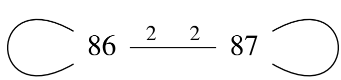
has its last disc as a B-orbit. Because of this we have calculated the geodesic closure for this GAB, the results of which are summarized in Theorem 2.1. All the others have the number of B-orbit ranging from 14 to 89. Indeed the more sporadic geometries studied in [Carr and Rowley 2018] and [Rowley 2009] come closer to buildings in this respect.

Notwithstanding the above comments on the last disc, we have looked at the induced graph on this disc. The most interesting (as far as we can see) are the GAB's from $G_2(3)$. Now we describe the B-collapsed adjacency graphs for the last disc of γ_0. The B-collapsed adjacency graph is formed by taking B-orbits, $B = \mathrm{Stab}_G\, \gamma_0$, as the vertices. We use j to stand for the B orbit of $B * DB[j]$ (where j is as given in the accompanying files). Two B-orbits, j and k are adjacent if and only if each chamber in j is adjacent to some chamber in k and we label the edge coming out from j with the number of chambers in k a chamber in j is adjacent with. If this number is 1 (as is mainly the case below) we omit this number.

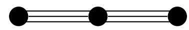
(i) If $G \cong P\Omega_6^-(3)$ and Γ has diagram



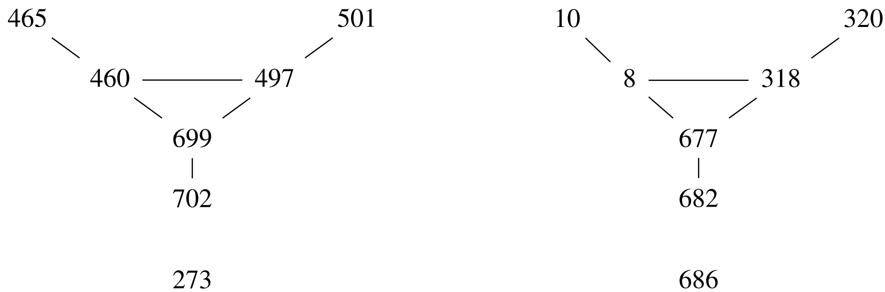
then the last disc of the B-collapsed adjacency graph is connected apart from 87 and 89, with 87 and 89 having the following adjacencies.



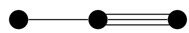
(ii) If $G \cong G_2(3)$ and Γ has diagram



then the 14 B-orbits in the last disc form the following collapsed B-adjacency graph.

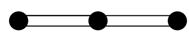


(iii) If $G \cong G_2(3)$ and Γ has diagram



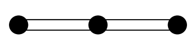
then there is only one B-orbit in the last disc and $\Delta_{13}(\gamma_0)$ is a co-clique.

(iv) If $G \cong U_6(2)$ and Γ has diagram



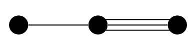
then the last disc of the B-collapsed adjacency graph is connected apart from 215 and 377, with 215 and 377 having the following adjacencies.

(v) If $G \cong \Omega_8^+(2)$ and Γ has diagram

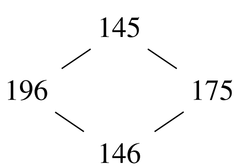


then the B-collapsed adjacency graph of $\Delta_9(\gamma_0)$ is connected.

(vi) If $G \cong Suz$ and Γ has diagram



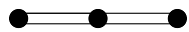
then the last disc of the B-collapsed adjacency graph is connected apart from 145, 146, 175 and 196, which have the following adjacencies.



Geodesic closure. For $\gamma, \gamma' \in \mathcal{C}$ a shortest path between them in $\mathcal{C}$ is called a geodesic. The geodesic closure of a set of chambers X is defined to be the set $\overline{X}$ of all chambers lying on some geodesic of γ, γ' for any pair $\gamma, \gamma' \in X$. The motivation for geodesic closures comes from the fact that in the chamber graph of a building, the geodesic closure of two chambers at maximal distance apart yields (the chambers of) an apartment.

Theorem 2.1. *Let G denote one of the groups $P\Omega_6^-(3)$ or $G_2(3)$, and let Γ denote a GAB associated to one of these groups. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let γ_0 be a fixed chamber of $\mathcal{C}$. Put $B = \mathrm{Stab}_G(\gamma_0)$.*

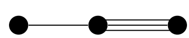
(i) *Suppose $G \cong P\Omega_6^-(3)$ and Γ has diagram*



and let $\gamma_i \in \Delta_{10}(\gamma_0)$, $i = 1, \ldots, 28$ be B-orbit representatives of $\Delta_{10}(\gamma_0)$. Set $n_{i,j} = |\overline{\{\gamma_0, \gamma_i\}} \cap \Delta_j(\gamma_0)|$. Then:

j	0	1	2	3	4	5	6	7	8	9	10
$n_{1,j}, n_{2,j}$	1	3	4	6	6	4	6	6	4	3	1
$n_{3,j}, n_{4,j}, n_{5,j}, n_{6,j}$	1	2	2	3	3	2	3	3	2	2	1
$n_{7,j}, n_{8,j}, n_{9,j}, n_{10,j}$	1	3	4	5	6	5	4	4	3	2	1
$n_{11,j}, n_{12,j}$	1	3	4	6	6	4	4	4	2	2	1
$n_{13,j}, n_{14,j}$	1	1	2	1	1	2	1	1	2	1	1
$n_{15,j}, n_{16,j}, n_{17,j}, n_{18,j}$	1	3	4	4	5	6	5	4	4	3	1
$n_{19,j}, n_{20,j}, n_{21,j}, n_{22,j}$	1	2	3	4	4	5	6	5	4	3	1
$n_{23,j}, n_{24,j}, n_{25,j}, n_{26,j}$	1	2	2	2	2	2	2	2	2	2	1
$n_{27,j}, n_{28,j}$	1	2	2	4	4	4	6	6	4	3	1

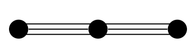
(ii) *Suppose $G \cong G_2(3)$ and Γ has diagram*



and let $\gamma' \in \Delta_{13}(\gamma_0)$. Set $n_j = |\overline{\{\gamma_0, \gamma'\}} \cap \Delta_j(\gamma_0)|$. Then:

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
n_j	1	6	15	23	24	26	25	25	26	24	23	15	6	1

(iii) *Suppose $G \cong G_2(3)$ and Γ has diagram*



and let $\gamma_i \in \Delta_{12}(\gamma_0)$, $i = 1, \ldots, 14$ be B-orbit representatives of $\Delta_{12}(\gamma_0)$. Set

$n_{i,j} = |\overline{\{\gamma_0, \gamma_i\}} \cap \Delta_j(\gamma_0)|$. *Then*:

j	0	1	2	3	4	5	6	7	8	9	10	11	12
$n_{1,j}, n_{2,j}$	1	3	6	9	9	10	12	10	9	9	6	3	1
$n_{3,j}, n_{4,j}$	1	5	9	13	13	13	18	13	13	13	9	5	1
$n_{5,j}, n_{6,j}$	1	6	14	17	25	29	26	29	25	17	14	6	1
$n_{7,j}, n_{8,j}$	1	3	5	6	6	7	7	8	7	7	5	3	1
$n_{9,j}, n_{10,j}$	1	5	12	15	18	18	16	18	18	15	12	5	1
$n_{11,j}, n_{12,j}$	1	3	5	7	7	8	7	7	6	6	5	3	1
$n_{13,j}, n_{14,j}$	1	5	8	12	12	13	16	13	12	12	8	5	1

Apartments of GABs associated with $U_6(2)$ and $\Omega_8^+(2)$. The GAB's for $U_6(2)$ and $\Omega_8^+(2)$ possesses apartments (see [Kantor 1981]), viewed as the fixed chambers of T. For $U_6(2)$ we take T to be a cyclic group of order 4, and for $\Omega_8^+(2)$ we take T to be an elementary abelian group order 4, see [Kantor 1981]. In both cases the apartments are isomorphic and have diameter 8. They also have the property that the distance between any two chambers in the apartment (as measured in the apartment) is the same as in the chamber graph. So this is something one expects from a building. However, for $\Omega_8^+(2)$ the diameter of its chamber graph is 9, so not equal to the diameter of the apartment — unlike the situation in a building.

Theorem 2.2. *Suppose $G \cong \Omega_8^+(2)$, let Γ denote a GAB associated to G. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let γ_0 be a fixed chamber of $\mathcal{C}$. Put $B = \mathrm{Stab}_G(\gamma_0)$.*

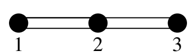
An apartment, $\mathcal{A}$, of Γ containing γ_0 cuts the discs as follows.

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8	9
$\lvert\mathcal{A} \cap \Delta_i(\gamma_0)\rvert$	1	3	5	8	11	13	13	8	2	0

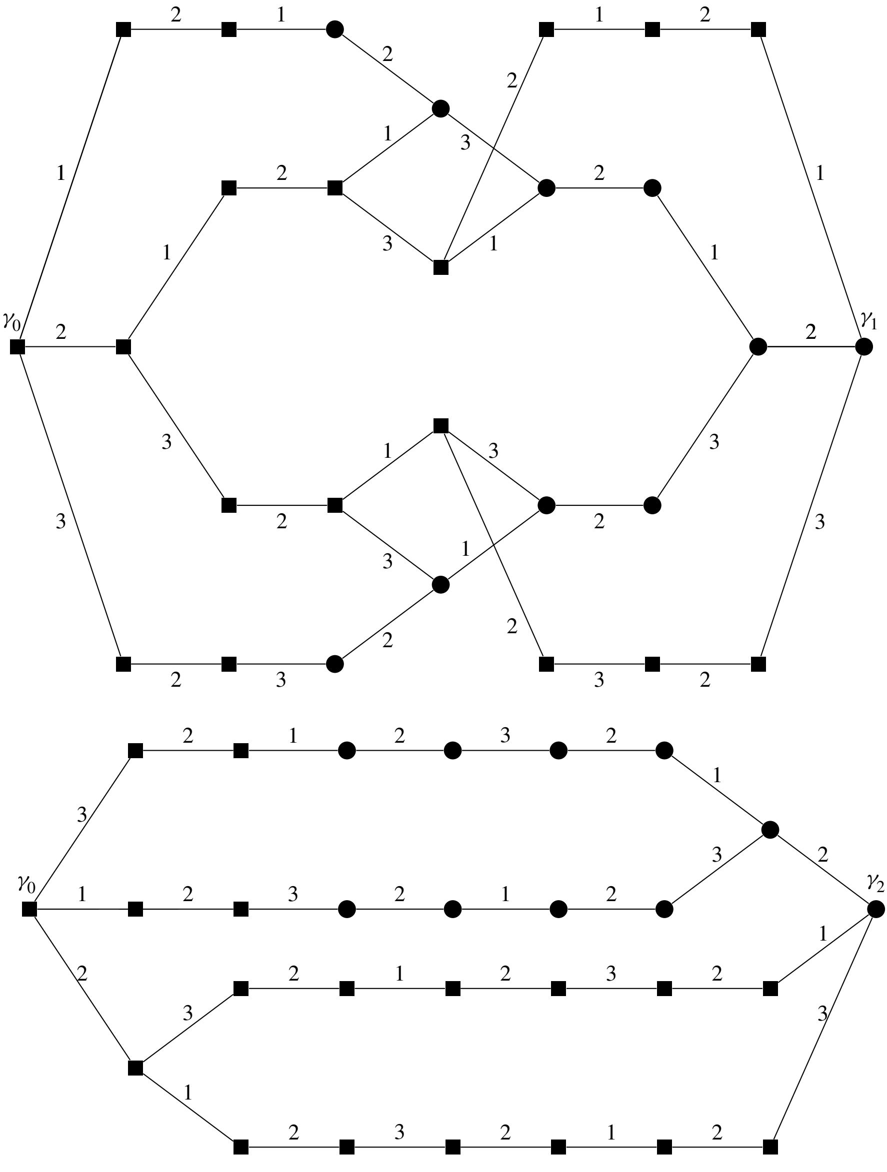
Let $\mathcal{A} \cap \Delta_8(\gamma_0) = \{\gamma_1, \gamma_2\}$. For $j = 1, 2$ the geodesic closure of the γ_0, γ_j cuts the discs as follows.

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8
$\lvert\overline{\{\gamma_0, \gamma_j\}} \cap \Delta_i(\gamma_0)\rvert$	1	3	4	4	4	4	4	3	1

The graphs on the next page are the geodesic closures $\overline{\{\gamma_0, \gamma_1\}}$ and $\overline{\{\gamma_0, \gamma_2\}}$. The type of adjacency between two connected chambers is shown by the labelling on the edges, where



The set of chambers in both geodesic closures are subsets of the apartment. The intersection between $\overline{\{\gamma_0, \gamma_1\}}$ and $\overline{\{\gamma_0, \gamma_2\}}$ has size 18 and the chambers that lie in both geodesic closures are labelled with squares rather than circles.



Geodesic closures (see Theorem 2.2).

Theorem 2.3. *Suppose $G \cong U_6(2)$, and let Γ denote a GAB associated to G. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let γ_0 be a fixed chamber of $\mathcal{C}$. Put $B = \mathrm{Stab}_G(\gamma_0)$.*

An apartment, $\mathcal{A}$, of Γ containing γ_0 cuts the discs as follows.

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8
$\lvert\mathcal{A} \cap \Delta_i(\gamma_0)\rvert$	1	3	5	8	11	13	13	9	1

Let $\mathcal{A} \cap \Delta_8(\gamma_0) = \{\gamma'\}$. The geodesic closure of γ_0, γ' cuts the discs as follows.

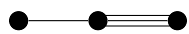
Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8
$\lvert\overline{\{\gamma_0, \gamma'\}} \cap \Delta_i(\gamma_0)\rvert$	1	3	4	4	4	4	4	3	1

The graph for the geodesic closure of the only B-orbit in the last disc of the apartment in the GAB of $U_6(2)$ is identical to the first diagram on page 197.

Again, the set of chambers in the geodesic closure in Theorem 2.3 is a proper subset of the apartment (once more not very building like).

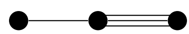
Maximal opposite sets. A maximal opposite set of chambers is a set of chambers of maximal size subject to having the property that any two chambers are opposite to each other, meaning that their distance apart is the diameter of the graph.

Theorem 2.4. *If $G \cong G_2(3)$ and Γ has diagram*



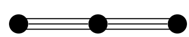
then a maximal opposite set of chambers consists of three chambers.

Proof. Suppose $G \cong G_2(3)$ and Γ has diagram



Since G_{γ_0} is transitive on $\Delta_{13}(\gamma_0)$, we may assume our maximal opposite set contains $\{\gamma_0, \gamma_1\}$, where $\gamma_1 \in \Delta_{13}(\gamma_0)$ is the chamber corresponding to $B * DB[149]$ (the right coset of B containing $DB[149]$). We identify a chamber γ with the triple $\{F_1(\gamma), F_2(\gamma), F_3(\gamma)\}$ which corresponds to a point-line-quad triple. Using the action of B, we determine $\Delta_{13}(\gamma_0)$, and by applying $DB[149]$ to this set we obtain $\Delta_{13}(\gamma_1)$. We can then see that $|\Delta_{13}(\gamma_0) \cap \Delta_{13}(\gamma_1)| = 1$. If we take $\gamma_2 \in \Delta_{13}(\gamma_0) \cap \Delta_{13}(\gamma_1)$ we can see that $|\Delta_{13}(\gamma_0) \cap \Delta_{13}(\gamma_1) \cap \Delta_{13}(\gamma_2)| = 0$, and so $\{\gamma_0, \gamma_1, \gamma_2\}$ is a maximal opposite set. □

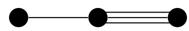
Theorem 2.5. *If $G \cong G_2(3)$ and Γ has diagram*



then each choice of the B-orbits in the last disc gives rise to a maximal opposite set of chambers consisting of four chambers. In particular all maximal opposite sets consist of four chambers.

Proof. We proceed as in Theorem 2.4, starting with γ_0 but then there are 14 possible choices of $\gamma_1 \in \Delta_{12}(\gamma_0)$ (one from each B-orbit in $\Delta_{12}(\gamma_0)$). We give the details for γ_1 being the chamber corresponding to $B * DB[8]$ (the right coset of B containing $DB[8]$). We use MAGMA to calculate $\Delta_{12}(\gamma_1)$ and find that $\Delta_{12}(\gamma_0) \cap \Delta_{12}(\gamma_1)$ is comprised of 21 chambers. One of these 21 chambers, γ_2, has the property that $|\Delta_{12}(\gamma_0) \cap \Delta_{12}(\gamma_1) \cap \Delta_{12}(\gamma_2)| = 2$. Two of the other twenty chambers give rise to an intersection of 1 and the others to 0. Taking γ_3 to be either of the chambers in $\Delta_{12}(\gamma_0)\cap\Delta_{12}(\gamma_1)\cap\Delta_{12}(\gamma_2)$ we find that $\Delta_{12}(\gamma_0)\cap\Delta_{12}(\gamma_1)\cap\Delta_{12}(\gamma_2)\cap\Delta_{12}(\gamma_3)=\varnothing$. Hence γ_1 is contained in a maximal opposite set with four chambers, so proving the theorem. □

Perhaps the most surprising overall result was how unalike the chamber graphs of buildings and the chamber graphs of these GABs appear. In [Carr and Rowley 2018] and [Rowley 2009] all the geometries investigated were in some sense "building like", indeed their chamber graphs had at most two B-orbits in their final disc. The only GAB investigated here displaying this type of behaviour was $G_2(3)$ with diagram



There were also differences by other measures. For the two groups, $\Omega_8^+(2)$ and $U_6(2)$ possessing apartments we found that the geodesic closures were proper subsets of the apartments rather than being equal. Furthermore the apartment of $\Omega_8^+(2)$ did not even span the whole diameter of the chamber graph as it would were it a building.

Perhaps it would be of interest to try and characterise why a limited number of these GABs have so few B-orbits in their last disc while most have so many. Could it be that there is a more unifying lens through which to view these chamber graphs that would justify the name "geometries that are almost buildings"?

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