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A classification of homogeneous compact Tits geometries of irreducible spherical type, with connected panels and admitting a compact flag-transitive automorphism group acting continuously on the geometry, has been obtained by Kramer and Lytchak (2014; 2019). According to their main result, all such geometries but two are quotients of buildings. The two exceptions are flat geometries of type C_3 and arise from polar actions on the Cayley plane over the division algebra of real octonions. The classification obtained by Kramer and Lytchak does not contain the claim that those two exceptional geometries are simply connected, but this holds true, as proved by Schillewaert and Struyve (2017). Their proof is of topological nature and relies on the main result of (Kramer and Lytchak 2014; 2019). In this paper we provide a combinatorial proof of that claim, independent of (Kramer and Lytchak 2014; 2019).

1. Introduction

We presume that the reader has some knowledge of diagram geometry, in particular Tits geometries, namely geometries belonging to Coxeter diagrams, and buildings. A celebrated theorem of Tits [1981] states that Tits geometries generally come from buildings. Explicitly, a Tits geometry of rank $n \geq 3$ is 2-covered by a building if and only if all of its residues of type C_3 or H_3 are covered by buildings; moreover, buildings of rank $n \geq 3$ are 2-simply connected.

Having mentioned coverings and simple connectedness, I recall that, for $1 \leq k \leq n$, a *k-covering* of geometries of rank n is a type-preserving morphism which induces isomorphims on rank k residues (with the convention that an n-covering is just an isomorphism), the domain of a k-covering being called a *k-cover* of the codomain. A geometry is said to be *k-simply connected* if it does not admit any proper k-cover [Pasini 1994, Chapter 12]. (It goes without saying that a k-covering

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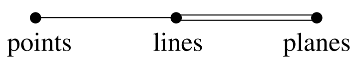
is *proper* if it is not an isomorphism.) I warn that $(n-1)$-coverings are usually called *coverings*, for short (which forbids us from using the word “covering" as a possible abbreviation for k-covering). Accordingly, a geometry of rank n is said to be *simply connected* if it is $(n-1)$-simply connected. In particular, coverings of geometries of rank 3 are 2-coverings and when we say that a geometry of rank 3 is simply connected we just mean it is 2-simply connected.

Turning back to the above theorem of Tits, that theorem shows the importance of the investigation of C_3 geometries. As noticed by Tits [1981], geometries of type C_3 that have no relation at all with buildings can be constructed by some kind of free construction, but more examples exist that are not covered by buildings. Classifying them all is perhaps hopeless. Nevertheless, with the help of some reasonable additional hypotheses, something can be done. For instance, the following is well known [Aschbacher 1984; Yoshiara 1996]:

Theorem 1.1. *There exists a unique flag-transitive finite thick C_3-geometry which is not a building. It is simply connected and its automorphism group is isomorphic to the alternating group* Alt(7).

The exceptional geometry of Theorem 1.1 is called the Alt(7)-*geometry* (also *Neumaier geometry* after its discoverer Neumaier [1984]). Calling the elements of a C_3 geometry *points*, *lines* and *planes* as explained by the picture



the Alt(7)-geometry has 7 points, 35 lines and 15 planes. Moreover, all of its points are incident with all of its planes; therefore, this geometry is *flat*. We refer to [Neumaier 1984] (also [Rees 1985; Pasini 1994, §6.4.2, §12.6.4]) for more details on the Alt(7) geometry.

A number of flag-transitive locally finite (even finite) thick Tits geometries of irreducible type are known that admit the Alt(7)-geometry as a proper residue (see, e.g., [Buekenhout and Pasini 1995, §3] for a survey), but none of them belongs to a diagram of spherical type. Indeed, as proved by Aschbacher [1984], the Alt(7)-geometry cannot occur as a rank-3 residue in any flag-transitive finite thick Tits geometry of irreducible spherical type and rank $n > 3$. Moreover, no finite thick geometry of type H_3 exists (as no finite thick generalized pentagons exist [Feit and Higman 1964]) and no finite thick building of irreducible type and rank at least 3 admits proper quotients [Brouwer and Cohen 1983]. Consequently:

Corollary 1.2. *Apart from the* Alt(7)-*geometry, all flag-transitive finite thick Tits geometries of irreducible spherical type are buildings.*

Results in the same vein as Theorem 1.1 and Corollary 1.2 have recently been obtained by Kramer and Lytchak [2014; 2019] for compact Tits geometries with connected panels admitting a flag-transitive and compact group of automorhisms acting continuously on Γ. Before reporting on those results, I must explain what a compact geometry is and what we mean when saying that it admits connected panels.

Let Γ be a geometry over a (finite) set of types I. Assume that for every $i \in I$ a compact Hausdorff topology is given on the set Γ_i of i-elements of Γ and let $\mathcal{V}_i$ be the topological space thus defined on Γ_i. For every $J \subseteq I$ the set Γ_J of J-flags of Γ is a subspace, say $\mathcal{V}_J$, of the product space $\prod_{j \in J} \mathcal{V}_j$. If $\mathcal{V}_J$ is closed (equivalently, compact) for every $J \subseteq I$, then Γ is said to be a *compact geometry*. (We warn that this definition is not literally the same as in [Kramer and Lytchak 2014, §2.1], but it is equivalent to it; see Remark 1.7 below.) When saying that Γ has *connected panels* we mean that, for every type $i \in I$, the i-panels of Γ are connected as subspaces of $\mathcal{V}_i$ (or of $\mathcal{V}_I$, if we regard panels as sets of chambers).

With Γ a compact geometry as defined above, let G be a flag-transitive group of type-preserving automorphisms of Γ. Suppose that G is a locally compact topological group (we recall that for topological groups local compactness entails Hausdorff, by convention) and that G acts continuously on $\mathcal{V}_i$ for every $i \in I$ (explicitly, the function $\rho : G \times \mathcal{V}_i \to \mathcal{V}_i$ that maps $(g, x) \in G \times \mathcal{V}_i$ onto $g(x) \in \mathcal{V}_i$ is continuous). Then the pair (Γ, G) is called a *homogeneous compact geometry* [Kramer and Lytchak 2014, §2.1]. We call Γ and G the *geometric support* and the *group* of (Γ, G).

If (Γ, G) is a homogeneous compact geometry, then G also acts continuously on $\mathcal{V}_J$ for every $J \subseteq I$. Consequently, for every flag $X \in \Gamma_J$, the stabilizer G_X of X in G is closed in G (recall that, as $\mathcal{V}_J$ is Hausdorff, the singleton $\{X\}$ is closed in $\mathcal{V}_J$). The function $\rho_X : G/G_X \to \mathcal{V}_J$ which maps every coset gG_X onto the flag $g(X) \in \mathcal{V}_J$ is a continuous bijection from the coset space G/G_X to $\mathcal{V}_J$. If moreover G/G_X is compact (which is obviously the case when G is compact), then ρ_X is a homeomorphism. Indeed every continuous bijective mapping from a compact space to a Hausdorff space is a homeomorphism.

Conversely, without assuming any topology on the sets Γ_i, let G be a flag-transitive automorphism group of Γ carrying the structure of a locally compact group such that G_X is closed and G/G_X is compact for every flag X of Γ. Note that, as G is Hausdorff and G_X is closed, the coset space G/G_X is Hausdorff (see, e.g., [Freudenthal and de Vries 1969, §4.8]). For every $i \in I$ and chosen $x \in \Gamma_i$, we can copy the topology of G/G_x on Γ_i via the bijection $\rho_x : G/G_x \to \Gamma_i$, thus defining a compact Hausdorff space $\mathcal{V}_i$ on Γ_i. As $G/G_x \approx G/G_y$ for any two elements $x, y \in \Gamma_i$, the space $\mathcal{V}_i$ does not depend on the particular choice $x \in \Gamma_x$. The group G acts continuously on the space $\mathcal{V}_i$. Thus, Γ is turned into a compact geometry and (Γ, G) is a homogeneous compact geometry. By the previous paragraph, we also have $G/G_X \approx \mathcal{V}_J$ for any $J \subseteq I$ and any flag $X \in \mathcal{V}_J$.

In this way, as noticed in [Kramer and Lytchak 2014], one can see that all buildings of spherical type associated to semisimple or reductive isotropic algebraic groups defined over local fields are (geometric supports of) homogeneous compact geometries.

We add one more definition and a few conventions. Given two homogeneous compact geometries $(\widetilde{\Gamma}, \widetilde{G})$ and (Γ, G) of rank $n \geq 2$ with compact groups $\widetilde{G}$ and G, a *compact covering* from $(\widetilde{\Gamma}, \widetilde{G})$ to (Γ, G) is a 2-covering $\gamma : \widetilde{\Gamma} \to \Gamma$ such that γ is continuous as a mapping from the space $\widetilde{\mathcal{V}}$ of elements of $\widetilde{\Gamma}$ to the space $\mathcal{V}$ of elements of Γ, the group $\widetilde{G}$ normalizes the deck group D of γ and γ induces a continuous isomorphism from the topological group $\widetilde{G}/\widetilde{G} \cap D$ to the topological group G. Clearly, $\widetilde{G} \cap D$ is compact.

The category of homogeneous compact geometries with compact groups and compact coverings as morphisms is named **HCG** in [Kramer and Lytchak 2014]. We have defined compact coverings only for homogenous compact geometries with compact groups since these are the objects of **HCG**. According to this restriction, when we say that a given homogeneous compact geometry (Γ, G) with G compact is compactly covered by another homogeneous compact geometry $(\widetilde{\Gamma}, \widetilde{G})$, it must be understood that $\widetilde{G}$ too is compact.

We warn the reader that the name "compact covering" is not used in [Kramer and Lytchak 2014]. We have introduced it with the hope that it can remind the reader of the objects and the morphisms of the category **HCG**.

We say that a homogeneous compact geometry is a Tits geometry (in particular, a building) if its geometric support is a Tits geometry (a building). Accordingly, when saying that a homogeneous compact geometry with compact group is compactly covered by a building, we mean that it is compactly covered by a homogenous compact geometry, the geometric support of which is a building. It goes without saying that, when speaking of coverings of geometric supports, we mean coverings in the usual "combinatorial" sense, recalled at the beginning of this Introduction.

More generally, when we say that (Γ, G) has some geometric property which neither refers to the topology of Γ nor to the group G (such as being a flat C_3-geometry, for instance) we mean that the geometric support Γ of (Γ, G) has that property as a diagram geometry.

We are now ready to state the main result of Kramer and Lytchak [2014; 2019].

Theorem 1.3. *Let* (Γ, G) *be a homogeneous compact Tits geometry of type* C_3 *with connected panels and compact group* G*. Then either* (Γ, G) *is compactly covered by a building or it is one of two exceptional flat geometries where* G *is either* $((\mathrm{SU}(3) \times \mathrm{SU}(3))/C_3) \rtimes C_2$ *or* $\mathrm{SO}(3) \times \mathrm{G}_2$, *respectively*, *in its polar action* *on the Cayley plane of real octonions. Moreover, the geometric supports of these two exceptional geometries are not covered by any building.*

It is convenient to have a name for the two exceptional geometries mentioned in Theorem 1.3. We shall call them $\mathbb{O}\mathrm{P}^2$-*geometries* where $\mathbb{O}$ stands for the octonion algebra over the reals and $\mathbb{O}\mathrm{P}^2$ is the Cayley plane, namely the projective plane over $\mathbb{O}$.

By exploiting Theorem 1.3, Kramer and Lytchak [2014; 2019] also obtain:

Corollary 1.4. *Apart from the two* $\mathbb{O}\mathrm{P}^2$*-geometries, all homogeneous compact Tits geometries of irreducible spherical type, rank at least* 2, *with connected panels and compact group, are compactly covered by buildings.*

The two $\mathbb{O}\mathrm{P}^2$-geometries, or rather the group actions giving rise to them, were first discovered by Podestà and Thorbergsson [1999] and Gorodski and Kollross [2016], in the context of an investigation of polar actions of Lie groups on symmetric spaces. A purely algebraic construction of (the geometric supports of) these two geometries is given by Schillewaert and Struyve [2017]. We shall report on that construction in the next section.

Let (Γ, G) be any of the two $\mathbb{O}\mathrm{P}^2$-geometries. The reader should be warned that in the final part of Theorem 1.3 it is not claimed that Γ is simply connected. It is only stated that the universal cover $\widetilde{\Gamma}$ of Γ is not a building. Thus, in view of the rest of the statement of Theorem 1.3, if $\widetilde{\Gamma} \neq \Gamma$, then either $\widetilde{\Gamma}$ is not the geometric support of any homogeneous compact geometry with compact group or, if it is such, no compact covering exists from that homogeneous compact geometry to (Γ, G). So, it is natural to ask if Γ is simply connected. The following theorem, due to Schillewaert and Struyve [2017], answers this question in the affirmative.

Theorem 1.5. *The geometric support of either of the two* $\mathbb{O}\mathrm{P}^2$*-geometries is simply connected.*

The proof that Schillewaert and Struyve give for this theorem is of topological nature. They prove that, if (Γ, G) is any of the two $\mathbb{O}\mathrm{P}^2$-geometries, then the universal cover $\widetilde{\Gamma}$ of Γ carries a compact Hausdorff topology and G lifts to a compact group $\widetilde{G} \leq \mathrm{Aut}(\widetilde{\Gamma})$, so that $(\widetilde{\Gamma}, \widetilde{G})$ is a compact cover of (Γ, G). Having proved this, the conclusion follows from Theorem 1.3: necessarily $\widetilde{\Gamma} = \Gamma$. However, Schillewaert and Struyve [2017] also collect a great deal of information of combinatorial nature on homotopies of closed paths of the two $\mathbb{O}\mathrm{P}^2$-geometries. In this paper we shall exploit that information to arrange a combinatorial proof of Theorem 1.5, with no use of [Kramer and Lytchak 2014] or [2019].

Remark 1.6. As the title of [Kramer and Lytchak 2019] makes clear, an error occurs in [2014]: the $\mathbb{O}\mathrm{P}^2$-geometry associated to $\mathrm{SO}(3) \times \mathrm{G}_2$ is missing in [2014]. That gap is filled in [2019].

Remark 1.7. In the definition of compact geometry as stated in [Kramer and Lytchak 2014, §2.1], a compact Hausdorff topology $\mathcal{V}$ is assumed on the set of elements of Γ such that for every $J \subseteq I$ the set Γ_J is closed in the power space $\mathcal{V}^J$. In particular, Γ_i is closed in $\mathcal{V}$ for every $i \in I$. So, $\{\Gamma_i\}_{i\in I}$ is a finite partition of $\mathcal{V}$ in closed sets. Accordingly, $\mathcal{V}$ is the "free" union of the spaces $\mathcal{V}_i$ induced by $\mathcal{V}$ on the sets Γ_i for $i \in I$, the open sets of $\mathcal{V}$ being just the unions $\bigcup_{i\in I} A_i$ with A_i open in $\mathcal{V}_i$. Clearly, $\mathcal{V}^J$ and its subspace $\prod_{j\in J} \mathcal{V}_j$ induce the same topology on Γ_J. Thus, we can forget about $\mathcal{V}$ and start from a compact Hausdorff space $\mathcal{V}_i$ defined on Γ_i for each $i \in I$, as we have done in our definition.

2. The two $\mathbb{O}\mathrm{P}^2$-geometries

A description of the two $\mathbb{O}\mathrm{P}^2$-geometries as coset geometries is given by Kramer and Lytchak [2014] (for the geometry with group $G = (\mathrm{SU}(3) \times \mathrm{SU}(3))/C_3 \rtimes C_2$) and in [2019] (for $G = \mathrm{SO}(3) \times \mathrm{G}_2$). On the other hand, Schillewaert and Struyve [2017] propose a purely algebraic construction for these geometries, which we are going to recall in this section.

2A. ***Algebraic background.*** Let $\mathbb{A}$ be a division algebra over the field $\mathbb{R}$ of real numbers. It is well known that $\mathbb{A}$ has dimension 1, 2, 4 or 8 over $\mathbb{R}$. Accordingly, $\mathbb{A}$ is either $\mathbb{R}$ itself or the field $\mathbb{C}$ of complex numbers or the division ring $\mathbb{H}$ or real quaternions or the Cayley–Dickson algebra $\mathbb{O}$ of real octonions. In any case, $\mathbb{A}$ comes with a *norm* $|\cdot| : \mathbb{A} \to \mathbb{R}$ and a *conjugation* $\bar{\cdot} : \mathbb{A} \to \mathbb{A}$.

Explicitly, when $\mathbb{A} = \mathbb{R}$, then $|\cdot|$ is the usual absolute value and $\bar{\cdot}$ is the identity; if $\mathbb{A} = \mathbb{C}$, then $|\cdot|$ and $\bar{\cdot}$ are the usual modulus and conjugation. When $\mathbb{A} = \mathbb{H}$, then $\mathbb{A}$ can also be regarded as a right $\mathbb{C}$-vector space with canonical basis $\{1, \boldsymbol{j}\}$. The $\mathbb{C}$-span $\mathbb{C} = 1 \cdot \mathbb{C}$ of 1 is a subring of $\mathbb{H}$, $\boldsymbol{j}^2 = -1$ and $x\boldsymbol{j} = \boldsymbol{j}\bar{x}$ for any $x \in \mathbb{C}$. The norm and the conjugation of $\mathbb{H}$ map $x + \boldsymbol{j}y$ onto $\sqrt{|x|^2 + |y|^2}$ and $\bar{x} - \boldsymbol{j}y$, respectively. The conjugation of $\mathbb{H}$ is an involutory antiautomorphism. Clearly, $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{j}\boldsymbol{i}\}$ is a basis of $\mathbb{H}$ over $\mathbb{R}$ (the canonical one), where $\boldsymbol{i}$ stands for any of the two square roots of -1 in $\mathbb{C}$.

Finally, $\mathbb{O}$ contains $\mathbb{H}$ as a subring and is generated by $\mathbb{H}$ together with an extra element $\boldsymbol{k}$ such that $\boldsymbol{k}^2 = -1$ and

$$u\boldsymbol{k} = \boldsymbol{k}\bar{u} \quad \text{for } u \in \mathbb{H}, \tag{1}$$

where $\bar{\cdot}$ denotes the conjugation in $\mathbb{H}$ as defined above. Moreover,

$$(\boldsymbol{k}u)v = \boldsymbol{k}(vu) = \bar{v}(\boldsymbol{k}u) \quad \text{and} \quad (\boldsymbol{k}u)(\boldsymbol{k}v) = -v\bar{u} \quad \text{for all } u, v \in \mathbb{H}. \tag{2}$$

Conditions (2) imply $(uv)\boldsymbol{k} = v(u\boldsymbol{k}) = v(\boldsymbol{k}\bar{u})$. Jointly with (1) they also imply that the elements of $\mathbb{O}$ admit the representation

$$u + \boldsymbol{k}v \quad \text{for } u, v \in \mathbb{H}. \tag{3}$$

In spite of (3), the multiplication of $\mathbb{O}$ does not yield an $\mathbb{H}$-vector space on $\mathbb{O}$, as it follows from the first equality of (2) and the fact that $\mathbb{H}$ is noncommutative. More precisely, $\mathbb{O}$ does carry an $\mathbb{H}$-vector space structure, as is clear from (3), but the scalar multiplication of that space is not the multiplication of $\mathbb{O}$ restricted to $\mathbb{O} \times \mathbb{H}$. On the other hand, for $x, y \in \mathbb{C}$ we have

$$\begin{aligned}(\boldsymbol{k}x)y &= \boldsymbol{k}(yx) = \boldsymbol{k}(xy),\\ (\boldsymbol{k}\boldsymbol{j}x)y &= (\boldsymbol{k}(x\boldsymbol{j}))y = \boldsymbol{k}(y(x\boldsymbol{j})) = \boldsymbol{k}((yx)\boldsymbol{j}) = (\boldsymbol{k}\boldsymbol{j})(yx) = (\boldsymbol{k}\boldsymbol{j})(xy).\end{aligned}$$

So, the multiplication of $\mathbb{O}$ restricted to $\mathbb{O} \times \mathbb{C}$ defines a 4-dimensional $\mathbb{C}$-vector space on $\mathbb{O}$, with $\{1, \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{kj}\}$ as the canonical basis. Needless to say, $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{ji}, \boldsymbol{k}, \boldsymbol{ki}, \boldsymbol{kj}, \boldsymbol{k}(\boldsymbol{ji})\}$ is a basis of $\mathbb{O}$ over $\mathbb{R}$ (the canonical one).

The norm and the conjugation of $\mathbb{O}$ map $u + \boldsymbol{k}v$ onto $\sqrt{|u|^2 + |v|^2}$ and $\bar{u} - \boldsymbol{k}v$, respectively. The conjugation of $\mathbb{O}$ is an involutory antiautomorphism.

In any case, the norm of $\mathbb{A}$ induces a positive definite $\mathbb{R}$-bilinear form $(\cdot \mid \cdot)_{\mathbb{R}}$ which maps $(x, y) \in \mathbb{A} \times \mathbb{A}$ onto the real part $\mathrm{Re}(\bar{x}y)$ of the product $\bar{x}y$. Clearly, $|x| = \sqrt{(x, x)_{\mathbb{R}}}$. We denote by $\perp_{\mathbb{R}} K$ the orthogonal complement of a subspace K of $\mathbb{A}$ with respect to $(\cdot \mid \cdot)_{\mathbb{R}}$.

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, with $\mathbb{F} = \mathbb{R}$ when $\mathbb{A} = \mathbb{R}$. Regarding $\mathbb{F}$ as a subfield of $\mathbb{A}$ in the usual way, namely as the $\mathbb{F}$-span of 1, we set $\mathrm{Pu}_{\mathbb{F}}(\mathbb{A}) := \perp_{\mathbb{R}} \mathbb{F}$ (in particular, $\mathrm{Pu}_{\mathbb{F}}(\mathrm{A}) = 0$ when $\mathbb{A} = \mathbb{F}$). Clearly, $\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ is a subspace of the $\mathbb{F}$-vector space $\mathbb{A}$ and $\mathbb{A} = \mathbb{F} \oplus \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$. The elements of $\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ are said to be $\mathbb{F}$-*pure*.

As $\mathbb{A} = \mathbb{F} \oplus \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$, every element $x \in \mathbb{A}$ splits in a unique way as a sum $x = x_1 + x_2$ with $x_1 \in \mathbb{F}$ and $x_2 \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$. We call x_1 and x_2 the $\mathbb{F}$-*part* and the $\mathbb{F}$-*pure part* of x.

When $\mathbb{F} = \mathbb{C}$ we also define a Hermitian inner product $(\cdot \mid \cdot)_{\mathbb{C}} : \mathbb{A} \times \mathbb{A} \to \mathbb{C}$ by taking $(x \mid y)_{\mathbb{C}}$ equal to the complex part of $\bar{x}y$. Obviously, $\mathrm{Re}((x \mid y)_{\mathbb{C}}) = (x \mid y)_{\mathbb{R}}$. Hence, we also have $|x| = \sqrt{(x \mid x)_{\mathbb{C}}}$ for every $x \in \mathbb{A}$.

The elements of $\mathbb{A}$ of norm 1 are called *unit elements*. Clearly, the set $\mathrm{Un}(\mathbb{A})$ of unit elements of $\mathbb{A}$ is closed under multiplication and taking inverses in $\mathbb{A}$ and

$$\mathbb{A} = \mathrm{Un}(\mathbb{A}) \cdot |\mathbb{R}| := \{x \cdot |t| \mid x \in \mathrm{Un}(\mathbb{A}),\ t \in \mathbb{R}\}.$$

We recall that a homomorphism of $\mathbb{F}$-algebras is an $\mathbb{F}$-linear mapping which also preserves multiplication. In the sequel we shall deal with a particular class of homorphisms of $\mathbb{F}$-algebras, which we shall call sharp $\mathbb{F}$-morphisms. We define them as follows:

Definition 2.1. With $\mathbb{F}$ equal to $\mathbb{R}$ or $\mathbb{C}$, let $\mathbb{A}$ and let $\mathbb{B}$ be two division algebras over $\mathbb{R}$ containing $\mathbb{F}$. When $\mathbb{F} = \mathbb{C}$ both $\mathbb{A}$ and $\mathbb{B}$ can also be regarded as algebras over $\mathbb{C}$. Thus, in any case, both $\mathbb{A}$ and $\mathbb{B}$ are $\mathbb{F}$-algebras.

A *sharp* $\mathbb{F}$*-morphism* from $\mathbb{A}$ to $\mathbb{B}$ is a homomorphism of $\mathbb{F}$-algebras from $\mathbb{A}$ to $\mathbb{B}$ which also preserves the inner product $(\cdot \mid \cdot)_{\mathbb{F}}$.

Let $\phi : \mathbb{A} \to \mathbb{B}$ be a sharp $\mathbb{F}$-morphism. Then ϕ is injective, since it preserves $(\cdot \mid \cdot)_{\mathbb{F}}$. Consequently, $\phi(1) = 1$; hence, $\phi(\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})) \subseteq \mathrm{Pu}_{\mathbb{F}}(\mathbb{B})$. Moreover, $\phi(\mathrm{Un}(\mathbb{A})) \subseteq \mathrm{Un}(\mathbb{B})$. We have $\bar{x} = x^{-1}$ for every unit element x. Therefore, $\phi(\bar{x}) = \overline{\phi(x)}$ for every $x \in \mathrm{Un}(\mathbb{A})$. Finally, ϕ also preserves conjugation.

As sharp $\mathbb{F}$-morphisms are injective, every sharp $\mathbb{F}$-morphism from $\mathbb{A}$ to $\mathbb{A}$ is an automorphism. We call it a *sharp* $\mathbb{F}$*-automorphism*.

Setting 2.2. From now on we assume that $\mathbb{A}$ and $\mathbb{F}$ are as follows: either $\mathbb{A} = \mathbb{H}$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{A} = \mathbb{O}$ and $\mathbb{F} = \mathbb{C}$.

The following is proved in [Schillewaert and Struyve 2017, Proposition 2.1]:

Lemma 2.3. *With $\mathbb{F}$ and $\mathbb{A}$ as in Setting 2.2, let $a_1, a_2 \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ and $b_1, b_2 \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{B})$ be such that $(a_1 \mid a_2)_{\mathbb{F}} = (b_1 \mid b_2)_{\mathbb{F}}$, $|a_i| = |b_i|$ for $i = 1, 2$ and $a_1\mathbb{F} \neq a_2\mathbb{F}$. Then there exists a unique sharp $\mathbb{F}$-morphism from $\mathbb{A}$ to $\mathbb{O}$ mapping a_i onto b_i for $i = 1, 2$.*

Lemma 2.4. *Every sharp $\mathbb{R}$-morphism from $\mathbb{H}$ to $\mathbb{O}$ can be extended to a sharp $\mathbb{R}$-automorphism of $\mathbb{O}$.*

Proof. Let $\phi : \mathbb{H} \to \mathbb{O}$ be a sharp $\mathbb{R}$-morphism. Put $\boldsymbol{i}' := \phi(\boldsymbol{i})$ and $\boldsymbol{j}' := \phi(\boldsymbol{i})$ and recall that $\phi(1) = 1$. Then $\phi(\mathbb{H})$ is the $\mathbb{R}$-span $\mathbb{H}' := \langle 1, \boldsymbol{i}', \boldsymbol{j}', \boldsymbol{j}'\boldsymbol{i}' \rangle_{\mathbb{R}}$ of $\{1, \boldsymbol{i}', \boldsymbol{j}', \boldsymbol{j}'\boldsymbol{i}'\}$ and ϕ is a sharp $\mathbb{R}$-isomorphism from $\mathbb{H}$ to $\mathbb{H}'$. We can construct a copy $\mathbb{O}'$ of $\mathbb{O}$ starting from $\mathbb{H}'$ instead of $\mathbb{H}$, and if $\boldsymbol{k}'$ is the element of $\mathbb{O}'$ corresponding to $\boldsymbol{k}$, a sharp $\mathbb{R}$-isomorphism $\psi : \mathbb{O} \to \mathbb{O}'$ is uniquely determined which maps $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ onto $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$, respectively, which coincides with ϕ in $\mathbb{H}$. If we can choose $\boldsymbol{k}' \in \mathbb{O}$, then ψ can also be regarded as a sharp $\mathbb{F}$-automorphism of $\mathbb{O}$ and we are done.

So it remains to prove that we can choose $\boldsymbol{k}' \in \mathbb{O}$, namely $\mathbb{O}$ contains an element $\boldsymbol{k}'$ orthogonal to $\mathbb{H}$ and such that $(\boldsymbol{k}')^2 = -1$. But this is obvious. Indeed every unit element orthogonal to $\mathbb{H}$ has this property. The conclusion follows. □

2B. *Construction of the geometries.* With $\mathbb{A}$ and $\mathbb{F}$ as in Setting 2.2, let $\mathrm{PG}(\mathbb{A})$ be the projective space of the $\mathbb{F}$-vector space $\mathbb{A}$. For every nonzero vector $x \in \mathbb{A}$, we denote by $[x]$ the corresponding point of $\mathrm{PG}(\mathbb{A})$, and for every subset X of $\mathbb{A}$ we put $[X] := \{[x] \mid x \in X \setminus \{0\}\}$. In particular, if X is a subspace of $\mathbb{A}$, then $[X]$ is the corresponding subspace of $\mathrm{PG}(\mathbb{A})$.

We write $(\cdot \mid \cdot)$ instead of $(\cdot \mid \cdot)_{\mathbb{F}}$ and $\perp$ instead of $\perp_{\mathbb{F}}$, for short. As usual, $\mathbb{F}^*$ stands for the multiplicative group of $\mathbb{F}$. Following Schillewaert and Struyve [2017], we construct a C_3-geometry $\Gamma_{\mathbb{F}}(\mathbb{A})$ as follows.

Definition 2.5. The elements (points, lines and planes) of $\Gamma_{\mathbb{F}}(\mathbb{A})$ are defined as follows:

(A1) The *points* are the points of $[\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})]$.

(A2) The *lines* are the sets of pairs $[x, u] := \{(xt, ut) \mid t \in \mathbb{F}^*\}$ with $x \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$, $u \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{O})$ and $|x| = |u| \neq 0$.

(A3) The *planes* are the sharp $\mathbb{F}$-morphisms $\phi : \mathbb{A} \to \mathbb{O}$.

The *incidence relation* is defined as follows:

(B1) Every point is incident with all planes.

(B2) A line $[x, u]$ and a point $[y]$ are declared to be incident when $y \in x^{\perp}$.

(B3) A line $[x, u]$ and a plane $\phi : \mathbb{A} \to \mathbb{O}$ are incident precisely when $\phi(x) = u$.

Clearly, the conditions defining point-line and line-plane incidences do not depend on the particular choice of the pair $(x, u) \in [x, u]$. It is proved in [Schillewaert and Struyve 2017, Proposition 4.3] that $\Gamma_{\mathbb{F}}(\mathbb{A})$ is indeed a C_3-geometry. According to clause (B1) of Definition 2.5, this geometry is flat.

Lemma 2.6. *Both the following hold*:

(1) *Two lines* $[x, u]$ *and* $[y, v]$ *are coplanar if and only if* $(x \mid y) = (u \mid v)$. *If this is the case*, *then the unique sharp* $\mathbb{F}$*-morphism* $\phi : \mathbb{A} \to \mathbb{O}$ *such that* $\phi(x) = u$ *and* $\phi(y) = v$ (*see* Lemma 2.3) *is the unique plane incident with both* $[x, u]$ *and* $[y, v]$.

(2) *If two lines have two distinct points in common*, *then they have the same set of points.*

Proof. Claim (1) immediately follows from Lemma 2.3 (see also [Schillewaert and Struyve 2017, Lemma 4.2]). Claim (2) follows from clause (B2) of Definition 2.5 and the fact that $\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ has dimension 3 over $\mathbb{F}$ (see also [Schillewaert and Struyve 2017, Lemma 5.1]). □

The set of points of a line $[x, u]$ is the line $x^{\perp} \cap \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ of $\mathrm{PG}(\mathrm{Pu}_{\mathbb{F}}(\mathbb{A}))$. We call it the *shadow* of $[x, u]$ and also a *shadow-line*. With this terminology, we can rephrase claim (2) of Lemma 2.6 as follows:

Corollary 2.7. *The set of points of* $\Gamma_{\mathbb{F}}(\mathbb{A})$ *equipped with the shadow lines as lines coincides with the projective plane* $\mathrm{PG}(\mathrm{Pu}_{\mathbb{F}}(\mathbb{A}))$.

2C. *Automorphism groups.* Let $\mathrm{Aut}_{\mathbb{F}}(\mathbb{A})$ and $\mathrm{Aut}_{\mathbb{F}}(\mathbb{O})$ be the groups of sharp $\mathbb{F}$-automorphisms of $\mathbb{A}$ and $\mathbb{O}$. The product $\mathrm{Aut}_{\mathbb{F}}(\mathbb{A}) \times \mathrm{Aut}_{\mathbb{F}}(\mathbb{O})$ acts on $\Gamma_{\mathbb{F}}(\mathbb{A})$ as a group of automorphisms. Explicitly, given an element $(\alpha, \omega) \in \mathrm{Aut}_{\mathbb{F}}(\mathbb{A}) \times \mathrm{Aut}_{\mathbb{F}}(\mathbb{O})$,

$$\begin{aligned}
(\alpha, \omega) &: [x] \to [\alpha(x)] && \text{for every point } [x] \text{ of } \Gamma_{\mathbb{F}}(\mathbb{A}),\\
(\alpha, \omega) &: [x, u] \to [\alpha(x), \omega(u)] && \text{for every line } [x, u] \text{ of } \Gamma_{\mathbb{F}}(\mathbb{A}),\\
(\alpha, \omega) &: \phi \to \omega\phi\alpha^{-1} && \text{for every plane } \phi \text{ of } \Gamma_{\mathbb{F}}(\mathbb{A}).
\end{aligned}$$

The first questions one may ask are whether this action is faithful and whether all automorphisms of $\Gamma_{\mathbb{R}}(\mathbb{A})$ arise in these way. Both questions are answered by Schillewaert and Struyve [2017], but the answers are different according to whether $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$ or $(\mathbb{F}, \mathbb{A}) = (\mathbb{C}, \mathbb{O})$.

Let $\mathbb{F} = \mathbb{R}$ and $\mathbb{A} = \mathbb{H}$. Then both questions are answered in the affirmative:

$$\mathrm{Aut}(\Gamma_{\mathbb{R}}(\mathbb{H})) = \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}) \times \mathrm{Aut}_{\mathbb{R}}(\mathbb{O}) = \mathrm{SO}(3) \times \mathrm{G}_2.$$

(Recall that $\mathrm{Aut}_{\mathbb{R}}(\mathbb{H}) = \mathrm{SO}(3)$ and $\mathrm{Aut}_{\mathbb{R}}(\mathbb{O}) = \mathrm{G}_2$.) When $\mathbb{F} = \mathbb{C}$ and $\mathbb{A} = \mathbb{O}$ the answer is sligthly different. Indeed $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$ acts nonfaithfully on $\Gamma_{\mathbb{C}}(\mathbb{O})$, with kernel a group C_3 of order 3 contributed by elements (ζ, ζ) with ζ in the center of $\mathrm{SU}(3)$ (recall that $\mathrm{SU}(3) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$). Moreover, the conjugation in $\mathbb{C}$ also induces an automorphism γ of $\Gamma_{\mathbb{C}}(\mathbb{O})$ which, being semilinear as a mapping of $\mathbb{O} \times \mathbb{O}$, does not belong to $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$. All automorphisms of $\Gamma_{\mathbb{C}}(\mathbb{O})$ belong to the group generated by $(\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{Aut}_{\mathbb{C}}(\mathbb{O}))/C_3$ and γ. To sum up,

$$\begin{aligned}
\mathrm{Aut}(\Gamma_{\mathbb{C}}(\mathbb{O})) &= ((\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{Aut}_{\mathbb{C}}(\mathbb{O}))/C_3) \rtimes C_2\\
&= ((\mathrm{SU}(3) \times \mathrm{SU}(3))/C_3) \rtimes C_2.
\end{aligned}$$

2D. ***Recognizing $\Gamma_{\mathbb{F}}(\mathbb{A})$ as an $\mathbb{O}\mathrm{P}^2$-geometry.*** Let $\Gamma := \Gamma_{\mathbb{F}}(\mathbb{A})$ and $G := \mathrm{Aut}(\Gamma)$. As shown by Schillewaert and Struyve [2017, §5], in either of the two cases that we have considered, (Γ, G) is a homogeneous compact geometry. They obtain this conclusion by noticing that in either case G is compact and the stabilizers in G of the flags of Γ are closed in G, but a direct proof is also possible. We shall briefly sketch it here.

In order to stick to the notation used in the Introduction of this paper, let Γ_1, Γ_2 and Γ_3, respectively, be the sets of points, lines and planes of Γ. In either case each of Γ_1, Γ_2 and Γ_3 can be equipped with a natural compact topology.

Explicitly, $\Gamma_1 = [\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})]$ carries the topology of the real projective plane $\mathbb{R}\mathrm{P}^2$ when $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$ and the topology of the complex projective plane $\mathbb{C}\mathrm{P}^2$ when $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$. Either of these spaces is both Hausdorff and compact.

When $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$, the line-set Γ_2 carries the topology of the quotient $(\mathbb{S}^2 \times \mathbb{S}^6)/Z$ of the product space $\mathbb{S}^2 \times \mathbb{S}^6 \subset \mathbb{R}^{10}$ over the center Z of $\mathrm{SL}(\mathbb{R}^{10})$. When $(\mathbb{F}, \mathbb{A}) = (\mathbb{C}, \mathbb{O})$ then Γ_2 carries the topology of the quotient $(U \times U)/\Lambda$ where $U := \{x \in \mathbb{C}^3 \mid |x| = 1\}$ is the standard unital of $\mathbb{C}^3$ and Λ is the group of scalar transformations $\lambda \cdot \mathrm{id}$ of $\mathbb{C}^6$ with $|\lambda| = 1$. Again, either of these spaces is Hausdorff and compact.

When $(\mathbb{F}, \mathbb{A}) = (\mathbb{C}, \mathbb{O})$ then Γ_3 carries the same topology as $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) = \mathrm{SU}(3)$, which is (Hausdorff and) compact. Finally, let $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$. Then every sharp $\mathbb{R}$-morphism from $\mathbb{H}$ to $\mathbb{O}$ can be regarded as the restriction of a sharp $\mathbb{R}$-automorphism of $\mathbb{O}$ (Lemma 2.4). Accordingly, the planes of Γ naturally correspond to the cosets ωH of the elementwise stabilizer H of $\mathbb{H}$ in $G := \mathrm{Aut}_{\mathbb{R}}(\mathbb{O}) = \mathrm{G}_2$. The group H is the intersection $H = \bigcap_{x \in \mathbb{H}} G_x$ of the stabilizers G_x for $x \in \mathbb{H}$, which are closed. Hence, H is closed as well. Thus, Γ_3 can be regarded as a copy of the quotient-space G/H, which is still compact and Hausdorff since H is closed.

As in the Introduction, let $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ be the spaces defined on Γ_1, Γ_2 and Γ_3 as above. It is straighforward to check that $\Gamma_{\{i,j\}}$ is closed in $\mathcal{V}_i \times \mathcal{V}_j$ for every choice of $1 \leq i < j \leq 3$ and the set of chambers $\Gamma_{\{1,2,3\}}$ is closed in $\mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3$. So Γ is a compact geometry. Each of the groups $\mathrm{Aut}(\Gamma_{\mathbb{R}}(\mathbb{H})) = \mathrm{SO}(3) \times \mathrm{G}_2$ and $\mathrm{Aut}(\Gamma_{\mathbb{C}}(\mathbb{O})) = ((\mathrm{SU}(3) \times \mathrm{SU}(3))/C_3) \rtimes C_2$ is compact and acts continuously on $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$.

It remains to show that the group G acts flag-transitively on Γ. Clearly, in either case G is transitive on the set of point-line flags of Γ. So in order to prove flag-transitivity, we only must show that the stabilizer in G of a given point-line flag $([u], [v, x])$ of Γ acts transitively on the set of sharp $\mathbb{F}$-morphisms ϕ of Γ such that $\phi(v) = x$. This follows from Lemma 2.4. So:

Result 2.8. *The pair* (Γ, G) *is indeed a homogeneous compact geometry.*

As G acts flag-transitively on Γ, we can recover Γ as a coset-geometry from G, where the flags naturally correspond to the cosets of the stabilizers of the flags contained in a selected chamber of Γ, two flags being incident precisely when the corresponding cosets meet nontrivally (see, e.g., [Tits 1974, §1.4] or [Pasini 1994, §10.1]). Accordingly, Γ is uniquely determined by the complex of the stabilizers in G of the subflags of a chamber of Γ. This complex, as described by Schillewaert and Struyve [2017] for the case $(\mathbb{F}, \mathbb{A}) = (\mathbb{C}, \mathbb{O})$, is the same as computed for G regarded as the automorphism group of the $\mathbb{O}\mathrm{P}^2$-geometry considered in [Kramer and Lytchak 2014] (see also [Schillewaert and Struyve 2017]). Similarly for the case $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$ and the $\mathbb{O}\mathrm{P}^2$-geometry of [Kramer and Lytchak 2019]. So:

Result 2.9. *The C_3-geometries* $\Gamma_{\mathbb{R}}(\mathbb{H})$ *and* $\Gamma_{\mathbb{C}}(\mathbb{O})$ *are the* (*geometric supports of the*) *two* $\mathbb{O}\mathrm{P}^2$*-geometries.*

Remark 2.10. The two cases of Setting 2.2 correspond to the two cases of [Schillewaert and Struyve 2017] with $\mathbb{B} = \mathbb{O}$. Schillewaert and Struyve [2017] also consider one more case, with $\mathbb{F} = \mathbb{R}$ and $\mathbb{A} = \mathbb{B} = \mathbb{H}$, which leads to a flat C_3-geometry which is a quotient of the building associated to the Chevalley group $\mathrm{O}(7, \mathbb{R})$ and admits $\mathrm{SO}(3) \times \mathrm{SO}(3)$ as a flag-transitive automorphism group. This geometry also appears in [Rees 1985, §1.6, (2.2)(ii)] as a member of a larger family of flag-transitive flat C_3-geometries, obtained as quotients from $O(7, K)$-buildings, with K any ordered field. Note that the construction used by Rees [1985] is primarily geometric.

This geometry is indeed worth further investigation, but I have preferred to leave it aside in order to stick to the subject of this paper.

3. A combinatorial proof of Theorem 1.5

3A. ***Preliminaries.*** We follow [Pasini 1994] for basics on diagram geometry. We recall that, according to [Pasini 1994], all geometries are residually connected, by definition. In particular, all geometries of rank at least 2 are connected.

Throughout this subsection Γ is a given geometry of rank $n \geq 2$. Recall that Γ can be regarded as a simplicial complex, where the vertices are the elements of the geometry and the simplices are the flags. Moreover, with $\{1, 2, \ldots, n\}$ chosen as the type-set of Γ, the vertices of the complex are marked by positive integers not greater than n, according to their type as elements of Γ. The incidence graph of Γ is just the skeleton of the complex Γ.

We firstly state some notation and recall a few basics on homotopy of paths. Given two paths $\alpha = (a_0, \ldots, a_k)$ and $\beta = (b_0, \ldots, b_h)$ of Γ with $a_k = b_0$, the *join* of α with β, also called the *product* of α and β, is the path:

$$\alpha \cdot \beta := (a_0, a_1, \ldots, a_k = b_0, b_1, \ldots, b_h).$$

A *null* path is a path of lenght 0. The *opposite* (also called the *inverse*) of a path $\alpha = (a_0, a_1, \ldots, a_k)$ is the path $\alpha^{-1} := (a_k, a_{k-1}, \ldots, a_0)$.

Two paths $\alpha = (a_0, a_1, \ldots, a_k)$ and $\beta = (b_0, b_1, \ldots, n_h)$ with $a_0 = b_0$ and $a_k = b_h$ are said to be *elementarily homotopic* if $\alpha = \gamma \cdot \alpha' \cdot \delta$ and $\beta = \gamma \cdot \beta' \cdot \delta$ for suitable subpaths γ, δ, α' and β' with α' and β' contained in the same simplex (namely flag) of Γ. More generally, two paths α and β are said to be *homotopic* if there exists a sequence $\alpha_0, \alpha_1, \ldots, \alpha_m$ of paths with $\alpha = \alpha_0$, $\beta = \alpha_m$ and such that α_{i-1} and α_i are elementarily homotopic for $i = 1, 2, \ldots, m$.

If α and β are homotopic we write $\alpha \sim \beta$. We say that a closed path α based at a vertex a is *null homotopic* if it is homotopic with the null path (a). Equivalently, α splits in triangles each of which is contained in a simplex and, possibly, paths of the form $\beta \cdot \beta^{-1}$.

Clearly, homotopy is an equivalence relation. We denote by $[\alpha]$ the homotopy class of a path α. Given a vertex a of Γ, the homotopy classes of closed paths of Γ based at a form a group $\pi_1(\Gamma, a)$, with $[(a)]$ as the identity element and multiplication defined as follows: $[\alpha] \cdot [\beta] := [\alpha \cdot \beta]$. The group $\pi_1(\Gamma, a)$ is called the *fundamental group* of Γ *based* at a. As Γ is connected, we have $\pi_1(\Gamma, a) \cong \pi_1(\Gamma, b)$

for any two vertices $a, b \in \Gamma$. Explicitly, for very choice of a path γ from a to b, the mapping

$$[\alpha] \in \pi_1(\Gamma, a) \mapsto [\gamma^{-1} \cdot \alpha \cdot \gamma] \in \pi_1(\Gamma, b)$$

is an isomorphims from $\pi_1(\Gamma, a)$ to $\pi_1(\Gamma, b)$. So, as far as we are interested only in the isomorphism type of $\pi_1(\Gamma, a)$, we are free not to keep a record of the base point a of $\pi_1(\Gamma, a)$ in our notation, thus writing $\pi_1(\Gamma)$ for $\pi_1(\Gamma, a)$ and calling $\pi_1(\Gamma)$ the *fundamental group* of Γ, for short.

It is well known (see, e.g, [Pasini 1994, §12.6.1]) that the geometry Γ is simply connected (namely $(n-1)$-simply connected) if and only it is simply connected as a complex, namely $\pi_1(\Gamma)$ is trivial; equivalently, every closed path is null-homotopic.

Lemma 3.1. *For $1 \le i < j \le n$, let $\Gamma_{i,j}$ be the $\{i, j\}$-truncation of Γ, namely the subgeometry induced by Γ on the set of elements of Γ of type i or j. Then every path of Γ starting and ending at $\Gamma_{i,j}$ (in particular, every closed path based at an element of type i or j) is homotopic to a path of $\Gamma_{i,j}$.*

Proof. Let $\alpha = (a_0, a_1, \dots, a_k)$ be a path of Γ with $a_0, a_k \in F_{i,j}$. We argue by induction on the length k of α. When $k \le 1$ there is nothing to prove. Let $k = 2$. If $a_1 \in \Gamma_{i,j}$ there is nothing to prove as well. Let $a_1 \notin \Gamma_{i,j}$. By the so-called strong connectedness property [Pasini 1994, Theorem 1.18], the intersection $\mathrm{Res}(a_1) \cap \Gamma_{i,j}$ of the residue $\mathrm{Res}(a_1)$ of a_1 with $\Gamma_{i,j}$ contains a path

$$\beta = (b_0 = a_0, b_1, \dots, b_{h-1}, b_h = a_2)$$

from a_0 to a_2. We have $(b_{i-1}, b_i) \sim (b_{i-1}, a_1, b_i)$ for every $i = 1, 2, \dots, h$, since $\{b_{i-1}, a_1, b_i\}$ is a flag. Moreover, $(a_1, b_i, a_1) \sim (a_1)$ for every $i = 1, 2, \dots, h$. Therefore

$$\beta \sim \gamma := (b_0, a_1, b_1, a_1, b_2, \dots, b_{h-1}, a_1, b_h) \sim (b_0, a_1, b_h) = (a_0, a_1, a_2) = \alpha.$$

The claim is proved. Let now $k > 2$. If $a_{k-1} \in \Gamma_{i,j}$ the claim follows by the inductive hypothesis on the subpath $(a_0, a_1, \dots, a_{k-1})$. Let $a_{k-1} \notin \Gamma_{i,j}$. If $a_{k-2} \in \Gamma_{i,j}$ then the conclusion follows by the above on the subpath (a_{k-2}, a_{k-1}, a_k) and the inductive hypothesis on $(a_0, a_1, \dots, a_{k-2})$. Let $a_{k-2} \notin \Gamma_{i,j}$. Then $\mathrm{Res}(a_{k-2}, a_{k-1}) \cap \Gamma_{i,j} \neq \varnothing$, since neither i nor j belong to the type of the flag $\{a_{k-2}, a_{k-1}\}$ and every flag is contained in a chamber. Pick an element $c \in \mathrm{Res}(a_{k-2}, a_{k-1}) \cap \Gamma_{i,j}$ and consider the paths

$$\alpha' := (a_0, a_1, \dots, a_{k-2}, c), \quad \alpha'' := (c, a_{k-1}, a_k).$$

The path α' has length $k-1$. So, by the inductive hypothesis, a path β' exists in $\Gamma_{i,j}$ from a_0 to c such that $\beta' \sim \alpha'$. Similarly, as we have already proved the claim for paths of length 2, a path β'' exists in $\Gamma_{i,j}$ from c to a_k such that $\beta'' \sim \alpha''$. So, $\beta := \beta' \cdot \beta'' \sim \alpha' \cdot \alpha'' \sim \alpha$ is a path of $\Gamma_{i,j}$ with the required properties. □

The following lemma is implicit in [Pasini 1994, Lemma 12.60].

Lemma 3.2. *Given two elements v and w of Γ, let α and β be two paths of Γ from v to w. If an element u exists in Γ such that its residue $\mathrm{Res}(u)$ contains both α and β, then $\alpha \sim \beta$.*

Proof. Let $\alpha = (a_0, a_1, \dots, a_k)$ with $a_0 = v$, $a_k = w$ and $\alpha \subseteq \mathrm{Res}(u)$. For every $i = 1, 2, \dots, k$ put $\alpha_i = (a_{i-1}, u, a_i)$. As $(a_{i-1}, a_i) \sim (a_{i-1}, u, a_i)$ and $(u, a_i, u) \sim (u)$, we have

$$\alpha \sim \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_k = (a_0, u, a_1, u, a_2, \dots, a_{k-1}, u, a_k) \sim (a_0, u, a_k).$$

So, $\alpha \sim (a_0, u, a_k) = (v, u, w)$. Similarly, $\beta \sim (v, u, w)$. Therefore $\alpha \sim \beta$. □

3B. *Peculiar properties of C_3-geometries.* From now on Γ is a geometry of type C_3. The integers 1, 2 and 3 are taken as types and stand for points, lines and planes respectively.

Definition 3.3. A *primitive path* of Γ is a closed path $\alpha := (p, L, q, M, p)$ where p and q are points and L and M lines. If $p = q$ or $L = M$ then α is said to be *degenerate*.

Clearly, degenerate primitive paths are null-homotopic. The following is also well known [Tits 1981, Proposition 9] (see also [Pasini 1994, Corollary 7.39]).

Lemma 3.4. *The geometry Γ is a building if and only if all of its primitive paths are degenerate.*

The proof of the next lemma is implicit in [Schillewaert and Struyve 2017, §6.6]. We make it explicit.

Lemma 3.5. *Every closed path of Γ based at a point is homotopic to a primitive path.*

Proof. Let α be a closed path based at a point p. In view of Lemma 3.1, we may assume that α is contained in $\Gamma_{1,2}$. So, $\alpha = (p_0, L_1, p_1, \dots, L_k, p_k)$ where $p_0 = p_k = p$ and, for $i = 1, \dots, k$, p_i is a point and L_i a line. We argue by induction on k. If $k = 1$ there is nothing to prove. Let $k > 1$. Suppose firstly that L_{i-1} and L_i are coplanar. Let ξ be the plane on L_{i-1} and L_i and let M be the line of $\mathrm{Res}(\xi)$ through p_{i-2} and p_i. Then $(p_{i-2}, L_{i-1}, p_{i-1}, L_i, p_i) \sim (p_{i-2}, M, p_i)$ by Lemma 3.2. Accordingly, $\alpha \sim \alpha' := (p_0, L_1, \dots, p_{i-2}, M, p_i, \dots, L_k, p_k)$. However α', being shorter than α, is homotopic to a primitive path, by the inductive hypothesis. Hence α too is homotopic to a primitive path.

Assume now that L_{i-1} and L_i are never coplanar, for any $i = 2, \dots, k$. Choose a plane ξ_2 on L_2. The residue $\mathrm{Res}(p_1)$ of p_1 contains a unique line-plane flag

(M_1, ξ_1) such that L_1 and M_1 are incident with ξ_1 and ξ_2 respectively. Similarly, $\mathrm{Res}(p_2)$ contains a unique line-plane flag (M_2, ξ_3) such that L_3 and M_2 are incident with ξ_3 and ξ_2 respectively. Let q be the meet-point of M_1 and M_2 in $\mathrm{Res}(\xi_2)$, let M_0 be the line through p_0 and q in $\mathrm{Res}(\xi_1)$ and let M_3 be the line through p_3 and q in $\mathrm{Res}(\xi_3)$. By Lemma 3.2 we have the following homotopies:

$$(p_0, L_1, p_1) \sim (p_0, M_0, q, M_1, p_1),$$
$$(p_1, L_2, p_2) \sim (p_1, M_1, q, M_2, p_2),$$
$$(p_2, L_3, p_3) \sim (p_2, M_2, q, M_3, p_3).$$

Therefore

$$\begin{aligned}(p_0, L_1, p_1, L_2, p_2, L_3, p_3) &= (p_0, L_1, p_1) \cdot (p_1, L_2, p_2) \cdot (p_2, L_3, p_3)\\ &\sim (p_0, M_0, q, M_1, p_1) \cdot (p_1, M_1, q, M_2, p_2) \cdot (p_2, M_2, q, M_3, p_3)\\ &= (p_0, M_0, q, M_1, p_1, M_1, q, M_2, p_2, M_2, q, M_3, p_3)\\ &\sim (p_0, M_0, q, M_3, p_3).\end{aligned}$$

Accordingly, α is homotopic to the path, say β, obtained by replacing the subpath $(p_0, L_1, p_1, L_2, p_2, L_3, p_3)$ of α with (p_0, M_0, q, M_3, p_3). The path β is shorther than α, whence it is homotopic to a primitive path by the inductive hypothesis. As $\alpha \sim \beta$, the same holds for α. □

By Lemma 3.5 we immediately obtain the following:

Corollary 3.6. *The geometry Γ is simply connected if and only if all of its primitive paths are null-homotopic.*

Let $\phi : \widetilde{\Gamma} \to \Gamma$ be the universal covering of Γ. As $\widetilde{\Gamma}$ is simply connected, all of its closed paths (in particular, all of its primitive paths) are null-homotopic. A closed path of Γ is null-homotopic if and only if it lifts through ϕ to a closed path of $\widetilde{\Gamma}$. In particular:

Corollary 3.7. *A primitive path of Γ is null-homotopic if and only if it is the ϕ-image of a primitive path of $\widetilde{\Gamma}$.*

Corollary 3.8. *The geometry Γ is covered by a building if and only if none of its nondegenerate primitive paths is null-homotopic.*

Proof. Let $\widetilde{\Gamma}$ be a building. Then, by Lemma 3.4, no nondegenerate primitive path occurs in $\widetilde{\Gamma}$. By Corollary 3.7, none of the nondegenerate primitive paths of Γ can be null-homotopic. On the other hand, let $\widetilde{\Gamma}$ be not a building. Then $\widetilde{\Gamma}$ admits at least one nondegenerate primitive path $\tilde{\alpha}$, necessarily null-homotopic since $\widetilde{\Gamma}$ is simply connected. Accordingly, $\alpha := \phi(\tilde{\alpha})$ is a null-homotopic nondegenerate primitive path of Γ. □

Definition 3.9. Let $\alpha = (p, L, q, M, p)$ be a nondegenerate primitive path. Recall that $\mathrm{Res}(q)$ is a generalized quadrangle, the lines L and M being points of this quadrangle. So, lines on q exist which are coplanar with each of L and M. Let N be such a line and r a point of N. The line N is different from each of L and M, as L and M are noncoplanar. Let ξ be the plane on N and L and let L' be the line of ξ through p and r. Similarly, if χ is the plane on N and M, let M' be the line of χ through p and r. Then (p, L', r, M', p) is a primitive path. We denote it by $\sigma^N_{q\to r}(\alpha)$ and call it the *shift* of α *from q to r along N*. We also say that N is *admissible* for the path α.

Lemma 3.10. *Let $\alpha = (p, L, q, M, p)$ be a nondegenerate primitive path, N a line admissible for α and r a point of N. Then*:

(1) *We have $\sigma^N_{q\to r}(\alpha) = \alpha$ if and only if $r = q$.*

(2) *The shift $\sigma^N_{q\to r}(\alpha)$ is a nondegenerate primitive path and the line N is admissible for it.*

(3) $\sigma^N_{r\to q}(\sigma^N_{q\to r}(\alpha)) = \alpha$.

(4) $\alpha \sim \sigma^N_{q\to r}(\alpha)$.

Proof. Claims (1), (2) and (3) are trivial. Claim (4) can be proved as follows:

$$\begin{aligned}(p, L, q, M, p) &\sim (p, \xi, L, q, M, \chi, p) \sim (p, \xi, q, \chi, p)\\ &\sim (p, \xi, N, q, N, \chi, p) \sim (p, \xi, N, \chi, p) \sim (p, \xi, N, r, N, \chi, p)\\ &\sim (p, \xi, r, \chi, p) \sim (p, L', \xi, r, \chi, M', p) \sim (p, L', r, M', p).\end{aligned}$$

(This is essentially the same argoment as used by Schillewaert and Struyve to prove Lemma 6.6 of [2017].) □

3C. *Primitive paths in $\mathbb{O}\mathbf{P}^2$-geometries.* Henceforth $\Gamma = \Gamma_{\mathbb{F}}(\mathbb{A})$ (see Section 2B). Recall that the point-line geometry with the same points as Γ and the shadow-lines as lines coincides with $\mathrm{PG}(\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})) \cong \mathrm{PG}(2, \mathbb{F})$ (Corollary 2.7). In particular, two lines of Γ either have just one point in common or have exactly the same points.

Definition 3.11. Let L and M be two lines of Γ with the same shadow, namely $L = [a, u]$ and $M = [b, v]$ for $a, b \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ and $u, v \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{O})$ with $|a| = |u| \neq 0$, $|b| = |v| \neq 0$ and $[a] = [b]$. Suppose we have chosen the pairs (a, u) and (b, v) in such a way that $a = b$, as we can. Then we put $(L \mid M) := (u \mid v)/|u||v|$.

Given a primitive path $\alpha = (p, L, q, M, p)$ we put $\ell(\alpha) := (L \mid M)$ and we call $\ell(\alpha)$ the *line-invariant* of α.

Clearly, $|(L \mid M)| \leq 1$ by Cauchy–Schwartz inequality, with equality if and only if u and v are proportional. Moreover $(L \mid M) = 1$ if and only if $L = M$. So, $\ell(\alpha) \neq 1$ whenever α is nondegenerate.

The hypothesis $a = b$ is necessary for the above definition of $(L \mid M)$ to make sense. Indeed, without it, only the modulus $|(u \mid v)|/|u||v|$ of $(u \mid v)/|u||v|$ is determined by the pair L and M. It is also clear that $(L \mid M)$ can be defined only when L and M have the same shadow. On the other hand, the particular choice of a in the representations $L = [a, u]$ and $M = [a, v]$ is irrelevant. Indeed, if we replace a with $a' = ta$ for some $t \in \mathbb{F} \setminus \{0\}$ then we must also replace u with $u' = tu$ and v with $v' = tv$. Accordingly, $(u' \mid v')/|u'||v'| = |t|^2(u \mid v)/|t^2||u||v| = (u \mid v)/|u||v|$.

Remark 3.12. Schillewaert and Struyve [2017] call $\ell(\alpha)$ the PL-invariant of α.

Definition 3.13. We say that a primitive path $\alpha = (p, L, q, M, p)$ is *orthogonal* if $p \perp q$. Assuming that α is nondegenerate but not that it is orthogonal, an *orthogonal shift* of α is a shift $\sigma^N_{q\to r}(\alpha)$ with $p \perp r$.

Lemma 3.14. *Every nondegenerate primitive path $\alpha = (p, L, q, M, p)$ admits orthogonal shifts along every line N admissible for it and, once N has been chosen, the orhogonal shift of α along N is uniquely determined. Moreover, if α is orthogonal, then α is its own orthogonal shift.*

Proof. As N is coplanar with either of L and M, it has at most one point in common with L or M. However N contains q. Hence it cannot contain p. By Corollary 2.7, the line $p^\perp \cap [\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})]$ of $\mathrm{PG}(\mathrm{Pu}_{\mathbb{F}}(\mathbb{A}))$ meets the shadow of N in just one point. (This argument is the same as in the proof of Lemma 6.6 of [Schillewaert and Struyve 2017].) The first part of the lemma is proved. The last claim of the lemma is obvious. □

Henceforth we denote by $\sigma^N_\perp(\alpha)$ the orthogonal shift of α along a line N admissible for α.

Lemma 3.15. *Given a nonorthogonal nondegenerate primitive path α of Γ and a line N admissible for α, let $\beta = \sigma^N_\perp(\alpha)$ be the orthogonal shift of α along N and let $\ell = \ell(\beta)$ be the line-invariant of β.*

We can always choose the line N in such a way that $\ell \neq -1$.

Proof. We must distinguish two cases and two subcases for each of them.

Case 1. $\Gamma = \Gamma_{\mathbb{R}}(\mathbb{H})$. Modulo automorphisms of Γ, we can always assume that

$$L = [\boldsymbol{j}, \boldsymbol{j}], \quad M = [\boldsymbol{j}, \boldsymbol{i}m_1 + \boldsymbol{j}m_2], \quad m_1^2 + m_2^2 = 1,$$
$$p = [\boldsymbol{i}], \qquad q = [\boldsymbol{i}q_1 + \boldsymbol{ji}q_3], \qquad q_1^2 + q_3^2 = 1.$$

So, $\ell(\alpha) = m_2$. Note that $q_1 \neq 0$ (otherwise $p \perp q$, while α is nonorthogonal by assumption) and $q_3 \neq 0$ (otherwise $p = q$). Let $N = [b, x]$ be admissible for α, where

$$b = \boldsymbol{i}b_1 + \boldsymbol{j}b_2 + \boldsymbol{ji}b_3, \qquad b_1^2 + b_2^2 + b_3^2 = 1,$$
$$x = \boldsymbol{i}x_1 + \boldsymbol{j}x_2 + \boldsymbol{ji}x_3 + \boldsymbol{k}x_4 + \boldsymbol{ki}x_5 + \boldsymbol{kj}x_6 + \boldsymbol{k}(\boldsymbol{ji})x_7, \qquad |x| = 1.$$

Modulo automorphisms of $\mathbb{O}$ that leave $\mathbb{H}$ elementwise fixed, we can always assume that

$$x = \boldsymbol{i}x_1 + \boldsymbol{j}x_2 + \boldsymbol{ji}x_3 + \boldsymbol{k}x_4, \quad (x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1).$$

For N to be admissible for α the following must hold: $(\boldsymbol{i}q_1 + \boldsymbol{ji}q_3 \mid b) = 0$ (namely q belongs to N) and $(\boldsymbol{j} \mid b) = (\boldsymbol{j} \mid x) = (\boldsymbol{i}m_1 + \boldsymbol{j}m_2 \mid x)$ (Lemma 2.6, claim (1)). Explicitly:

$$b_1q_1 + b_3q_3 = 0, \tag{4}$$

and $b_2 = x_2 = m_1x_1 + m_2x_2$, namely

$$b_2 = x_2, \quad m_1x_1 = (1 - m_2)b_2. \tag{5}$$

Let $r = [\boldsymbol{i}r_1 + \boldsymbol{j}r_2 + \boldsymbol{ji}r_3]$ be the unique point of $\{[b], p\}^\perp$. So, $r_1 = 0$, namely $r = [\boldsymbol{j}r_2 + \boldsymbol{ji}r_3]$, and

$$b_2r_2 + b_3r_3 = 0. \tag{6}$$

Moreover we assume $r_2^2 + r_3^2 = 1$, as we can. We have already noticed that $q_1 \neq 0$. We also have $r_2 \neq 0$, otherwise equations (4) and (6) force $b_1 = b_3 = 0$, hence $b = \pm\boldsymbol{j}$, contrary to the fact that N is coplanar with L and M. Thus, by (4) and (6) we obtain

$$b_1 = -b_3q_3q_1^{-1}, \quad b_2 = -b_3r_3r_2^{-1}. \tag{7}$$

These equations show that $b_3 \neq 0$ (otherwise $b = 0$, which is ridiculous). Recalling that $b_1^2 + b_2^2 + b_3^2 = 1$ now we get

$$b_3 = \pm\frac{q_1r_2}{\sqrt{q_1^2 + r_2^2 - q_1^2r_2^2}} = \pm\frac{q_1r_2}{\sqrt{q_1^2r_3^2 + 1 - r_3^2}} = \pm\frac{q_1r_2}{\sqrt{1 - q_3^2r_3^2}}. \tag{8}$$

Equation (8) is equivalent to the following

$$r_2 = \pm\frac{b_3}{\sqrt{b_2^2 + b_3^2}},$$

which better shows that the point r depends on the choice of the line N but, in view of the sequel, (8) is more convenient. We shall now consider two subcases: either $m_2 = -1$ or $-1 < m_2 < 1$ (note that $m_2 = 1$ is impossible, since $m_2 = (L \mid M)$ and $(L \mid M) \neq 1$ because $L \neq M$).

Subcase 1.1. $m_2 = -1$. Equivalently, $m_1 = 0$. Then $b_2 = x_2 = 0$ by (5), $r_3 = 0$ by (7) and since $b_3 \neq 0$, whence $r_2 = \pm 1$ (as $r_2^2 + r_3^2 = 1$) and $b_3 = \pm q_1$ by (8). Consequently, $b_1 = \pm q_3$, since $b_1^2 + b_3^2 = q_1^2 + q_3^2 = 1$. Summarizing:

m_1	m_2	r_2	r_3	b_1	b_2	b_3	x_2
0	-1	± 1	0	$\pm q_3$	0	$\pm q_1$	0.

Let now ξ be the plane on L and N and χ the plane on M and N. Then ξ and χ, regarded as sharp $\mathbb{R}$-morphisms from $\mathbb{H}$ to $\mathbb{O}$, are uniquely determined by the following conditions (Lemma 2.3): $\xi(\boldsymbol{j}) = \boldsymbol{j}$, $\chi(\boldsymbol{j}) = \boldsymbol{i}m_1 + \boldsymbol{j}m_2$ and $\xi(b) = \chi(b) = x$. By entering the above values for m_1, m_2 and x_2 we get

$$\xi(\boldsymbol{j}) = \boldsymbol{j}, \quad \chi(\boldsymbol{j}) = -\boldsymbol{j}, \quad \xi(b) = \chi(b) = \boldsymbol{i}x_1 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4. \tag{9}$$

Clearly $\boldsymbol{i} = \boldsymbol{i}(b_1 - \boldsymbol{j}b_3)(b_1 - \boldsymbol{j}b_3)^{-1} = b(b_1 + \boldsymbol{j}b_3)$. Therefore, and taking (9) into account,

$$\begin{aligned}\xi(\boldsymbol{i}) &= (\boldsymbol{i}x_1 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4)(b_1 + \boldsymbol{j}b_3),\\ \chi(\boldsymbol{i}) &= (\boldsymbol{i}x_1 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4)(b_1 - \boldsymbol{j}b_3).\end{aligned} \tag{10}$$

Let now L' and M' be the lines through p and r in ξ and χ respectively. Then $L' = [a, \xi(a)]$ and $M' = [a, \chi(a)]$ where $a = \boldsymbol{i}a_1 + \boldsymbol{j}a_2 + \boldsymbol{k}a_3$ is orthogonal with both p and r and we assume $a_1^2 + a_2^2 + a_3^2 = 1$, as we can. Orthogonality with p and r forces $a_1 = 0 = a_2$. Therefore $a = \pm\boldsymbol{j}\boldsymbol{i}$. Accordingly, and recalling (10),

$$\begin{aligned}\xi(a) &= \pm\boldsymbol{j}(\boldsymbol{i}x_1 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4)(b_1 + \boldsymbol{j}b_3),\\ \chi(a) &= \mp\boldsymbol{j}(\boldsymbol{i}x_1 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4)(b_1 - \boldsymbol{j}b_3).\end{aligned} \tag{11}$$

With $\beta = \sigma_\perp^N(\alpha) = (p, L', r, M', p)$ we have $\ell(\beta) = (\xi(a) \mid \chi(a))$. Equations (11) allow to explicitly compute the inner product $(\xi(a) \mid \chi(a))$. We obtain:

$$\begin{aligned}(\xi(a) \mid \chi(a)) &= x_1^2(b_3^2 - b_1)^2 + x_3^2(b_3^2 - b_1^2) + x_4^2(b_3^2 - b_1^2)\\ &= (x_1^2 + x_3^2 + x_4^2)(b_3^2 - b_1)^2 = b_3^2 - b_1^2 = q_1^2 - q_3^2.\end{aligned} \tag{12}$$

So, $(\xi(a) \mid \chi(a)) = q_1^2 - q_3^2$. As $q_1, q_3 \neq 0$, we have $-1 < (\xi(a) \mid \chi(a)) < 1$.

Subcase 1.2. $m_1 \neq 0$, namely $m_2 \neq -1$. In this case the second equation of (5) yields

$$x_1 = \frac{1 - m_2}{m_1} b_2. \tag{13}$$

The planes ξ and χ on L and N and on M and N are determined by the following conditions:

$$\begin{aligned}&\xi(\boldsymbol{j}) = \boldsymbol{j}, \quad \chi(\boldsymbol{j}) = \boldsymbol{i}m_1 + \boldsymbol{j}m_2,\\ &\xi(b) = \chi(b) = \boldsymbol{i}x_1 + \boldsymbol{j}x_2 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4 = \left(\boldsymbol{i}\frac{1 - m_2}{m_1} + \boldsymbol{j}\right)b_2 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4.\end{aligned} \tag{14}$$

Moreover, $x_3^2 + x_4^2 = 1 - ((1 - m_2)^2 m_1^{-2} + 1)b_2^2 = 1 - 2(1 + m_2)^{-1}b_2^2$. Therefore

$$x_3^2 + x_4^2 = 1 - \frac{2}{1 + m_2} b_2^2. \tag{15}$$

Now $\boldsymbol{i} = (b - \boldsymbol{j}b_2)(b_1 - \boldsymbol{j}b_3)^{-1} = (b - \boldsymbol{j}b_2)(b_1 + \boldsymbol{j}b_3)(b_1^2 + b_3^2)^{-1}$. Recalling equations (7), we obtain

$$\boldsymbol{i} = \left(b + \boldsymbol{j}\frac{r_3}{r_2}b_3\right)(\boldsymbol{j}q_1 - q_3)q_1 b_3^{-1}. \tag{16}$$

As in Subcase 1.1, let $L' = [a, \xi(a)]$ and $M' = [a, \chi(a)]$ be the lines through p and r in ξ and χ respectively, where $a = \boldsymbol{i}a_1 + \boldsymbol{j}a_2 + \boldsymbol{k}a_3$ with $|a| = 1$. The vector a is orthogonal with both p and r. Orthogonality with p still forces $a_1 = 0$ but orthogonality with r only implies $a_2r_2 + a_3r_3 = 0$. So $a_2 = -a_3r_3r_2^{-1}$ and the condition $|a| = 1$ implies $a_3 = \pm r_2$. Hence $a_2 = \pm r_3$. Summarizing

$$a = \pm(\boldsymbol{j}r_3 + \boldsymbol{j}\boldsymbol{i}r_2). \tag{17}$$

Exploiting (14), (16) and (17), we can compute $\xi(a)$ and $\chi(a)$ explicitly, whence $(\xi(a) \mid \chi(a))$ too. We firstly obtain $(\xi(a) \mid \chi(a)) = A(x_3^3 + x_4^2) + B$ where

$$\begin{aligned} A &= (q_3^2 m_2 + q_1^2)q_1 r_2^2 b_3^{-2}, \\ B &= (-m_1 r_3 + (x_1 - m_1 b_2)q_1^2 r_2 b_3^{-1})x_1 q_1^2 b_3^{-1} + r_3^2 m_2 \\ &\qquad + (m_2 - 1)r_3 r_2 q_1^2 b_2 b_3^{-1} + (x_1 m_2 q_3 - m_1 q_3 b_2)x_1 q_3 q_1^2 r_2^2 b_3^{-2}. \end{aligned}$$

By exploiting (7), (8) and (15) we eventually obtain the following:

$$(\xi(a) \mid \chi(a)) = -r_3^2 \frac{q_1^4 m_2^2}{1 + m_2} + q_3^2 m_2 + q_1^2. \tag{18}$$

In this equation $(\xi(a) \mid \chi(a))$ is expressed as a function of r_3 rather than b_3, but recall that r is uniquely determined by b. Note that the coefficient of r_3^2 in (18) is negative except when $m_2 = 0$. If $m_2 = 0$ then $(\xi(a) \mid \chi(a)) = q_1^2$, which is strictly positive and less than 1, since neither q_1 nor q_3 are zero.

Case 2. $\Gamma = \Gamma_{\mathbb{C}}(\mathbb{O})$. As in Case 1, we can assume that

$$\begin{aligned} &L = [\boldsymbol{k}, \boldsymbol{k}], \quad M = [\boldsymbol{k}, \boldsymbol{j}m_1 + \boldsymbol{k}m_2], \quad |m_1|^2 + |m_2|^2 = 1, \\ &p = [\boldsymbol{j}], \qquad q = [\boldsymbol{j}q_1 + \boldsymbol{k}\boldsymbol{j}q_3], \qquad |q_1|^2 + |q_3|^2 = 1. \end{aligned}$$

So, $\ell(\alpha) = m_2$. As in Case 1, we have $q_1 \neq 0 \neq q_3$. Let $N = [b, x]$ be admissible for α, where

$$\begin{aligned} b &= \boldsymbol{j}b_1 + \boldsymbol{k}b_2 + \boldsymbol{k}\boldsymbol{j}b_3, \quad |b_1|^2 + |b_2|^2 + |b_3|^2 = 1, \\ x &= \boldsymbol{j}x_1 + \boldsymbol{k}x_2 + \boldsymbol{k}\boldsymbol{j}x_3, \quad |x_1|^2 + |x_2|^2 + |x_3|^2 = 1. \end{aligned}$$

For N to be admissible for α the following must hold: $(\boldsymbol{j}q_1 + \boldsymbol{k}\boldsymbol{j}q_3 \mid b) = 0$ and $(\boldsymbol{k} \mid b) = (\boldsymbol{k} \mid x) = (\boldsymbol{j}m_1 + \boldsymbol{k}m_2 \mid x)$. Explicitly:

$$\overline{q_1}b_1 + \overline{q_3}b_3 = 0, \tag{19}$$

and $b_2 = x_2 = \overline{m_1}x_1 + \overline{m_2}x_2$, namely

$$b_2 = x_2, \quad \overline{m_1}x_1 = (1-\overline{m_2})b_2. \tag{20}$$

Let $r = [\boldsymbol{j}r_1 + \boldsymbol{k}r_2 + \boldsymbol{kj}r_3]$ be such that $\{r\} = \{[b], p\}^{\perp}$. So, $r = [\boldsymbol{k}r_2 + \boldsymbol{kj}r_3]$, where we assume $|r_2|^2 + |r_3|^2 = 1$, and

$$\overline{r_2}b_2 + \overline{r_3}b_3 = 0. \tag{21}$$

Recall that $q_1 \neq 0$ because $p \not\perp q$ by assumption. We also have $r_2 \neq 0$, otherwise N cannot be coplanar with either of L and M. Thus, by (19) and (21) we obtain

$$b_1 = -b_3\frac{\overline{q_3}}{\overline{q_1}}, \quad b_2 = -b_3\frac{\overline{r_3}}{\overline{r_2}}. \tag{22}$$

These equations show that $b_3 \neq 0$. Recalling that $|b_1|^2 + |b_2|^2 + |b_3|^2 = 1$ we get

$$b_3 = \varepsilon \cdot \frac{q_1 r_2}{\sqrt{|q_1|^2 + |r_2|^2 - |q_1|^2|r_2|^2}} = \varepsilon \frac{q_1 r_2}{\sqrt{1-|q_3|^2|r_3|^2}} \tag{23}$$

for a suitable multiplier ε with $|\varepsilon| = 1$. We shall now consider two subcases: either $|m_2| = 1$ or $|m_1| < 1$.

Subcase 2.1. $|m_2| = 1$. Equivalently, $m_1 = 0$. Then $b_2 = x_2 = 0$ by (20), $r_3 = 0$ by (22) and since $b_3 \neq 0$, whence $|r_2| = 1$ and $|b_3| = |q_1|$ by (23). Consequently, $|b_1| = |q_3|$.

Let now ξ be the plane on L and N and χ the plane on M and N. Then ξ and χ, regarded as sharp $\mathbb{C}$-autorphisms of $\mathbb{O}$, are uniquely determined by the following conditions: $\xi(\boldsymbol{k}) = \boldsymbol{k}$, $\chi(\boldsymbol{k}) = \boldsymbol{j}m_1 + \boldsymbol{k}m_2$ and $\xi(b) = \chi(b) = x$. In view of the above:

$$\xi(\boldsymbol{k}) = \boldsymbol{k}, \quad \chi(\boldsymbol{k}) = \boldsymbol{k}m_2, \quad \xi(b) = \chi(b) = \boldsymbol{j}x_1 + \boldsymbol{kj}x_3. \tag{24}$$

It is easy to check that

$$\boldsymbol{j} = (\boldsymbol{j}b_1 + \boldsymbol{kj}b_3)(\overline{b_1} + \boldsymbol{k}\overline{b_3}) = b(\overline{b_1} + \boldsymbol{k}\overline{b_3}).$$

By this and (24) we get

$$\begin{aligned} \xi(\boldsymbol{j}) &= (\boldsymbol{j}x_1 + \boldsymbol{kj}x_3)(\overline{b_1} + \boldsymbol{k}\overline{b_3}), \\ \chi(\boldsymbol{j}) &= (\boldsymbol{j}x_1 + \boldsymbol{jkj}x_3)(\overline{b_1} + \boldsymbol{k}m_2\overline{b_3}). \end{aligned} \tag{25}$$

Let $L' = [a, \xi(a)]$ and $M' = [a, \chi(a)]$ be the lines through p and r in ξ and χ respectively, where $a = \boldsymbol{j}a_1 + \boldsymbol{k}a_2 + \boldsymbol{kj}a_3$ is orthogonal with both p and r and $|a_1|^2 + |a_2| + |a_3|^2 = 1$. Orthogonality with p and r forces $a_1 = 0 = a_2$. Therefore $a = \boldsymbol{kj}\eta$ for a suitable η with $|\eta| = 1$. By this and (25),

$$\begin{aligned} \xi(a) &= \boldsymbol{k}((\boldsymbol{j}x_1 + \boldsymbol{kj}x_3 + \boldsymbol{k})(\overline{b_1} + \boldsymbol{k}\overline{b_3}))\eta, \\ \chi(a) &= \boldsymbol{k}m_2((\boldsymbol{j}x_1 + \boldsymbol{kj}x_3 + \boldsymbol{k}x_4)(\overline{b_1} + \boldsymbol{k}m_2\overline{b_3}))\eta. \end{aligned} \tag{26}$$

Equations (26) allow to explicitly compute the inner product $(\xi(a) \mid \chi(a))$. We obtain:

$$(\xi(a) \mid \chi(a)) = |b_3|^2 + |b_1|^2\overline{m_2} = |q_1|^2 + |q_3|^2\overline{m_2}. \tag{27}$$

So, $|(\xi(a) \mid \chi(a))| = |q_1|^4 + |q_3|^4 + |q_1|^2|q_3|^2(m_2 + \overline{m_2}) < 1$, as $m_2 + \overline{m_2}$ is a real number not less than -2 and less than 2 (because $|m_2| = 1$ but $m_2 \neq 1$) and $|q_1|^2 + |q_3|^2 = 1$.

Subcase 2.2. $m_1 \neq 0$, namely $|m_2| < 1$. In this case the second equation of (20) yields

$$x_1 = \frac{1 - \overline{m_2}}{\overline{m_1}} b_2. \tag{28}$$

The planes ξ and χ on L and N and on M and N are determined by the following conditions:

$$\begin{aligned} &\xi(\boldsymbol{k}) = \boldsymbol{k}, \quad \chi(\boldsymbol{k}) = \boldsymbol{j}m_1 + \boldsymbol{k}m_2, \\ &\xi(b) = \chi(b) = \boldsymbol{j}x_1 + \boldsymbol{k}x_2 + \boldsymbol{kj}x_3 = \left(\boldsymbol{j}\frac{1 - m_2}{m_1} + \boldsymbol{k}\right)b_2 + \boldsymbol{kj}x_3. \end{aligned} \tag{29}$$

Moreover, $|x_3|^2 = 1 - (1 + |1 - m_2|^2|m_1|^{-2})|b_2|^2$ by (28) and $x_2 = b_2$. Therefore

$$|x_3|^2 = 1 - \frac{2 - m_2 - \overline{m_2}}{|m_1|^2}|b_2|^2. \tag{30}$$

Now $\boldsymbol{j} = (b - \boldsymbol{k}b_2)((\overline{b_1} + \boldsymbol{k}\overline{b_3})(1 - |b_2|^2)^{-1})$. Recalling equations (22), we obtain

$$\boldsymbol{j} = \left(b + \boldsymbol{k}\frac{\overline{r_3}}{\overline{r_2}}b_3\right)((\boldsymbol{k}q_1 - q_3)\overline{q_1}b_3^{-1}). \tag{31}$$

Let $L' = [a, \xi(a)]$ and $M' = [a, \chi(a)]$ be the lines through p and r in ξ and χ respectively, where $a = \boldsymbol{j}a_1 + \boldsymbol{k}a_2 + \boldsymbol{kj}a_3$ is orthogonal with both p and r and $|a| = 1$. Orthogonality with p forces $a_1 = 0$ but orthogonality with r only implies $\overline{r}_2a_2 + \overline{r_3}a_3 = 0$. So $a_2 = -a_3\overline{r_3}\overline{r_2}^{-1}$ and the condition $|a| = 1$ implies $|a_3| = |r_2|$, namely $a_3 = \overline{r_2}\eta$ for some η with $|\eta| = 1$. Hence

$$a = (-\boldsymbol{k}\overline{r_3} + \boldsymbol{kj}\overline{r_2})\eta = (\boldsymbol{k}(-\overline{r_3} + \overline{r_2}\boldsymbol{j}))\eta = (\boldsymbol{k}(-\overline{r_3} + \boldsymbol{j}r_2))\eta. \tag{32}$$

By exploiting (29), (31) and (32) as well as (22) and (30) one can compute $\xi(a)$ and $\chi(a)$ explicitly, whence $(\xi(a) \mid \chi(a))$ too, but these computations are terribly toilsome. However, in order to prove the lemma, we do not need to perform them. It is enough to show that, for a lucky choice of $N = [b, x]$, whence of r, satisfying the above conditions, we get $\ell \neq -1$. We will go on in this way, referring the interested reader to Remark 3.16 for a way to express $(\xi(a) \mid \chi(a))$ in the general case.

The previous conditions on r, b and x allow to choose $r_3 = 0$. Accordingly, $|r_2| = 1$. Hence $b_2 = 0$ by the second equation of (22) and $b_3 = \lambda\overline{q_1}$ for some λ with $|\lambda| = 1$ by (23). Therefore $b_1 = -\lambda\overline{q}_3$ by the first equation of (22). Moreover $x_1 = x_2 = 0$ by (20) and (28), whence $|x_3| = 1$. Accordingly,

$$\boldsymbol{j} = b((\boldsymbol{k}q_1 - q_3)\lambda^{-1}) \tag{33}$$

by (31) and since $b_1 = \lambda\overline{q}_1$ and

$$a = \boldsymbol{kj}\overline{r_2}\eta \tag{34}$$

by (32) and since $r_3 = 0$. By (33), recalling that $x_1 = x_2 = 0$, we obtain

$$\begin{aligned}
\xi(\boldsymbol{j}) &= x((\boldsymbol{k}q_1 - q_3)\lambda^{-1}) = \boldsymbol{kj}x_3(\boldsymbol{k}q_1\lambda^{-1} - q_3\lambda^{-1}) \\
&= \boldsymbol{j}\overline{q_1x_3}\lambda - \boldsymbol{kj}q_3x_3\overline{\lambda}, \\
\chi(\boldsymbol{j}) &= x(((\boldsymbol{j}m_1 + \boldsymbol{k}m_2)q_1 - q_3)\lambda^{-1}) \\
&= \boldsymbol{kj}x_3(\boldsymbol{j}m_1q_1\lambda^{-1} + \boldsymbol{k}m_2q_1\lambda^{-1} - q_3\lambda^{-1}) \\
&= \boldsymbol{j}\overline{m_2q_1x_3}\lambda - \boldsymbol{k}\overline{m_1q_1x_3}\lambda - \boldsymbol{kj}q_3x_3\overline{\lambda}.
\end{aligned} \tag{35}$$

(Recall that $\lambda^{-1} = \overline{\lambda}$ since $|\lambda| = 1$.) By combining (34) with (35) we obtain

$$\begin{aligned}
\xi(a) &= (\boldsymbol{k}(\boldsymbol{j}\overline{q_1x_3}\lambda - \boldsymbol{kj}q_3x_3\overline{\lambda}))\overline{r_2}\eta \\
&= \boldsymbol{j}\overline{q_3x_3r_2}\lambda\eta + \boldsymbol{kj}q_1x_3\overline{r_2}\overline{\lambda}\eta, \\
\chi(a) &= ((\boldsymbol{j}m_1 + \boldsymbol{k}m_2)(\boldsymbol{j}\overline{m_2q_1x_3}\lambda - \boldsymbol{k}\overline{m_1q_1x_3}\lambda - \boldsymbol{kj}q_3x_3\overline{\lambda}))\overline{r_2}\eta \\
&= \boldsymbol{j}\overline{m_2q_3x_3r_2}\lambda\eta - \boldsymbol{k}\overline{m_1q_3x_3r_2}\lambda\eta + \boldsymbol{kj}q_1x_3\overline{r_2}\overline{\lambda}\eta.
\end{aligned}$$

Therefore $(\xi(a) \mid \chi(a)) = (|q_3|^2\overline{m_2} + |\, q_1^2)(|x_3|^2|r_2|^2|\lambda|^2|\eta|^2$. Finally, recalling that $|x_3| = |r_2| = |\lambda| = |\eta| = 1$,

$$(\xi(a) \mid \chi(a)) = |q_3|^2\overline{m_2} + |q_1|^2. \tag{36}$$

The right side of (36) is equal to -1 only if $q_1 = 0$ and $m_2 = -1$. However, $q_1 \neq 0$ because $p \not\perp q$. Therefore $(\xi(a) \mid \chi(a)) \neq -1$. □

Remark 3.16. In Subcase 2.2 of the above proof, with no additional hypotheses on $[b, x]$ we get

$$(\xi(a) \mid \chi(a)) = A|r_2|^2|b_3|^{-2} - 2\,\mathrm{Im}(m_1\overline{q_1}q_3|q_3|^2r_2\overline{r_3}b_3^{-1}) + |r_3|^2B$$

where Im(.) stands for imaginary part and

$$\begin{aligned}
A &= |q_1q_3|^2\overline{m_2} + |q_1|^4, \\
B &= m_2 - A - |q_1q_3|^2 + |q_1|^4(m_2^3 + \overline{m_2} - 2)|m_1|^{-2}.
\end{aligned}$$

This shows that $(\xi(a) \mid \chi(b))$ depends on r_2, r_3 and x_2 nontrivially. Thus, we can always choose the line $N = [b, x]$ in such a way that $|(\xi(a) \mid \chi(a))| < 1$. Accordingly, Lemma 3.15 can be given a stronger formulation: we can always choose N in such a way that $|\ell| < 1$.

Remark 3.17. It follows from above proof that when $|m_2| = 1$ then $|\ell| < 1$ for every choice of the admissible line $N = [b, x]$. However, for certain values of m_2 we can also choose N in such a way that $\ell = -1$. For instance, when $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$, this is possible in the following cases:

(1) $q_1^4 = q_3^2$ (namely $q_1^2 = (\sqrt{5} - 1)/2$) and $-1 \le m_2 \le -(\sqrt{5} + 1)/4$;

(2) $q_1^2 > q_3^2$ and $-1 \le m_2 \le (1 - \sqrt{4q_1^6 + 8q_1^4 - 3})/(q_1^4 - q_3^2)$;

(3) $q_1^2 < q_3^2$ and $1 \ge m_2 \ge (-1 + \sqrt{4q_1^6 + 8q_1^2 - 3})/(q_3^2 - q_1^4)$.

Lemma 3.18. *Every orthogonal nondegenerate primitive path α of $\Gamma_{\mathbb{C}}(\mathbb{O})$ such that $|\ell(\alpha)| = 1$ but $\ell(\alpha) \neq -1$ is homotopic with an orthogonal nondegenerate primitive path β such that $|\ell(\beta)| < 1$.*

See [Schillewaert and Struyve 2017, Lemma 6.7] for the above. The following lemma is also proved in [Schillewaert and Struyve 2017, Lemma 6.8].

Lemma 3.19. *Let $\ell \in \mathbb{F}$ such that $|\ell| < 1$. Then, for every choice of two distinct lines L and M with the same shadow, there exists a sequence $L_0 = L, L_1, \ldots, L_n = M$ of lines with the same shadow as L and M and such that $(L_{i-1} \mid L_i) = \ell$ for every $i = 1, 2, \ldots, n$.*

The next statement is implicit in what Schillewaert and Struyve say to justify [2017, Remark 6.9]. We make it explicit.

Corollary 3.20. *Let $\ell \in \mathbb{F}$ such that $|\ell| < 1$ and let $\alpha = (p, L, q, M, p)$ be a nondegenerate primitive path of $\Gamma = \Gamma_{\mathbb{F}}(\mathbb{A})$. Then $\alpha \sim \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n$ for a suitable sequence of nondegenerate primitive paths $\alpha_1, \alpha_2, \ldots, \alpha_n$ of Γ with the same points p and q as α and such that $\ell(\alpha_i) = \ell$ for every $i = 1, 2, \ldots, n$.*

Proof. By Lemma 3.19 there exist lines $L_0 = L, L_1, \ldots, L_n = M$ such that $(L_{i-1} \mid L_i) = \ell$ for $i = 1, 2, \ldots, n$. For $i = 1, 2, \ldots, n$ put $\alpha_i = (p, L_{i-1}, q, L_i)$. Thus, the product $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n$ is well defined. Note that

$$\alpha_{n-1} \cdot \alpha_n = (p, L_{n-2}, q, L_{n-1}, p, L_{n-1}, q, L_n, p) \sim (p, L_{n-2}, q, L_n) =: \alpha'_{n-1}.$$

So, $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_{n-1} \cdot \alpha_n \sim \alpha_1 \cdot \alpha_3 \cdot \cdots \cdot \alpha'_{n-1}$. By iterating this argument we eventually obtain $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n \sim (p, L_0, q, L_n, p) = \alpha$. □

We can now prove the main theorem of this subsection.

Theorem 3.21. *Either $\Gamma_{\mathbb{F}}(\mathbb{A})$ is simply connected or it is covered by a building.*

Proof. Suppose that $\Gamma = \Gamma_{\mathbb{F}}(\mathbb{A})$ is not covered by a building. Then, by Corollary 3.8, at least one of its nondegenerate primitive paths is null-homotopic. By Lemma 3.10 (claim (4)) and Lemma 3.14, at least one orthogonal nondegenerate primitive path, say α, is null-homotopic. Let $\ell = \ell(\alpha)$ be its line-invariant. The action of $G := \mathrm{Aut}(\Gamma)$ on $\mathbb{A}$ and $\mathbb{O}$ makes it clear that G acts transitively on the set of orthogonal primitive paths with line-invariant equal to ℓ. Thus, all orthogonal primitive paths with line invariant ℓ are null-homotopic.

Suppose firstly that $|\ell| < 1$. Then every orthogonal primitive path β is null homotopic, by Corollary 3.20 and the above remark. In this case Γ is simply connected by Lemmas 3.10 and 3.14 and Corollary 3.6.

Let $|\ell| = 1$. If $\ell \neq -1$ (whence $\Gamma = \Gamma_{\mathbb{C}}(\mathbb{O})$) then $\alpha \sim \beta$ for some orthogonal primitive path β with $|\ell(\beta)| < 1$, by Lemma 3.18. Thus, we can replace α with β and we are back to the previous case.

Finally, let $\ell(\alpha) = -1$. Clearly α admits a nonorthogonal shift $\beta \sim \alpha$, necessarily nondegenerate (Lemma 3.10). In its turn β admits an orthogonal shift γ with $\ell(\gamma) \neq -1$, by Lemma 3.15. Moreover $\beta \sim \gamma$ by Lemma 3.10. Hence $\alpha \sim \gamma$. Therefore γ is null-homotopic. We can now replace α with γ and we are back to the first or second one of the two previous cases, according to whether $|\ell(\gamma)| < 1$ or $|\ell(\gamma)| = 1$. □

Remark 3.22. What Schillewaert and Struyve say to explain their Remark 6.9 in [2017] amounts to a sketch of the first three paragraphs of the above proof. However, as they had nothing like Lemma 3.15 at their disposal, they could only refer to the case $\ell \neq -1$ in that remark.

3D. ***End of the proof of Theorem 1.5.*** Let $\widetilde{\Gamma}$ be the universal cover of $\Gamma = \Gamma_{\mathbb{F}}(\mathbb{A})$. In view of Theorem 3.21, either $\widetilde{\Gamma} = \Gamma$ or $\widetilde{\Gamma}$ is a building. In order to finish the proof of Theorem 1.5 it only remains to prove that $\widetilde{\Gamma}$ cannot be a building. This immediately follows from the last claim of Theorem 1.3. However, as we have promised not to use that theorem, we shall give an explicit proof of this claim.

We firstly recall a few general properties of universal coverings and state some notation for quadratic and hermitian forms and related polar spaces.

3D1. *Lifting automorphisms through universal coverings.* Let $\phi : \widetilde{\Gamma} \to \Gamma$ be the universal k-covering of a geometry Γ of rank $n > k$. Let $G := \mathrm{Aut}(\Gamma)$ and $\widehat{G} := \mathrm{Aut}(\widetilde{\Gamma})$.

Pick a chamber C of Γ and a chamber $\widetilde{C} \in \phi^{-1}(C)$. For every $g \in G$ and every chamber $\widetilde{X} \in \phi^{-1}(g(C))$ there exists a unique $\tilde{g} \in \widehat{G}$, called a *lifting* of g to $\widetilde{\Gamma}$ *through* ϕ, such that $\phi \cdot \tilde{g} = g \cdot \phi$ and $\tilde{g}(\widetilde{C}) = \widetilde{X}$ [Pasini 1994, Theorem 12.13]. The set of all liftings of the elements $g \in G$ is a subgroup $\widetilde{G}$ of $\widehat{G}$ and the function $p_\phi : \widetilde{G} \to G$ which maps every $\tilde{g} \in \widetilde{G}$ onto the unique $g \in G$ such that $\phi \cdot \tilde{g} = g \cdot \phi$ is a (surjective) homomorphism of groups. The kernel of p_ϕ, namely the group of all liftings of the identity automorphisms of Γ, is the *deck group* $D(\phi)$ of ϕ and $\Gamma \cong \widetilde{\Gamma}/D(\phi)$ [Pasini 1994, Theorem 12.13].

Given a subflag $F \subset C$ of rank k, let $\widetilde{F}$ be the corresponding subflag of $\widetilde{C}$ and let G_F be the stabilizer F in G. The stabilizer $\widetilde{G}_{\widetilde{F}}$ of $\widetilde{F}$ in $\widetilde{G}$ meets $D(\phi)$ trivially. Hence p_ϕ induces an isomorphism from $\widetilde{G}_{\widetilde{F}}$ to G_F. We call $\widetilde{G}_{\widetilde{F}}$ the *lifting* of G_F to $\widetilde{\Gamma}$ *through* ϕ *based* at $\widetilde{F}$.

Moreover, let $K_F \trianglelefteq G_F$ be the elementwise stabilizer in G_F of the residue $\mathrm{Res}_\Gamma(F)$ of F in Γ. Similarly, let $\widetilde{K}_{\widetilde{F}}$ be the elementwise stabilizer of $\mathrm{Res}_{\widetilde{\Gamma}}(\widetilde{F})$ in $\widetilde{G}_{\widetilde{F}}$. Then p_ϕ isomorphically maps $\widetilde{K}_{\widetilde{F}}$ onto K_F.

In order to complete the notation adopted above, we denote by $\widehat{G}_{\widetilde{F}}$ and $\widehat{K}_{\widetilde{F}}$ the stabilizer of $\widetilde{F}$ in $\widehat{G}$ and the elementwise stabilizer of $\mathrm{Res}_{\widetilde{\Gamma}}(\widetilde{F})$ in $\widehat{G}_{\widetilde{F}}$. Needless to say, $\widetilde{G}_{\widetilde{F}}$ and $\widetilde{K}_{\widetilde{F}}$ are subgroups of $\widehat{G}_{\widetilde{F}}$ and $\widehat{K}_{\widetilde{F}}$ respectively and $\widetilde{K}_{\widetilde{F}} = \widehat{K}_{\widetilde{F}} \cap \widetilde{G}_{\widetilde{F}}$.

The group K_F (respectively $\widetilde{K}_{\widetilde{F}}$ or $\widehat{K}_{\widetilde{F}}$) is often called the *kernel* of G_F (respectively $\widetilde{G}_{\widetilde{F}}$ or $\widehat{G}_{\widetilde{F}}$), as a shortening for "kernel of the action of G_F on $\mathrm{Res}_\Gamma(F)$". We shall adopt this terminology too in the sequel.

3D2. *Some notation for quadratic and hermitian forms.* For a positive integer n, let $f_n^{\mathbb{F}}$ be the usual scalar product on $\mathbb{F}^n$ and let $\mathrm{L}(f_n^{\mathbb{F}})$ be the group of all linear mappings preserving $f_n^{\mathbb{F}}$. So, $\mathrm{L}(f_n^{\mathbb{R}}) = \mathrm{O}(n)$ and $\mathrm{L}(f_n^{\mathbb{C}}) = \mathrm{U}(n)$ (notation as usual for Lie groups).

Given two positive integers n, m with $n \le m$, let $f_{n,m}^{\mathbb{F}} := (-f_n^{\mathbb{F}}) \oplus f_m^{\mathbb{F}}$. Namely, $f_{n,m}^{\mathbb{F}}$ admits the following representations, according to whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, where $x = (x_i)_{i=1}^{n+m}$ and $y = (y_i)_{i=1}^{n+m}$ (vectors of $\mathbb{F}^{n+m}$):

$$(\mathbb{F} = \mathbb{R}) \quad f_{n,m}^{\mathbb{R}}(x, y) := -\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{m} x_{i+n} y_{i+m},$$

$$(\mathbb{F} = \mathbb{C}) \quad f_{n.m}^{\mathbb{C}}(x, y) := -\sum_{i=1}^{n} \overline{x_i} y_i + \sum_{i=1}^{m} \overline{x_{i+n}} y_{i+m}.$$

Clearly, n is the Witt index of $f_{n,m}^{\mathbb{F}}$. We also recall that, by Sylvester's law of inertia, every nondegenerate bilinear form on $\mathbb{R}^{n+m}$ of Witt index $n \le m$ can be expressed as $f_{n,m}^{\mathbb{R}}$ or its opposite, modulo a suitable choice of the basis of $\mathbb{R}^{n+m}$ (see, e.g., [Bourbaki 1959, §7, n.2]). The same for hermitian forms of $\mathbb{C}^{n+m}$.

Let $\mathrm{L}(f_{n.m}^{\mathbb{F}})$ be the group of linear trasformations of $\mathbb{F}^{n+m}$ preserving $f_{n,m}^{\mathbb{F}}$. So we have $\mathrm{L}(f_{n,n}^{\mathbb{R}}) = \mathrm{O}^+(2n, \mathbb{R})$, $\mathrm{L}(f_{n,n+1}^{\mathbb{R}}) = \mathrm{O}(2n+1, \mathbb{R})$, $\mathrm{L}(f_{n,n}^{\mathbb{C}}) = \mathrm{U}(2n, \mathbb{C})$ and $\mathrm{L}(f_{n,n+1}^{\mathbb{C}}) = \mathrm{U}(2n+1, \mathbb{C})$ (notation as usual for Chevalley groups).

Let $\Gamma(f_{n,m}^{\mathbb{F}})$ be the polar space associated to $f_{n,m}^{\mathbb{F}}$. Recall that its full automorphims group $\mathrm{Aut}(\Gamma(f_{n,m}^{\mathbb{F}}))$ is the projectivization $\mathrm{PL}(f_{n,m}^{\mathbb{F}})$ of $\mathrm{L}(f_{n,m}^{\mathbb{F}})$, extended by two (possibly trivial) outer automorphism groups, henceforth denoted by $\boldsymbol{d}$ and $\boldsymbol{f}$. The group $\boldsymbol{d}$ is contributed by linear transformations of $\mathbb{F}^{n+m}$ which do not preserve

$f^{\mathbb{F}}_{n,m}$ but multiply it by a scalar. However, as we deal with $\mathrm{PL}(f^{\mathbb{F}}_{n,m})$ rather than $\mathrm{L}(f^{\mathbb{F}}_{n,m})$, it turns ut that $\boldsymbol{d}$ is either trivial or isomorphic to the group C_2 of order 2, according to whether $n+m$ is odd or even. The group $\boldsymbol{f}$ is trivial when $\mathbb{F}=\mathbb{R}$ and isomorphic to C_2 when $\mathbb{F}=\mathbb{C}$. In the latter case, the unique nontrivial involution of $\boldsymbol{f}$ is contributed by the usual conjugation of $\mathbb{C}$ and the extension $(\mathrm{PL}(f^{\mathbb{C}}_{n,m})\cdot\boldsymbol{d})\cdot\boldsymbol{f}$ is split: it can be realized as the semidirect product $(\mathrm{PL}(f^{\mathbb{C}}_{n,m})\cdot\boldsymbol{d})\rtimes\langle\iota\rangle$ of $\mathrm{PL}(f^{\mathbb{C}}_{n,m})\cdot\boldsymbol{d}$ with the group $\langle\iota\rangle$ generated by a suitable involutory semilinear transformation ι of $\mathbb{C}^{n+m}$.

3D3. *The case* $(\mathbb{F},\mathbb{A})=(\mathbb{C},\mathbb{O})$. Let $\phi:\widetilde{\Gamma}\to\Gamma$ be the universal covering of $\Gamma=\Gamma_{\mathbb{C}}(\mathbb{O})$. We already know that either $\widetilde{\Gamma}=\Gamma$ or $\widetilde{\Gamma}$ is a building. We want show that $\widetilde{\Gamma}$ cannot be a building.

By contradiction, suppose that $\widetilde{\Gamma}$ is a building, namely a polar space of rank 3. We know that the residues of the planes of Γ are isomorphic to the complex projective plane $\mathbb{C}\mathrm{P}^2=\mathrm{PG}(2,\mathbb{C})$ while the panels of type 3 (namely the residues of the point-line flags) are homeomorphic to the 3-dimensional sphere $\mathbb{S}^3$ [Kramer and Lytchak 2014]. The same properties hold for $\widetilde{\Gamma}$. So, in view of Tits's classification of polar spaces [Tits 1974, Chapter 8], necessarily $\widetilde{\Gamma}=\Gamma(f^{\mathbb{C}}_{3,4})$, with full automorphism group

$$\widehat{G}:=\mathrm{Aut}(\Gamma(f^{\mathbb{C}}_{3,4}))=\mathrm{PU}(7,\mathbb{C})\rtimes\boldsymbol{f}\cong\mathrm{PSU}(7,\mathbb{C})\rtimes C_2.$$

We set $G:=\mathrm{Aut}(\Gamma)=((\mathrm{SU}(3)\times\mathrm{SU}(3))/C_3)\rtimes C_2$ (see Section 2C).

Let $\tilde{\xi}$ be a plane of $\widetilde{\Gamma}$ and $\xi=\phi(\tilde{\xi})$. With the notation and the terminology of Section 3D1, let G_ξ, $\widehat{G}_{\tilde{\xi}}$ and $\widetilde{G}_{\tilde{\xi}}$ be respectively the stabilizer of ξ in G, the stabilizer of $\tilde{\xi}$ in $\widehat{G}$ and the lifting of G_ξ to $\widetilde{\Gamma}$ through ϕ at $\tilde{\xi}$ and let K_ξ, $\widehat{K}_{\tilde{\xi}}$ and $\widetilde{K}_{\tilde{\xi}}$ be their kernels. It is not difficult to check that

$$G_\xi=\mathrm{PSU}(3)\rtimes C_2 \quad \text{with } K_\xi=1.$$

(See also [Schillewaert and Struyve 2017].) Hence $\widetilde{G}_{\tilde{\xi}}\cong\mathrm{PSU}(3)\rtimes C_2$ and $\widetilde{K}_{\tilde{\xi}}=1$. On the other hand, $\widehat{G}_{\tilde{\xi}}$ is the semidirect product $\widehat{G}_{\tilde{\xi}}=U\rtimes L$ of its unipotent radical U and a Levi complement L, where $U\cong\mathbb{C}^6\times\mathbb{R}^3\cong\mathbb{R}^{15}$, with $\mathbb{C}^6$, $\mathbb{R}^3$ and $\mathbb{R}^{15}$ being regarded as additive groups, and $L\cong\mathrm{GL}(3,\mathbb{C})\rtimes\boldsymbol{f}=\Gamma\mathrm{L}(3,\mathbb{C})$. Moreover $\widehat{K}_{\tilde{\xi}}=U\rtimes Z$ where $Z=Z(L)$ is the center of L (see, e.g., [Weiss 2003, Chapter 11]). The group $\widetilde{G}_{\tilde{\xi}}\cong\mathrm{PSU}(3)\rtimes C_2$ is contained in $\widehat{G}_{\tilde{\xi}}=U\rtimes L$ but, as its kernel is trivial, it meets $\widehat{K}_{\tilde{\xi}}=U\rtimes Z$ trivially. Accordingly, the group $L\cong\Gamma\mathrm{L}(3,\mathbb{C})$ contains a copy of $\widetilde{G}_{\tilde{\xi}}=\mathrm{PSU}(3)\rtimes C_2$. The group L indeed contains copies of $\mathrm{SU}(3)\rtimes C_2$, but no copy of $\mathrm{PSU}(3)\rtimes C_2$. Indeed $\mathrm{SU}(3)$ is not a semidirect product of its center C_3 and a copy of $\mathrm{PSU}(3)$.

We have reached a contradiction. Hence in this case $\widetilde{\Gamma}=\Gamma$.

3D4. *The case* $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$. Let now $\phi : \widetilde{\Gamma} \to \Gamma$ be the universal covering of $\Gamma = \Gamma_{\mathbb{R}}(\mathbb{H})$. By contradiction, suppose that $\widetilde{\Gamma}$ is a building. The residues of the planes of Γ are isomorphic to the real projective plane $\mathrm{PG}(2, \mathbb{R})$ and the panels of type 3 are homeomorphic to the 5-dimensional sphere $\mathbb{S}^5$ [Kramer and Lytchak 2019]. By Tits's classification of polar spaces [1974] we see that $\widetilde{\Gamma} = \Gamma(f^{\mathbb{R}}_{3,8})$, with full automorphism group $\widehat{G} := \mathrm{Aut}(\Gamma(f^{\mathbb{R}}_{3,8})) = \mathrm{PL}(f^{\mathbb{R}}_{3,8})$. We set $G := \mathrm{Aut}(\Gamma) = \mathrm{SO}(3) \times \mathrm{G}_2$ (see Section 2C). As in the previous case, let $\tilde{\xi}$ be a plane of $\widetilde{\Gamma}$ and $\xi := \phi(\tilde{\xi})$. We now have

$$\begin{aligned} G_\xi &= (\mathrm{SU}(2) \times \mathrm{SU}(2))/\langle(-\iota, -\iota)\rangle = 2^{\cdot}(\mathrm{PSU}(2) \times \mathrm{PSU}(2)), \\ K_\xi &= 2^{\cdot}\,\mathrm{PSU}(2) = \mathrm{SU}(2), \\ G_\xi / K_\xi &\cong \mathrm{PSU}(2) \cong \mathrm{SO}(3). \end{aligned}$$

Here ι stands for the identity element of $\mathrm{SU}(2)$, whence (ι, ι) is the identity element of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The extension $2^{\cdot}(\mathrm{PSU}(2) \times \mathrm{PSU}(2))$ is nonsplit.

On the other hand, $\widehat{G}_\xi = U \rtimes L$ where $L \cong \mathrm{GL}(3, \mathbb{R}) \times \mathrm{SO}(5)$ and $U = U_0{}^{\cdot}U_1$ with U_0 and U_1 isomorphic to the additive groups of $\mathbb{R}^3$ and $\mathbb{R}^{15}$ respectively. The group U_0 is both the center and the commutator subgroup of U. Moreover, $\widehat{K}_{\tilde{\xi}} = U \rtimes (Z \times \mathrm{SO}(5))$, where Z is the center of $\mathrm{GL}(3, \mathbb{R})$.

We have $\widetilde{G}_{\tilde{\xi}} \cong G_\xi = 2^{\cdot}(\mathrm{PSU}(2) \times \mathrm{PSU}(2))$, $\widetilde{K}_{\tilde{\xi}} \cong K_\xi = \mathrm{SU}(2)$ and $\widetilde{K}_{\tilde{\xi}}$ must be placed in $\widehat{K}_{\tilde{\xi}}$. As $U \trianglelefteq \widehat{K}_{\tilde{\xi}}$, the intersection $\widetilde{K}_{\tilde{\xi}} \cap U$ is normal in $\widetilde{K}_{\tilde{\xi}}$. However $\widetilde{K}_{\tilde{\xi}} \cong \mathrm{SU}(2)$ is quasisimple as an abstract group, with center of order 2, while every nontrivial subgroup of U is infinite. Therefore $\widetilde{K}_{\tilde{\xi}} \cap U = 1$, namely $\widetilde{K}_{\tilde{\xi}} \leq L \cap \widehat{K}_{\tilde{\xi}} = Z \times \mathrm{SO}(5)$. Moreover $\widetilde{K}_{\tilde{\xi}} \leq \mathrm{SO}(5)$, since $\mathrm{SU}(2)$ doesn't split as the direct product of its center and a copy of $\mathrm{PSU}(2)$. So far, no contradiction has arised; indeed $\mathrm{SO}(5)$ actually contains copies of $\mathrm{SU}(2)$.

Similarly, $\widetilde{G}_{\tilde{\xi}}/\widetilde{K}_{\tilde{\xi}} \cong \mathrm{PSU}(2)$ must be placed in $\widehat{G}_{\tilde{\xi}}/\widehat{K}_{\tilde{\xi}} = L/(Z \times \mathrm{SO}(5)) = \mathrm{PGL}(3, \mathbb{R})$. This can be done as well, since $\mathrm{PGL}(3, \mathbb{R})$ contains copies of $\mathrm{SO}(3) \cong \mathrm{PSU}(2)$. However these copies of $\mathrm{SO}(3)$ inside $\mathrm{GL}(3, \mathbb{R})$ meet the center Z of $\mathrm{GL}(3, \mathbb{R})$ trivially. It follows that $\widetilde{G}_{\tilde{\xi}}$ is the direct product $\widetilde{G}_{\tilde{\xi}} = \widetilde{K}_{\tilde{\xi}} \times X$ for a subgroup $X \cong \mathrm{SO}(3) \cong \mathrm{PSU}(2)$ of $\mathrm{GL}(3, \mathbb{R})$. In short, $\widetilde{G}_{\tilde{\xi}} = \mathrm{SU}(2) \times \mathrm{PSU}(2)$. However $\widetilde{G}_{\tilde{\xi}} \cong G_\xi = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\langle(-\iota, -\iota)\rangle$, which is not a direct product of $\mathrm{SU}(2)$ and $\mathrm{PSU}(2)$. Eventually, we have reached a contradiction.

Therefore $\widetilde{\Gamma} = \Gamma$ in this case too. The proof of Theorem 1.5 is complete.

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ANTONIO PASINI:

antonio.pasini@unisi.it
Department of Information Engineering and Mathematics, University of Siena, Siena, Italy

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