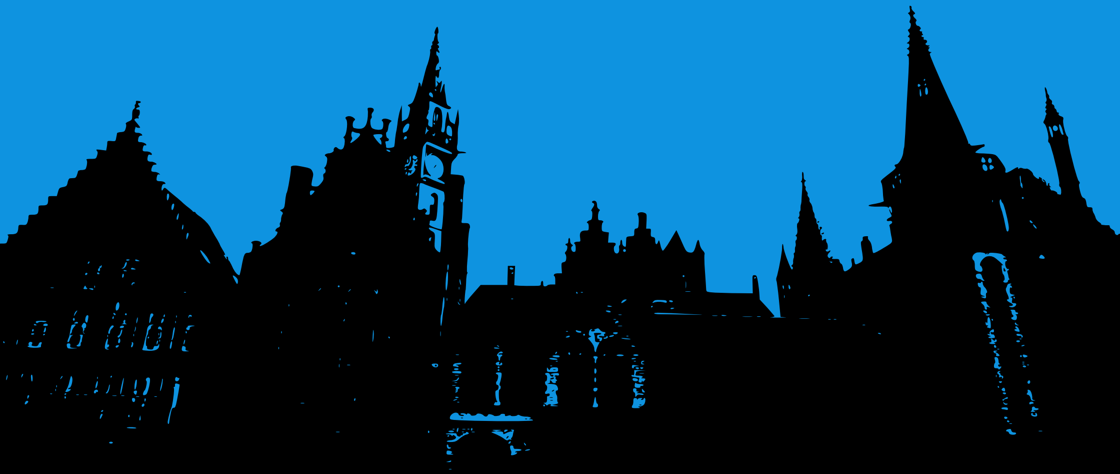
Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial



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Vol. 17 No. 3 2019

Innovations in Incidence Geometry

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Innovations in Incidence Geometry: Algebraic, Topological and Combinatorial (ISSN 2640-7345 electronic, 2640-7337 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

IIG peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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Innovations in Incidence Geometry
Algebraic, Topological and Combinatorial

vol. 17, no. 3, 2019
dx.doi.org/10.2140/iig.2019.17.189

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Chamber graphs of some geometries that are almost buildings

Veronica Kelsey and Peter Rowley

The global structure of the chamber graph of certain rank 3 geometries that are almost buildings is determined. Computer files containing extensive details of these graphs accompany this paper.

1. Introduction

The study of geometries that are almost buildings was instigated by Tits [1981]. The acronym "GAB" was bestowed upon them in [Kantor 1981], and they also go under the names of "geometries of type M" or "Tits geometries of type M". These geometries are Buekenhout–Tits geometries [Buekenhout 1979a] all of whose rank-2 residue geometries are generalized polygons (though they are not required to satisfy the intersection property). That is, they are incidence geometries satisfying axioms (1) and (2) but not necessarily (3) of [Buekenhout 1979a].

We recall that an incidence geometry over a set I is a triple $(\Gamma, *, \tau)$ where Γ is a set, τ an onto map from Γ to I and $*$ is an incidence relation on Γ such that if $x, y \in \Gamma$ and $x*y$ then $\tau(x) \neq \tau(y)$. The map τ is called the type map and $|I|$ the rank of Γ. As is customary, we shall abbreviate $(\Gamma, *, \tau)$ to Γ. A flag F of Γ is a subset of Γ such that $x*y$ for all $x, y \in F$, $x \neq y$ and the type of F is $\{\tau(x) \mid x \in F\}$. The residue of F in Γ, Γ_F, is the (subgeometry) given by $\{x \in \Gamma \mid y*x \text{ for all } y \in F\}$. If $F = \{x\}$, then we write Γ_x instead of $\Gamma_{\{x\}}$. We shall call a maximal flag of Γ a chamber of Γ. Note that, by axiom (1) of [Buekenhout 1979a], the type of a chamber of a GAB is I. The chamber graph $\mathcal{C}(\Gamma)$ is defined as follows. The vertices are the chambers of Γ with distinct chambers γ and γ' deemed adjacent in $\mathcal{C}(\Gamma)$ if $|\gamma \cap \gamma'| = |I| - 1$. We sometimes also say that γ and γ' are i-adjacent if $I = \{i\} \cup \{\tau(x) \mid x \in \gamma \cap \gamma'\}$. Let γ be a chamber of Γ. The i-th disc of γ, denoted by $\Delta_i(\gamma)$, consists of all the chambers which are distance i from γ in the graph $\mathcal{C}(\Gamma)$. We shall use $d(\ ,\ )$ for

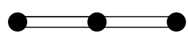
Kelsey was supported by LMS Undergraduate Research Bursary 16-17 01.
MSC2010: primary 51E24; secondary 05B25.
Keywords: chamber graphs, geometries, almost buildings.

the distance metric on $\mathcal{C}(\Gamma)$ and $\mathrm{Diam}(\mathcal{C}(\Gamma))$ for the diameter of $\mathcal{C}(\Gamma)$. For more on incidence geometries, consult [Buekenhout 1979b; 1995], while for GAB's the survey paper [Kantor 1986] contains much interesting material.

The chamber graph of a building contains all the important geometric information about the building. For example, the (chambers of the) apartments of the building can be detected in the chamber graph. The sets $\Delta_i(\gamma)$, for γ a chamber, encode data relating to the Weyl group of the building. Further, if d is the diameter of the chamber graph and G is the automorphism group of the building, then G_γ, a Borel subgroup of G, acts transitively on $\Delta_d(\gamma)$. See [Ronan 2009; Tits 1974; 1981] for more on buildings. It is natural to wonder about chamber graphs of other geometries associated with groups which are, in some sense, close to buildings. This has prompted a number of papers which have focussed on analyzing the disc structure of such chamber graphs. See [Carr and Rowley 2018; Rowley 1998; 2009; 2010]. Most of the geometries of interest have a large number of chambers and so these investigations have necessarily involved extensive computation using packages such as MAGMA [Cannon and Playoust 1997]. Here we continue this line of work, examining the chamber graphs of rank 3 GAB's. The examples we look at have been drawn from [Aschbacher and Smith 1983; Cooperstein 1989; Kantor 1981; Ronan and Smith 1980] (see also [Connor 2011; Kantor 1985; Yoshiara 1988]). We now state our main results on the disc structure of these GAB's.

Theorem 1.1. *Let G denote one of the five groups $P\Omega_6^-(3)$, $G_2(3)$, $U_6(2)$, $\Omega_8^+(2)$ and Suz, and let Γ denote a GAB associated to one of these groups. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let γ_0 be a fixed chamber of $\mathcal{C}$. Put $B = \mathrm{Stab}_G(\gamma_0)$.*

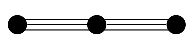
(i) *If $G \cong P\Omega_6^-(3)$ and Γ has diagram*



then $\mathcal{C}$ has 25515 *chambers*, 196 *B-orbits*, *diameter* 10 *and disc structure*

i-th disc	1	2	3	4	5	6	7	8	9	10
$\lvert\Delta_i(\gamma_0)\rvert$	6	20	64	176	416	1024	2432	5120	9088	7168
# of B-orbits	3	5	8	12	15	19	27	35	43	28

(ii) *If $G \cong G_2(3)$ and Γ has diagram*



then $\mathcal{C}$ has 66339 *chambers*, 1144 *B-orbits*, *diameter* 12 *and disc structure*

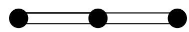
i-th disc	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert\Delta_i(\gamma_0)\rvert$	6	20	64	208	600	1728	4640	10368	17920	20416	9472	896
# of B-orbits	3	6	10	18	27	42	90	176	288	321	148	14

(iii) *If* $G \cong G_2(3)$ *and* Γ *has diagram*

then $\mathcal{C}$ *has* 66339 *chambers*, 1144 B*-orbits*, *diameter* 13 *and disc structure*

i-th disc	1	2	3	4	5	6	7	8	9	10	11	12	13
$\|\Delta_i(\gamma_0)\|$	6	20	56	144	384	960	2176	4864	10368	19072	21248	6976	64
# of B-orbits	3	6	9	14	21	31	51	92	172	302	332	109	1

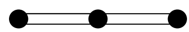
(iv) *If* $G \cong U_6(2)$ *and* Γ *has diagram*



then $\mathcal{C}$ *has* 1576960 *chambers*, 505 B*-orbits*, *diameter* 8 *and disc structure*

i-th disc	1	2	3	4	5	6	7	8
$\|\Delta_i(\gamma_0)\|$	15	117	972	6075	35721	203391	875043	455625
# of B-orbits	3	6	10	17	35	98	246	89

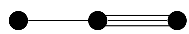
(v) *If* $G \cong \Omega_8^+(2)$ *and* Γ *has diagram*



then $\mathcal{C}$ *has* 179200 *chambers*, 317 B*-orbits*, *diameter* 9 *and disc structure*

i-th disc	1	2	3	4	5	6	7	8	9
$\|\Delta_i(\gamma_0)\|$	9	45	216	891	3159	11421	37098	80676	45684
# of B-orbits	3	6	10	16	26	43	68	95	49

(vi) *If* $G \cong Suz$ *and* Γ *has diagram*



then $\mathcal{C}$ *has* 18243225 *chambers*, 1276 B*-orbits*, *diameter* 16 *and disc structure*

i-th disc	1	2	3	4	5	6	7	8
$\|\Delta_i(\gamma_0)\|$	8	32	128	432	1216	3712	11008	29184
# of B-orbits	3	5	8	12	15	19	26	33
i-th disc	9	10	11	12	13	14	15	16
$\|\Delta_i(\gamma_0)\|$	81920	229376	598016	1576960	3595264	5410816	5304320	1400832
# of B-orbits	44	66	99	155	241	270	222	57

The GAB associated with the Lyons sporadic simple group is beyond our computational reach having 207060716016 chambers. However, we can give bounds on the diameter of the chamber graph.

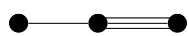
Theorem 1.2. *Let* Γ *be the GAB for Ly. Then* $10 \leq \mathrm{Diam}(\mathcal{C}(\Gamma)) \leq 16$.

2. Properties of $\mathcal{C}(\Gamma)$

The information collated in Theorem 1.1 was obtained using the code available with [Carr and Rowley 2018] and employing MAGMA. In fact, much more intricate details about $\mathcal{C}(\Gamma)$ were obtained, and these are available in the files in the online supplement (see article web page, doi 10.2140/iig.2019.17.189). We give a brief summary of such things.

The chambers of Γ are viewed as the right cosets of B. The panel stabilizers will be denoted by P_1, P_2 and P_3 (recall we are only looking at rank 3 geometries). The data obtained and program code is underpinned by DB, a sequence containing the (B, B) double coset representatives. So for $g = DB[j]$, the Bg coset is a representative for the B-orbits on the chambers of Γ. To minimise storage, we record j rather than $DB[j]$ whenever possible.The important output files are BorbitsDiscs and Neighbours. The first is a sequence where BorbitsDiscs[i] tells us the B-orbits making up $\Delta_i(\gamma_0)$ (where γ_0 is identified with the coset B). Here we give B-orbit representatives Bg, where $g = DB[k]$, by recording k. Neighbours is also a sequence where Neighbours[j] is giving information on the neighbours of Bg (where $g = DB[j]$). Suppose we have $[P_i : B] = 3$ for $i = 1, 2, 3$ (as happens for the GAB associated with $P\Omega_6^-(3)$, for example), so $\mathcal{C}(\Gamma)$ has valency 6. Returning to Neighbours[j], in this case this would be a 6-tuple $[k_1, k_2, k_3, k_4, k_5, k_6]$. This is saying that the six neighbours of Bg are in the B-orbits of $B * DB[k_i]$ $(i = 1, \ldots, 6)$. More than this we are also keeping track of the kind of adjacency. So the neighbours in the B-orbits of $B * DB[k_1]$ and $B * DB[k_2]$ are 1-adjacent to Bg, those in the B orbits of $B * DB[k_3]$ and $B * DB[k_4]$ are 2-adjacent to Bg, and those in the B-orbits of $B * DB[k_5]$ and $B * DB[k_6]$ are 3-adjacent to Bg.

Proof of Theorem 1.2. Let $G = Ly$ and let γ_0 be a chamber of $\mathcal{C}(\Gamma)$, and put $B = \mathrm{Stab}_G(\gamma_0)$. Recall that the diagram for Γ is



Let x be a point of Γ. Then by Section 6 of [Kantor 1981], Γ_x is a generalized hexagon dual to the usual $G_2(5)$ generalized hexagon. In particular, for any two chambers γ, γ' of Γ containing x we have $d(\gamma, \gamma') \leq 6$. Let the point, line and plane of γ_0 be respectively x_0, l_0, p_0 and γ_1 a chamber whose point, line and plane are respectively x_1, l_0, p_1 where $x_0 \neq x_1$. So x_0 and x_1 are collinear in Γ. Now $\gamma_0 = \{x_0, l_0, p_0\}, \{x_0, l_0, p_1\}, \{x_1, l_0, p_1\} = \gamma_1$ is a path in $\mathcal{C}(\Gamma)$, whence $d(\gamma_0, \gamma_1) \leq 2$. Since the point-line collinearity graph of Γ has diameter 2 (see Section 6 of [Kantor 1981] again), we infer that $\mathrm{Diam}(\mathcal{C}(\Gamma)) \leq 2 + 6 + 2 + 6 = 16$.

The number of chambers in the GAB associated with the Lyons group is

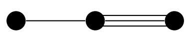
$$\frac{|G|}{N_G(S)} = \frac{|G|}{5^6 \cdot 2^4} = 207060716016,$$

where $S \in Syl_5(G)$. We find a lower bound for the diameter of the $\mathcal{C}(\Gamma)$ by working out the maximum number of chambers that can be in each disc. We have $[P_i : B] = 6$, $i = 1, 2, 3$, and so the valency of $\mathcal{C}(\Gamma)$ is 15. Therefore each chamber γ in $\Delta_1(\gamma_0)$ is joined to 5 chambers in $\Delta_1(\Gamma_0) \cup \{\gamma_0\}$. Hence $|\Delta_1(\gamma) \cap \Delta_2(\gamma)| = 10$. Of course for $i \geq 2$, a chamber in $\Delta_i(\gamma_0)$ can have at most 14 neighbours in $\Delta_{i+1}(\gamma_0)$. Thus, letting $d = \mathrm{Diam}(\mathcal{C}(\Gamma))$,

$$207060716016 \leq 1 + 15 + 150 + 150 \cdot 14 + \cdots + 150 \cdot 14^{d-2} = 16 + 150\left(\tfrac{14^{d-1}-1}{14-1}\right).$$

This gives $d - 1 \geq \log_{14}\left(\frac{13}{150}(207060716001) + 1\right)$, whence $d - 1 \geq 8.947$. Consequently, $\mathrm{Diam}(\mathcal{C}(\Gamma)) \geq 10$, which completes the proof of Theorem 1.2. □

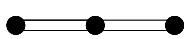
Collapsed adjacency graphs. For a GAB with diameter of say d, we call $\Delta_d(\gamma_0)$ the last disc (of γ_0) of the chamber graph. When examining the number of B-orbits which comprise the last disc we see, from the point of the chamber graph, the appellation of "almost building" is something of a misnomer. Of the GAB's investigated here only the GAB associated with $G_2(3)$, diagram



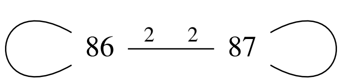
has its last disc as a B-orbit. Because of this we have calculated the geodesic closure for this GAB, the results of which are summarized in Theorem 2.1. All the others have the number of B-orbit ranging from 14 to 89. Indeed the more sporadic geometries studied in [Carr and Rowley 2018] and [Rowley 2009] come closer to buildings in this respect.

Notwithstanding the above comments on the last disc, we have looked at the induced graph on this disc. The most interesting (as far as we can see) are the GAB's from $G_2(3)$. Now we describe the B-collapsed adjacency graphs for the last disc of γ_0. The B-collapsed adjacency graph is formed by taking B-orbits, $B = \mathrm{Stab}_G\, \gamma_0$, as the vertices. We use j to stand for the B orbit of $B * DB[j]$ (where j is as given in the accompanying files). Two B-orbits, j and k are adjacent if and only if each chamber in j is adjacent to some chamber in k and we label the edge coming out from j with the number of chambers in k a chamber in j is adjacent with. If this number is 1 (as is mainly the case below) we omit this number.

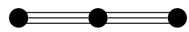
(i) If $G \cong P\Omega_6^-(3)$ and Γ has diagram



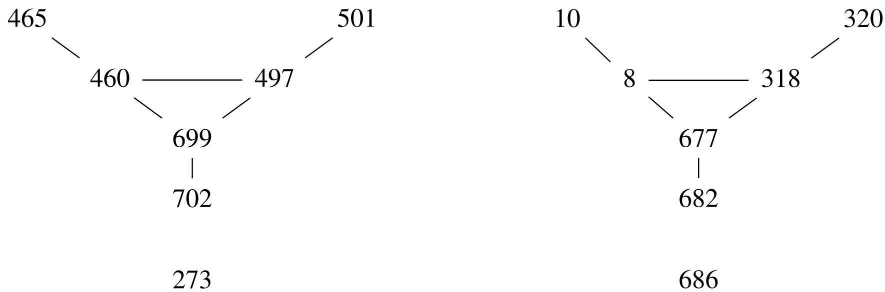
then the last disc of the B-collapsed adjacency graph is connected apart from 87 and 89, with 87 and 89 having the following adjacencies.



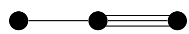
(ii) If $G \cong G_2(3)$ and Γ has diagram



then the 14 B-orbits in the last disc form the following collapsed B-adjacency graph.

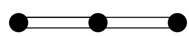


(iii) If $G \cong G_2(3)$ and Γ has diagram



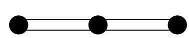
then there is only one B-orbit in the last disc and $\Delta_{13}(\gamma_0)$ is a co-clique.

(iv) If $G \cong U_6(2)$ and Γ has diagram



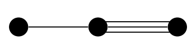
then the last disc of the B-collapsed adjacency graph is connected apart from 215 and 377, with 215 and 377 having the following adjacencies.

(v) If $G \cong \Omega_8^+(2)$ and Γ has diagram

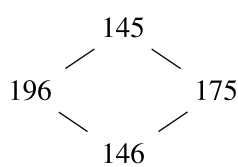


then the B-collapsed adjacency graph of $\Delta_9(\gamma_0)$ is connected.

(vi) If $G \cong Suz$ and Γ has diagram



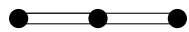
then the last disc of the B-collapsed adjacency graph is connected apart from 145, 146, 175 and 196, which have the following adjacencies.



Geodesic closure. For $\gamma, \gamma' \in \mathcal{C}$ a shortest path between them in $\mathcal{C}$ is called a geodesic. The geodesic closure of a set of chambers X is defined to be the set $\overline{X}$ of all chambers lying on some geodesic of γ, γ' for any pair $\gamma, \gamma' \in X$. The motivation for geodesic closures comes from the fact that in the chamber graph of a building, the geodesic closure of two chambers at maximal distance apart yields (the chambers of) an apartment.

Theorem 2.1. *Let G denote one of the groups $P\Omega_6^-(3)$ or $G_2(3)$, and let Γ denote a GAB associated to one of these groups. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let γ_0 be a fixed chamber of $\mathcal{C}$. Put $B = \mathrm{Stab}_G(\gamma_0)$.*

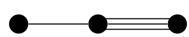
(i) *Suppose $G \cong P\Omega_6^-(3)$ and Γ has diagram*



and let $\gamma_i \in \Delta_{10}(\gamma_0)$, $i = 1, \ldots, 28$ be B-orbit representatives of $\Delta_{10}(\gamma_0)$. Set $n_{i,j} = |\overline{\{\gamma_0, \gamma_i\}} \cap \Delta_j(\gamma_0)|$. Then:

j	0	1	2	3	4	5	6	7	8	9	10
$n_{1,j}, n_{2,j}$	1	3	4	6	6	4	6	6	4	3	1
$n_{3,j}, n_{4,j}, n_{5,j}, n_{6,j}$	1	2	2	3	3	2	3	3	2	2	1
$n_{7,j}, n_{8,j}, n_{9,j}, n_{10,j}$	1	3	4	5	6	5	4	4	3	2	1
$n_{11,j}, n_{12,j}$	1	3	4	6	6	4	4	4	2	2	1
$n_{13,j}, n_{14,j}$	1	1	2	1	1	2	1	1	2	1	1
$n_{15,j}, n_{16,j}, n_{17,j}, n_{18,j}$	1	3	4	4	5	6	5	4	4	3	1
$n_{19,j}, n_{20,j}, n_{21,j}, n_{22,j}$	1	2	3	4	4	5	6	5	4	3	1
$n_{23,j}, n_{24,j}, n_{25,j}, n_{26,j}$	1	2	2	2	2	2	2	2	2	2	1
$n_{27,j}, n_{28,j}$	1	2	2	4	4	4	6	6	4	3	1

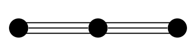
(ii) *Suppose $G \cong G_2(3)$ and Γ has diagram*



and let $\gamma' \in \Delta_{13}(\gamma_0)$. Set $n_j = |\overline{\{\gamma_0, \gamma'\}} \cap \Delta_j(\gamma_0)|$. Then:

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
n_j	1	6	15	23	24	26	25	25	26	24	23	15	6	1

(iii) *Suppose $G \cong G_2(3)$ and Γ has diagram*



and let $\gamma_i \in \Delta_{12}(\gamma_0)$, $i = 1, \ldots, 14$ be B-orbit representatives of $\Delta_{12}(\gamma_0)$. Set

$n_{i,j} = |\overline{\{\gamma_0, \gamma_i\}} \cap \Delta_j(\gamma_0)|$. *Then*:

j	0	1	2	3	4	5	6	7	8	9	10	11	12
$n_{1,j}, n_{2,j}$	1	3	6	9	9	10	12	10	9	9	6	3	1
$n_{3,j}, n_{4,j}$	1	5	9	13	13	13	18	13	13	13	9	5	1
$n_{5,j}, n_{6,j}$	1	6	14	17	25	29	26	29	25	17	14	6	1
$n_{7,j}, n_{8,j}$	1	3	5	6	6	7	7	8	7	7	5	3	1
$n_{9,j}, n_{10,j}$	1	5	12	15	18	18	16	18	18	15	12	5	1
$n_{11,j}, n_{12,j}$	1	3	5	7	7	8	7	7	6	6	5	3	1
$n_{13,j}, n_{14,j}$	1	5	8	12	12	13	16	13	12	12	8	5	1

Apartments of GABs associated with $U_6(2)$ and $\Omega_8^+(2)$. The GAB's for $U_6(2)$ and $\Omega_8^+(2)$ possesses apartments (see [Kantor 1981]), viewed as the fixed chambers of T. For $U_6(2)$ we take T to be a cyclic group of order 4, and for $\Omega_8^+(2)$ we take T to be an elementary abelian group order 4, see [Kantor 1981]. In both cases the apartments are isomorphic and have diameter 8. They also have the property that the distance between any two chambers in the apartment (as measured in the apartment) is the same as in the chamber graph. So this is something one expects from a building. However, for $\Omega_8^+(2)$ the diameter of its chamber graph is 9, so not equal to the diameter of the apartment — unlike the situation in a building.

Theorem 2.2. *Suppose $G \cong \Omega_8^+(2)$, let Γ denote a GAB associated to G. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let γ_0 be a fixed chamber of $\mathcal{C}$. Put $B = \mathrm{Stab}_G(\gamma_0)$.*

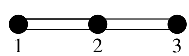
An apartment, $\mathcal{A}$, of Γ containing γ_0 cuts the discs as follows.

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8	9
$\lvert\mathcal{A} \cap \Delta_i(\gamma_0)\rvert$	1	3	5	8	11	13	13	8	2	0

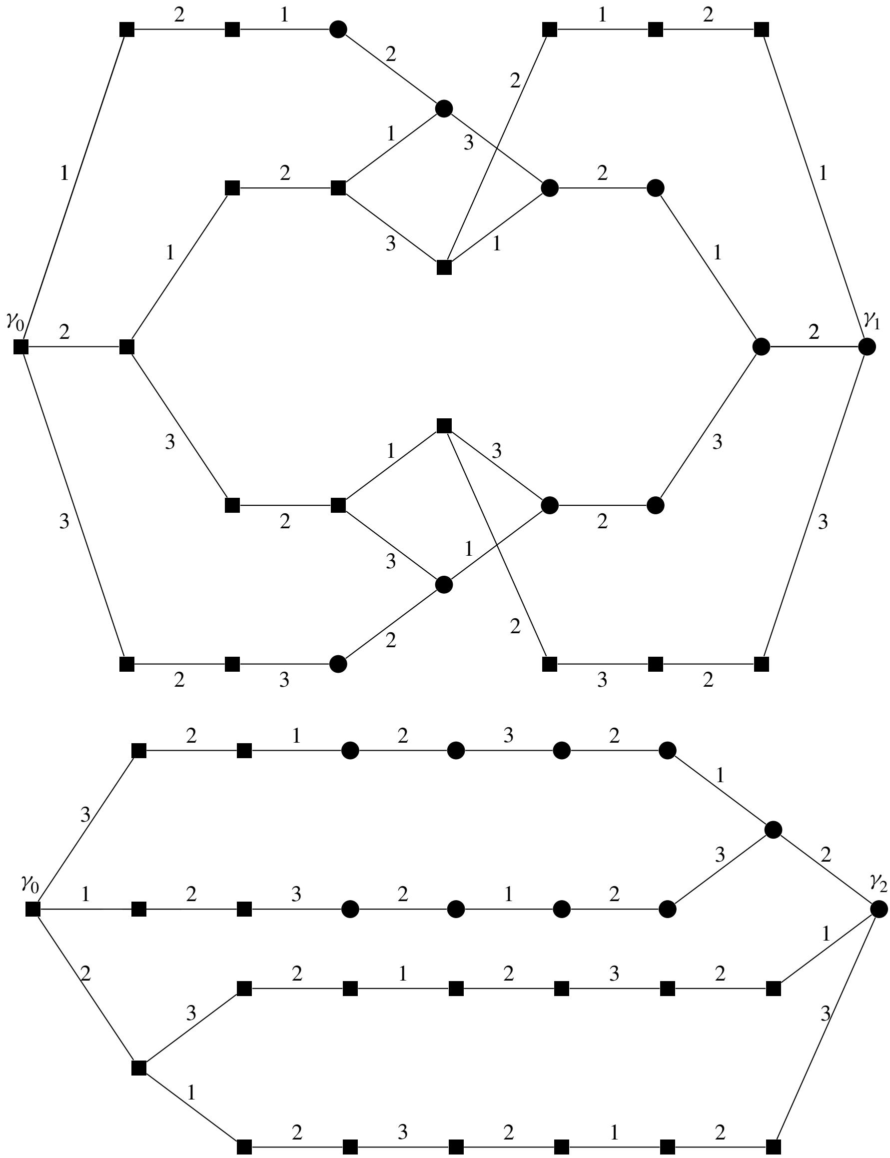
Let $\mathcal{A} \cap \Delta_8(\gamma_0) = \{\gamma_1, \gamma_2\}$. For $j = 1, 2$ the geodesic closure of the γ_0, γ_j cuts the discs as follows.

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8
$\lvert\overline{\{\gamma_0, \gamma_j\}} \cap \Delta_i(\gamma_0)\rvert$	1	3	4	4	4	4	4	3	1

The graphs on the next page are the geodesic closures $\overline{\{\gamma_0, \gamma_1\}}$ and $\overline{\{\gamma_0, \gamma_2\}}$. The type of adjacency between two connected chambers is shown by the labelling on the edges, where



The set of chambers in both geodesic closures are subsets of the apartment. The intersection between $\overline{\{\gamma_0, \gamma_1\}}$ and $\overline{\{\gamma_0, \gamma_2\}}$ has size 18 and the chambers that lie in both geodesic closures are labelled with squares rather than circles.



Geodesic closures (see Theorem 2.2).

Theorem 2.3. *Suppose $G \cong U_6(2)$, and let Γ denote a GAB associated to G. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let γ_0 be a fixed chamber of $\mathcal{C}$. Put $B = \mathrm{Stab}_G(\gamma_0)$.*

An apartment, $\mathcal{A}$, of Γ containing γ_0 cuts the discs as follows.

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8
$\lvert\mathcal{A} \cap \Delta_i(\gamma_0)\rvert$	1	3	5	8	11	13	13	9	1

Let $\mathcal{A} \cap \Delta_8(\gamma_0) = \{\gamma'\}$. The geodesic closure of γ_0, γ' cuts the discs as follows.

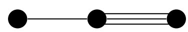
Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8
$\lvert\overline{\{\gamma_0, \gamma'\}} \cap \Delta_i(\gamma_0)\rvert$	1	3	4	4	4	4	4	3	1

The graph for the geodesic closure of the only B-orbit in the last disc of the apartment in the GAB of $U_6(2)$ is identical to the first diagram on page 197.

Again, the set of chambers in the geodesic closure in Theorem 2.3 is a proper subset of the apartment (once more not very building like).

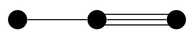
Maximal opposite sets. A maximal opposite set of chambers is a set of chambers of maximal size subject to having the property that any two chambers are opposite to each other, meaning that their distance apart is the diameter of the graph.

Theorem 2.4. *If $G \cong G_2(3)$ and Γ has diagram*



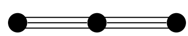
then a maximal opposite set of chambers consists of three chambers.

Proof. Suppose $G \cong G_2(3)$ and Γ has diagram



Since G_{γ_0} is transitive on $\Delta_{13}(\gamma_0)$, we may assume our maximal opposite set contains $\{\gamma_0, \gamma_1\}$, where $\gamma_1 \in \Delta_{13}(\gamma_0)$ is the chamber corresponding to $B * DB[149]$ (the right coset of B containing $DB[149]$). We identify a chamber γ with the triple $\{F_1(\gamma), F_2(\gamma), F_3(\gamma)\}$ which corresponds to a point-line-quad triple. Using the action of B, we determine $\Delta_{13}(\gamma_0)$, and by applying $DB[149]$ to this set we obtain $\Delta_{13}(\gamma_1)$. We can then see that $|\Delta_{13}(\gamma_0) \cap \Delta_{13}(\gamma_1)| = 1$. If we take $\gamma_2 \in \Delta_{13}(\gamma_0) \cap \Delta_{13}(\gamma_1)$ we can see that $|\Delta_{13}(\gamma_0) \cap \Delta_{13}(\gamma_1) \cap \Delta_{13}(\gamma_2)| = 0$, and so $\{\gamma_0, \gamma_1, \gamma_2\}$ is a maximal opposite set. □

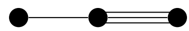
Theorem 2.5. *If $G \cong G_2(3)$ and Γ has diagram*



then each choice of the B-orbits in the last disc gives rise to a maximal opposite set of chambers consisting of four chambers. In particular all maximal opposite sets consist of four chambers.

Proof. We proceed as in Theorem 2.4, starting with γ_0 but then there are 14 possible choices of $\gamma_1 \in \Delta_{12}(\gamma_0)$ (one from each B-orbit in $\Delta_{12}(\gamma_0)$). We give the details for γ_1 being the chamber corresponding to $B * DB[8]$ (the right coset of B containing $DB[8]$). We use MAGMA to calculate $\Delta_{12}(\gamma_1)$ and find that $\Delta_{12}(\gamma_0) \cap \Delta_{12}(\gamma_1)$ is comprised of 21 chambers. One of these 21 chambers, γ_2, has the property that $|\Delta_{12}(\gamma_0) \cap \Delta_{12}(\gamma_1) \cap \Delta_{12}(\gamma_2)| = 2$. Two of the other twenty chambers give rise to an intersection of 1 and the others to 0. Taking γ_3 to be either of the chambers in $\Delta_{12}(\gamma_0)\cap\Delta_{12}(\gamma_1)\cap\Delta_{12}(\gamma_2)$ we find that $\Delta_{12}(\gamma_0)\cap\Delta_{12}(\gamma_1)\cap\Delta_{12}(\gamma_2)\cap\Delta_{12}(\gamma_3)=\varnothing$. Hence γ_1 is contained in a maximal opposite set with four chambers, so proving the theorem. □

Perhaps the most surprising overall result was how unalike the chamber graphs of buildings and the chamber graphs of these GABs appear. In [Carr and Rowley 2018] and [Rowley 2009] all the geometries investigated were in some sense "building like", indeed their chamber graphs had at most two B-orbits in their final disc. The only GAB investigated here displaying this type of behaviour was $G_2(3)$ with diagram



There were also differences by other measures. For the two groups, $\Omega_8^+(2)$ and $U_6(2)$ possessing apartments we found that the geodesic closures were proper subsets of the apartments rather than being equal. Furthermore the apartment of $\Omega_8^+(2)$ did not even span the whole diameter of the chamber graph as it would were it a building.

Perhaps it would be of interest to try and characterise why a limited number of these GABs have so few B-orbits in their last disc while most have so many. Could it be that there is a more unifying lens through which to view these chamber graphs that would justify the name "geometries that are almost buildings"?

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Received 13 Sep 2017. Revised 10 Feb 2019.

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Innovations in Incidence Geometry
Algebraic, Topological and Combinatorial

vol. 17, no. 3, 2019
dx.doi.org/10.2140/iig.2019.17.201

msp

Groups of compact 8-dimensional planes: conditions implying the Lie property

Helmut R. Salzmann

The automorphism group Σ of a compact topological projective plane with an 8-dimensional point space is a locally compact group. If the dimension of Σ is at least 12, then Σ is known to be a Lie group. For the connected component Δ of Σ it is shown that $\dim \Delta \geq 10$ suffices, if Δ is semisimple or does not fix exactly a nonincident point-line pair or a double-flag. Δ is also a Lie group, if Δ has a compact connected 1-dimensional normal subgroup and $\dim \Delta \geq 11$.

1. Introduction

A systematic study of compact 8-dimensional projective planes began with [Salzmann 1979]. Many of the results obtained in the following 15 years are presented in Chapter 8 of the treatise *Compact projective planes* [Salzmann et al. 1995]. An up-to-date account of more recent contributions to the theme can be found in [Salzmann 2014]. The *classical* model, the projective plane over the quaternion field $\mathbb{H}$, has the automorphism group $\mathrm{PSL}_3\mathbb{H}$ of dimension 35. If $\mathcal{P} = (P, \mathfrak{L})$ is any other compact 8-dimensional plane, then its automorphism group $\Sigma = \mathrm{Aut}\,\mathcal{P}$, taken with the compact-open topology, is a locally compact transformation group of the point space P as well as of the line space $\mathfrak{L}$, and $\dim \Sigma \leq 18$. All planes $\mathcal{P}$ such that $\dim \Sigma \geq 17$ have been described explicitly [Hähl 1986; Salzmann 2014]. The goal is to extend these results and to determine all pairs $(\mathcal{P}, \Delta)$, where Δ is a suitable *connected* subgroup of $\mathrm{Aut}\,\mathcal{P}$. As in the cases of finite projective planes or compact connected planes of smaller dimension, such a classification is possible only if the group — in our case its dimension — is not too small. An important step is to show that Δ is a Lie group. In all known examples, lines are homeomorphic to the 4-sphere $\mathbb{S}_4$, each closed proper subplane is connected and has a point space of dimension 2 or 4, and Σ is even a Lie group. In general, however, it is only known that lines are homotopy equivalent to $\mathbb{S}_4$; it is conceivable that some planes

MSC2010: 22D05, 51H10.
Keywords: topological plane, Lie group.

have compact 0-dimensional subplanes; and it is an open problem whether or not Σ is always a Lie group. According to [Priwitzer 1994], the following theorem holds: *if* $\dim\Sigma \geq 12$, *then* Σ *is a Lie group*. Depending on the structure of a connected subgroup Δ and the configuration $\mathcal{F}_\Delta$ of its fixed elements (points and lines), sharper bounds will be obtained here.

2. Preliminaries and background

This section contains a collection of basic facts. $\mathcal{P} = (P, \mathfrak{L})$ will always be a compact 8-dimensional projective plane if not stated otherwise; Δ denotes a connected closed subgroup of $\operatorname{Aut}\mathcal{P}$.

Notation. The notation is more or less standard and agrees with that in the book [Salzmann et al. 1995]. A *flag* is an incident point-line pair; a *double flag* consists of two points, say u, v, their join uv, and a second line in the *pencil* $\mathfrak{L}_v$. Homeomorphism is indicated by $\approx$. As customary, $\mathrm{Cs}_\Delta\,\Gamma$ or just $\mathrm{Cs}\,\Gamma$ is the centralizer of Γ in Δ. Distinguish between the commutator subgroup Γ' and the connected component Γ^1 of the topological group Γ. The coset space $\Delta/\Gamma = \{\Gamma\delta \mid \delta \in \Delta\}$ has the (covering) dimension $\Delta : \Gamma = \dim\Delta - \dim\Gamma$. The group $\Delta_{[c,A]}$ consists of the axial collineations in Δ with axis A and center c. A collineation group Γ is said to be *straight* if each orbit x^Γ is contained in some line. In this case a theorem of Baer [1946] asserts that either $\Gamma = \Gamma_{[c,A]}$ is a group of axial collineations or the fixed configuration $\mathcal{F}_\Gamma$ is a Baer subplane.

2.1. ***Baer subplanes.*** It is known that *each* 4*-dimensional closed subplane* $\mathcal{B}$ *of a compact* 8*-dimensional plane* $\mathcal{P}$ *is a Baer subplane*; i.e., each point of $\mathcal{P}$ is incident with a line of $\mathcal{B}$ (and dually, each line of $\mathcal{P}$ contains a point of $\mathcal{B}$); see [Salzmann 2003, §3] or [Salzmann et al. 1995, 55.5] for details. Lines of a Baer subplane are homeomorphic to $\mathbb{S}_2$. If $\mathcal{P}$ contains a closed Baer subplane $\mathcal{B}$, it follows easily that the pencil of lines through a point outside $\mathcal{B}$ is a manifold, and hence, the lines of $\mathcal{P}$ are homeomorphic to $\mathbb{S}_4$; see [Salzmann et al. 1995, 53.10] or [Salzmann 2003, 3.7]. By a result of Löwen [1999], any two closed Baer subplanes of $\mathcal{P}$ have a point and a line in common. Generally, $\langle\mathcal{M}\rangle$ will denote the smallest *closed* subplane of $\mathcal{P}$ containing the set $\mathcal{M}$ of points and lines. We write $\mathcal{B} \lessdot \mathcal{P}$ if $\mathcal{B}$ is a Baer subplane.

2.2. ***Stiffness.*** In the classical plane $\mathcal{H}$, the stabilizer $\Lambda = \Sigma_\mathfrak{e}$ of any *frame* $\mathfrak{e}$ (= nondegenerate quadrangle) is isomorphic to $\mathrm{SO}_3\mathbb{R}$; in particular, Λ is compact and $\dim\Lambda = 3$. In any plane, Λ can be identified with the automorphism group of the ternary field H_τ defined with respect to $\mathfrak{e}$. The fixed elements of Λ form a closed subplane $\mathcal{E} = \mathcal{F}_\Lambda$. It is not known if $\mathcal{E}$ is always connected or if Λ is compact in general. Therefore, the following *stiffness* results play an important role:

(1) $\dim \Lambda \le 4$ [Bödi 1994].

(2) *If* $\mathcal{F}_\Lambda$ *is connected or if* Λ *is compact, then* $\dim \Lambda \le 3$ [Salzmann et al. 1995, 83.12–13].

(3) *If* $\mathcal{F}_\Lambda$ *is contained in a Baer subplane* $\mathcal{B}$, *then* $\mathcal{F}_\Lambda$ *is connected and the connected component* Λ^1 *of* Λ *is compact* ([Salzmann et al. 1995, 55.4 and 83.9] *or* [Salzmann 1979, (*)]).

(4) *If, moreover,* $\mathcal{B}$ *is* Λ*-invariant, then* $\dim \Lambda \le 1$ [Salzmann et al. 1995, 83.11],

($\hat{4}$) *if* $\mathcal{F}_\Lambda$ *itself is a Baer subplane, then* Λ *is compact* [Salzmann et al. 1995, 83.6].

(5) *If* Λ *is compact, then* Λ *is commutative or* $\Lambda^1 \cong \mathrm{SO}_3\mathbb{R}$ [Salzmann 1979, 2(1)].

(6) *The stabilizer* Ω *of a degenerate quadrangle has dimension at most* 7 [Salzmann et al. 1995, 83.17].

(7) *If* $\dim \Omega = 7$, *then* $\Omega^1 \cong e^{\mathbb{R}} \cdot \mathrm{SO}_4\mathbb{R}$ *and lines are* 4*-spheres* [Salzmann 1979, (**)].

(8) *If a subgroup* $\Phi \cong \mathrm{SO}_3\mathbb{R}$ *of* Δ *fixes a line* W, *then each involution in* Φ *is planar. Either* Φ *has no fixed point on* W *or* $\mathcal{F}_\Phi$ *is a* 2*-dimensional subplane* [Salzmann 2010, Observation].

2.3. ***Fixed elements.*** *The Lefschetz fixed-point theorem implies that each homeomorphism* $\varphi : P \to P$ *has a fixed point.*

(a) *By duality, each automorphism of* $\mathcal{P}$ *fixes a point and a line* [Salzmann et al. 1995, 55.19].

(b) *The solvable radical* $\mathrm{P} = \sqrt{\Delta}$ *of* Δ *fixes some element of* $\mathcal{P}$.

(c) *If* $\mathcal{F}_\Delta = \varnothing$, *then* Δ *is semisimple with trivial center, or* Δ *induces a simple group on some connected closed* Δ*-invariant subplane.*

Proof. Argument (A) If Θ is a commutative connected normal subgroup of Δ and if $\mathbb{1} \ne \zeta \in \mathrm{Cs}\,\Theta$, then $p^\zeta = p$ for some point p, and either $p^\Theta = p$, or p^Θ is contained in a fixed line of Θ, or p^Θ generates a connected (closed) subplane $\mathcal{S} = \langle p^\Theta \rangle$ and $\zeta|_\mathcal{S} = \mathbb{1}$. In the latter case, $\overline{\Theta} = \Theta|_\mathcal{S} \ne \mathbb{1}$, and $\mathcal{S}$ is a proper subplane of $\mathcal{P}$.

(b) The claim will be proved by induction over the solvable length. Suppose that Δ itself is solvable and that the normal subgroup Θ has no fixed element. Let $\mathcal{S}$ be a proper subplane as given by (A). If $\dim \mathcal{S} = 2$, then $\mathcal{S}$ has no proper closed subplane [Salzmann et al. 1995, 32.7], and Θ has a fixed element in $\mathcal{S}$. If $\mathcal{S}$ is a Baer subplane, then (A) can be applied to $\overline{\Theta}$; again $\mathcal{F}_\Theta \ne \varnothing$, say $p^\Theta = p$. Then $\Theta|_{p^\Delta} = \mathbb{1}$. Either Δ fixed some element or $\mathcal{D} = \langle p^\Delta \rangle$ is a proper subplane. In the latter case, $\Delta|_\mathcal{D} = (\Delta/\Theta)|_\mathcal{D}$ has a fixed element by induction.

(c) This will be proved successively for planes $\mathcal{R}$ of dimension 2, 4, and 8. If Δ is not semisimple, then $\mathrm{P} = \sqrt{\Delta} \neq \mathbb{1}$ by definition, and P fixes some element by step (b), say $p^{\mathrm{P}} = p$. Assume also that $\mathcal{F}_\Delta = \varnothing$. Then p^Δ is not contained in a line and $\langle p^\Delta \rangle = \mathcal{S} \leq \mathcal{R}$ is a closed subplane; normality of P implies $\mathrm{P}|_{\mathcal{S}} = \mathbb{1}$. If $\zeta \neq \mathbb{1}$ is a central element of Δ, then (A) yields a common fixed element p of ζ and P, and $\zeta|_{\mathcal{S}} = \mathrm{P}|_{\mathcal{S}} = \mathbb{1}$.

If $\dim \mathcal{R} = 2$, there is no proper closed subplane, $\mathrm{P}|_{\mathcal{R}} = \mathbb{1} = \zeta|_{\mathcal{R}}$, and Δ is semisimple with trivial center, and hence Δ is strictly simple; see [Salzmann et al. 1995, 33.7] or [Salzmann 1967, 5.2]. If $\dim \mathcal{R} = 4$, then $\mathrm{P} \neq \mathbb{1}$ or $\zeta \neq \mathbb{1}$ implies $\mathcal{S} \neq \mathcal{R}$, $\dim \mathcal{S} = 2$, and $\bar{\Delta} = \Delta|_{\mathcal{S}} \neq \mathbb{1}$ is simple. Finally, let $\dim \mathcal{R} = 8$. Then $\mathcal{S} = \langle p^\Delta \rangle < \mathcal{R}$, $\dim \mathcal{S} \leq 4$, and $\mathcal{F}_{\bar{\Delta}} = \varnothing$. Either $\dim \mathcal{S} = 2$ and $\Delta|_{\mathcal{S}}$ is simple by what has just been proved, or $\dim \mathcal{S} = 4$ and $\bar{\Delta}$ is semisimple with trivial center. In the latter case $\bar{\Delta}$ is simple by [Salzmann et al. 1995, 71.8]. □

2.4. ***Dimension formula.*** By [Halder 1971] or [Salzmann et al. 1995, 96.10], the following holds for the action of Δ on P or on any closed Δ-invariant subset M of P, and for any point $a \in M$:

$$\dim \Delta = \dim \Delta_a + \dim a^\Delta \quad \text{or} \quad \dim a^\Delta = \Delta : \Delta_a.$$

2.5. ***Approximation theorem***, see [Salzmann et al. 1995, 93.8].

(a) *Every locally compact group* Γ *has an open subgroup* Δ *which is an extension of its connected component* Δ^1 *by a compact group.*

(b) *If* Δ *is locally compact and* Δ/Δ^1 *is compact, then* Δ *has arbitrarily small compact normal subgroups* N *such that* Δ/N *is a Lie group.*

(c) *If, moreover,* $\dim \Delta$ *is finite, then* $\dim \mathrm{N} = 0$ *for each sufficiently small subgroup* $\mathrm{N} \leq \Delta$.

2.6. ***Groups with open orbits.*** *Let L be a line of the* 8*-dimensional plane* $\mathcal{P}$, *and let* Δ *be a closed subgroup of* $\mathrm{Aut}\,\mathcal{P}$ *with* $L^\Delta = L$. *If* $U \subseteq L$ *is a* Δ*-orbit which is open or, equivalently, satisfies* $\dim U = \dim L$, *then* L *is a manifold and* Δ *induces a Lie group on* U. *It follows that all lines are manifolds homeomorphic to* $\mathbb{S}_4$ (adapted from [Salzmann et al. 1995, 53.2]).

2.7. ***Compact groups on*** $\mathbb{S}_4$ (Richardson). *If a compact connected group* Φ *acts effectively on the* 4*-sphere* S, *and if* Φ *has an orbit of dimension* > 1, *then* Φ *is a Lie group and* (Φ, S) *is equivalent to the obvious standard action of a subgroup of* $\mathrm{SO}_5\mathbb{R}$ *on* $\mathbb{S}_4$ *or* $\Phi \cong \mathrm{SO}_3\mathbb{R}$ *has no fixed point on* S [Salzmann et al. 1995, 96.34].

2.8. ***Theorem*** (Löwen). *If the connected subgroup* Δ *of* $\mathrm{Aut}\,\mathcal{P}$ *fixes the line* W *and if* Δ_x *is a Lie group for each* $x \notin W$, *then* Δ *itself is a Lie group.*

Proof. The following has been shown in [Löwen 1976]. Let (Γ, M) be a locally compact connected transformation group of finite dimension, where $X = M \cup \infty$ is a Peano continuum, all cohomology groups $H^q(X, \mathbb{Q})$ are finite-dimensional, and $H^q(X, \mathbb{Q}) = 0$ for some n and all $q \geq n$; moreover, the Euler characteristic $\chi(X, \mathbb{Q}) \neq 0, 1$. If all stabilizers Γ_x with $x \in M$ are Lie groups, then Γ is a Lie group. This result applies to $(\Delta, P \setminus W)$: by [Salzmann et al. 1995, 51.6, 51.8, 52.12], the one-point compactification X of $P \setminus W$ is homeomorphic to the quotient space P/W, and X is a Peano continuum (i.e., a continuous image of the unit interval); moreover, X is homotopy equivalent to $\mathbb{S}_8$, and X has Euler characteristic $\chi(X) = 2$. □

2.9. *Compact groups.* *Each compact connected group is of the form* $(\mathrm{A}\times\Lambda)/\mathrm{N}$, *where* A *is the connected component of the center and* Λ *is a direct product of compact simply connected almost simple Lie groups*; *the* kernel N *is a compact central subgroup of dimension* $\dim \mathrm{N} = 0$. *A compact connected commutative normal subgroup* Θ *of a connected group* Δ *is contained in the center of* Δ [Salzmann et al. 1995, 93.11, 93.19].

2.10. *Groups of subplanes.* *The automorphism group of every proper connected closed subplane is a Lie group by* [Salzmann et al. 1995, 32.21, 71.2].

2.11. *Lemma.* *Suppose that* Φ *is a compact connected Lie group and that the compact connected* 1*-dimensional group* Θ *is not a Lie group. If* $\Gamma = \Phi\Theta$ *acts effectively on a subspace* M *of the plane, if* $\mathrm{H} = \Phi \cap \Theta$ *is finite, and if* $\Theta_a = \mathbb{1}$ *and* Φ_a *is finite for some* $a \in M$, *then* $\dim a^\Gamma > \dim a^\Phi$.

Proof. First, let $\mathrm{H} = \mathbb{1}$, so that $\Gamma = \Phi\times\Theta$. If $\dim a^\Gamma = \dim a^\Phi$, then the connected component Ξ of $(\Phi\Theta)_a$ satisfies $\dim \Xi = 1$. Consider the restrictions of the projection maps $\pi : \Xi \to \Phi$ and $\varrho : \Xi \to \Theta$. Both maps are continuous homomorphisms. The kernel $\ker \pi$ is contained in $\Theta_a = \mathbb{1}$ and π is injective. Compactness of Φ implies that Ξ is isomorphic to a closed subgroup of Φ; hence, Ξ is a Lie group. From $\ker \varrho \leq \Phi_a$ we infer that $\ker \varrho$ is finite, and [Salzmann et al. 1995, 93.12] shows that ϱ is surjective, but then Θ would be a Lie group contrary to the assumption. In the general case analogous arguments apply to the natural maps $\pi : \Xi \to \Phi/\mathrm{H}$ and $\varrho : \Xi \to \Theta/\mathrm{H}$. □

2.12. *Definition.* For the remainder of this article, we shall call a compact, connected 1-dimensional subgroup of Δ a *serpentine* subgroup. The letter Θ will be reserved for such subgroups. They are 1-tori or, more frequently, solenoids; the latter are not Lie groups.

3. No fixed elements

Suppose in this section that $\mathcal{F}_\Delta = \varnothing$.

3.1. ***Theorem.*** *If* $\dim \Delta \geq 10$, *or if* Δ *is semisimple and* $\dim \Delta \geq 9$, *then* Δ *is a Lie group.*

Proof. By the approximation theorem, there is a compact 0-dimensional central subgroup N such that Δ/N is a Lie group. Suppose that $\mathbb{1} \neq \zeta \in \mathrm{N}$, and let $p^\zeta = p$ be a fixed point of ζ. A slight variation of argument (A) in the proof of 2.3 shows that $\mathcal{E} = \langle p^\Delta \rangle$ is a connected proper subplane.

(a) If $\dim \mathcal{E} = 2$, then Δ induces on $\mathcal{E}$ a group $\Delta^* = \Delta/\mathrm{K}$ of dimension at most 8, and stiffness yields $\dim \mathrm{K} \leq 3$. Hence, $\dim \mathrm{K} \geq 1$ and $\Delta : \mathrm{K} \geq 6$. In particular, $\mathcal{E}$ is isomorphic to the classical real projective plane [Salzmann et al. 1995, 33.6], and Δ^* is a subgroup of $\mathrm{SL}_3\mathbb{R}$. As Δ^* has no fixed element, Δ^* is simple by [Salzmann 1967, 5.2] or [Salzmann et al. 1995, 33.1] (see also 2.3 above), and then $\dim \Delta^* = 8$, $\Delta^* \cong \mathrm{SL}_3\mathbb{R}$. If Δ is semisimple, the kernel K is also simple, and $\dim \mathrm{K} = 3$. In any case, $\dim \Delta \geq 10$ and $\dim \mathrm{K} \geq 2$. Because N induces a Lie group on $\mathcal{E}$ (see 2.10 or [Salzmann et al. 1995, 32.21]), we may assume that $\mathrm{N} < \mathrm{K}$. Either $\mathcal{F}_\zeta \lessdot \mathcal{P}$ for some $\zeta \in \mathrm{N} \setminus \{\mathbb{1}\}$, or N acts freely on the set of *exterior* points (points not belonging to $\mathcal{E}$). In the first case, the stiffness result (4) would imply $\dim \mathrm{K} \leq 1$. Hence, $\mathrm{N}_z = \mathbb{1}$ for each exterior point z on an *interior* line L (a line of $\mathcal{E}$). If $\dim \Delta_L - \dim \Delta_z = 4$, then Δ_L induces a Lie group on the orbit z^{Δ_L} by 2.6. Therefore, N is finite, and Δ would be a Lie group. Consequently $\Delta : \Delta_z \leq 2+3$. Choose two interior points $a, b \notin L$ and consider the stabilizer $\Omega = \Delta_{z,a,b}$; it fixes also the point $L \cap ab$ and hence 3 collinear points of $\mathcal{E}$. Linear algebra shows that Ω fixes all interior points of ab; moreover, $\dim \Omega \geq 1$ and $\Omega|_{z^\mathrm{N}} = \mathbb{1}$. Thus, $\mathcal{F}_\Omega$ is a connected proper subplane of dimension 2 or 4, and N acts effectively on $\mathcal{F}_\Omega$. From 2.10 it follows that N is a Lie group, and so is Δ.

(b) Finally, let $\mathcal{E} \lessdot \mathcal{P}$ and note that $\Delta^* = \Delta|_\mathcal{E}$ has no fixed element. According to [Salzmann et al. 1995, 71.4, 71.8], the group $\Delta^* = \Delta/\mathrm{K}$ is strictly simple. Stiffness shows $\dim \mathrm{K} \leq 1$ and $\Delta : \mathrm{K} > 8$ (since $\dim \Delta \geq 10$ or $\dim \mathrm{K} = 0$). All possibilities for Δ^* are listed in [Salzmann et al. 1995, 71.8]; only $\mathrm{PSL}_3\mathbb{C}$ has dimension > 8. Hence, $\dim \Delta \geq 16$, and Δ is a Lie group by [Priwitzer 1994]. □

Remark. Previously 3.1 was only known for $\dim \Delta \geq 11$; see [Salzmann 2010, Theorem 1.1] or [Salzmann 2014, 2.1].

3.2. ***Compact normal subgroup.*** *If* Δ *has a serpentine normal subgroup* Θ *and if* $\dim \Delta \geq 9$, *then* Δ *is a Lie group or, conceivably,* $\Delta \cong \mathrm{SL}_3\mathbb{R} \times \Theta$ *induces the full collineation group on some invariant* 2*-dimensional desarguesian subplane.*

Proof. The proof follows the scheme of the previous one, and the same notation will be used. If Δ is not a Lie group, then $\dim \Delta = 9$ by 3.1.

(a) Let $\mathcal{E} = \langle p^\Delta \rangle$ be a 2-dimensional subplane. Again $\mathcal{E}$ is the classical real plane and $\Delta^* = \Delta/\mathrm{K} \cong \mathrm{SL}_3\mathbb{R}$ is simple. Hence, $\Theta\mathrm{N} \le \mathrm{K}$. Either $\Delta' \cong \Delta^*$ or Δ' is a twofold covering of $\mathrm{SL}_3\mathbb{R}$. In the *first case*, each involution in Δ' is a reflection of $\mathcal{P}$ (if $\mathcal{F}_\beta \lessdot \mathcal{P}$ for some involution β, then N induces a Lie group on $\mathcal{F}_\beta$ by 2.10, the induced map $\beta|_\mathcal{E}$ is a reflection, $\langle \mathcal{E}, \mathcal{F}_\beta \rangle = \mathcal{P}$, and Δ would be a Lie group). Consequently, there is a translation group $\Delta_{[L,L]} \cong \mathbb{R}^2$ for each *interior* axis L. It remains an open problem whether or not Θ must be a Lie group in this situation.

In the *second case*, the center of Δ' contains an involution ι such that $\mathcal{F}_\iota \lessdot \mathcal{P}$, and the lines of $\mathcal{P}$ are homeomorphic to $\mathbb{S}_4$ (see 2.1). Moreover, $\Theta|_{\mathcal{F}_\iota} = \mathbb{1}$ by the stiffness property [Salzmann et al. 1995, 71.7(a)] or by [Grundhöfer and Salzmann 1990, XI.9.3] (recall that $\Theta|_\mathcal{E} = \mathbb{1}$). Hence, Θ acts freely on the set of points not belonging to $\mathcal{F}_\iota$. Let L be a line of $\mathcal{E}$ and put $L' = L \setminus \mathcal{F}_\iota$. The group Δ' has a subgroup $\Upsilon \cong \mathrm{SU}_2\mathbb{C}$, and the connected component Φ of Υ_L is a torus. As L' is dense in L, it follows that Φ acts effectively on L' (note that $\mathcal{E}$ is classical). Let $p \in L'$ such that $p^\Phi \ne p$, $\dim \Phi_p = 0$, and Φ_p is finite. We have $\Phi \cap \Theta = \langle \iota \rangle$. Therefore, Lemma 2.11 applies and shows that $1 = \dim p^\Phi < \dim p^{\Phi\Theta} = 2$. By [Salzmann et al. 1995, 96.24] or 2.7 above Δ is a Lie group.

(b) If $\langle p^\Delta \rangle = \mathcal{C} \lessdot \mathcal{P}$, the lines of $\mathcal{P}$ are 4-spheres. From 2.3 and 2.9 it follows that $\Theta|_\mathcal{C} = \mathbb{1}$ and that $\Delta|_\mathcal{C}$ is semisimple of dimension 8. By 2.3(c) and [Salzmann et al. 1995, 71.8], the group $\Delta^* = \Delta|_\mathcal{C}$ is isomorphic to $\mathrm{SL}_3\mathbb{R}$ or to $\mathrm{PSU}_3(\mathbb{C}, r)$, $r \le 1$. For each of the unitary groups, there is an interior line L such that $\mathrm{SU}_2\mathbb{C}$ acts nontrivially on the set L' of exterior points of L. In particular, a maximal compact subgroup of Δ_L has an orbit of dimension > 1 on L'. Recall that N acts freely on the set of exterior points. By Richardson's theorem, Δ_L induces a Lie group on L'. Hence, N and Δ are Lie groups. If $\Delta^* \cong \mathrm{SL}_3\mathbb{R}$, then there exists a Δ-invariant 2-dimensional subplane of $\mathcal{C}$ [Salzmann et al. 1995, 72.3], and $\dim L^\Delta = 2$ for a suitable line L. Hence, $\Delta'|_{L'}$ contains a circle group Φ. Again Φ acts effectively on L'. The proof can now be completed exactly as at the end of step (a). □

3.3. *Normal vector group.* *If* $\mathcal{F}_\Delta = \varnothing$, *if* Δ *has a minimal normal vector subgroup* Ξ, *and if* $\dim \Delta \ge 7$, *then* Δ *is a Lie group.*

Proof. From 2.3 it follows that $\mathcal{F} = \mathcal{F}_\Xi$ is a proper connected Δ-invariant subplane. There is a compact group $\mathrm{N} \lhd \Delta$ such that Δ/N is a Lie group. We may assume that $\dim \mathrm{N} = 0$, that N is not a Lie group, and that $\mathrm{N}|_\mathcal{F} = \mathbb{1}$. Note that $\dim \Delta \le 9$ by 3.1. If $\mathcal{F} \lessdot \mathcal{P}$, then $\Xi|_\mathcal{F} = \mathbb{1}$ by definition, and Ξ would be compact by stiffness. Hence, $\mathcal{F}$ is a 2-dimensional subplane.

(a) First, let $\dim \Delta = 9$. Then the induced group $\Delta|_{\mathcal{F}} = \Delta/\mathrm{K}$ is simple by 2.3; in fact, $\Delta/\mathrm{K} \cong \mathrm{SL}_3\mathbb{R}$. A maximal compact subgroup Φ of Δ is connected by the Malcev–Iwasawa theorem, and $\mathrm{N} < \Phi$. Consequently, $\dim \Phi = 4$ (or $\Phi \cong \mathrm{SO}_3\mathbb{R}$ by [Salzmann et al. 1995, 93.12]), Φ is a product of Φ' and a compact group $\Theta = \mathrm{K}^1$, $\Xi \cap \Theta = \mathbb{1}$, and Ξ would be contained in Δ/K^1, which is locally isomorphic to $\mathrm{SL}_3\mathbb{R}$.

(b) In the cases $\dim \Delta \in \{7, 8\}$ the induced group Δ/K is simple by 2.3, and then $\Delta : \mathrm{K} = 3$ [Salzmann et al. 1995, 33.6,7], but $\dim \mathrm{K} \le 3$ by stiffness. □

Remark. If $\mathcal{F}_\Delta$ is not empty, Ξ can be a group of axial collineations, and in the case $\Xi \lhd \Delta$ there are no sharper results than in general.

4. Exactly one fixed element

Up to duality, we may assume that $\mathcal{F}_\Delta$ consists of a line W.

4.1. ***Semisimple groups.*** *If the semisimple group Δ fixes exactly one line and possibly some points on this line, and if* $\dim \Delta > 3$, *then Δ is a Lie group* [Salzmann 2010, Theorem 1.3].

4.2. ***Theorem.*** *If $\mathcal{F}_\Delta = \{W\}$ and if* $\dim \Delta \ge 9$, *then Δ is a Lie group.*

Proof. (a) Again there exist arbitrarily small compact central subgroups $\mathrm{N} \le \Delta$ of dimension 0 such that Δ/N is a Lie group; see 2.5. If N acts freely on $P \setminus W$, then each stabilizer Δ_x with $x \notin W$ is a Lie group because $\Delta_x \cap \mathrm{N} = \mathbb{1}$, and Δ is a Lie group by 2.8.

(b) If $x^\zeta = x \notin W$ for some $\zeta \in \mathrm{N} \setminus \{\mathbb{1}\}$, then x^Δ is not contained in a line, $\zeta|_{x^\Delta} = \mathbb{1}$, and $\mathcal{E} = \langle x^\Delta, W\rangle$ is a proper connected subplane. Assume in this step that $\mathcal{E}$ is 2-dimensional. In this case the claim follows by similar arguments as in 3.1(a): let $\Delta^* = \Delta|_{\mathcal{E}} = \Delta/\mathrm{K}$. Then $\Delta : \mathrm{K} \le 6$ by [Salzmann 1967, 3.19] or [Salzmann et al. 1995, 33.6] together with the dimension formula 2.4, and $\dim \mathrm{K} \le 3$ by stiffness. It follows that $\dim \mathrm{K} = 3$, $\Delta : \mathrm{K} = 6$, and $\mathcal{E} \setminus W$ is the classical real affine plane [Salzmann 1967, 4.3]. As Δ^* is a Lie group, we may assume that $\mathrm{N} < \mathrm{K}$. Again N acts freely on the set of exterior points. The remainder of the proof is as in 3.1(a) with W instead of L.

(c) If Δ is not a Lie group, the case $\mathcal{E} \lessdot \mathcal{P}$ will lead to a contradiction. Write again $\Delta^* = \Delta|_{\mathcal{E}} = \Delta/\mathrm{K}$. Note that K is compact and acts freely on the set of points not in $\mathcal{E}$. If Δ is transitive on $W \cap \mathcal{E} \approx \mathbb{S}_2$, then a maximal compact subgroup of Δ induces a Lie group on W by 2.7. Hence, K and Δ are Lie groups. Therefore, Δ has a 1-dimensional orbit $V \subset W \cap \mathcal{E}$. Brouwer's theorem [Salzmann et al. 1995, 96.30] (see also [Hofmann 1965]) shows that $\Delta|_V = \Delta/\Gamma$ has dimension at most 3. Consequently $\dim \Gamma \ge 6$. Choose a point $v \in V$, a line L in $\mathcal{E}$ with $v \in L$, and an *exterior* point $z \in L$. By 2.6 we have $\dim z^{\Gamma_L} < 4$. Note that $\Lambda = (\Gamma_{L,z})^1$ fixes V pointwise and that $\dim \Lambda > 0$. Because N acts freely on $L \setminus \mathcal{E}$ and $\mathrm{N} \leq \mathrm{Cs}\,\Delta$, it follows that $\mathcal{F}_\Lambda$ is a proper connected subplane. Now N is a Lie group by 2.10. □

5. Collinear fixed points

Suppose in this section that Δ fixes a unique line W and one or more points on W.

5.1. *Theorem.* *Let $\mathcal{F}_\Delta = \{v, W\}$ be a flag. If* $\dim \Delta \geq 10$, *then Δ is a Lie group.*

Proof. By the approximation theorem, there is a compact 0-dimensional normal subgroup N such that Δ/N is a Lie group. Because of 2.8 we may assume that $x^\zeta = x$ for some $\zeta \in \mathrm{N} \setminus \{\mathbb{1}\}$ and some $x \notin W$. As x^Δ is not contained in a line and $\zeta|_{x^\Delta} = \mathbb{1}$, it follows that $\mathcal{C} = \langle x^\Delta, v, W\rangle$ is a proper connected subplane. If $\mathcal{C}$ is 2-dimensional, then $\dim \Delta|_\mathcal{C} \leq 5$ and $\dim \Delta \leq 8$ by stiffness. Therefore, $\mathcal{C}$ is a Δ-invariant Baer subplane. The induced group $\Delta|_\mathcal{C} = \Delta/\mathrm{K}$ is a Lie group by 2.10. Hence, it may be supposed that $\mathrm{N} \leq \mathrm{K}$. Obviously, K acts freely on the set of *exterior* points (points not in $\mathcal{C}$), and $\dim \mathrm{K} \leq 1$ by stiffness. Thus, $\Delta : \mathrm{K} \geq 9$, and $\mathcal{C}$ is isomorphic to the classical complex plane [Salzmann et al. 1995, 72.8]. Choose *interior* points $u, w \in W$, an interior line L in the pencil $\mathfrak{L}_v$, and an *exterior* point $z \in L$. If N is not a Lie group, then the connected component Λ of $\Delta_{u,w,z}$ has positive dimension by 2.6, because $\Delta : \Delta_{u,w,L} \leq 6$. Note that $z^\mathrm{N} \subset \mathcal{F}_\Lambda$ and that Λ fixes all interior points of W, so that $\mathcal{F}_\Lambda$ is a connected proper subplane. Now N is a Lie group by 2.10. □

5.2. *Theorem.* *If $\mathcal{F}_\Delta = \langle u, v\rangle$ and if* $\dim \Delta \geq 8$, *then Δ is a Lie group.*

For a proof see [Salzmann 2017, Lemma 6.0′].

5.3. *Proposition.* *If Δ fixes at least* 3 *distinct points and exactly* 1 *line, and if* $\dim \Delta \geq 8$, *then Δ is a Lie group.*

Remark. This follows from 5.2. An easy proof is given in [Salzmann 2017, 7.0′].

5.4. *Compact normal subgroup.* *Suppose that $\mathcal{F}_\Delta$ is a flag and that Δ has a serpentine normal subgroup Θ. If* $\dim \Delta \geq 9$, *then Δ is a Lie group.*

Proof. This can be proved in a similar way as 5.1 and the first arguments are the same. Again there is a Δ-invariant Baer subplane $\mathcal{C}$ and $\Delta|_\mathcal{C} = \Delta/\mathrm{K}$ is a Lie group. Note that $\Theta \leq \mathrm{Cs}\,\Delta$ by 2.9 and that $\Theta|_\mathcal{C}$ is a Lie group.

(a) $\Theta^* = \Theta|_\mathcal{C}$ *does not contain any involution*: as $\mathcal{F}_\Delta$ is a flag, there is no reflection in Θ^*. If ι is a planar involution in Θ^*, then $\mathcal{C} \cap \mathcal{F}_\iota$ is a Δ-invariant 2-dimensional subplane and stiffness implies $\dim \Delta \leq 5 + 1$. Hence, $\Theta^* = \mathbb{1}$ and $\Theta \leq \mathrm{K}$.

(b) Choose an *interior* line $L \in \mathfrak{L}_v$, and *exterior* points $x \in L$ and $z \in W$. The kernel K acts freely on the set of all exterior points. Result 2.6 implies that $1 \le \dim x^{\mathrm{K}}, \dim z^{\mathrm{K}} \le 3$, so that $\Lambda = (\Delta_{x,z})^1$ has positive dimension. Recall that $\mathrm{N} \le \mathrm{K}$. Put $\Gamma = \Theta\mathrm{N}$ and $\mathcal{E} = \langle x^\Gamma, z^\Gamma \rangle$. Then $\mathcal{E} \le \mathcal{F}_\Lambda$ is a proper connected subplane, Γ acts faithfully on $\mathcal{E}$, and Γ, N, and Δ are Lie groups (2.10). □

5.5. ***Compact normal subgroup.*** *Assume that* $\mathcal{F}_\Delta = \langle u, v, w \rangle$. *If* $\dim \Delta \ge 7$, *and if* Δ *has a serpentine normal subgroup* Θ, *then* Δ *is a Lie group.*

Proof. If Δ is not a Lie group, there exists a point $p \notin W = uv$ such that $\mathcal{E} = \langle p^\Delta, u, v, w \rangle$ is a 2- or 4-dimensional subplane; see steps (a) and (b) in the proof of 4.2. Put $\Delta|_\mathcal{E} = \Delta/\mathrm{K}$. In the first case, $\Delta : \mathrm{K} \le 3$ and $\dim \mathrm{K} \le 3$ by the dimension formula and stiffness. Therefore, $\mathcal{E} \lessdot \mathcal{P}$ and lines are homeomorphic to $\mathbb{S}_4$. Recall that $\Theta \le \mathrm{Cs}\,\Delta$ and that $\Theta|_\mathcal{E}$ is a Lie group, either a torus or trivial. A torus would contain a reflection [Salzmann et al. 1995, 55.21(c)], and Δ would fix some point $c \notin W$. Hence, $\mathcal{E} = \mathcal{F}_\Theta$ and $\Theta \le \mathrm{K}$. There is a compact central subgroup $\mathrm{N} < \Delta$ such that Δ/N is a Lie group and $\mathrm{N} \le \mathrm{K}$. As $\mathcal{E}$ is maximal in $\mathcal{P}$, the kernel K acts freely on the set of points outside $\mathcal{E}$ (the exterior points). Let x be an exterior point on an *interior* line L in the pencil $\mathfrak{L}_v$. Because of 2.6, we have $\Delta_L : \Delta_x < 4$. Hence, $\Lambda = (\Delta_x)^1$ satisfies $\dim \Lambda \ge 2$. Stiffness implies that $\mathcal{F}_\Lambda$ is 2-dimensional. KN acts freely on $\mathcal{F}_\Lambda$, and N is a Lie group by 2.10, but then Δ is also a Lie group. □

Arguments a little more intricate show that even the following is true:

5.6. ***Compact normal subgroup.*** *Assume that* $\mathcal{F}_\Delta = \langle u, v \rangle$. *If* $\dim \Delta \ge 7$, *and if* Δ *has a serpentine normal subgroup* Θ, *then* Δ *is a Lie group.*

Proof. Suppose that Δ is not a Lie group. Again there is a point $p \notin W = uv$ such that $\mathcal{E} = \langle p^\Delta, u, v \rangle$ is a proper connected subplane; see steps (a) and (b) in the proof of 4.2. Put $\Delta|_\mathcal{E} = \Delta/\mathrm{K}$. There is a compact central subgroup $\mathrm{N} < \Delta$ of dimension $\dim \mathrm{N} = 0$ such that Δ/N is a Lie group and $\mathrm{N} \le \mathrm{K}$.

(a) If $\mathcal{E}$ is 2-dimensional, then $\dim \mathrm{K} = 3$ and $\Delta : \mathrm{K} = 4$. From [Salzmann et al. 1995, 33.9] it follows that $\mathcal{E}$ is the classical real plane; moreover, each compact subgroup of $\Delta|_\mathcal{E}$ is trivial, and $\Theta|_\mathcal{E} = \mathbb{1}$. Let L be a line of $\mathcal{E}$ in the pencil $\mathfrak{L}_v$ and consider a point $x \in L \setminus \mathcal{E}$ and a third point $w \in uv \cap \mathcal{E}$. Then $\Lambda = \Delta_{x,w}$ has positive dimension and fixes each point of $uv \cap \mathcal{E}$. Hence, $\mathcal{F}_\Lambda$ is a proper connected subplane, and $\mathrm{N}|_{\mathcal{F}_\Lambda}$ is a Lie group by 2.10. This is true for each choice of x. As $\mathcal{P}$ is generated by $\mathcal{E}$ and at most two of such subplanes, N itself is a Lie group, and so is Δ.

(b) Thus, $\mathcal{E} \lessdot \mathcal{P}$ and lines are homeomorphic to $\mathbb{S}_4$ by 2.1. Recall that $\Theta \le \mathrm{Cs}\,\Delta$. Again $\Theta|_\mathcal{E}$ is a compact Lie group by 2.10, and $\Theta|_\mathcal{E}$ is either a torus or trivial. In the first case, the involution in $\Theta|_\mathcal{E}$ is a reflection by [Salzmann et al. 1995, 55.21(c)], and Δ would fix its center and axis. Hence, $\Theta|_{\mathcal{E}} = \mathbb{1}$ and $\mathcal{F}_\Theta = \mathcal{E}$. Choose L, x, and w as in step (a). Because of 2.6, we have $\dim \Delta_x \geq 2$. Put $\Lambda = \Delta_{x,w}$ and note that ΘN acts freely on $L \setminus \mathcal{E}$. It follows that $\mathcal{F}_\Lambda$ is connected and that N acts effectively on $\mathcal{F}_\Lambda$. Hence, $\mathcal{F}_\Lambda = \mathcal{P}$ and $\Delta_{x,w} = \mathbb{1}$ for each admissible w. Therefore, Δ_x is sharply transitive on a cylinder and Δ_x has a torus subgroup Ψ. If the involution $\iota \in \Psi$ is planar, then ΘN acts effectively on $\mathcal{F}_\iota$, and N would be a Lie group. Thus, ι is a reflection, its axis is L and its center is u. Interchanging the roles of u and v, we find also a torus subgroup $\Phi < \Delta$ such that the involution $\sigma \in \Phi$ has the center v. We have $\Delta_{w,L} : \Delta_{w,x} \leq \dim x^\Delta \leq 3$ and $\dim L^{\Delta_w} = \Delta_w : \Delta_{w,L} \geq 5-3$. Consequently Δ is transitive on the set of admissible lines L, which is homeomorphic to $\mathbb{R}^2$. Therefore, Φ fixes one of the lines L. This follows, e.g., from the much more general result [Poncet 1959, Théorème a]. The axis of σ is an interior line in $\mathfrak{L}_u$ and $\sigma \notin \Phi_x$ so that Φ_x is finite. As $L^\Delta \approx \mathbb{R}^2$ is simply connected, a maximal compact subgroup X of Δ_L is connected [Salzmann et al. 1995, 93.10], and X induces a connected group $\overline{\mathrm{X}}$ on $L \setminus \mathcal{E}$. The group Φ yields a torus $\overline{\Phi} \leq \overline{\mathrm{X}}$. If $\dim \overline{\mathrm{X}} = 2$, then $\overline{\mathrm{X}} = \overline{\Phi}\Theta$ by [Salzmann et al. 1995, 93.12], and $\mathrm{N} < \Theta$. Moreover, $\overline{\Phi} \cap \Theta = \mathbb{1}$ because Φ acts effectively on $\mathcal{E}$, and $\dim x^{\Phi\Theta} > 1$ by 2.11. If $\dim \overline{\mathrm{X}} > 2$, then $\dim x^{\mathrm{X}} \geq 2$ because $\mathrm{X}_x \leq \Delta_x$ and X_x is a torus. In both cases, X is a Lie group by [Salzmann et al. 1995, 96.24], and then Δ is also a Lie group. □

6. Nonincident fixed elements

If Δ fixes a nonincident point-line pair (and possibly further elements), then Löwen's criterion 2.8 does not apply.

6.1. *Proposition.* *If Δ fixes a line W and if Δ is transitive on W, then Δ is a Lie group* [Priwitzer 1994, 2.1].

Alternative proof. By [Hofmann and Kramer 2015, Corollary 5.5], the induced group $\Delta|_W$ is a Lie group and W is a manifold; in fact, $W \approx \mathbb{S}_4$ [Salzmann et al. 1995, 52.3]. From [Salzmann et al. 1995, 96.19–22] it follows that $\Delta|_W$ has a transitive subgroup $\mathrm{SO}_5\mathbb{R}$. The Malcev–Iwasawa theorem [Salzmann et al. 1995, 93.10] implies that a maximal compact subgroup Φ of Δ is connected. The result [Salzmann et al. 1995, 55.40] shows that Φ has a subgroup $\Upsilon \cong \mathrm{Spin}_5\mathbb{R}$. The central involution in Υ is a reflection with some center $a \notin W$. It suffices to show that Φ is a Lie group. By the approximation theorem, there is an arbitrarily small central subgroup $\mathrm{N} < \Phi$ such that Φ/N is a Lie group. As N centralizes each stabilizer Υ_z with $z \in W$, we conclude that $\mathrm{N}|_W = \mathbb{1}$, i.e., N consists of homologies with axis W and center a. Select a point $v \in W$ and consider the action of Φ_v on the line av. Note that $\Upsilon_v \cong \mathrm{Spin}_4\mathbb{R}$ fixes a second point $u \in W$, and that Υ_v has no subgroup of dimension 5. Put $\Upsilon_v|_{av} = \Upsilon_v/\mathrm{K}$. The homology group K has dimension at most 3. Hence, Υ_v has an orbit on av of dimension > 1, and

Richardson's theorem applies to $\Phi_v|_{av}$. In particular, Φ_v induces a Lie group on av, and then N is a Lie group. □

6.2. *Semisimple groups.* *Suppose that $\mathcal{F}_\Delta$ is a nonincident point-line pair $\{a, W\}$, Δ is semisimple, and* $\dim \Delta \geq 10$. *Then Δ is a Lie group.*

Proof. By [Priwitzer 1994] we may assume that $\dim \Delta < 12$.

Case 1 ($\dim \Delta = 11$). Then $\Delta = \Gamma\Psi$ is a product of two almost simple factors, where $\dim \Gamma = 3$.

(a) Suppose that Δ is not a Lie group, and denote the center of Δ by Z. If $\Gamma Z|_W \neq \mathbb{1}$, then there is a point p such that $\mathcal{G} = \langle p^{\Gamma Z}, a, W\rangle$ is a connected subplane (note that $\Gamma|_W = \mathbb{1}$ implies $p^\Gamma \neq p$). If $\dim p^\Psi = 8$, then Δ would be a Lie group by [Salzmann et al. 1995, 53.2]. Therefore, $\Psi_p \neq \mathbb{1}$ and $\Psi_p|_{\mathcal{G}} = \mathbb{1}$, so that $\mathcal{G}$ is a proper subplane (in fact a Baer subplane) and $\Gamma Z|_{\mathcal{G}}$ is a Lie group (see 2.10). Thus, $\mathcal{G} = \mathcal{F}_\zeta$ for some $\zeta \in Z$. Consequently $\mathcal{G}^\Delta = \mathcal{G}$, but Δ cannot act on the 4-dimensional plane $\mathcal{G}$ [Salzmann et al. 1995, 71.8].

(b) Hence, $\Gamma Z \leq \Delta_{[a,W]}$. From [Salzmann et al. 1995, 61.2] it follows that the almost simple group Γ is compact. By [Salzmann et al. 1995, 55.32(ii)], the homology group Γ does not contain a pair of commuting involutions. Hence, $\Gamma \cong \mathrm{SU}_2\mathbb{C}$. Moreover, Γ has 3-dimensional orbits on any line av, $v \in W$. The group Ψ acts almost effectively on W and Ψ is not a Lie group. Therefore, $\Psi|_W \cong \mathrm{PSU}_3(\mathbb{C}, 1)$. In fact, $\Psi|_W$ is strictly simple because $Z|_W = \mathbb{1}$, and $\Psi|_W$ is different from $\mathrm{PSL}_3\mathbb{R}$ and from the compact group $\mathrm{PSU}_3(\mathbb{C}, 0)$ because these groups admit only finite coverings and Ψ is not a Lie group. The kernel K of the canonical map $\kappa : \Psi \to \Psi|_W$ is contained in Z. Let Φ be a maximal compact subgroup of Ψ. Then Φ is connected, $\Phi^\kappa \cong \mathrm{U}_2\mathbb{C}$, and $\dim \Phi = 4$. As Ψ is not a Lie group, it follows that K is compact. If lines are manifolds, then Richardson's theorem as stated in [Salzmann et al. 1995, 96.34] applies and shows that Φ has two fixed points on W. Let $v^\Phi = v \in W$. Then a maximal compact subgroup Ω of Δ fixes v, and Ω is connected by the Malcev–Iwasawa theorem [Salzmann et al. 1995, 93.10]. Now $\Omega|_{av}$ is a Lie group by 2.7, and so are $Z \leq \Omega$ and Δ. Thus, lines are not manifolds, and 2.6 implies that all orbits of Δ on W have dimension < 4.

(c) The structure theorem 2.9 shows that Φ' is a Lie group. In fact, $\Phi' \cong \mathrm{SU}_2\mathbb{C}$ because $\Phi'^\kappa \ncong \mathrm{SO}_3\mathbb{R}$. The restriction of κ to Φ' is an isomorphism, the involution $\omega \in \Phi'$ is in the center of Φ, and ω is not planar (or lines would be manifolds); moreover, ω is not a reflection with axis W. Hence, $\omega \in \Delta_{[u,av]}$ for suitable points $u, v \in W$. Choose a maximal compact subgroup Ω of Δ such that $\Phi \leq \Omega$, so that Ω fixes u and v. Both Φ' and Γ act effectively on au; the product of their involutions is a reflection in $\Delta_{[v,au]}$. Hence, $\Phi'\Gamma|_{au} \cong \mathrm{SO}_4\mathbb{R}$. From $\dim \Phi = 4$ it follows that $\dim \Omega = 7$. The structure theorem of compact groups [Salzmann et al. 1995, 93.11] shows that Ω is a product of the connected component Θ of its center and the groups Φ' and Γ. Let U be some nontrivial orbit of Ω on au and note that $\dim U < 4$; in fact, $\dim U = 3$ because Γ acts freely on U. By [Salzmann et al. 1995, 96.13] we have $\dim \Omega|_U \leq 6$. Consequently Ω has a 1-dimensional normal subgroup acting trivially on U. The only possible kernel contains Θ, but $\Theta|_U \neq \mathbb{1}$ since Z acts freely on U. This contradiction proves that $\dim \Delta \neq 11$.

Case 2 ($\dim \Delta = 10$). Then $\Delta/\mathrm{Z} \cong \mathrm{PSp}_4\mathbb{R} \cong \mathrm{O}'_5(\mathbb{R}, 2)$; note that the other two 10-dimensional simple groups have simply connected double coverings [Salzmann et al. 1995, 94.33] and hence cannot be images of non-Lie groups.

(a) The center Z acts freely on $C = \{x \in P \setminus W \mid x \neq a\}$: suppose that $p^\zeta = p$ for some $p \in C$ and $\zeta \in \mathrm{Z} \setminus \{\mathbb{1}\}$. Then $\zeta|_{p^\Delta} = \mathbb{1}$, by assumption p^Δ is not contained in a line, and $\mathcal{D} = \langle a, p^\Delta, W\rangle$ is a proper connected subplane. The induced group $\Delta|_\mathcal{D}$ is locally isomorphic to $\mathrm{Sp}_4\mathbb{R}$, and $\mathcal{D}$ is a Baer subplane, but then $\dim \Delta|_\mathcal{D} \leq 8$ because Δ fixes $a, W \in \mathcal{D}$. (According to [Salzmann et al. 1995, 72.8] a 4-dimensional plane with a group of dimension > 8 is classical, and $\Delta|_\mathcal{D}$ would be contained in $\mathrm{GL}_2\mathbb{C}$; see also [Salzmann 1971, 8.1].)

(b) If Δ contains a planar involution β, then Z induces a Lie group on $\mathcal{F}_\beta$, $\mathcal{F}_\beta = \mathcal{F}_\zeta$ for some $\zeta \in \mathrm{Z}$, and $\mathcal{F}_\zeta$ would be a Δ-invariant Baer subplane. This is impossible for the same reasons as in step (a).

(c) As Δ/Z has a subgroup $\mathrm{SO}_3\mathbb{R}$, the structure theorem 2.9 shows that Δ has a subgroup $\Phi \cong \mathrm{SU}_2\mathbb{C}$: in the case $\Phi \cong \mathrm{SO}_3\mathbb{R}$ one of 3 pairwise commuting reflections of Φ would have the axis W [Salzmann et al. 1995, 55.35], but $\mathrm{SO}_3\mathbb{R}$ is simple.

(d) Suppose that lines are manifolds. Then $W \approx \mathbb{S}_4$ by [Salzmann et al. 1995, 52.3]. Some orbit of Φ on W has dimension at least 2. Consequently Δ induces a Lie group Δ/K on W (use Richardson's theorem 2.7). The structure of Δ shows that a maximal compact subgroup Ω of Δ is 4-dimensional. As $\mathrm{K} \leq \mathrm{Z}$ and $\dim \mathrm{Z} = 0$, it follows that $\dim \Omega/\mathrm{K} = 4$. Note that $\Omega' = \Phi \cong \mathrm{SU}_2\mathbb{C}$. Richardson's theorem as stated in [Salzmann et al. 1995, 96.34] shows that either $\Phi|_W \cong \Phi$ has exactly two fixed points $u, v \in W$, where v is the center of the involution $\iota \in \Phi$, or $\Phi|_W \cong \mathrm{SO}_3\mathbb{R}$ has a circle of fixed points and the central involution $\iota \in \Phi$ is a reflection with axis W. In any case, there is a point $v \in W$ such that $v^\Phi = v$ and $\Phi|_{av} \cong \Phi$. By 2.7 each orbit c^Φ with $a, v \neq c \in av$ is a 3-sphere. It follows that the orbit space av/Φ is a closed interval J. The compact group $\mathrm{K} \leq \Delta_{[a,W]}$ induces a group of order-preserving homeomorphisms on J. Each endpoint $b = c^\Phi$ of an orbit $x^\mathrm{K} \subset J$ is a fixed element of K. Hence, K maps c^Φ onto itself. As K is central, $c^\kappa = c^{\varphi(\kappa)}$ defines an injective continuous homomorphism $\mathrm{K} \to \Phi$. Consequently K is finite and Ω would be a Lie group.

(e) Thus, lines are not manifolds, and by 2.6 each orbit of (a subgroup of) Δ on a line has dimension at most 3. The group Ω is a product $\Theta\Phi$, where Θ is the connected component of the center of Ω, $\Theta \cap \Phi \leq \langle\sigma\rangle$ is trivial or generated by the involution $\sigma \in \Phi$, and Θ is not a Lie group.

(f) Suppose that σ is not a reflection with axis W. Step (b) shows that σ has some center $u \in W$ and an axis av with $v \in W$. Consider an arbitrary point $z \in Y := W \setminus \{u, v\}$. We have $\dim \Phi_z = 0$, and Φ_z is finite. With [Salzmann et al. 1995, 93.6] it follows that $\dim \Delta_z = 7$ and $\dim \Delta_z\Phi = 10$. Therefore, $\Delta = \Delta_z\Phi$ and $z^\Delta = z^\Phi$. Thus, $Y^\Delta = Y$ and $\{u, v\}$ would be Δ-invariant, but $\mathcal{F}_\Delta = \{a, W\}$.

(g) Hence, $\sigma \in \Delta_{[a,W]}$. Recall that a maximal compact subgroup $\Omega = \Theta\Phi$ of Δ has dimension $\dim\Omega = 4$. If $z^\Delta \subseteq W$ is a nontrivial orbit, and if $\Delta|_{z^\Delta} = \Delta/\mathrm{K}$, then the kernel K is contained in Z (because Δ is almost simple). Therefore, Ω acts almost effectively on z^Δ. By [Salzmann et al. 1995, 96.13(a)] either $z^\Omega = z$ or $\dim z^\Omega = 3$. Consequently, $\dim z^\Delta = 3$ for each $z \in W$. (Note that $z^\Delta \neq z$. If $\dim z^\Delta < 3$, then $\Omega^\delta|_{z^\Delta} = \mathbb{1}$ for all $\delta \in \Delta$. As Δ is generated by all conjugates of Ω, this is impossible.)

(h) Θ *has* (at least) 2 *fixed points* $u, v \in W$. This follows from [Löwen 1976, Lemma 1 or 2]; see also 2.8 above.

(i) By 2.5, there is a sufficiently small compact central subgroup Ξ of Δ such that Δ/Ξ is a Lie group. Put $\mathrm{N} = \Theta \cap \Xi$. Then Θ/N is a Lie group, and so is Ω/N. Hence, Δ/N is also a Lie group. Denote the canonical map $\Delta \to \Delta/\mathrm{N}$ by λ. The quotient space $M = \Delta^\lambda/(\Delta_v)^\lambda$ is a manifold, and M can be written in the form

$$\{\{\mathrm{N}\gamma \mid \gamma \in \Delta_v\}\mathrm{N}\delta \mid \delta \in \Delta\} = \{\Delta_v\delta \mid \delta \in \Delta\} = \Delta/\Delta_v \approx v^\Delta,$$

since $\mathrm{N} < \Theta < \Delta_v$. Therefore, v^Δ is a 3-manifold. If $v^\Omega \neq v$, then [Salzmann et al. 1995, 96.11(a)] implies $v^\Omega = v^\Delta$. As $\Theta \leq \mathrm{Cs}\,\Omega$, we have $\Theta|_{v^\Omega} = \mathbb{1}$ and hence $\Theta|_{v^\Delta} = \mathbb{1}$, i.e., Θ is in the kernel of the action of Δ on M. This kernel is contained in Z because Δ is almost simple. Consequently $\dim\Theta = 0$, a contradiction showing that $v^\Omega = v$.

(j) Consider the action of Ω and of Φ on $K := av \setminus \{a, v\}$. The only involution in Φ is the reflection σ with axis W. Therefore, $\dim \Phi_c = 0$ for each $c \in K$, and the compact group Φ_c is finite. Let $\Gamma = (\Delta_v)^1$ and note that $\dim\Gamma = 7$, $\dim c^\Gamma = \dim c^\Phi = 3$, $\dim \Gamma_c = 4$, $\dim \Gamma_c\Phi = 7$, and hence $\Gamma = \Gamma_c\Phi$, $c^\Theta \subseteq c^\Gamma = c^\Phi$. As Δ/Z is a Lie group and $\mathrm{Z}_c = \mathbb{1}$ by step (a), it follows that the stabilizer $\Pi = \Theta_c$ is a Lie group. The condition $c^\vartheta = c^{\varphi(\vartheta)}$ defines a continuous injective isomorphism of the compact group Θ/Π onto a closed subgroup of Φ. Hence, Θ/Π is a Lie group, and so are Θ and Δ. □

6.3. ***Compact normal subgroup.*** *Suppose that* $\mathcal{F}_\Delta = \{a, W\}$ *is a nonincident point-line pair. If* Δ *has a serpentine normal subgroup* Θ *and if* $\dim \Delta \geq 11$, *then* Δ *is a Lie group.*

Proof. (a) Θ is contained in the center $\mathrm{Z} = \mathrm{Cs}\,\Delta$ (see 2.9), and Δ/Z is a Lie group. Assume that Z is not a Lie group. If $\mathrm{Z}|_W \neq \mathbb{1}$, there is some point $p \notin W$ such that $p^{\mathrm{Z}} \not\subseteq ap$, and $\Delta_p|_{\langle p^{\mathrm{Z}}\rangle} = \mathbb{1}$. From [Salzmann et al. 1995, 53.2] it follows that $\dim \Delta_p \geq 4$. Thus, $\mathcal{E} = \langle p^{\mathrm{Z}}\rangle$ is a proper connected subplane, and $\mathrm{Z}|_{\mathcal{E}}$ is a Lie group by 2.10. Therefore, $\zeta|_{\mathcal{E}} = \mathbb{1}$ for some $\zeta \in \mathrm{Z} \setminus \{\mathbb{1}\}$. In particular, $p^\zeta = p$, $\zeta|_{p^\Delta} = \mathbb{1}$, $\dim p^\Delta \leq 4$, and $\dim \Delta_p \geq 7$. This contradicts stiffness and proves that $\mathrm{Z} \leq \Delta_{[a,W]}$.

(b) By assumption, Δ has no fixed point on W, and 6.1 shows that Δ is not transitive on W. Hence, there is some orbit $V = v^\Delta \subset W$ such that $0 < \dim V < 4$. Choose points $u, w \in V$ and $c \in av \setminus \{a, v\}$ and note that $\dim c^{\Delta_v} < 4$ by 2.6. If $\Lambda = \Delta_{c,u,w} \neq \mathbb{1}$, then $\mathcal{F}_\Lambda$ is a proper connected subplane, Z acts freely on $\mathcal{F}_\Lambda$, and Z would be a Lie group by 2.10. We have $\dim \Delta_c \geq 5$ and $\dim u^{\Delta_c} = 3$ for each $u \in V \setminus \{v\}$. Consequently Δ is doubly transitive on V.

(c) By [Salzmann et al. 1995, 96.16–17], either V is compact and the induced group $\Delta^* = \Delta|_V$ is isomorphic to one of the simple groups $\mathrm{PSL}_4\mathbb{R}$, $\mathrm{O}'_5(\mathbb{R}, 1)$, or $\mathrm{PSU}_3(\mathbb{C}, 1)$, or Δ^* is an extension of $\mathbb{R}^3 \approx V$ by a transitive linear group. In the first case $\dim \Delta > 15$ and Δ is a Lie group. In the last case, $\dim w^{\Delta_{u,v}} \leq 1$, $\Lambda \neq \mathbb{1}$, and Δ is also a Lie group. Only two possibilities remain: Δ^* is a simple group of dimension 10 or 8.

(d) If $\dim \Delta^* = 10$, then a maximal semisimple subgroup Ψ of Δ is isomorphic to the simple group $\mathrm{O}'_5(\mathbb{R}, 1)$ or to its double cover $\mathrm{U}_2(\mathbb{H}, 1)$; a maximal compact subgroup Φ of Ψ is isomorphic to $\mathrm{SO}_4\mathbb{R}$ or to $\mathrm{Spin}_4\mathbb{R}$. Accordingly $\Phi_v \cong \mathrm{SO}_3\mathbb{R}$ or $\Phi_v \cong \mathrm{Spin}_3\mathbb{R}$. In the first case, Φ_v would contain a reflection with axis W, but $\mathrm{SO}_3\mathbb{R}$ is simple. Hence, $\Upsilon = \Phi_v$ is simply connected. The involution $\omega \in \Upsilon$ is contained in $\Delta_{[a,W]}$, and each orbit c^Υ, $c \in av \setminus \{a, v\}$, is 3-dimensional. Hence, $\omega \notin \Upsilon_c$ and Υ_c is finite. Moreover, $\Theta_c = \mathbb{1}$ and $\Upsilon \cap \Theta \leq \langle\omega\rangle$. Lemma 2.11, applied to $\Upsilon\Theta$, shows that $\dim c^{\Upsilon\Theta} = 4$. By 2.7 the group Θ is a Lie group and so is Δ.

(e) Finally, let $\Delta^* = \Delta/\mathrm{K} \cong \mathrm{PSU}_3(\mathbb{C}, 1)$. Note that the central group Θ is contained in K. There exists an 8-dimensional semisimple subgroup Ψ of Δ (see [Salzmann et al. 1995, 94.27] or apply Levi's theorem [Salzmann et al. 1995, 94.28] to a Lie approximation of Δ). Consequently $\mathrm{K} = \sqrt{\Delta}$ is the radical, $\Delta = \Psi\mathrm{K}$, and $\mathrm{K} \leq \mathrm{Cs}_\Delta\,\Psi$. Suppose that $z^{\mathrm{K}} \neq z \in W$, let $c \in az \setminus \{a, z\}$, and put $\Lambda = \Psi_c$. If $\dim \Lambda = 0$, then $\dim c^\Delta = 8$, and Δ would be a Lie group by [Salzmann et al. 1995, 53.2]. As Λ fixes a connected set of points on W, it follows that $\mathcal{E} = \mathcal{F}_\Lambda$ is a connected proper subplane, and $\mathcal{E}^\Theta = \mathcal{E}$ because $\Theta \leq \mathrm{Cs}\,\Lambda$. The fact that $\Theta|_V = \mathbb{1}$ implies that Θ acts effectively on $\mathcal{E}$, so that Θ would be a Lie group by 2.10 above. Therefore, $\mathrm{K} \leq \Delta_{[a,W]}$, and K contains a compact connected subgroup of dimension at least 2 by [Salzmann et al. 1995, 61.2]. If lines are manifolds, the claim follows from Richardson's theorem 2.7. In the other case, 2.6 shows $\dim z^\Delta < 4$ for each $z \in W$. In fact, Δ is doubly transitive on each orbit $z^\Delta \subseteq W$; see step (b) of the present proof. Moreover, all transformation groups $(\Delta/\mathrm{K}, U)$, where U is an orbit of Δ on W, are equivalent to $(\mathrm{PSU}_3(\mathbb{C}, 1), \mathbb{S}_3)$ by [Salzmann et al. 1995, 96.17(b)]. Consequently, Δ_v has a fixed point in each of these orbits. Let again $c \in av \setminus \{a, v\}$. Then $\dim c^{\Delta_v} < 4$, $\Lambda = \Delta_c$ fixes a quadrangle, and $\dim \Lambda \geq 5$. This contradicts stiffness and completes the proof. □

7. Fixed double flag

Throughout this section, let $\mathcal{F}_\Delta = \langle u, v, av\rangle$ be a *double flag*.

7.0. ***Fact.*** *If a semisimple group* Δ *fixes a double flag*, *then* $\dim \Delta \leq 10$ [Salzmann 2014, 6.1].

7.1. ***Semisimple groups.*** *Suppose that* $\mathcal{F}_\Delta$ *is a double flag. If* Δ *is semisimple and if* $\dim \Delta \geq 10$, *then* Δ *is a Lie group*.

Proof. (a) We have $\dim \Delta = 10$ by 7.0, and Δ is almost simple. Let Φ be a maximal compact subgroup of Δ. If Δ is not a Lie group, then Δ maps onto $\mathrm{PSp}_4\mathbb{R}$ (or else Φ is locally isomorphic to $\mathrm{SO}_k\mathbb{R}$, $k \in \{4, 5\}$, and Δ would be a Lie group). Hence, Φ' is locally isomorphic to $\mathrm{SU}_2\mathbb{C}$. The center Z of Δ is an infinite compact 0-dimensional subgroup, and Z acts freely on $P \setminus (uv \cup av)$: if $x^\zeta = x$ for some x not on a fixed line and $\zeta \in \mathrm{Z} \setminus \{\mathbb{1}\}$, then $\zeta|_{\langle x^\Delta\rangle} = \mathbb{1}$ and $\langle x^\Delta\rangle$ is a proper connected subplane, but the almost simple group Δ cannot act on this subplane [Salzmann et al. 1995, 71.8]. By the Malcev–Iwasawa theorem $\mathrm{Z} \leq \Phi$.

(b) *Any involution* $\sigma \in \Phi$ *is a reflection with axis* av; *in particular*, $\Phi' \cong \mathrm{SU}_2\mathbb{C}$ *and* $\Phi'|_{av} \cong \mathrm{SO}_3\mathbb{R}$. In fact, σ is not planar (or else Z would induce a Lie group on $\mathcal{F}_\sigma$ and the kernel of the induced action would not act freely on $P \setminus (uv \cup av)$). If $\sigma \in \Delta_{[a,uv]}$, then $\sigma^\Delta\sigma$ would be a normal subgroup of translations of dimension $\Delta : \Delta_a$. Hence, $\sigma \in \Delta_{[u,av]}$.

(c) Z *consists of homologies with axis* av. Suppose that $a^{\mathrm{Z}} \neq a$. Then $\dim \Delta_a \leq 7$ by [Salzmann 1979, (*)] or [Salzmann et al. 1995, 83.17], and $d = \dim a^\Delta \geq 3$. It follows that $av \approx \mathbb{S}_4$: in the case $d = 3$, $\dim \Delta_a = 7$ [Salzmann 1979, (**)]; otherwise apply 2.6. Moreover, 2.6 implies that $\Phi|_{av}$ is a Lie group, since Φ' has an orbit of dimension > 1 on av. More precisely, $\Phi|_{av} \cong \mathrm{SO}_3\mathbb{R}$ and $\Theta = \sqrt{\Phi}$ acts trivially on av; see the explicit form of Richardson's theorem in [Salzmann et al. 1995, 96.34].

(d) Δ *acts faithfully on* uv, *in particular*, $\Phi_{[uv]} = \mathbb{1}$: this holds since Δ is almost simple and $\Delta_{[uv]} \leq \mathrm{Z} \leq \Delta_{[av]}$.

(e) Recall that $\Phi' \cong \mathrm{SU}_2\mathbb{C}$ and that $\Phi = \Phi'\Theta$ is not a Lie group. If $\dim z^{\Phi'} = 2$ for some $z \in uv \setminus \{u, v\}$, then Φ'_z would contain an involution σ, but σ is a reflection in $\Phi_{[u,av]}$. Hence, $\dim z^{\Phi'} = 3 \leq \dim z^{\Phi}$. Note that all the assumptions of Lemma 2.11 are satisfied by Φ instead of Γ; in fact, $\Phi' \cap \Theta \leq \langle\sigma\rangle$, $\Theta \leq \Delta_{[u,av]}$, and $\Theta_z = \mathbb{1}$; moreover, $\dim \Phi'_z = 0$ and Φ'_z is finite. Consequently $\dim z^{\Phi} = 4$ and 2.6 implies that Δ is a Lie group. □

7.2. ***Compact normal subgroup.*** *If Δ has a serpentine normal subgroup Θ, and if $\dim \Delta \geq 11$, then Δ is a Lie group.*

Proof. Assume that Δ is not a Lie group. By the approximation theorem, there is a compact subgroup $\mathrm{N} \lhd \Delta$ such that Δ/N is a Lie group and $\dim \mathrm{N} = 0$. From 2.9 it follows that $\Gamma := \Theta\mathrm{N} \leq \mathrm{Cs}\,\Delta$.

(a) If Γ is straight, then $\mathcal{F}_\Gamma \lessdot \mathcal{P}$ or Γ is a group of axial collineations with fixed center and axis in $\mathcal{F}_\Delta$ [Baer 1946]. In the first case, Δ induces on $\mathcal{F}_\Gamma$ a group of dimension at most 6, and $\dim \Delta \leq 7$ by stiffness. Letting $a \in \mathcal{F}_\Gamma$, we get $\dim \Delta_a \leq 5$.

(b) If Γ has the center v, then the axis passes through u and is fixed by Δ, i.e., $\Gamma \leq \Delta_{[v,uv]}$ and $\Gamma_a = \mathbb{1}$. From 2.6 it follows that there is a suitable point a such that $\dim a^{\Delta} < 4$. Let $z \in uv \setminus \{u, v\}$. The group Γ acts effectively on the *connected* subplane $\mathcal{D} = \langle a^{\Gamma}, z, u\rangle$ and $\Delta_{a,z}|_{\mathcal{D}} = \mathbb{1}$. In the cases $\mathcal{D} < \mathcal{P}$ both Γ and Δ would be Lie groups by 2.10. Therefore, $\Delta_{a,z} = \mathbb{1}$, $\dim \Delta \leq 7$, and $\dim \Delta_a \leq 4$.

(c) If Γ has the center u, then the axis of Γ is av. For a given point a there are points $z \in uv$ and $b \in au$ such that $\dim z^{\Delta}, \dim b^{\Delta} < 4$. As Γ is not a Lie group, the connected subplane $\mathcal{D} = \langle a, b, v, z^{\Gamma}\rangle$ coincides with $\mathcal{P}$. Consequently $\Delta_{a,b,z} = \mathbb{1}$, so that $\dim \Delta_a \leq 6$ and $\dim \Delta \leq 10$.

(d) If Γ is not straight, there is a point x such that $\mathcal{E} = \langle x^{\Gamma}, u, v, av\rangle$ is a connected subplane and $\Delta_x|_{\mathcal{E}} = \mathbb{1}$. In particular, $\Gamma_x|_{\mathcal{E}} = \mathbb{1}$ and Γ acts effectively on $\mathcal{E}$. Again $\mathcal{E} = \mathcal{P}$, and then $\dim \Delta \leq 7$ by 2.6. Similarly, $\dim \Delta_a \leq 6$. □

Remark. In any case, $\dim \Delta_a \leq 6$. This proves 8.2.

8. Fixed triangle

Let $\mathcal{F}_\Delta = \{a, u, v\}$ be a triangle.

8.0. ***Theorem.*** *If $\dim \Delta \geq 10$, then Δ is a Lie group.*

Proof. If Δ is not a Lie group, then 2.6 implies that Δ has only orbits of dimension at most 3 on two sides of the fixed triangle, say on uv and av. Hence, $\dim \Delta_z = 7$ for $z \in uv \setminus \{u, v\}$, and [Salzmann 1979, (**)] applies to Δ_z. Choose $c \in av \setminus \{a, v\}$ and put $x = az \cap cu$. Then $\dim \Delta_{c,z} \geq 4$, but 2.2(7) or [Salzmann 1979, (**)] asserts that $\Delta_x \cong \mathrm{SO}_3\mathbb{R}$, a contradiction. □

8.1. ***Semisimple groups.*** *If* $\mathcal{F}_\Delta$ *is a triangle*, *if* Δ *is semisimple*, *and if* $\dim\Delta \geq 9$, *then* Δ *is a Lie group.*

Proof. Suppose that Δ is not a Lie group. Only the case $\dim\Delta = 9$ has to be considered. Then Δ has a 3-dimensional factor Γ which is not a Lie group. Either the complement Ψ of Γ is locally isomorphic to $\mathrm{SL}_2\mathbb{C}$, or Ψ is a product of two 3-dimensional factors. Let $D = P \setminus (au \cup av \cup uv)$.

(a) *The center* Z *of* Δ *acts freely on* D: if $x^\zeta = x \in D$ for some $\zeta \in \mathrm{Z} \setminus \{\mathbb{1}\}$, then $\langle x^\Delta \rangle$ is a proper subplane, and $\dim\Delta_x \geq 5$ contrary to stiffness 2.2.

(b) $\Gamma|_{uv} \neq \mathbb{1}$ *and* $\Gamma/\mathrm{Z} \cong \mathrm{PSL}_2\mathbb{R}$: in the case $\Gamma \leq \Delta_{[a,uv]}$ it would follow from [Salzmann et al. 1995, 61.2] that Γ is compact and hence a Lie group. For the same reason, Γ acts nontrivially on the other sides of the fixed triangle.

(c) *There is at most one fixed line*, *say* uv, *such that* $\mathrm{Z}|_{uv}$ *is a Lie group*: otherwise Γ itself would be a Lie group.

(d) $\dim x^\Delta \leq 6$ *for each* $x \in D$, *and* $\dim\Delta_x \geq 3$: as $\mathrm{Z}|_{au}$ and $\mathrm{Z}|_{av}$ are not Lie groups, 2.6 implies that all orbits on these two sides of the fixed triangle have dimension < 4.

(e) *There is some* $p \in D$ *such that* $(\mathrm{Z}\Psi)_p = \mathbb{1}$, *and* $\Lambda = (\Delta_p)^1$ *satisfies* $\dim\Lambda = 3$; *moreover*, $(\Gamma\mathrm{Z})_p = \mathbb{1}$: if $p^\Gamma \not\subseteq ap$ (such a point p exists by step (b)), then $\langle p^\Gamma \rangle$ is a connected subplane, and $\langle p^{\Gamma\mathrm{Z}} \rangle = \mathcal{P}$, or else Z would be a Lie group by 2.10. On the other hand, $(\mathrm{Z}\Psi)_p|_{p^{\Gamma\mathrm{Z}}} = \mathbb{1}$, $\dim p^\Psi = 6$, $\dim\Delta_p = 3$ by step (d), $\langle p^\Psi \rangle = \mathcal{P}$, and $(\Gamma\mathrm{Z})_p|_{p^\Psi} = \mathbb{1}$.

(f) $\Lambda \cong \Gamma/\mathrm{Z}$ *and any involution* $\iota \in \Lambda$ *is planar*: consider the canonical epimorphism $\kappa : \Delta \to \Delta/\mathrm{Z}$ and note that $\Delta^\kappa = \Gamma^\kappa \times \Psi^\kappa$. Let π be the projection onto the first factor. Then $\kappa : \Lambda \cong \Lambda^\kappa$ since $\Lambda \cap \mathrm{Z} = \mathbb{1}$. The restriction $\pi : \Lambda^\kappa \to \Gamma^\kappa$ is injective because $\Lambda \cap \Psi\mathrm{Z} = \mathbb{1}$, and it is surjective since $\dim\Lambda = \dim\Gamma = 3$ [Salzmann et al. 1995, 93.12]. A reflection in Λ would have one of the fixed lines as axis, but Λ is simple; moreover, ι fixes a nondegenerate quadrangle. Therefore, ι is indeed planar. Now Z acts effectively on $\mathcal{F}_\iota$ by step (a), and Z is a Lie group contrary to the assumption. □

8.2. ***Compact normal subgroup.*** *If* Δ *has a serpentine normal subgroup* Θ, *and if* $\dim\Delta \geq 7$, *then* Δ *is a Lie group* (see the remark after 7.2).

Summary

The following table lists our conditions implying that Δ is a Lie group. There are always three conditions to be combined: the first column specifies the fixed configuration $\mathcal{F}_\Delta$, the first row lists possible assumptions on the structure of Δ, and in the body of the table, a lower bound for $\dim\Delta$ is given. The abbreviations in the first line mean, in this order, that Δ is semisimple, that Δ contains a serpentine normal subgroup in the sense of 2.12, that Δ contains a normal vector group, or that no condition is imposed on the structure of Δ.

$\mathcal{F}_\Delta$	Δ s-s	$\Theta \lhd \Delta$	$\mathbb{R}^t \lhd \Delta$	Δ arbitr.	references
$\varnothing$	9	9*	7	10	3.1, 3.2, 3.3
$\{W\}$	4			9	4.1, 4.2
flag	4	9		10	4.1, 5.4, 5.1
$\langle u, v\rangle$	4	7		8	4.1, 5.6, 5.2
$\langle u, v, w\rangle$	4	7		8	4.1, 5.5, 5.3
$\{o, W\}$	10	11		12	6.2, 6.3, [Priwitzer 1994]
$\langle u, v, ov\rangle$	10	11		12	7.1, 7.2, [Priwitzer 1994]
$\langle o, u, v\rangle$	9	7		10	8.1, 8.2, 8.0
arbitrary	10	11		12	[Priwitzer 1994]

Here 9* means that also $\Delta \cong \mathrm{SL}_3\mathbb{R} \times \Theta$ is conceivable.

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Received 18 Nov 2018. Revised 19 Mar 2019.

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Innovations in Incidence Geometry
Algebraic, Topological and Combinatorial
vol. 17, no. 3, 2019
dx.doi.org/10.2140/iig.2019.17.221

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On two nonbuilding but simply connected compact Tits geometries of type C_3

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A classification of homogeneous compact Tits geometries of irreducible spherical type, with connected panels and admitting a compact flag-transitive automorphism group acting continuously on the geometry, has been obtained by Kramer and Lytchak (2014; 2019). According to their main result, all such geometries but two are quotients of buildings. The two exceptions are flat geometries of type C_3 and arise from polar actions on the Cayley plane over the division algebra of real octonions. The classification obtained by Kramer and Lytchak does not contain the claim that those two exceptional geometries are simply connected, but this holds true, as proved by Schillewaert and Struyve (2017). Their proof is of topological nature and relies on the main result of (Kramer and Lytchak 2014; 2019). In this paper we provide a combinatorial proof of that claim, independent of (Kramer and Lytchak 2014; 2019).

1. Introduction

We presume that the reader has some knowledge of diagram geometry, in particular Tits geometries, namely geometries belonging to Coxeter diagrams, and buildings. A celebrated theorem of Tits [1981] states that Tits geometries generally come from buildings. Explicitly, a Tits geometry of rank $n \geq 3$ is 2-covered by a building if and only if all of its residues of type C_3 or H_3 are covered by buildings; moreover, buildings of rank $n \geq 3$ are 2-simply connected.

Having mentioned coverings and simple connectedness, I recall that, for $1 \leq k \leq n$, a *k-covering* of geometries of rank n is a type-preserving morphism which induces isomorphims on rank k residues (with the convention that an n-covering is just an isomorphism), the domain of a k-covering being called a *k-cover* of the codomain. A geometry is said to be *k-simply connected* if it does not admit any proper k-cover [Pasini 1994, Chapter 12]. (It goes without saying that a k-covering

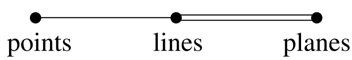
MSC2010: 20E42, 51E24, 57S15.
Keywords: compact geometries, composition algebras, diagram geometries.

is *proper* if it is not an isomorphism.) I warn that $(n-1)$-coverings are usually called *coverings*, for short (which forbids us from using the word “covering" as a possible abbreviation for k-covering). Accordingly, a geometry of rank n is said to be *simply connected* if it is $(n-1)$-simply connected. In particular, coverings of geometries of rank 3 are 2-coverings and when we say that a geometry of rank 3 is simply connected we just mean it is 2-simply connected.

Turning back to the above theorem of Tits, that theorem shows the importance of the investigation of C_3 geometries. As noticed by Tits [1981], geometries of type C_3 that have no relation at all with buildings can be constructed by some kind of free construction, but more examples exist that are not covered by buildings. Classifying them all is perhaps hopeless. Nevertheless, with the help of some reasonable additional hypotheses, something can be done. For instance, the following is well known [Aschbacher 1984; Yoshiara 1996]:

Theorem 1.1. *There exists a unique flag-transitive finite thick C_3-geometry which is not a building. It is simply connected and its automorphism group is isomorphic to the alternating group* Alt(7).

The exceptional geometry of Theorem 1.1 is called the Alt(7)-*geometry* (also *Neumaier geometry* after its discoverer Neumaier [1984]). Calling the elements of a C_3 geometry *points*, *lines* and *planes* as explained by the picture



the Alt(7)-geometry has 7 points, 35 lines and 15 planes. Moreover, all of its points are incident with all of its planes; therefore, this geometry is *flat*. We refer to [Neumaier 1984] (also [Rees 1985; Pasini 1994, §6.4.2, §12.6.4]) for more details on the Alt(7) geometry.

A number of flag-transitive locally finite (even finite) thick Tits geometries of irreducible type are known that admit the Alt(7)-geometry as a proper residue (see, e.g., [Buekenhout and Pasini 1995, §3] for a survey), but none of them belongs to a diagram of spherical type. Indeed, as proved by Aschbacher [1984], the Alt(7)-geometry cannot occur as a rank-3 residue in any flag-transitive finite thick Tits geometry of irreducible spherical type and rank $n > 3$. Moreover, no finite thick geometry of type H_3 exists (as no finite thick generalized pentagons exist [Feit and Higman 1964]) and no finite thick building of irreducible type and rank at least 3 admits proper quotients [Brouwer and Cohen 1983]. Consequently:

Corollary 1.2. *Apart from the* Alt(7)-*geometry*, *all flag-transitive finite thick Tits geometries of irreducible spherical type are buildings.*

Results in the same vein as Theorem 1.1 and Corollary 1.2 have recently been obtained by Kramer and Lytchak [2014; 2019] for compact Tits geometries with connected panels admitting a flag-transitive and compact group of automorhisms acting continuously on Γ. Before reporting on those results, I must explain what a compact geometry is and what we mean when saying that it admits connected panels.

Let Γ be a geometry over a (finite) set of types I. Assume that for every $i \in I$ a compact Hausdorff topology is given on the set Γ_i of i-elements of Γ and let $\mathcal{V}_i$ be the topological space thus defined on Γ_i. For every $J \subseteq I$ the set Γ_J of J-flags of Γ is a subspace, say $\mathcal{V}_J$, of the product space $\prod_{j \in J} \mathcal{V}_j$. If $\mathcal{V}_J$ is closed (equivalently, compact) for every $J \subseteq I$, then Γ is said to be a *compact geometry*. (We warn that this definition is not literally the same as in [Kramer and Lytchak 2014, §2.1], but it is equivalent to it; see Remark 1.7 below.) When saying that Γ has *connected panels* we mean that, for every type $i \in I$, the i-panels of Γ are connected as subspaces of $\mathcal{V}_i$ (or of $\mathcal{V}_I$, if we regard panels as sets of chambers).

With Γ a compact geometry as defined above, let G be a flag-transitive group of type-preserving automorphisms of Γ. Suppose that G is a locally compact topological group (we recall that for topological groups local compactness entails Hausdorff, by convention) and that G acts continuously on $\mathcal{V}_i$ for every $i \in I$ (explicitly, the function $\rho : G \times \mathcal{V}_i \to \mathcal{V}_i$ that maps $(g, x) \in G \times \mathcal{V}_i$ onto $g(x) \in \mathcal{V}_i$ is continuous). Then the pair (Γ, G) is called a *homogeneous compact geometry* [Kramer and Lytchak 2014, §2.1]. We call Γ and G the *geometric support* and the *group* of (Γ, G).

If (Γ, G) is a homogeneous compact geometry, then G also acts continuously on $\mathcal{V}_J$ for every $J \subseteq I$. Consequently, for every flag $X \in \Gamma_J$, the stabilizer G_X of X in G is closed in G (recall that, as $\mathcal{V}_J$ is Hausdorff, the singleton $\{X\}$ is closed in $\mathcal{V}_J$). The function $\rho_X : G/G_X \to \mathcal{V}_J$ which maps every coset gG_X onto the flag $g(X) \in \mathcal{V}_J$ is a continuous bijection from the coset space G/G_X to $\mathcal{V}_J$. If moreover G/G_X is compact (which is obviously the case when G is compact), then ρ_X is a homeomorphism. Indeed every continuous bijective mapping from a compact space to a Hausdorff space is a homeomorphism.

Conversely, without assuming any topology on the sets Γ_i, let G be a flag-transitive automorphism group of Γ carrying the structure of a locally compact group such that G_X is closed and G/G_X is compact for every flag X of Γ. Note that, as G is Hausdorff and G_X is closed, the coset space G/G_X is Hausdorff (see, e.g., [Freudenthal and de Vries 1969, §4.8]). For every $i \in I$ and chosen $x \in \Gamma_i$, we can copy the topology of G/G_x on Γ_i via the bijection $\rho_x : G/G_x \to \Gamma_i$, thus defining a compact Hausdorff space $\mathcal{V}_i$ on Γ_i. As $G/G_x \approx G/G_y$ for any two elements $x, y \in \Gamma_i$, the space $\mathcal{V}_i$ does not depend on the particular choice $x \in \Gamma_x$. The group G acts continuously on the space $\mathcal{V}_i$. Thus, Γ is turned into a compact geometry and (Γ, G) is a homogeneous compact geometry. By the previous paragraph, we also have $G/G_X \approx \mathcal{V}_J$ for any $J \subseteq I$ and any flag $X \in \mathcal{V}_J$.

In this way, as noticed in [Kramer and Lytchak 2014], one can see that all buildings of spherical type associated to semisimple or reductive isotropic algebraic groups defined over local fields are (geometric supports of) homogeneous compact geometries.

We add one more definition and a few conventions. Given two homogeneous compact geometries $(\widetilde{\Gamma}, \widetilde{G})$ and (Γ, G) of rank $n \geq 2$ with compact groups $\widetilde{G}$ and G, a *compact covering* from $(\widetilde{\Gamma}, \widetilde{G})$ to (Γ, G) is a 2-covering $\gamma : \widetilde{\Gamma} \to \Gamma$ such that γ is continuous as a mapping from the space $\widetilde{\mathcal{V}}$ of elements of $\widetilde{\Gamma}$ to the space $\mathcal{V}$ of elements of Γ, the group $\widetilde{G}$ normalizes the deck group D of γ and γ induces a continuous isomorphism from the topological group $\widetilde{G}/\widetilde{G} \cap D$ to the topological group G. Clearly, $\widetilde{G} \cap D$ is compact.

The category of homogeneous compact geometries with compact groups and compact coverings as morphisms is named **HCG** in [Kramer and Lytchak 2014]. We have defined compact coverings only for homogenous compact geometries with compact groups since these are the objects of **HCG**. According to this restriction, when we say that a given homogeneous compact geometry (Γ, G) with G compact is compactly covered by another homogeneous compact geometry $(\widetilde{\Gamma}, \widetilde{G})$, it must be understood that $\widetilde{G}$ too is compact.

We warn the reader that the name "compact covering" is not used in [Kramer and Lytchak 2014]. We have introduced it with the hope that it can remind the reader of the objects and the morphisms of the category **HCG**.

We say that a homogeneous compact geometry is a Tits geometry (in particular, a building) if its geometric support is a Tits geometry (a building). Accordingly, when saying that a homogeneous compact geometry with compact group is compactly covered by a building, we mean that it is compactly covered by a homogenous compact geometry, the geometric support of which is a building. It goes without saying that, when speaking of coverings of geometric supports, we mean coverings in the usual "combinatorial" sense, recalled at the beginning of this Introduction.

More generally, when we say that (Γ, G) has some geometric property which neither refers to the topology of Γ nor to the group G (such as being a flat C_3-geometry, for instance) we mean that the geometric support Γ of (Γ, G) has that property as a diagram geometry.

We are now ready to state the main result of Kramer and Lytchak [2014; 2019].

Theorem 1.3. *Let* (Γ, G) *be a homogeneous compact Tits geometry of type* C_3 *with connected panels and compact group* G. *Then either* (Γ, G) *is compactly covered by a building or it is one of two exceptional flat geometries where* G *is either* $((\mathrm{SU}(3) \times \mathrm{SU}(3))/C_3) \rtimes C_2$ *or* $\mathrm{SO}(3) \times \mathrm{G}_2$, *respectively, in its polar action* *on the Cayley plane of real octonions. Moreover, the geometric supports of these two exceptional geometries are not covered by any building.*

It is convenient to have a name for the two exceptional geometries mentioned in Theorem 1.3. We shall call them $\mathbb{O}\mathrm{P}^2$-*geometries* where $\mathbb{O}$ stands for the octonion algebra over the reals and $\mathbb{O}\mathrm{P}^2$ is the Cayley plane, namely the projective plane over $\mathbb{O}$.

By exploiting Theorem 1.3, Kramer and Lytchak [2014; 2019] also obtain:

Corollary 1.4. *Apart from the two* $\mathbb{O}\mathrm{P}^2$*-geometries, all homogeneous compact Tits geometries of irreducible spherical type, rank at least* 2, *with connected panels and compact group, are compactly covered by buildings.*

The two $\mathbb{O}\mathrm{P}^2$-geometries, or rather the group actions giving rise to them, were first discovered by Podestà and Thorbergsson [1999] and Gorodski and Kollross [2016], in the context of an investigation of polar actions of Lie groups on symmetric spaces. A purely algebraic construction of (the geometric supports of) these two geometries is given by Schillewaert and Struyve [2017]. We shall report on that construction in the next section.

Let (Γ, G) be any of the two $\mathbb{O}\mathrm{P}^2$-geometries. The reader should be warned that in the final part of Theorem 1.3 it is not claimed that Γ is simply connected. It is only stated that the universal cover $\widetilde{\Gamma}$ of Γ is not a building. Thus, in view of the rest of the statement of Theorem 1.3, if $\widetilde{\Gamma} \neq \Gamma$, then either $\widetilde{\Gamma}$ is not the geometric support of any homogeneous compact geometry with compact group or, if it is such, no compact covering exists from that homogeneous compact geometry to (Γ, G). So, it is natural to ask if Γ is simply connected. The following theorem, due to Schillewaert and Struyve [2017], answers this question in the affirmative.

Theorem 1.5. *The geometric support of either of the two* $\mathbb{O}\mathrm{P}^2$*-geometries is simply connected.*

The proof that Schillewaert and Struyve give for this theorem is of topological nature. They prove that, if (Γ, G) is any of the two $\mathbb{O}\mathrm{P}^2$-geometries, then the universal cover $\widetilde{\Gamma}$ of Γ carries a compact Hausdorff topology and G lifts to a compact group $\widetilde{G} \leq \mathrm{Aut}(\widetilde{\Gamma})$, so that $(\widetilde{\Gamma}, \widetilde{G})$ is a compact cover of (Γ, G). Having proved this, the conclusion follows from Theorem 1.3: necessarily $\widetilde{\Gamma} = \Gamma$. However, Schillewaert and Struyve [2017] also collect a great deal of information of combinatorial nature on homotopies of closed paths of the two $\mathbb{O}\mathrm{P}^2$-geometries. In this paper we shall exploit that information to arrange a combinatorial proof of Theorem 1.5, with no use of [Kramer and Lytchak 2014] or [2019].

Remark 1.6. As the title of [Kramer and Lytchak 2019] makes clear, an error occurs in [2014]: the $\mathbb{O}\mathrm{P}^2$-geometry associated to $\mathrm{SO}(3) \times \mathrm{G}_2$ is missing in [2014]. That gap is filled in [2019].

Remark 1.7. In the definition of compact geometry as stated in [Kramer and Lytchak 2014, §2.1], a compact Hausdorff topology $\mathscr{V}$ is assumed on the set of elements of Γ such that for every $J \subseteq I$ the set Γ_J is closed in the power space $\mathscr{V}^J$. In particular, Γ_i is closed in $\mathscr{V}$ for every $i \in I$. So, $\{\Gamma_i\}_{i\in I}$ is a finite partition of $\mathscr{V}$ in closed sets. Accordingly, $\mathscr{V}$ is the "free" union of the spaces $\mathscr{V}_i$ induced by $\mathscr{V}$ on the sets Γ_i for $i \in I$, the open sets of $\mathscr{V}$ being just the unions $\bigcup_{i\in I} A_i$ with A_i open in $\mathscr{V}_i$. Clearly, $\mathscr{V}^J$ and its subspace $\prod_{j\in J} \mathscr{V}_j$ induce the same topology on Γ_J. Thus, we can forget about $\mathscr{V}$ and start from a compact Hausdorff space $\mathscr{V}_i$ defined on Γ_i for each $i \in I$, as we have done in our definition.

2. The two $\mathbb{O}\mathrm{P}^2$-geometries

A description of the two $\mathbb{O}\mathrm{P}^2$-geometries as coset geometries is given by Kramer and Lytchak [2014] (for the geometry with group $G = (\mathrm{SU}(3) \times \mathrm{SU}(3))/C_3 \rtimes C_2$) and in [2019] (for $G = \mathrm{SO}(3) \times \mathrm{G}_2$). On the other hand, Schillewaert and Struyve [2017] propose a purely algebraic construction for these geometries, which we are going to recall in this section.

2A. ***Algebraic background.*** Let $\mathbb{A}$ be a division algebra over the field $\mathbb{R}$ of real numbers. It is well known that $\mathbb{A}$ has dimension 1, 2, 4 or 8 over $\mathbb{R}$. Accordingly, $\mathbb{A}$ is either $\mathbb{R}$ itself or the field $\mathbb{C}$ of complex numbers or the division ring $\mathbb{H}$ or real quaternions or the Cayley–Dickson algebra $\mathbb{O}$ of real octonions. In any case, $\mathbb{A}$ comes with a *norm* $|\cdot| : \mathbb{A} \to \mathbb{R}$ and a *conjugation* $\bar{\cdot} : \mathbb{A} \to \mathbb{A}$.

Explicitly, when $\mathbb{A} = \mathbb{R}$, then $|\cdot|$ is the usual absolute value and $\bar{\cdot}$ is the identity; if $\mathbb{A} = \mathbb{C}$, then $|\cdot|$ and $\bar{\cdot}$ are the usual modulus and conjugation. When $\mathbb{A} = \mathbb{H}$, then $\mathbb{A}$ can also be regarded as a right $\mathbb{C}$-vector space with canonical basis $\{1, \boldsymbol{j}\}$. The $\mathbb{C}$-span $\mathbb{C} = 1 \cdot \mathbb{C}$ of 1 is a subring of $\mathbb{H}$, $\boldsymbol{j}^2 = -1$ and $x\boldsymbol{j} = \boldsymbol{j}\bar{x}$ for any $x \in \mathbb{C}$. The norm and the conjugation of $\mathbb{H}$ map $x + \boldsymbol{j}y$ onto $\sqrt{|x|^2 + |y|^2}$ and $\bar{x} - \boldsymbol{j}y$, respectively. The conjugation of $\mathbb{H}$ is an involutory antiautomorphism. Clearly, $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{ji}\}$ is a basis of $\mathbb{H}$ over $\mathbb{R}$ (the canonical one), where $\boldsymbol{i}$ stands for any of the two square roots of -1 in $\mathbb{C}$.

Finally, $\mathbb{O}$ contains $\mathbb{H}$ as a subring and is generated by $\mathbb{H}$ together with an extra element $\boldsymbol{k}$ such that $\boldsymbol{k}^2 = -1$ and

$$u\boldsymbol{k} = \boldsymbol{k}\bar{u} \quad \text{for } u \in \mathbb{H}, \tag{1}$$

where $\bar{\cdot}$ denotes the conjugation in $\mathbb{H}$ as defined above. Moreover,

$$(\boldsymbol{k}u)v = \boldsymbol{k}(vu) = \bar{v}(\boldsymbol{k}u) \quad \text{and} \quad (\boldsymbol{k}u)(\boldsymbol{k}v) = -v\bar{u} \quad \text{for all } u, v \in \mathbb{H}. \tag{2}$$

Conditions (2) imply $(uv)\boldsymbol{k} = v(u\boldsymbol{k}) = v(\boldsymbol{k}\bar{u})$. Jointly with (1) they also imply that the elements of $\mathbb{O}$ admit the representation

$$u + \boldsymbol{k}v \quad \text{for } u, v \in \mathbb{H}. \tag{3}$$

In spite of (3), the multiplication of $\mathbb{O}$ does not yield an $\mathbb{H}$-vector space on $\mathbb{O}$, as it follows from the first equality of (2) and the fact that $\mathbb{H}$ is noncommutative. More precisely, $\mathbb{O}$ does carry an $\mathbb{H}$-vector space structure, as is clear from (3), but the scalar multiplication of that space is not the multiplication of $\mathbb{O}$ restricted to $\mathbb{O} \times \mathbb{H}$. On the other hand, for $x, y \in \mathbb{C}$ we have

$$(\boldsymbol{k}x)y = \boldsymbol{k}(yx) = \boldsymbol{k}(xy),$$
$$(\boldsymbol{kj}x)y = (\boldsymbol{k}(x\boldsymbol{j}))y = \boldsymbol{k}(y(x\boldsymbol{j})) = \boldsymbol{k}((yx)\boldsymbol{j}) = (\boldsymbol{kj})(yx) = (\boldsymbol{kj})(xy).$$

So, the multiplication of $\mathbb{O}$ restricted to $\mathbb{O} \times \mathbb{C}$ defines a 4-dimensional $\mathbb{C}$-vector space on $\mathbb{O}$, with $\{1, \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{kj}\}$ as the canonical basis. Needless to say, $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{ji}, \boldsymbol{k}, \boldsymbol{ki}, \boldsymbol{kj}, \boldsymbol{k}(\boldsymbol{ji})\}$ is a basis of $\mathbb{O}$ over $\mathbb{R}$ (the canonical one).

The norm and the conjugation of $\mathbb{O}$ map $u + \boldsymbol{k}v$ onto $\sqrt{|u|^2 + |v|^2}$ and $\bar{u} - \boldsymbol{k}v$, respectively. The conjugation of $\mathbb{O}$ is an involutory antiautomorphism.

In any case, the norm of $\mathbb{A}$ induces a positive definite $\mathbb{R}$-bilinear form $(\cdot \mid \cdot)_{\mathbb{R}}$ which maps $(x, y) \in \mathbb{A} \times \mathbb{A}$ onto the real part $\mathrm{Re}(\bar{x}y)$ of the product $\bar{x}y$. Clearly, $|x| = \sqrt{(x, x)_{\mathbb{R}}}$. We denote by $\perp_{\mathbb{R}} K$ the orthogonal complement of a subspace K of $\mathbb{A}$ with respect to $(\cdot \mid \cdot)_{\mathbb{R}}$.

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, with $\mathbb{F} = \mathbb{R}$ when $\mathbb{A} = \mathbb{R}$. Regarding $\mathbb{F}$ as a subfield of $\mathbb{A}$ in the usual way, namely as the $\mathbb{F}$-span of 1, we set $\mathrm{Pu}_{\mathbb{F}}(\mathbb{A}) := \perp_{\mathbb{R}} \mathbb{F}$ (in particular, $\mathrm{Pu}_{\mathbb{F}}(\mathrm{A}) = 0$ when $\mathbb{A} = \mathbb{F}$). Clearly, $\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ is a subspace of the $\mathbb{F}$-vector space $\mathbb{A}$ and $\mathbb{A} = \mathbb{F} \oplus \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$. The elements of $\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ are said to be $\mathbb{F}$-*pure*.

As $\mathbb{A} = \mathbb{F} \oplus \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$, every element $x \in \mathbb{A}$ splits in a unique way as a sum $x = x_1 + x_2$ with $x_1 \in \mathbb{F}$ and $x_2 \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$. We call x_1 and x_2 the $\mathbb{F}$-*part* and the $\mathbb{F}$-*pure part* of x.

When $\mathbb{F} = \mathbb{C}$ we also define a Hermitian inner product $(\cdot \mid \cdot)_{\mathbb{C}} : \mathbb{A} \times \mathbb{A} \to \mathbb{C}$ by taking $(x \mid y)_{\mathbb{C}}$ equal to the complex part of $\bar{x}y$. Obviously, $\mathrm{Re}((x \mid y)_{\mathbb{C}}) = (x \mid y)_{\mathbb{R}}$. Hence, we also have $|x| = \sqrt{(x \mid x)_{\mathbb{C}}}$ for every $x \in \mathbb{A}$.

The elements of $\mathbb{A}$ of norm 1 are called *unit elements*. Clearly, the set $\mathrm{Un}(\mathbb{A})$ of unit elements of $\mathbb{A}$ is closed under multiplication and taking inverses in $\mathbb{A}$ and

$$\mathbb{A} = \mathrm{Un}(\mathbb{A}) \cdot |\mathbb{R}| := \{x \cdot |t| \mid x \in \mathrm{Un}(\mathbb{A}),\ t \in \mathbb{R}\}.$$

We recall that a homomorphism of $\mathbb{F}$-algebras is an $\mathbb{F}$-linear mapping which also preserves multiplication. In the sequel we shall deal with a particular class of homorphisms of $\mathbb{F}$-algebras, which we shall call sharp $\mathbb{F}$-morphisms. We define them as follows:

Definition 2.1. With $\mathbb{F}$ equal to $\mathbb{R}$ or $\mathbb{C}$, let $\mathbb{A}$ and let $\mathbb{B}$ be two division algebras over $\mathbb{R}$ containing $\mathbb{F}$. When $\mathbb{F}=\mathbb{C}$ both $\mathbb{A}$ and $\mathbb{B}$ can also be regarded as algebras over $\mathbb{C}$. Thus, in any case, both $\mathbb{A}$ and $\mathbb{B}$ are $\mathbb{F}$-algebras.

A *sharp* $\mathbb{F}$*-morphism* from $\mathbb{A}$ to $\mathbb{B}$ is a homomorphism of $\mathbb{F}$-algebras from $\mathbb{A}$ to $\mathbb{B}$ which also preserves the inner product $(\cdot\mid\cdot)_{\mathbb{F}}$.

Let $\phi:\mathbb{A}\to\mathbb{B}$ be a sharp $\mathbb{F}$-morphism. Then ϕ is injective, since it preserves $(\cdot\mid\cdot)_{\mathbb{F}}$. Consequently, $\phi(1)=1$; hence, $\phi(\mathrm{Pu}_{\mathbb{F}}(\mathbb{A}))\subseteq\mathrm{Pu}_{\mathbb{F}}(\mathbb{B})$. Moreover, $\phi(\mathrm{Un}(\mathbb{A}))\subseteq\mathrm{Un}(\mathbb{B})$. We have $\bar{x}=x^{-1}$ for every unit element x. Therefore, $\phi(\bar{x})=\overline{\phi(x)}$ for every $x\in\mathrm{Un}(\mathbb{A})$. Finally, ϕ also preserves conjugation.

As sharp $\mathbb{F}$-morphisms are injective, every sharp $\mathbb{F}$-morphism from $\mathbb{A}$ to $\mathbb{A}$ is an automorphism. We call it a *sharp* $\mathbb{F}$*-automorphism*.

Setting 2.2. From now on we assume that $\mathbb{A}$ and $\mathbb{F}$ are as follows: either $\mathbb{A}=\mathbb{H}$ and $\mathbb{F}=\mathbb{R}$ or $\mathbb{A}=\mathbb{O}$ and $\mathbb{F}=\mathbb{C}$.

The following is proved in [Schillewaert and Struyve 2017, Proposition 2.1]:

Lemma 2.3. *With* $\mathbb{F}$ *and* $\mathbb{A}$ *as in Setting 2.2, let* $a_1,a_2\in\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ *and* $b_1,b_2\in\mathrm{Pu}_{\mathbb{F}}(\mathbb{B})$ *be such that* $(a_1\mid a_2)_{\mathbb{F}}=(b_1\mid b_2)_{\mathbb{F}}$, $|a_i|=|b_i|$ *for* $i=1,2$ *and* $a_1\mathbb{F}\neq a_2\mathbb{F}$. *Then there exists a unique sharp* $\mathbb{F}$*-morphism from* $\mathbb{A}$ *to* $\mathbb{O}$ *mapping* a_i *onto* b_i *for* $i=1,2$.

Lemma 2.4. *Every sharp* $\mathbb{R}$*-morphism from* $\mathbb{H}$ *to* $\mathbb{O}$ *can be extended to a sharp* $\mathbb{R}$*-automorphism of* $\mathbb{O}$.

Proof. Let $\phi:\mathbb{H}\to\mathbb{O}$ be a sharp $\mathbb{R}$-morphism. Put $\boldsymbol{i}':=\phi(\boldsymbol{i})$ and $\boldsymbol{j}':=\phi(\boldsymbol{i})$ and recall that $\phi(1)=1$. Then $\phi(\mathbb{H})$ is the $\mathbb{R}$-span $\mathbb{H}':=\langle 1,\boldsymbol{i}',\boldsymbol{j}',\boldsymbol{j}'\boldsymbol{i}'\rangle_{\mathbb{R}}$ of $\{1,\boldsymbol{i}',\boldsymbol{j}',\boldsymbol{j}'\boldsymbol{i}'\}$ and ϕ is a sharp $\mathbb{R}$-isomorphism from $\mathbb{H}$ to $\mathbb{H}'$. We can construct a copy $\mathbb{O}'$ of $\mathbb{O}$ starting from $\mathbb{H}'$ instead of $\mathbb{H}$, and if $\boldsymbol{k}'$ is the element of $\mathbb{O}'$ corresponding to $\boldsymbol{k}$, a sharp $\mathbb{R}$-isomorphism $\psi:\mathbb{O}\to\mathbb{O}'$ is uniquely determined which maps $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ onto $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$, respectively, which coincides with ϕ in $\mathbb{H}$. If we can choose $\boldsymbol{k}'\in\mathbb{O}$, then ψ can also be regarded as a sharp $\mathbb{F}$-automorphism of $\mathbb{O}$ and we are done.

So it remains to prove that we can choose $\boldsymbol{k}'\in\mathbb{O}$, namely $\mathbb{O}$ contains an element $\boldsymbol{k}'$ orthogonal to $\mathbb{H}$ and such that $(\boldsymbol{k}')^2=-1$. But this is obvious. Indeed every unit element orthogonal to $\mathbb{H}$ has this property. The conclusion follows. □

2B. ***Construction of the geometries.*** With $\mathbb{A}$ and $\mathbb{F}$ as in Setting 2.2, let $\mathrm{PG}(\mathbb{A})$ be the projective space of the $\mathbb{F}$-vector space $\mathbb{A}$. For every nonzero vector $x\in\mathbb{A}$, we denote by $[x]$ the corresponding point of $\mathrm{PG}(\mathbb{A})$, and for every subset X of $\mathbb{A}$ we put $[X]:=\{[x]\mid x\in X\setminus\{0\}\}$. In particular, if X is a subspace of $\mathbb{A}$, then $[X]$ is the corresponding subspace of $\mathrm{PG}(\mathbb{A})$.

We write $(\cdot\mid\cdot)$ instead of $(\cdot\mid\cdot)_{\mathbb{F}}$ and $\perp$ instead of $\perp_{\mathbb{F}}$, for short. As usual, $\mathbb{F}^*$ stands for the multiplicative group of $\mathbb{F}$. Following Schillewaert and Struyve [2017], we construct a C_3-geometry $\Gamma_{\mathbb{F}}(\mathbb{A})$ as follows.

Definition 2.5. The elements (points, lines and planes) of $\Gamma_{\mathbb{F}}(\mathbb{A})$ are defined as follows:

(A1) The *points* are the points of $[\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})]$.

(A2) The *lines* are the sets of pairs $[x, u] := \{(xt, ut) \mid t \in \mathbb{F}^*\}$ with $x \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$, $u \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{O})$ and $|x| = |u| \neq 0$.

(A3) The *planes* are the sharp $\mathbb{F}$-morphisms $\phi : \mathbb{A} \to \mathbb{O}$.

The *incidence relation* is defined as follows:

(B1) Every point is incident with all planes.

(B2) A line $[x, u]$ and a point $[y]$ are declared to be incident when $y \in x^{\perp}$.

(B3) A line $[x, u]$ and a plane $\phi : \mathbb{A} \to \mathbb{O}$ are incident precisely when $\phi(x) = u$.

Clearly, the conditions defining point-line and line-plane incidences do not depend on the particular choice of the pair $(x, u) \in [x, u]$. It is proved in [Schillewaert and Struyve 2017, Proposition 4.3] that $\Gamma_{\mathbb{F}}(\mathbb{A})$ is indeed a C_3-geometry. According to clause (B1) of Definition 2.5, this geometry is flat.

Lemma 2.6. *Both the following hold*:

(1) *Two lines* $[x, u]$ *and* $[y, v]$ *are coplanar if and only if* $(x \mid y) = (u \mid v)$. *If this is the case*, *then the unique sharp* $\mathbb{F}$*-morphism* $\phi : \mathbb{A} \to \mathbb{O}$ *such that* $\phi(x) = u$ *and* $\phi(y) = v$ (*see* Lemma 2.3) *is the unique plane incident with both* $[x, u]$ *and* $[y, v]$.

(2) *If two lines have two distinct points in common*, *then they have the same set of points.*

Proof. Claim (1) immediately follows from Lemma 2.3 (see also [Schillewaert and Struyve 2017, Lemma 4.2]). Claim (2) follows from clause (B2) of Definition 2.5 and the fact that $\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ has dimension 3 over $\mathbb{F}$ (see also [Schillewaert and Struyve 2017, Lemma 5.1]). □

The set of points of a line $[x, u]$ is the line $x^{\perp} \cap \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ of $\mathrm{PG}(\mathrm{Pu}_{\mathbb{F}}(\mathbb{A}))$. We call it the *shadow* of $[x, u]$ and also a *shadow-line*. With this terminology, we can rephrase claim (2) of Lemma 2.6 as follows:

Corollary 2.7. *The set of points of* $\Gamma_{\mathbb{F}}(\mathbb{A})$ *equipped with the shadow lines as lines coincides with the projective plane* $\mathrm{PG}(\mathrm{Pu}_{\mathbb{F}}(\mathbb{A}))$.

2C. *Automorphism groups.* Let $\mathrm{Aut}_{\mathbb{F}}(\mathbb{A})$ and $\mathrm{Aut}_{\mathbb{F}}(\mathbb{O})$ be the groups of sharp $\mathbb{F}$-automorphisms of $\mathbb{A}$ and $\mathbb{O}$. The product $\mathrm{Aut}_{\mathbb{F}}(\mathbb{A}) \times \mathrm{Aut}_{\mathbb{F}}(\mathbb{O})$ acts on $\Gamma_{\mathbb{F}}(\mathbb{A})$ as a group of automorphisms. Explicitly, given an element $(\alpha, \omega) \in \mathrm{Aut}_{\mathbb{F}}(\mathbb{A}) \times \mathrm{Aut}_{\mathbb{F}}(\mathbb{O})$,

$$\begin{array}{ll} (\alpha, \omega) : [x] \to [\alpha(x)] & \text{for every point } [x] \text{ of } \Gamma_{\mathbb{F}}(\mathbb{A}), \\ (\alpha, \omega) : [x, u] \to [\alpha(x), \omega(u)] & \text{for every line } [x, u] \text{ of } \Gamma_{\mathbb{F}}(\mathbb{A}), \\ (\alpha, \omega) : \phi \to \omega\phi\alpha^{-1} & \text{for every plane } \phi \text{ of } \Gamma_{\mathbb{F}}(\mathbb{A}). \end{array}$$

The first questions one may ask are whether this action is faithful and whether all automorphisms of $\Gamma_{\mathbb{R}}(\mathbb{A})$ arise in these way. Both questions are answered by Schillewaert and Struyve [2017], but the answers are different according to whether $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$ or $(\mathbb{F}, \mathbb{A}) = (\mathbb{C}, \mathbb{O})$.

Let $\mathbb{F} = \mathbb{R}$ and $\mathbb{A} = \mathbb{H}$. Then both questions are answered in the affirmative:

$$\mathrm{Aut}(\Gamma_{\mathbb{R}}(\mathbb{H})) = \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}) \times \mathrm{Aut}_{\mathbb{R}}(\mathbb{O}) = \mathrm{SO}(3) \times \mathrm{G}_2.$$

(Recall that $\mathrm{Aut}_{\mathbb{R}}(\mathbb{H}) = \mathrm{SO}(3)$ and $\mathrm{Aut}_{\mathbb{R}}(\mathbb{O}) = \mathrm{G}_2$.) When $\mathbb{F} = \mathbb{C}$ and $\mathbb{A} = \mathbb{O}$ the answer is sligthly different. Indeed $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$ acts nonfaithfully on $\Gamma_{\mathbb{C}}(\mathbb{O})$, with kernel a group C_3 of order 3 contributed by elements (ζ, ζ) with ζ in the center of $\mathrm{SU}(3)$ (recall that $\mathrm{SU}(3) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$). Moreover, the conjugation in $\mathbb{C}$ also induces an automorphism γ of $\Gamma_{\mathbb{C}}(\mathbb{O})$ which, being semilinear as a mapping of $\mathbb{O} \times \mathbb{O}$, does not belong to $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$. All automorphisms of $\Gamma_{\mathbb{C}}(\mathbb{O})$ belong to the group generated by $(\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{Aut}_{\mathbb{C}}(\mathbb{O}))/C_3$ and γ. To sum up,

$$\begin{aligned} \mathrm{Aut}(\Gamma_{\mathbb{C}}(\mathbb{O})) &= ((\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{Aut}_{\mathbb{C}}(\mathbb{O}))/C_3) \rtimes C_2 \\ &= ((\mathrm{SU}(3) \times \mathrm{SU}(3))/C_3) \rtimes C_2. \end{aligned}$$

2D. ***Recognizing*** $\mathbf{\Gamma}_{\mathbb{F}}(\mathbb{A})$ ***as an*** $\mathbb{O}\mathbf{P}^2$***-geometry.*** Let $\Gamma := \Gamma_{\mathbb{F}}(\mathbb{A})$ and $G := \mathrm{Aut}(\Gamma)$. As shown by Schillewaert and Struyve [2017, §5], in either of the two cases that we have considered, (Γ, G) is a homogeneous compact geometry. They obtain this conclusion by noticing that in either case G is compact and the stabilizers in G of the flags of Γ are closed in G, but a direct proof is also possible. We shall briefly sketch it here.

In order to stick to the notation used in the Introduction of this paper, let Γ_1, Γ_2 and Γ_3, respectively, be the sets of points, lines and planes of Γ. In either case each of Γ_1, Γ_2 and Γ_3 can be equipped with a natural compact topology.

Explicitly, $\Gamma_1 = [\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})]$ carries the topology of the real projective plane $\mathbb{R}\mathrm{P}^2$ when $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$ and the topology of the complex projective plane $\mathbb{C}\mathrm{P}^2$ when $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$. Either of these spaces is both Hausdorff and compact.

When $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$, the line-set Γ_2 carries the topology of the quotient $(\mathbb{S}^2 \times \mathbb{S}^6)/Z$ of the product space $\mathbb{S}^2 \times \mathbb{S}^6 \subset \mathbb{R}^{10}$ over the center Z of $\mathrm{SL}(\mathbb{R}^{10})$. When $(\mathbb{F}, \mathbb{A}) = (\mathbb{C}, \mathbb{O})$ then Γ_2 carries the topology of the quotient $(U \times U)/\Lambda$ where $U := \{x \in \mathbb{C}^3 \mid |x| = 1\}$ is the standard unital of $\mathbb{C}^3$ and Λ is the group of scalar transformations $\lambda \cdot \mathrm{id}$ of $\mathbb{C}^6$ with $|\lambda| = 1$. Again, either of these spaces is Hausdorff and compact.

When $(\mathbb{F}, \mathbb{A}) = (\mathbb{C}, \mathbb{O})$ then Γ_3 carries the same topology as $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) = \mathrm{SU}(3)$, which is (Hausdorff and) compact. Finally, let $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$. Then every sharp $\mathbb{R}$-morphism from $\mathbb{H}$ to $\mathbb{O}$ can be regarded as the restriction of a sharp $\mathbb{R}$-automorphism of $\mathbb{O}$ (Lemma 2.4). Accordingly, the planes of Γ naturally correspond to the cosets ωH of the elementwise stabilizer H of $\mathbb{H}$ in $G := \mathrm{Aut}_{\mathbb{R}}(\mathbb{O}) = \mathrm{G}_2$. The group H is the intersection $H = \bigcap_{x \in \mathbb{H}} G_x$ of the stabilizers G_x for $x \in \mathbb{H}$, which are closed. Hence, H is closed as well. Thus, Γ_3 can be regarded as a copy of the quotient-space G/H, which is still compact and Hausdorff since H is closed.

As in the Introduction, let $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ be the spaces defined on Γ_1, Γ_2 and Γ_3 as above. It is straighforward to check that $\Gamma_{\{i,j\}}$ is closed in $\mathcal{V}_i \times \mathcal{V}_j$ for every choice of $1 \le i < j \le 3$ and the set of chambers $\Gamma_{\{1,2,3\}}$ is closed in $\mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3$. So Γ is a compact geometry. Each of the groups $\mathrm{Aut}(\Gamma_{\mathbb{R}}(\mathbb{H})) = \mathrm{SO}(3) \times \mathrm{G}_2$ and $\mathrm{Aut}(\Gamma_{\mathbb{C}}(\mathbb{O})) = ((\mathrm{SU}(3) \times \mathrm{SU}(3))/C_3) \rtimes C_2$ is compact and acts continuously on $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$.

It remains to show that the group G acts flag-transitively on Γ. Clearly, in either case G is transitive on the set of point-line flags of Γ. So in order to prove flag-transitivity, we only must show that the stabilizer in G of a given point-line flag $([u], [v, x])$ of Γ acts transitively on the set of sharp $\mathbb{F}$-morphisms ϕ of Γ such that $\phi(v) = x$. This follows from Lemma 2.4. So:

Result 2.8. *The pair* (Γ, G) *is indeed a homogeneous compact geometry.*

As G acts flag-transitively on Γ, we can recover Γ as a coset-geometry from G, where the flags naturally correspond to the cosets of the stabilizers of the flags contained in a selected chamber of Γ, two flags being incident precisely when the corresponding cosets meet nontrivally (see, e.g., [Tits 1974, §1.4] or [Pasini 1994, §10.1]). Accordingly, Γ is uniquely determined by the complex of the stabilizers in G of the subflags of a chamber of Γ. This complex, as described by Schillewaert and Struyve [2017] for the case $(\mathbb{F}, \mathbb{A}) = (\mathbb{C}, \mathbb{O})$, is the same as computed for G regarded as the automorphism group of the $\mathbb{O}\mathrm{P}^2$-geometry considered in [Kramer and Lytchak 2014] (see also [Schillewaert and Struyve 2017]). Similarly for the case $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$ and the $\mathbb{O}\mathrm{P}^2$-geometry of [Kramer and Lytchak 2019]. So:

Result 2.9. *The* C_3*-geometries* $\Gamma_{\mathbb{R}}(\mathbb{H})$ *and* $\Gamma_{\mathbb{C}}(\mathbb{O})$ *are the* (*geometric supports of the*) *two* $\mathbb{O}\mathrm{P}^2$*-geometries.*

Remark 2.10. The two cases of Setting 2.2 correspond to the two cases of [Schillewaert and Struyve 2017] with $\mathbb{B} = \mathbb{O}$. Schillewaert and Struyve [2017] also consider one more case, with $\mathbb{F} = \mathbb{R}$ and $\mathbb{A} = \mathbb{B} = \mathbb{H}$, which leads to a flat C_3-geometry which is a quotient of the building associated to the Chevalley group $\mathrm{O}(7, \mathbb{R})$ and admits $\mathrm{SO}(3) \times \mathrm{SO}(3)$ as a flag-transitive automorphism group. This geometry also appears in [Rees 1985, §1.6, (2.2)(ii)] as a member of a larger family of flag-transitive flat C_3-geometries, obtained as quotients from O(7, K)-buildings, with K any ordered field. Note that the construction used by Rees [1985] is primarily geometric.

This geometry is indeed worth further investigation, but I have preferred to leave it aside in order to stick to the subject of this paper.

3. A combinatorial proof of Theorem 1.5

3A. ***Preliminaries.*** We follow [Pasini 1994] for basics on diagram geometry. We recall that, according to [Pasini 1994], all geometries are residually connected, by definition. In particular, all geometries of rank at least 2 are connected.

Throughout this subsection Γ is a given geometry of rank $n \geq 2$. Recall that Γ can be regarded as a simplicial complex, where the vertices are the elements of the geometry and the simplices are the flags. Moreover, with $\{1, 2, \ldots, n\}$ chosen as the type-set of Γ, the vertices of the complex are marked by positive integers not greater than n, according to their type as elements of Γ. The incidence graph of Γ is just the skeleton of the complex Γ.

We firstly state some notation and recall a few basics on homotopy of paths. Given two paths $\alpha = (a_0, \ldots, a_k)$ and $\beta = (b_0, \ldots, b_h)$ of Γ with $a_k = b_0$, the *join* of α with β, also called the *product* of α and β, is the path:

$$\alpha \cdot \beta := (a_0, a_1, \ldots, a_k = b_0, b_1, \ldots, b_h).$$

A *null* path is a path of lenght 0. The *opposite* (also called the *inverse*) of a path $\alpha = (a_0, a_1, \ldots, a_k)$ is the path $\alpha^{-1} := (a_k, a_{k-1}, \ldots, a_0)$.

Two paths $\alpha = (a_0, a_1, \ldots, a_k)$ and $\beta = (b_0, b_1, \ldots, n_h)$ with $a_0 = b_0$ and $a_k = b_h$ are said to be *elementarily homotopic* if $\alpha = \gamma \cdot \alpha' \cdot \delta$ and $\beta = \gamma \cdot \beta' \cdot \delta$ for suitable subpaths γ, δ, α' and β' with α' and β' contained in the same simplex (namely flag) of Γ. More generally, two paths α and β are said to be *homotopic* if there exists a sequence $\alpha_0, \alpha_1, \ldots, \alpha_m$ of paths with $\alpha = \alpha_0$, $\beta = \alpha_m$ and such that α_{i-1} and α_i are elementarily homotopic for $i = 1, 2, \ldots, m$.

If α and β are homotopic we write $\alpha \sim \beta$. We say that a closed path α based at a vertex a is *null homotopic* if it is homotopic with the null path (a). Equivalently, α splits in triangles each of which is contained in a simplex and, possibly, paths of the form $\beta \cdot \beta^{-1}$.

Clearly, homotopy is an equivalence relation. We denote by $[\alpha]$ the homotopy class of a path α. Given a vertex a of Γ, the homotopy classes of closed paths of Γ based at a form a group $\pi_1(\Gamma, a)$, with $[(a)]$ as the identity element and multiplication defined as follows: $[\alpha]\cdot[\beta] := [\alpha \cdot \beta]$. The group $\pi_1(\Gamma, a)$ is called the *fundamental group* of Γ *based* at a. As Γ is connected, we have $\pi_1(\Gamma, a) \cong \pi_1(\Gamma, b)$

for any two vertices $a, b \in \Gamma$. Explicitly, for very choice of a path γ from a to b, the mapping

$$[\alpha] \in \pi_1(\Gamma, a) \mapsto [\gamma^{-1} \cdot \alpha \cdot \gamma] \in \pi_1(\Gamma, b)$$

is an isomorphims from $\pi_1(\Gamma, a)$ to $\pi_1(\Gamma, b)$. So, as far as we are interested only in the isomorphism type of $\pi_1(\Gamma, a)$, we are free not to keep a record of the base point a of $\pi_1(\Gamma, a)$ in our notation, thus writing $\pi_1(\Gamma)$ for $\pi_1(\Gamma, a)$ and calling $\pi_1(\Gamma)$ the *fundamental group* of Γ, for short.

It is well known (see, e.g, [Pasini 1994, §12.6.1]) that the geometry Γ is simply connected (namely $(n-1)$-simply connected) if and only it is simply connected as a complex, namely $\pi_1(\Gamma)$ is trivial; equivalently, every closed path is null-homotopic.

Lemma 3.1. *For $1 \le i < j \le n$, let $\Gamma_{i,j}$ be the $\{i, j\}$-truncation of Γ, namely the subgeometry induced by Γ on the set of elements of Γ of type i or j. Then every path of Γ starting and ending at $\Gamma_{i,j}$ (in particular, every closed path based at an element of type i or j) is homotopic to a path of $\Gamma_{i,j}$.*

Proof. Let $\alpha = (a_0, a_1, \dots, a_k)$ be a path of Γ with $a_0, a_k \in F_{i,j}$. We argue by induction on the length k of α. When $k \le 1$ there is nothing to prove. Let $k = 2$. If $a_1 \in \Gamma_{i,j}$ there is nothing to prove as well. Let $a_1 \notin \Gamma_{i,j}$. By the so-called strong connectedness property [Pasini 1994, Theorem 1.18], the intersection $\mathrm{Res}(a_1) \cap \Gamma_{i,j}$ of the residue $\mathrm{Res}(a_1)$ of a_1 with $\Gamma_{i,j}$ contains a path

$$\beta = (b_0 = a_0, b_1, \dots, b_{h-1}, b_h = a_2)$$

from a_0 to a_2. We have $(b_{i-1}, b_i) \sim (b_{i-1}, a_1, b_i)$ for every $i = 1, 2, \dots, h$, since $\{b_{i-1}, a_1, b_i\}$ is a flag. Moreover, $(a_1, b_i, a_1) \sim (a_1)$ for every $i = 1, 2, \dots, h$. Therefore

$$\beta \sim \gamma := (b_0, a_1, b_1, a_1, b_2, \dots, b_{h-1}, a_1, b_h) \sim (b_0, a_1, b_h) = (a_0, a_1, a_2) = \alpha.$$

The claim is proved. Let now $k > 2$. If $a_{k-1} \in \Gamma_{i,j}$ the claim follows by the inductive hypothesis on the subpath $(a_0, a_1, \dots, a_{k-1})$. Let $a_{k-1} \notin \Gamma_{i,j}$. If $a_{k-2} \in \Gamma_{i,j}$ then the conclusion follows by the above on the subpath (a_{k-2}, a_{k-1}, a_k) and the inductive hypothesis on $(a_0, a_1, \dots, a_{k-2})$. Let $a_{k-2} \notin \Gamma_{i,j}$. Then $\mathrm{Res}(a_{k-2}, a_{k-1}) \cap \Gamma_{i,j} \neq \varnothing$, since neither i nor j belong to the type of the flag $\{a_{k-2}, a_{k-1}\}$ and every flag is contained in a chamber. Pick an element $c \in \mathrm{Res}(a_{k-2}, a_{k-1}) \cap \Gamma_{i,j}$ and consider the paths

$$\alpha' := (a_0, a_1, \dots, a_{k-2}, c), \quad \alpha'' := (c, a_{k-1}, a_k).$$

The path α' has length $k-1$. So, by the inductive hypothesis, a path β' exists in $\Gamma_{i,j}$ from a_0 to c such that $\beta' \sim \alpha'$. Similarly, as we have already proved the claim for paths of length 2, a path β'' exists in $\Gamma_{i,j}$ from c to a_k such that $\beta'' \sim \alpha''$. So, $\beta := \beta' \cdot \beta'' \sim \alpha' \cdot \alpha'' \sim \alpha$ is a path of $\Gamma_{i,j}$ with the required properties. □

The following lemma is implicit in [Pasini 1994, Lemma 12.60].

Lemma 3.2. *Given two elements v and w of Γ, let α and β be two paths of Γ from v to w. If an element u exists in Γ such that its residue $\mathrm{Res}(u)$ contains both α and β, then $\alpha \sim \beta$.*

Proof. Let $\alpha = (a_0, a_1, \dots, a_k)$ with $a_0 = v$, $a_k = w$ and $\alpha \subseteq \mathrm{Res}(u)$. For every $i = 1, 2, \dots, k$ put $\alpha_i = (a_{i-1}, u, a_i)$. As $(a_{i-1}, a_i) \sim (a_{i-1}, u, a_i)$ and $(u, a_i, u) \sim (u)$, we have

$$\alpha \sim \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_k = (a_0, u, a_1, u, a_2, \dots, a_{k-1}, u, a_k) \sim (a_0, u, a_k).$$

So, $\alpha \sim (a_0, u, a_k) = (v, u, w)$. Similarly, $\beta \sim (v, u, w)$. Therefore $\alpha \sim \beta$. □

3B. *Peculiar properties of C_3-geometries.* From now on Γ is a geometry of type C_3. The integers 1, 2 and 3 are taken as types and stand for points, lines and planes respectively.

Definition 3.3. A *primitive path* of Γ is a closed path $\alpha := (p, L, q, M, p)$ where p and q are points and L and M lines. If $p = q$ or $L = M$ then α is said to be *degenerate*.

Clearly, degenerate primitive paths are null-homotopic. The following is also well known [Tits 1981, Proposition 9] (see also [Pasini 1994, Corollary 7.39]).

Lemma 3.4. *The geometry Γ is a building if and only if all of its primitive paths are degenerate.*

The proof of the next lemma is implicit in [Schillewaert and Struyve 2017, §6.6]. We make it explicit.

Lemma 3.5. *Every closed path of Γ based at a point is homotopic to a primitive path.*

Proof. Let α be a closed path based at a point p. In view of Lemma 3.1, we may assume that α is contained in $\Gamma_{1,2}$. So, $\alpha = (p_0, L_1, p_1, \dots, L_k, p_k)$ where $p_0 = p_k = p$ and, for $i = 1, \dots, k$, p_i is a point and L_i a line. We argue by induction on k. If $k = 1$ there is nothing to prove. Let $k > 1$. Suppose firstly that L_{i-1} and L_i are coplanar. Let ξ be the plane on L_{i-1} and L_i and let M be the line of $\mathrm{Res}(\xi)$ through p_{i-2} and p_i. Then $(p_{i-2}, L_{i-1}, p_{i-1}, L_i, p_i) \sim (p_{i-2}, M, p_i)$ by Lemma 3.2. Accordingly, $\alpha \sim \alpha' := (p_0, L_1, \dots, p_{i-2}, M, p_i, \dots, L_k, p_k)$. However α', being shorter than α, is homotopic to a primitive path, by the inductive hypothesis. Hence α too is homotopic to a primitive path.

Assume now that L_{i-1} and L_i are never coplanar, for any $i = 2, \dots, k$. Choose a plane ξ_2 on L_2. The residue $\mathrm{Res}(p_1)$ of p_1 contains a unique line-plane flag

(M_1, ξ_1) such that L_1 and M_1 are incident with ξ_1 and ξ_2 respectively. Similarly, $\mathrm{Res}(p_2)$ contains a unique line-plane flag (M_2, ξ_3) such that L_3 and M_2 are incident with ξ_3 and ξ_2 respectively. Let q be the meet-point of M_1 and M_2 in $\mathrm{Res}(\xi_2)$, let M_0 be the line through p_0 and q in $\mathrm{Res}(\xi_1)$ and let M_3 be the line through p_3 and q in $\mathrm{Res}(\xi_3)$. By Lemma 3.2 we have the following homotopies:

$$(p_0, L_1, p_1) \sim (p_0, M_0, q, M_1, p_1),$$
$$(p_1, L_2, p_2) \sim (p_1, M_1, q, M_2, p_2),$$
$$(p_2, L_3, p_3) \sim (p_2, M_2, q, M_3, p_3).$$

Therefore

$$\begin{aligned}(p_0, L_1, p_1, L_2, p_2, L_3, p_3) &= (p_0, L_1, p_1) \cdot (p_1, L_2, p_2) \cdot (p_2, L_3, p_3) \\ &\sim (p_0, M_0, q, M_1, p_1) \cdot (p_1, M_1, q, M_2, p_2) \cdot (p_2, M_2, q, M_3, p_3) \\ &= (p_0, M_0, q, M_1, p_1, M_1, q, M_2, p_2, M_2, q, M_3, p_3) \\ &\sim (p_0, M_0, q, M_3, p_3).\end{aligned}$$

Accordingly, α is homotopic to the path, say β, obtained by replacing the subpath $(p_0, L_1, p_1, L_2, p_2, L_3, p_3)$ of α with (p_0, M_0, q, M_3, p_3). The path β is shorter than α, whence it is homotopic to a primitive path by the inductive hypothesis. As $\alpha \sim \beta$, the same holds for α. □

By Lemma 3.5 we immediately obtain the following:

Corollary 3.6. *The geometry Γ is simply connected if and only if all of its primitive paths are null-homotopic.*

Let $\phi : \widetilde{\Gamma} \to \Gamma$ be the universal covering of Γ. As $\widetilde{\Gamma}$ is simply connected, all of its closed paths (in particular, all of its primitive paths) are null-homotopic. A closed path of Γ is null-homotopic if and only if it lifts through ϕ to a closed path of $\widetilde{\Gamma}$. In particular:

Corollary 3.7. *A primitive path of Γ is null-homotopic if and only if it is the ϕ-image of a primitive path of $\widetilde{\Gamma}$.*

Corollary 3.8. *The geometry Γ is covered by a building if and only if none of its nondegenerate primitive paths is null-homotopic.*

Proof. Let $\widetilde{\Gamma}$ be a building. Then, by Lemma 3.4, no nondegenerate primitive path occurs in $\widetilde{\Gamma}$. By Corollary 3.7, none of the nondegenerate primitive paths of Γ can be null-homotopic. On the other hand, let $\widetilde{\Gamma}$ be not a building. Then $\widetilde{\Gamma}$ admits at least one nondegenerate primitive path $\tilde{\alpha}$, necessarily null-homotopic since $\widetilde{\Gamma}$ is simply connected. Accordingly, $\alpha := \phi(\tilde{\alpha})$ is a null-homotopic nondegenerate primitive path of Γ. □

Definition 3.9. Let $\alpha = (p, L, q, M, p)$ be a nondegenerate primitive path. Recall that $\mathrm{Res}(q)$ is a generalized quadrangle, the lines L and M being points of this quadrangle. So, lines on q exist which are coplanar with each of L and M. Let N be such a line and r a point of N. The line N is different from each of L and M, as L and M are noncoplanar. Let ξ be the plane on N and L and let L' be the line of ξ through p and r. Similarly, if χ is the plane on N and M, let M' be the line of χ through p and r. Then (p, L', r, M', p) is a primitive path. We denote it by $\sigma^N_{q\to r}(\alpha)$ and call it the *shift* of α *from q to r along N*. We also say that N is *admissible* for the path α.

Lemma 3.10. *Let $\alpha = (p, L, q, M, p)$ be a nondegenerate primitive path, N a line admissible for α and r a point of N. Then*:

(1) *We have $\sigma^N_{q\to r}(\alpha) = \alpha$ if and only if $r = q$.*

(2) *The shift $\sigma^N_{q\to r}(\alpha)$ is a nondegenerate primitive path and the line N is admissible for it.*

(3) $\sigma^N_{r\to q}(\sigma^N_{q\to r}(\alpha)) = \alpha$.

(4) $\alpha \sim \sigma^N_{q\to r}(\alpha)$.

Proof. Claims (1), (2) and (3) are trivial. Claim (4) can be proved as follows:

$$\begin{aligned}(p, L, q, M, p) &\sim (p, \xi, L, q, M, \chi, p) \sim (p, \xi, q, \chi, p)\\ &\sim (p, \xi, N, q, N, \chi, p) \sim (p, \xi, N, \chi, p) \sim (p, \xi, N, r, N, \chi, p)\\ &\sim (p, \xi, r, \chi, p) \sim (p, L', \xi, r, \chi, M', p) \sim (p, L', r, M', p).\end{aligned}$$

(This is essentially the same argoment as used by Schillewaert and Struyve to prove Lemma 6.6 of [2017].) □

3C. *Primitive paths in $\mathbb{O}\mathbf{P}^2$-geometries.* Henceforth $\Gamma = \Gamma_{\mathbb{F}}(\mathbb{A})$ (see Section 2B). Recall that the point-line geometry with the same points as Γ and the shadow-lines as lines coincides with $\mathrm{PG}(\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})) \cong \mathrm{PG}(2, \mathbb{F})$ (Corollary 2.7). In particular, two lines of Γ either have just one point in common or have exactly the same points.

Definition 3.11. Let L and M be two lines of Γ with the same shadow, namely $L = [a, u]$ and $M = [b, v]$ for $a, b \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{A})$ and $u, v \in \mathrm{Pu}_{\mathbb{F}}(\mathbb{O})$ with $|a| = |u| \neq 0$, $|b| = |v| \neq 0$ and $[a] = [b]$. Suppose we have chosen the pairs (a, u) and (b, v) in such a way that $a = b$, as we can. Then we put $(L \mid M) := (u \mid v)/|u||v|$.

Given a primitive path $\alpha = (p, L, q, M, p)$ we put $\ell(\alpha) := (L \mid M)$ and we call $\ell(\alpha)$ the *line-invariant* of α.

Clearly, $|(L \mid M)| \leq 1$ by Cauchy–Schwartz inequality, with equality if and only if u and v are proportional. Moreover $(L \mid M) = 1$ if and only if $L = M$. So, $\ell(\alpha) \neq 1$ whenever α is nondegenerate.

The hypothesis $a = b$ is necessary for the above definition of $(L \mid M)$ to make sense. Indeed, without it, only the modulus $|(u \mid v)|/|u||v|$ of $(u \mid v)/|u||v|$ is determined by the pair L and M. It is also clear that $(L \mid M)$ can be defined only when L and M have the same shadow. On the other hand, the particular choice of a in the representations $L = [a, u]$ and $M = [a, v]$ is irrelevant. Indeed, if we replace a with $a' = ta$ for some $t \in \mathbb{F} \setminus \{0\}$ then we must also replace u with $u' = tu$ and v with $v' = tv$. Accordingly, $(u' \mid v')/|u'||v'| = |t|^2(u \mid v)/|t^2||u||v| = (u \mid v)/|u||v|$.

Remark 3.12. Schillewaert and Struyve [2017] call $\ell(\alpha)$ the PL-invariant of α.

Definition 3.13. We say that a primitive path $\alpha = (p, L, q, M, p)$ is *orthogonal* if $p \perp q$. Assuming that α is nondegenerate but not that it is orthogonal, an *orthogonal shift* of α is a shift $\sigma^N_{q \to r}(\alpha)$ with $p \perp r$.

Lemma 3.14. *Every nondegenerate primitive path $\alpha = (p, L, q, M, p)$ admits orthogonal shifts along every line N admissible for it and, once N has been chosen, the orhogonal shift of α along N is uniquely determined. Moreover, if α is orthogonal, then α is its own orthogonal shift.*

Proof. As N is coplanar with either of L and M, it has at most one point in common with L or M. However N contains q. Hence it cannot contain p. By Corollary 2.7, the line $p^\perp \cap [\mathrm{Pu}_{\mathbb{F}}(\mathbb{A})]$ of $\mathrm{PG}(\mathrm{Pu}_{\mathbb{F}}(\mathbb{A}))$ meets the shadow of N in just one point. (This argument is the same as in the proof of Lemma 6.6 of [Schillewaert and Struyve 2017].) The first part of the lemma is proved. The last claim of the lemma is obvious. □

Henceforth we denote by $\sigma^N_\perp(\alpha)$ the orthogonal shift of α along a line N admissible for α.

Lemma 3.15. *Given a nonorthogonal nondegenerate primitive path α of Γ and a line N admissible for α, let $\beta = \sigma^N_\perp(\alpha)$ be the orthogonal shift of α along N and let $\ell = \ell(\beta)$ be the line-invariant of β.*

We can always choose the line N in such a way that $\ell \neq -1$.

Proof. We must distinguish two cases and two subcases for each of them.

Case 1. $\Gamma = \Gamma_{\mathbb{R}}(\mathbb{H})$. Modulo automorphisms of Γ, we can always assume that

$$L = [\boldsymbol{j}, \boldsymbol{j}], \quad M = [\boldsymbol{j}, \boldsymbol{i}m_1 + \boldsymbol{j}m_2], \quad m_1^2 + m_2^2 = 1,$$
$$p = [\boldsymbol{i}], \qquad q = [\boldsymbol{i}q_1 + \boldsymbol{j}\boldsymbol{i}q_3], \qquad q_1^2 + q_3^2 = 1.$$

So, $\ell(\alpha) = m_2$. Note that $q_1 \neq 0$ (otherwise $p \perp q$, while α is nonorthogonal by assumption) and $q_3 \neq 0$ (otherwise $p = q$). Let $N = [b, x]$ be admissible for α, where

$$b = \boldsymbol{i}b_1 + \boldsymbol{j}b_2 + \boldsymbol{j}\boldsymbol{i}b_3, \qquad b_1^2 + b_2^2 + b_3^2 = 1,$$
$$x = \boldsymbol{i}x_1 + \boldsymbol{j}x_2 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4 + \boldsymbol{k}\boldsymbol{i}x_5 + \boldsymbol{k}\boldsymbol{j}x_6 + \boldsymbol{k}(\boldsymbol{j}\boldsymbol{i})x_7, \qquad |x| = 1.$$

Modulo automorphisms of $\mathbb{O}$ that leave $\mathbb{H}$ elementwise fixed, we can always assume that

$$x = \boldsymbol{i}x_1 + \boldsymbol{j}x_2 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4, \quad (x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1).$$

For N to be admissible for α the following must hold: $(\boldsymbol{i}q_1 + \boldsymbol{j}\boldsymbol{i}q_3 \mid b) = 0$ (namely q belongs to N) and $(\boldsymbol{j} \mid b) = (\boldsymbol{j} \mid x) = (\boldsymbol{i}m_1 + \boldsymbol{j}m_2 \mid x)$ (Lemma 2.6, claim (1)). Explicitly:

$$b_1q_1 + b_3q_3 = 0, \tag{4}$$

and $b_2 = x_2 = m_1x_1 + m_2x_2$, namely

$$b_2 = x_2, \quad m_1x_1 = (1 - m_2)b_2. \tag{5}$$

Let $r = [\boldsymbol{i}r_1 + \boldsymbol{j}r_2 + \boldsymbol{j}\boldsymbol{i}r_3]$ be the unique point of $\{[b], p\}^\perp$. So, $r_1 = 0$, namely $r = [\boldsymbol{j}r_2 + \boldsymbol{j}\boldsymbol{i}r_3]$, and

$$b_2r_2 + b_3r_3 = 0. \tag{6}$$

Moreover we assume $r_2^2 + r_3^2 = 1$, as we can. We have already noticed that $q_1 \neq 0$. We also have $r_2 \neq 0$, otherwise equations (4) and (6) force $b_1 = b_3 = 0$, hence $b = \pm\boldsymbol{j}$, contrary to the fact that N is coplanar with L and M. Thus, by (4) and (6) we obtain

$$b_1 = -b_3q_3q_1^{-1}, \quad b_2 = -b_3r_3r_2^{-1}. \tag{7}$$

These equations show that $b_3 \neq 0$ (otherwise $b = 0$, which is ridiculous). Recalling that $b_1^2 + b_2^2 + b_3^2 = 1$ now we get

$$b_3 = \pm\frac{q_1r_2}{\sqrt{q_1^2 + r_2^2 - q_1^2r_2^2}} = \pm\frac{q_1r_2}{\sqrt{q_1^2r_3^2 + 1 - r_3^2}} = \pm\frac{q_1r_2}{\sqrt{1 - q_3^2r_3^2}}. \tag{8}$$

Equation (8) is equivalent to the following

$$r_2 = \pm\frac{b_3}{\sqrt{b_2^2 + b_3^2}},$$

which better shows that the point r depends on the choice of the line N but, in view of the sequel, (8) is more convenient. We shall now consider two subcases: either $m_2 = -1$ or $-1 < m_2 < 1$ (note that $m_2 = 1$ is impossible, since $m_2 = (L \mid M)$ and $(L \mid M) \neq 1$ because $L \neq M$).

Subcase 1.1. $m_2 = -1$. Equivalently, $m_1 = 0$. Then $b_2 = x_2 = 0$ by (5), $r_3 = 0$ by (7) and since $b_3 \neq 0$, whence $r_2 = \pm 1$ (as $r_2^2 + r_3^2 = 1$) and $b_3 = \pm q_1$ by (8). Consequently, $b_1 = \pm q_3$, since $b_1^2 + b_3^2 = q_1^2 + q_3^2 = 1$. Summarizing:

m_1	m_2	r_2	r_3	b_1	b_2	b_3	x_2
0	-1	± 1	0	$\pm q_3$	0	$\pm q_1$	0.

Let now ξ be the plane on L and N and χ the plane on M and N. Then ξ and χ, regarded as sharp $\mathbb{R}$-morphisms from $\mathbb{H}$ to $\mathbb{O}$, are uniquely determined by the following conditions (Lemma 2.3): $\xi(\boldsymbol{j}) = \boldsymbol{j}$, $\chi(\boldsymbol{j}) = \boldsymbol{i}m_1 + \boldsymbol{j}m_2$ and $\xi(b) = \chi(b) = x$. By entering the above values for m_1, m_2 and x_2 we get

$$\xi(\boldsymbol{j}) = \boldsymbol{j}, \quad \chi(\boldsymbol{j}) = -\boldsymbol{j}, \quad \xi(b) = \chi(b) = \boldsymbol{i}x_1 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4. \tag{9}$$

Clearly $\boldsymbol{i} = \boldsymbol{i}(b_1 - \boldsymbol{j}b_3)(b_1 - \boldsymbol{j}b_3)^{-1} = b(b_1 + \boldsymbol{j}b_3)$. Therefore, and taking (9) into account,

$$\begin{aligned} \xi(\boldsymbol{i}) &= (\boldsymbol{i}x_1 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4)(b_1 + \boldsymbol{j}b_3), \\ \chi(\boldsymbol{i}) &= (\boldsymbol{i}x_1 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4)(b_1 - \boldsymbol{j}b_3). \end{aligned} \tag{10}$$

Let now L' and M' be the lines through p and r in ξ and χ respectively. Then $L' = [a, \xi(a)]$ and $M' = [a, \chi(a)]$ where $a = \boldsymbol{i}a_1 + \boldsymbol{j}a_2 + \boldsymbol{k}a_3$ is orthogonal with both p and r and we assume $a_1^2 + a_2^2 + a_3^2 = 1$, as we can. Orthogonality with p and r forces $a_1 = 0 = a_2$. Therefore $a = \pm\boldsymbol{j}\boldsymbol{i}$. Accordingly, and recalling (10),

$$\begin{aligned} \xi(a) &= \pm\boldsymbol{j}(\boldsymbol{i}x_1 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4)(b_1 + \boldsymbol{j}b_3), \\ \chi(a) &= \mp\boldsymbol{j}(\boldsymbol{i}x_1 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4)(b_1 - \boldsymbol{j}b_3). \end{aligned} \tag{11}$$

With $\beta = \sigma_{\perp}^{N}(\alpha) = (p, L', r, M', p)$ we have $\ell(\beta) = (\xi(a) \mid \chi(a))$. Equations (11) allow to explicitly compute the inner product $(\xi(a) \mid \chi(a))$. We obtain:

$$\begin{aligned} (\xi(a) \mid \chi(a)) &= x_1^2(b_3^2 - b_1)^2 + x_3^2(b_3^2 - b_1^2) + x_4^2(b_3^2 - b_1^2) \\ &= (x_1^2 + x_3^2 + x_4^2)(b_3^2 - b_1)^2 = b_3^2 - b_1^2 = q_1^2 - q_3^2. \end{aligned} \tag{12}$$

So, $(\xi(a) \mid \chi(a)) = q_1^2 - q_3^2$. As $q_1, q_3 \neq 0$, we have $-1 < (\xi(a) \mid \chi(a)) < 1$.

Subcase 1.2. $m_1 \neq 0$, namely $m_2 \neq -1$. In this case the second equation of (5) yields

$$x_1 = \frac{1 - m_2}{m_1} b_2. \tag{13}$$

The planes ξ and χ on L and N and on M and N are determined by the following conditions:

$$\begin{aligned} &\xi(\boldsymbol{j}) = \boldsymbol{j}, \quad \chi(\boldsymbol{j}) = \boldsymbol{i}m_1 + \boldsymbol{j}m_2, \\ &\xi(b) = \chi(b) = \boldsymbol{i}x_1 + \boldsymbol{j}x_2 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4 = \left(\boldsymbol{i}\frac{1 - m_2}{m_1} + \boldsymbol{j}\right) b_2 + \boldsymbol{j}\boldsymbol{i}x_3 + \boldsymbol{k}x_4. \end{aligned} \tag{14}$$

Moreover, $x_3^2 + x_4^2 = 1 - ((1 - m_2)^2 m_1^{-2} + 1)b_2^2 = 1 - 2(1 + m_2)^{-1} b_2^2$. Therefore

$$x_3^2 + x_4^2 = 1 - \frac{2}{1 + m_2} b_2^2. \tag{15}$$

Now $\boldsymbol{i} = (b - \boldsymbol{j}b_2)(b_1 - \boldsymbol{j}b_3)^{-1} = (b - \boldsymbol{j}b_2)(b_1 + \boldsymbol{j}b_3)(b_1^2 + b_3^2)^{-1}$. Recalling equations (7), we obtain

$$\boldsymbol{i} = \left(b + \boldsymbol{j}\frac{r_3}{r_2}b_3\right)(\boldsymbol{j}q_1 - q_3)q_1 b_3^{-1}. \tag{16}$$

As in Subcase 1.1, let $L' = [a, \xi(a)]$ and $M' = [a, \chi(a)]$ be the lines through p and r in ξ and χ respectively, where $a = \boldsymbol{i}a_1 + \boldsymbol{j}a_2 + \boldsymbol{k}a_3$ with $|a| = 1$. The vector a is orthogonal with both p and r. Orthogonality with p still forces $a_1 = 0$ but orthogonality with r only implies $a_2r_2 + a_3r_3 = 0$. So $a_2 = -a_3r_3r_2^{-1}$ and the condition $|a| = 1$ implies $a_3 = \pm r_2$. Hence $a_2 = \pm r_3$. Summarizing

$$a = \pm(\boldsymbol{j}r_3 + \boldsymbol{j}\boldsymbol{i}r_2). \tag{17}$$

Exploiting (14), (16) and (17), we can compute $\xi(a)$ and $\chi(a)$ explicitly, whence $(\xi(a) \mid \chi(a))$ too. We firstly obtain $(\xi(a) \mid \chi(a)) = A(x_3^3 + x_4^2) + B$ where

$$\begin{aligned} A &= (q_3^2 m_2 + q_1^2)q_1 r_2^2 b_3^{-2}, \\ B &= (-m_1 r_3 + (x_1 - m_1 b_2)q_1^2 r_2 b_3^{-1})x_1 q_1^2 b_3^{-1} + r_3^2 m_2 \\ &\qquad + (m_2 - 1)r_3 r_2 q_1^2 b_2 b_3^{-1} + (x_1 m_2 q_3 - m_1 q_3 b_2)x_1 q_3 q_1^2 r_2^2 b_3^{-2}. \end{aligned}$$

By exploiting (7), (8) and (15) we eventually obtain the following:

$$(\xi(a) \mid \chi(a)) = -r_3^2 \frac{q_1^4 m_2^2}{1 + m_2} + q_3^2 m_2 + q_1^2. \tag{18}$$

In this equation $(\xi(a) \mid \chi(a))$ is expressed as a function of r_3 rather than b_3, but recall that r is uniquely determined by b. Note that the coefficient of r_3^2 in (18) is negative except when $m_2 = 0$. If $m_2 = 0$ then $(\xi(a) \mid \chi(a)) = q_1^2$, which is strictly positive and less than 1, since neither q_1 nor q_3 are zero.

Case 2. $\Gamma = \Gamma_{\mathbb{C}}(\mathbb{O})$. As in Case 1, we can assume that

$$\begin{aligned} &L = [\boldsymbol{k}, \boldsymbol{k}], \quad M = [\boldsymbol{k}, \boldsymbol{j}m_1 + \boldsymbol{k}m_2], \quad |m_1|^2 + |m_2|^2 = 1, \\ &p = [\boldsymbol{j}], \qquad q = [\boldsymbol{j}q_1 + \boldsymbol{k}\boldsymbol{j}q_3], \qquad |q_1|^2 + |q_3|^2 = 1. \end{aligned}$$

So, $\ell(\alpha) = m_2$. As in Case 1, we have $q_1 \neq 0 \neq q_3$. Let $N = [b, x]$ be admissible for α, where

$$\begin{aligned} b &= \boldsymbol{j}b_1 + \boldsymbol{k}b_2 + \boldsymbol{k}\boldsymbol{j}b_3, \quad |b_1|^2 + |b_2|^2 + |b_3|^2 = 1, \\ x &= \boldsymbol{j}x_1 + \boldsymbol{k}x_2 + \boldsymbol{k}\boldsymbol{j}x_3, \quad |x_1|^2 + |x_2|^2 + |x_3|^2 = 1. \end{aligned}$$

For N to be admissible for α the following must hold: $(\boldsymbol{j}q_1 + \boldsymbol{k}\boldsymbol{j}q_3 \mid b) = 0$ and $(\boldsymbol{k} \mid b) = (\boldsymbol{k} \mid x) = (\boldsymbol{j}m_1 + \boldsymbol{k}m_2 \mid x)$. Explicitly:

$$\overline{q_1}b_1 + \overline{q_3}b_3 = 0, \tag{19}$$

and $b_2 = x_2 = \overline{m_1}x_1 + \overline{m_2}x_2$, namely

$$b_2 = x_2, \quad \overline{m_1}x_1 = (1-\overline{m_2})b_2. \tag{20}$$

Let $r = [\boldsymbol{j}r_1 + \boldsymbol{k}r_2 + \boldsymbol{kj}r_3]$ be such that $\{r\} = \{[b], p\}^{\perp}$. So, $r = [\boldsymbol{k}r_2 + \boldsymbol{kj}r_3]$, where we assume $|r_2|^2 + |r_3|^2 = 1$, and

$$\overline{r_2}b_2 + \overline{r_3}b_3 = 0. \tag{21}$$

Recall that $q_1 \neq 0$ because $p \not\perp q$ by assumption. We also have $r_2 \neq 0$, otherwise N cannot be coplanar with either of L and M. Thus, by (19) and (21) we obtain

$$b_1 = -b_3\frac{\overline{q_3}}{\overline{q_1}}, \quad b_2 = -b_3\frac{\overline{r_3}}{\overline{r_2}}. \tag{22}$$

These equations show that $b_3 \neq 0$. Recalling that $|b_1|^2 + |b_2|^2 + |b_3|^2 = 1$ we get

$$b_3 = \varepsilon\cdot\frac{q_1r_2}{\sqrt{|q_1|^2+|r_2|^2-|q_1|^2|r_2|^2}} = \varepsilon\frac{q_1r_2}{\sqrt{1-|q_3|^2|r_3|^2}} \tag{23}$$

for a suitable multiplier ε with $|\varepsilon| = 1$. We shall now consider two subcases: either $|m_2| = 1$ or $|m_1| < 1$.

Subcase 2.1. $|m_2| = 1$. Equivalently, $m_1 = 0$. Then $b_2 = x_2 = 0$ by (20), $r_3 = 0$ by (22) and since $b_3 \neq 0$, whence $|r_2| = 1$ and $|b_3| = |q_1|$ by (23). Consequently, $|b_1| = |q_3|$.

Let now ξ be the plane on L and N and χ the plane on M and N. Then ξ and χ, regarded as sharp $\mathbb{C}$-autorphisms of $\mathbb{O}$, are uniquely determined by the following conditions: $\xi(\boldsymbol{k}) = \boldsymbol{k}$, $\chi(\boldsymbol{k}) = \boldsymbol{j}m_1 + \boldsymbol{k}m_2$ and $\xi(b) = \chi(b) = x$. In view of the above:

$$\xi(\boldsymbol{k}) = \boldsymbol{k}, \quad \chi(\boldsymbol{k}) = \boldsymbol{k}m_2, \quad \xi(b) = \chi(b) = \boldsymbol{j}x_1 + \boldsymbol{kj}x_3. \tag{24}$$

It is easy to check that

$$\boldsymbol{j} = (\boldsymbol{j}b_1 + \boldsymbol{kj}b_3)(\overline{b_1} + \boldsymbol{k}\overline{b_3}) = b(\overline{b_1} + \boldsymbol{k}\overline{b_3}).$$

By this and (24) we get

$$\begin{aligned}\xi(\boldsymbol{j}) &= (\boldsymbol{j}x_1 + \boldsymbol{kj}x_3)(\overline{b_1} + \boldsymbol{k}\overline{b_3}),\\ \chi(\boldsymbol{j}) &= (\boldsymbol{j}x_1 + \boldsymbol{jkj}x_3)(\overline{b_1} + \boldsymbol{k}m_2\overline{b_3}).\end{aligned} \tag{25}$$

Let $L' = [a, \xi(a)]$ and $M' = [a, \chi(a)]$ be the lines through p and r in ξ and χ respectively, where $a = \boldsymbol{j}a_1 + \boldsymbol{k}a_2 + \boldsymbol{kj}a_3$ is orthogonal with both p and r and $|a_1|^2 + |a_2| + |a_3|^2 = 1$. Orthogonality with p and r forces $a_1 = 0 = a_2$. Therefore $a = \boldsymbol{kj}\eta$ for a suitable η with $|\eta| = 1$. By this and (25),

$$\begin{aligned}\xi(a) &= \boldsymbol{k}((\boldsymbol{j}x_1 + \boldsymbol{kj}x_3 + \boldsymbol{k})(\overline{b_1} + \boldsymbol{k}\overline{b_3}))\eta,\\ \chi(a) &= \boldsymbol{k}m_2((\boldsymbol{j}x_1 + \boldsymbol{kj}x_3 + \boldsymbol{k}x_4)(\overline{b_1} + \boldsymbol{k}\overline{m_2b_3}))\eta.\end{aligned} \tag{26}$$

Equations (26) allow to explicitly compute the inner product $(\xi(a) \mid \chi(a))$. We obtain:

$$(\xi(a) \mid \chi(a)) = |b_3|^2 + |b_1|^2\overline{m_2} = |q_1|^2 + |q_3|^2\overline{m_2}. \quad (27)$$

So, $|(\xi(a) \mid \chi(a))| = |q_1|^4 + |q_3|^4 + |q_1|^2|q_3|^2(m_2 + \overline{m_2}) < 1$, as $m_2 + \overline{m_2}$ is a real number not less than -2 and less than 2 (because $|m_2| = 1$ but $m_2 \neq 1$) and $|q_1|^2 + |q_3|^2 = 1$.

Subcase 2.2. $m_1 \neq 0$, namely $|m_2| < 1$. In this case the second equation of (20) yields

$$x_1 = \frac{1 - \overline{m_2}}{\overline{m_1}} b_2. \quad (28)$$

The planes ξ and χ on L and N and on M and N are determined by the following conditions:

$$\begin{aligned} &\xi(\boldsymbol{k}) = \boldsymbol{k}, \quad \chi(\boldsymbol{k}) = \boldsymbol{j}m_1 + \boldsymbol{k}m_2, \\ &\xi(b) = \chi(b) = \boldsymbol{j}x_1 + \boldsymbol{k}x_2 + \boldsymbol{kj}x_3 = \left(\boldsymbol{j}\frac{1 - m_2}{m_1} + \boldsymbol{k}\right)b_2 + \boldsymbol{kj}x_3. \end{aligned} \quad (29)$$

Moreover, $|x_3|^2 = 1 - (1 + |1 - m_2|^2|m_1|^{-2})|b_2|^2$ by (28) and $x_2 = b_2$. Therefore

$$|x_3|^2 = 1 - \frac{2 - m_2 - \overline{m_2}}{|m_1|^2}|b_2|^2. \quad (30)$$

Now $\boldsymbol{j} = (b - \boldsymbol{k}b_2)((\overline{b_1} + \boldsymbol{k}\overline{b_3})(1 - |b_2|^2)^{-1})$. Recalling equations (22), we obtain

$$\boldsymbol{j} = \left(b + \boldsymbol{k}\frac{\overline{r_3}}{\overline{r_2}}b_3\right)((\boldsymbol{k}q_1 - q_3)\overline{q_1}b_3^{-1}). \quad (31)$$

Let $L' = [a, \xi(a)]$ and $M' = [a, \chi(a)]$ be the lines through p and r in ξ and χ respectively, where $a = \boldsymbol{j}a_1 + \boldsymbol{k}a_2 + \boldsymbol{kj}a_3$ is orthogonal with both p and r and $|a| = 1$. Orthogonality with p forces $a_1 = 0$ but orthogonality with r only implies $\overline{r}_2 a_2 + \overline{r_3}a_3 = 0$. So $a_2 = -a_3\overline{r_3}\overline{r_2}^{-1}$ and the condition $|a| = 1$ implies $|a_3| = |r_2|$, namely $a_3 = \overline{r_2}\eta$ for some η with $|\eta| = 1$. Hence

$$a = (-\boldsymbol{k}\overline{r_3} + \boldsymbol{kj}\overline{r_2})\eta = (\boldsymbol{k}(-\overline{r_3} + \overline{r_2}\boldsymbol{j}))\eta = (\boldsymbol{k}(-\overline{r_3} + \boldsymbol{j}r_2))\eta. \quad (32)$$

By exploiting (29), (31) and (32) as well as (22) and (30) one can compute $\xi(a)$ and $\chi(a)$ explicitly, whence $(\xi(a) \mid \chi(a))$ too, but these computations are terribly toilsome. However, in order to prove the lemma, we do not need to perform them. It is enough to show that, for a lucky choice of $N = [b, x]$, whence of r, satisfying the above conditions, we get $\ell \neq -1$. We will go on in this way, referring the interested reader to Remark 3.16 for a way to express $(\xi(a) \mid \chi(a))$ in the general case.

The previous conditions on r, b and x allow to choose $r_3 = 0$. Accordingly, $|r_2| = 1$. Hence $b_2 = 0$ by the second equation of (22) and $b_3 = \lambda\overline{q_1}$ for some λ with $|\lambda| = 1$ by (23). Therefore $b_1 = -\lambda\overline{q}_3$ by the first equation of (22). Moreover $x_1 = x_2 = 0$ by (20) and (28), whence $|x_3| = 1$. Accordingly,

$$\boldsymbol{j} = b((\boldsymbol{k}q_1 - q_3)\lambda^{-1}) \tag{33}$$

by (31) and since $b_1 = \lambda\overline{q}_1$ and

$$a = \boldsymbol{kj}\overline{r_2}\eta \tag{34}$$

by (32) and since $r_3 = 0$. By (33), recalling that $x_1 = x_2 = 0$, we obtain

$$\begin{aligned}
\xi(\boldsymbol{j}) &= x((\boldsymbol{k}q_1 - q_3)\lambda^{-1}) = \boldsymbol{kj}x_3(\boldsymbol{k}q_1\lambda^{-1} - q_3\lambda^{-1}) \\
&= \boldsymbol{j}\overline{q_1x_3}\lambda - \boldsymbol{kj}q_3x_3\overline{\lambda}, \\
\chi(\boldsymbol{j}) &= x(((\boldsymbol{j}m_1 + \boldsymbol{k}m_2)q_1 - q_3)\lambda^{-1}) \\
&= \boldsymbol{kj}x_3(\boldsymbol{j}m_1q_1\lambda^{-1} + \boldsymbol{k}m_2q_1\lambda^{-1} - q_3\lambda^{-1}) \\
&= \boldsymbol{j}\overline{m_2q_1x_3}\lambda - \boldsymbol{k}\overline{m_1q_1x_3}\lambda - \boldsymbol{kj}q_3x_3\overline{\lambda}.
\end{aligned} \tag{35}$$

(Recall that $\lambda^{-1} = \overline{\lambda}$ since $|\lambda| = 1$.) By combining (34) with (35) we obtain

$$\begin{aligned}
\xi(a) &= (\boldsymbol{k}(\boldsymbol{j}\overline{q_1x_3}\lambda - \boldsymbol{kj}q_3x_3\overline{\lambda}))\overline{r_2}\eta \\
&= \boldsymbol{j}\overline{q_3x_3r_2}\lambda\eta + \boldsymbol{kj}q_1x_3\overline{r_2}\overline{\lambda}\eta, \\
\chi(a) &= ((\boldsymbol{j}m_1 + \boldsymbol{k}m_2)(\boldsymbol{j}\overline{m_2q_1x_3}\lambda - \boldsymbol{k}\overline{m_1q_1x_3}\lambda - \boldsymbol{kj}q_3x_3\overline{\lambda}))\overline{r_2}\eta \\
&= \boldsymbol{j}\overline{m_2q_3x_3r_2}\lambda\eta - \boldsymbol{k}\overline{m_1q_3x_3r_2}\lambda\eta + \boldsymbol{kj}q_1x_3\overline{r_2}\overline{\lambda}\eta.
\end{aligned}$$

Therefore $(\xi(a) \mid \chi(a)) = (|q_3|^2\overline{m_2} + |\, q_1^2)(|x_3|^2|r_2|^2|\lambda|^2|\eta|^2$. Finally, recalling that $|x_3| = |r_2| = |\lambda| = |\eta| = 1$,

$$(\xi(a) \mid \chi(a)) = |q_3|^2\overline{m_2} + |q_1|^2. \tag{36}$$

The right side of (36) is equal to -1 only if $q_1 = 0$ and $m_2 = -1$. However, $q_1 \neq 0$ because $p \not\perp q$. Therefore $(\xi(a) \mid \chi(a)) \neq -1$. □

Remark 3.16. In Subcase 2.2 of the above proof, with no additional hypotheses on $[b, x]$ we get

$$(\xi(a) \mid \chi(a)) = A|r_2|^2|b_3|^{-2} - 2\,\mathrm{Im}(m_1\overline{q_1}q_3|q_3|^2r_2\overline{r_3}b_3^{-1}) + |r_3|^2B$$

where Im(.) stands for imaginary part and

$$\begin{aligned}
A &= |q_1q_3|^2\overline{m_2} + |q_1|^4, \\
B &= m_2 - A - |q_1q_3|^2 + |q_1|^4(m_2^3 + \overline{m_2} - 2)|m_1|^{-2}.
\end{aligned}$$

This shows that $(\xi(a) \mid \chi(b))$ depends on r_2, r_3 and x_2 nontrivially. Thus, we can always choose the line $N = [b, x]$ in such a way that $|(\xi(a) \mid \chi(a))| < 1$. Accordingly, Lemma 3.15 can be given a stronger formulation: we can always choose N in such a way that $|\ell| < 1$.

Remark 3.17. It follows from above proof that when $|m_2| = 1$ then $|\ell| < 1$ for every choice of the admissible line $N = [b, x]$. However, for certain values of m_2 we can also choose N in such a way that $\ell = -1$. For instance, when $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$, this is possible in the following cases:

(1) $q_1^4 = q_3^2$ (namely $q_1^2 = (\sqrt{5}-1)/2$) and $-1 \leq m_2 \leq -(\sqrt{5}+1)/4$;

(2) $q_1^2 > q_3^2$ and $-1 \leq m_2 \leq (1 - \sqrt{4q_1^6 + 8q_1^4 - 3})/(q_1^4 - q_3^2)$;

(3) $q_1^2 < q_3^2$ and $1 \geq m_2 \geq (-1 + \sqrt{4q_1^6 + 8q_1^2 - 3})/(q_3^2 - q_1^4)$.

Lemma 3.18. *Every orthogonal nondegenerate primitive path α of $\Gamma_{\mathbb{C}}(\mathbb{O})$ such that $|\ell(\alpha)| = 1$ but $\ell(\alpha) \neq -1$ is homotopic with an orthogonal nondegenerate primitive path β such that $|\ell(\beta)| < 1$.*

See [Schillewaert and Struyve 2017, Lemma 6.7] for the above. The following lemma is also proved in [Schillewaert and Struyve 2017, Lemma 6.8].

Lemma 3.19. *Let $\ell \in \mathbb{F}$ such that $|\ell| < 1$. Then, for every choice of two distinct lines L and M with the same shadow, there exists a sequence $L_0 = L, L_1, \dots, L_n = M$ of lines with the same shadow as L and M and such that $(L_{i-1} \mid L_i) = \ell$ for every $i = 1, 2, \dots, n$.*

The next statement is implicit in what Schillewaert and Struyve say to justify [2017, Remark 6.9]. We make it explicit.

Corollary 3.20. *Let $\ell \in \mathbb{F}$ such that $|\ell| < 1$ and let $\alpha = (p, L, q, M, p)$ be a nondegenerate primitive path of $\Gamma = \Gamma_{\mathbb{F}}(\mathbb{A})$. Then $\alpha \sim \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n$ for a suitable sequence of nondegenerate primitive paths $\alpha_1, \alpha_2, \dots, \alpha_n$ of Γ with the same points p and q as α and such that $\ell(\alpha_i) = \ell$ for every $i = 1, 2, \dots, n$.*

Proof. By Lemma 3.19 there exist lines $L_0 = L, L_1, \dots, L_n = M$ such that $(L_{i-1} \mid L_i) = \ell$ for $i = 1, 2, \dots, n$. For $i = 1, 2, \dots, n$ put $\alpha_i = (p, L_{i-1}, q, L_i)$. Thus, the product $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n$ is well defined. Note that

$$\alpha_{n-1} \cdot \alpha_n = (p, L_{n-2}, q, L_{n-1}, p, L_{n-1}, q, L_n, p) \sim (p, L_{n-2}, q, L_n) =: \alpha'_{n-1}.$$

So, $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_{n-1} \cdot \alpha_n \sim \alpha_1 \cdot \alpha_3 \cdot \dots \cdot \alpha'_{n-1}$. By iterating this argument we eventually obtain $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n \sim (p, L_0, q, L_n, p) = \alpha$. □

We can now prove the main theorem of this subsection.

Theorem 3.21. *Either $\Gamma_{\mathbb{F}}(\mathbb{A})$ is simply connected or it is covered by a building.*

Proof. Suppose that $\Gamma = \Gamma_{\mathbb{F}}(\mathbb{A})$ is not covered by a building. Then, by Corollary 3.8, at least one of its nondegenerate primitive paths is null-homotopic. By Lemma 3.10 (claim (4)) and Lemma 3.14, at least one orthogonal nondegenerate primitive path, say α, is null-homotopic. Let $\ell = \ell(\alpha)$ be its line-invariant. The action of $G := \mathrm{Aut}(\Gamma)$ on $\mathbb{A}$ and $\mathbb{O}$ makes it clear that G acts transitively on the set of orthogonal primitive paths with line-invariant equal to ℓ. Thus, all orthogonal primitive paths with line invariant ℓ are null-homotopic.

Suppose firstly that $|\ell| < 1$. Then every orthogonal primitive path β is null homotopic, by Corollary 3.20 and the above remark. In this case Γ is simply connected by Lemmas 3.10 and 3.14 and Corollary 3.6.

Let $|\ell| = 1$. If $\ell \neq -1$ (whence $\Gamma = \Gamma_{\mathbb{C}}(\mathbb{O})$) then $\alpha \sim \beta$ for some orthogonal primitive path β with $|\ell(\beta)| < 1$, by Lemma 3.18. Thus, we can replace α with β and we are back to the previous case.

Finally, let $\ell(\alpha) = -1$. Clearly α admits a nonorthogonal shift $\beta \sim \alpha$, necessarily nondegenerate (Lemma 3.10). In its turn β admits an orthogonal shift γ with $\ell(\gamma) \neq -1$, by Lemma 3.15. Moreover $\beta \sim \gamma$ by Lemma 3.10. Hence $\alpha \sim \gamma$. Therefore γ is null-homotopic. We can now replace α with γ and we are back to the first or second one of the two previous cases, according to whether $|\ell(\gamma)| < 1$ or $|\ell(\gamma)| = 1$. □

Remark 3.22. What Schillewaert and Struyve say to explain their Remark 6.9 in [2017] amounts to a sketch of the first three paragraphs of the above proof. However, as they had nothing like Lemma 3.15 at their disposal, they could only refer to the case $\ell \neq -1$ in that remark.

3D. ***End of the proof of Theorem 1.5.*** Let $\widetilde{\Gamma}$ be the universal cover of $\Gamma = \Gamma_{\mathbb{F}}(\mathbb{A})$. In view of Theorem 3.21, either $\widetilde{\Gamma} = \Gamma$ or $\widetilde{\Gamma}$ is a building. In order to finish the proof of Theorem 1.5 it only remains to prove that $\widetilde{\Gamma}$ cannot be a building. This immediately follows from the last claim of Theorem 1.3. However, as we have promised not to use that theorem, we shall give an explicit proof of this claim.

We firstly recall a few general properties of universal coverings and state some notation for quadratic and hermitian forms and related polar spaces.

3D1. *Lifting automorphisms through universal coverings.* Let $\phi : \widetilde{\Gamma} \to \Gamma$ be the universal k-covering of a geometry Γ of rank $n > k$. Let $G := \mathrm{Aut}(\Gamma)$ and $\widehat{G} := \mathrm{Aut}(\widetilde{\Gamma})$.

Pick a chamber C of Γ and a chamber $\widetilde{C} \in \phi^{-1}(C)$. For every $g \in G$ and every chamber $\widetilde{X} \in \phi^{-1}(g(C))$ there exists a unique $\tilde{g} \in \widehat{G}$, called a *lifting* of g to $\widetilde{\Gamma}$ *through* ϕ, such that $\phi \cdot \tilde{g} = g \cdot \phi$ and $\tilde{g}(\widetilde{C}) = \widetilde{X}$ [Pasini 1994, Theorem 12.13]. The set of all liftings of the elements $g \in G$ is a subgroup $\widetilde{G}$ of $\widehat{G}$ and the function $p_\phi : \widetilde{G} \to G$ which maps every $\tilde{g} \in \widetilde{G}$ onto the unique $g \in G$ such that $\phi \cdot \tilde{g} = g \cdot \phi$ is a (surjective) homomorphism of groups. The kernel of p_ϕ, namely the group of all liftings of the identity automorphisms of Γ, is the *deck group* $D(\phi)$ of ϕ and $\Gamma \cong \widetilde{\Gamma}/D(\phi)$ [Pasini 1994, Theorem 12.13].

Given a subflag $F \subset C$ of rank k, let $\widetilde{F}$ be the corresponding subflag of $\widetilde{C}$ and let G_F be the stabilizer F in G. The stabilizer $\widetilde{G}_{\widetilde{F}}$ of $\widetilde{F}$ in $\widetilde{G}$ meets $D(\phi)$ trivially. Hence p_ϕ induces an isomorphism from $\widetilde{G}_{\widetilde{F}}$ to G_F. We call $\widetilde{G}_{\widetilde{F}}$ the *lifting* of G_F to $\widetilde{\Gamma}$ *through* ϕ *based* at $\widetilde{F}$.

Moreover, let $K_F \trianglelefteq G_F$ be the elementwise stabilizer in G_F of the residue $\mathrm{Res}_\Gamma(F)$ of F in Γ. Similarly, let $\widetilde{K}_{\widetilde{F}}$ be the elementwise stabilizer of $\mathrm{Res}_{\widetilde{\Gamma}}(\widetilde{F})$ in $\widetilde{G}_{\widetilde{F}}$. Then p_ϕ isomorphically maps $\widetilde{K}_{\widetilde{F}}$ onto K_F.

In order to complete the notation adopted above, we denote by $\widehat{G}_{\widetilde{F}}$ and $\widehat{K}_{\widetilde{F}}$ the stabilizer of $\widetilde{F}$ in $\widehat{G}$ and the elementwise stabilizer of $\mathrm{Res}_{\widetilde{\Gamma}}(\widetilde{F})$ in $\widehat{G}_{\widetilde{F}}$. Needless to say, $\widetilde{G}_{\widetilde{F}}$ and $\widetilde{K}_{\widetilde{F}}$ are subgroups of $\widehat{G}_{\widetilde{F}}$ and $\widehat{K}_{\widetilde{F}}$ respectively and $\widetilde{K}_{\widetilde{F}} = \widehat{K}_{\widetilde{F}} \cap \widetilde{G}_{\widetilde{F}}$.

The group K_F (respectively $\widetilde{K}_{\widetilde{F}}$ or $\widehat{K}_{\widetilde{F}}$) is often called the *kernel* of G_F (respectively $\widetilde{G}_{\widetilde{F}}$ or $\widehat{G}_{\widetilde{F}}$), as a shortening for “kernel of the action of G_F on $\mathrm{Res}_\Gamma(F)$". We shall adopt this terminology too in the sequel.

3D2. *Some notation for quadratic and hermitian forms.* For a positive integer n, let $f_n^{\mathbb{F}}$ be the usual scalar product on $\mathbb{F}^n$ and let $\mathrm{L}(f_n^{\mathbb{F}})$ be the group of all linear mappings preserving $f_n^{\mathbb{F}}$. So, $\mathrm{L}(f_n^{\mathbb{R}}) = \mathrm{O}(n)$ and $\mathrm{L}(f_n^{\mathbb{C}}) = \mathrm{U}(n)$ (notation as usual for Lie groups).

Given two positive integers n, m with $n \le m$, let $f_{n,m}^{\mathbb{F}} := (-f_n^{\mathbb{F}}) \oplus f_m^{\mathbb{F}}$. Namely, $f_{n,m}^{\mathbb{F}}$ admits the following representations, according to whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, where $x = (x_i)_{i=1}^{n+m}$ and $y = (y_i)_{i=1}^{n+m}$ (vectors of $\mathbb{F}^{n+m}$):

$$(\mathbb{F} = \mathbb{R}) \quad f_{n,m}^{\mathbb{R}}(x, y) := -\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{m} x_{i+n} y_{i+m},$$

$$(\mathbb{F} = \mathbb{C}) \quad f_{n.m}^{\mathbb{C}}(x, y) := -\sum_{i=1}^{n} \overline{x_i} y_i + \sum_{i=1}^{m} \overline{x_{i+n}} y_{i+m}.$$

Clearly, n is the Witt index of $f_{n,m}^{\mathbb{F}}$. We also recall that, by Sylvester's law of inertia, every nondegenerate bilinear form on $\mathbb{R}^{n+m}$ of Witt index $n \le m$ can be expressed as $f_{n,m}^{\mathbb{R}}$ or its opposite, modulo a suitable choice of the basis of $\mathbb{R}^{n+m}$ (see, e.g., [Bourbaki 1959, §7, n.2]). The same for hermitian forms of $\mathbb{C}^{n+m}$.

Let $\mathrm{L}(f_{n.m}^{\mathbb{F}})$ be the group of linear trasformations of $\mathbb{F}^{n+m}$ preserving $f_{n,m}^{\mathbb{F}}$. So we have $\mathrm{L}(f_{n,n}^{\mathbb{R}}) = \mathrm{O}^+(2n, \mathbb{R})$, $\mathrm{L}(f_{n,n+1}^{\mathbb{R}}) = \mathrm{O}(2n+1, \mathbb{R})$, $\mathrm{L}(f_{n,n}^{\mathbb{C}}) = \mathrm{U}(2n, \mathbb{C})$ and $\mathrm{L}(f_{n,n+1}^{\mathbb{C}}) = \mathrm{U}(2n+1, \mathbb{C})$ (notation as usual for Chevalley groups).

Let $\Gamma(f_{n,m}^{\mathbb{F}})$ be the polar space associated to $f_{n,m}^{\mathbb{F}}$. Recall that its full automorphims group $\mathrm{Aut}(\Gamma(f_{n,m}^{\mathbb{F}}))$ is the projectivization $\mathrm{PL}(f_{n,m}^{\mathbb{F}})$ of $\mathrm{L}(f_{n,m}^{\mathbb{F}})$, extended by two (possibly trivial) outer automorphism groups, henceforth denoted by $\boldsymbol{d}$ and $\boldsymbol{f}$. The group $\boldsymbol{d}$ is contributed by linear transformations of $\mathbb{F}^{n+m}$ which do not preserve

$f^{\mathbb{F}}_{n,m}$ but multiply it by a scalar. However, as we deal with $\mathrm{PL}(f^{\mathbb{F}}_{n,m})$ rather than $\mathrm{L}(f^{\mathbb{F}}_{n,m})$, it turns ut that $\boldsymbol{d}$ is either trivial or isomorphic to the group C_2 of order 2, according to whether $n+m$ is odd or even. The group $\boldsymbol{f}$ is trivial when $\mathbb{F}=\mathbb{R}$ and isomorphic to C_2 when $\mathbb{F}=\mathbb{C}$. In the latter case, the unique nontrivial involution of $\boldsymbol{f}$ is contributed by the usual conjugation of $\mathbb{C}$ and the extension $(\mathrm{PL}(f^{\mathbb{C}}_{n,m})\cdot\boldsymbol{d})\cdot\boldsymbol{f}$ is split: it can be realized as the semidirect product $(\mathrm{PL}(f^{\mathbb{C}}_{n,m})\cdot\boldsymbol{d})\rtimes\langle\iota\rangle$ of $\mathrm{PL}(f^{\mathbb{C}}_{n,m})\cdot\boldsymbol{d}$ with the group $\langle\iota\rangle$ generated by a suitable involutory semilinear transformation ι of $\mathbb{C}^{n+m}$.

3D3. *The case* $(\mathbb{F},\mathbb{A})=(\mathbb{C},\mathbb{O})$. Let $\phi:\widetilde{\Gamma}\to\Gamma$ be the universal covering of $\Gamma=\Gamma_{\mathbb{C}}(\mathbb{O})$. We already know that either $\widetilde{\Gamma}=\Gamma$ or $\widetilde{\Gamma}$ is a building. We want show that $\widetilde{\Gamma}$ cannot be a building.

By contradiction, suppose that $\widetilde{\Gamma}$ is a building, namely a polar space of rank 3. We know that the residues of the planes of Γ are isomorphic to the complex projective plane $\mathbb{C}\mathrm{P}^2=\mathrm{PG}(2,\mathbb{C})$ while the panels of type 3 (namely the residues of the point-line flags) are homeomorphic to the 3-dimensional sphere $\mathbb{S}^3$ [Kramer and Lytchak 2014]. The same properties hold for $\widetilde{\Gamma}$. So, in view of Tits's classification of polar spaces [Tits 1974, Chapter 8], necessarily $\widetilde{\Gamma}=\Gamma(f^{\mathbb{C}}_{3,4})$, with full automorphism group

$$\widehat{G}:=\mathrm{Aut}(\Gamma(f^{\mathbb{C}}_{3,4}))=\mathrm{PU}(7,\mathbb{C})\rtimes\boldsymbol{f}\cong\mathrm{PSU}(7,\mathbb{C})\rtimes C_2.$$

We set $G:=\mathrm{Aut}(\Gamma)=((\mathrm{SU}(3)\times\mathrm{SU}(3))/C_3)\rtimes C_2$ (see Section 2C).

Let $\tilde{\xi}$ be a plane of $\widetilde{\Gamma}$ and $\xi=\phi(\tilde{\xi})$. With the notation and the terminology of Section 3D1, let G_ξ, $\widehat{G}_{\tilde{\xi}}$ and $\widetilde{G}_{\tilde{\xi}}$ be respectively the stabilizer of ξ in G, the stabilizer of $\tilde{\xi}$ in $\widehat{G}$ and the lifting of G_ξ to $\widetilde{\Gamma}$ through ϕ at $\tilde{\xi}$ and let K_ξ, $\widehat{K}_{\tilde{\xi}}$ and $\widetilde{K}_{\tilde{\xi}}$ be their kernels. It is not difficult to check that

$$G_\xi=\mathrm{PSU}(3)\rtimes C_2\quad\text{with } K_\xi=1.$$

(See also [Schillewaert and Struyve 2017].) Hence $\widetilde{G}_{\tilde{\xi}}\cong\mathrm{PSU}(3)\rtimes C_2$ and $\widetilde{K}_{\tilde{\xi}}=1$. On the other hand, $\widehat{G}_{\tilde{\xi}}$ is the semidirect product $\widehat{G}_{\tilde{\xi}}=U\rtimes L$ of its unipotent radical U and a Levi complement L, where $U\cong\mathbb{C}^6\times\mathbb{R}^3\cong\mathbb{R}^{15}$, with $\mathbb{C}^6$, $\mathbb{R}^3$ and $\mathbb{R}^{15}$ being regarded as additive groups, and $L\cong\mathrm{GL}(3,\mathbb{C})\rtimes\boldsymbol{f}=\Gamma\mathrm{L}(3,\mathbb{C})$. Moreover $\widehat{K}_{\tilde{\xi}}=U\rtimes Z$ where $Z=Z(L)$ is the center of L (see, e.g., [Weiss 2003, Chapter 11]). The group $\widetilde{G}_{\tilde{\xi}}\cong\mathrm{PSU}(3)\rtimes C_2$ is contained in $\widehat{G}_{\tilde{\xi}}=U\rtimes L$ but, as its kernel is trivial, it meets $\widehat{K}_{\tilde{\xi}}=U\rtimes Z$ trivially. Accordingly, the group $L\cong\Gamma\mathrm{L}(3,\mathbb{C})$ contains a copy of $\widetilde{G}_{\tilde{\xi}}=\mathrm{PSU}(3)\rtimes C_2$. The group L indeed contains copies of $\mathrm{SU}(3)\rtimes C_2$, but no copy of $\mathrm{PSU}(3)\rtimes C_2$. Indeed $\mathrm{SU}(3)$ is not a semidirect product of its center C_3 and a copy of $\mathrm{PSU}(3)$.

We have reached a contradiction. Hence in this case $\widetilde{\Gamma}=\Gamma$.

3D4. *The case* $(\mathbb{F}, \mathbb{A}) = (\mathbb{R}, \mathbb{H})$. Let now $\phi : \widetilde{\Gamma} \to \Gamma$ be the universal covering of $\Gamma = \Gamma_{\mathbb{R}}(\mathbb{H})$. By contradiction, suppose that $\widetilde{\Gamma}$ is a building. The residues of the planes of Γ are isomorphic to the real projective plane $\mathrm{PG}(2, \mathbb{R})$ and the panels of type 3 are homeomorphic to the 5-dimensional sphere $\mathbb{S}^5$ [Kramer and Lytchak 2019]. By Tits's classification of polar spaces [1974] we see that $\widetilde{\Gamma} = \Gamma(f^{\mathbb{R}}_{3,8})$, with full automorphism group $\widehat{G} := \mathrm{Aut}(\Gamma(f^{\mathbb{R}}_{3,8})) = \mathrm{PL}(f^{\mathbb{R}}_{3,8})$. We set $G := \mathrm{Aut}(\Gamma) = \mathrm{SO}(3) \times \mathrm{G}_2$ (see Section 2C). As in the previous case, let $\tilde{\xi}$ be a plane of $\widetilde{\Gamma}$ and $\xi := \phi(\tilde{\xi})$. We now have

$$\begin{aligned} G_\xi &= (\mathrm{SU}(2) \times \mathrm{SU}(2))/\langle(-\iota, -\iota)\rangle = 2^{\cdot}(\mathrm{PSU}(2) \times \mathrm{PSU}(2)), \\ K_\xi &= 2^{\cdot}\,\mathrm{PSU}(2) = \mathrm{SU}(2), \\ G_\xi/K_\xi &\cong \mathrm{PSU}(2) \cong \mathrm{SO}(3). \end{aligned}$$

Here ι stands for the identity element of $\mathrm{SU}(2)$, whence (ι, ι) is the identity element of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The extension $2^{\cdot}(\mathrm{PSU}(2) \times \mathrm{PSU}(2))$ is nonsplit.

On the other hand, $\widehat{G}_\xi = U \rtimes L$ where $L \cong \mathrm{GL}(3, \mathbb{R}) \times \mathrm{SO}(5)$ and $U = U_0{}^{\cdot}U_1$ with U_0 and U_1 isomorphic to the additive groups of $\mathbb{R}^3$ and $\mathbb{R}^{15}$ respectively. The group U_0 is both the center and the commutator subgroup of U. Moreover, $\widehat{K}_{\tilde{\xi}} = U \rtimes (Z \times \mathrm{SO}(5))$, where Z is the center of $\mathrm{GL}(3, \mathbb{R})$.

We have $\widetilde{G}_{\tilde{\xi}} \cong G_\xi = 2^{\cdot}(\mathrm{PSU}(2) \times \mathrm{PSU}(2))$, $\widetilde{K}_{\tilde{\xi}} \cong K_\xi = \mathrm{SU}(2)$ and $\widetilde{K}_{\tilde{\xi}}$ must be placed in $\widehat{K}_{\tilde{\xi}}$. As $U \trianglelefteq \widehat{K}_{\tilde{\xi}}$, the intersection $\widetilde{K}_{\tilde{\xi}} \cap U$ is normal in $\widetilde{K}_{\tilde{\xi}}$. However $\widetilde{K}_{\tilde{\xi}} \cong \mathrm{SU}(2)$ is quasisimple as an abstract group, with center of order 2, while every nontrivial subgroup of U is infinite. Therefore $\widetilde{K}_{\tilde{\xi}} \cap U = 1$, namely $\widetilde{K}_{\tilde{\xi}} \leq L \cap \widehat{K}_{\tilde{\xi}} = Z \times \mathrm{SO}(5)$. Moreover $\widetilde{K}_{\tilde{\xi}} \leq \mathrm{SO}(5)$, since $\mathrm{SU}(2)$ doesn't split as the direct product of its center and a copy of $\mathrm{PSU}(2)$. So far, no contradiction has arised; indeed $\mathrm{SO}(5)$ actually contains copies of $\mathrm{SU}(2)$.

Similarly, $\widetilde{G}_{\tilde{\xi}}/\widetilde{K}_{\tilde{\xi}} \cong \mathrm{PSU}(2)$ must be placed in $\widehat{G}_{\tilde{\xi}}/\widehat{K}_{\tilde{\xi}} = L/(Z \times \mathrm{SO}(5)) = \mathrm{PGL}(3, \mathbb{R})$. This can be done as well, since $\mathrm{PGL}(3, \mathbb{R})$ contains copies of $\mathrm{SO}(3) \cong \mathrm{PSU}(2)$. However these copies of $\mathrm{SO}(3)$ inside $\mathrm{GL}(3, \mathbb{R})$ meet the center Z of $\mathrm{GL}(3, \mathbb{R})$ trivially. It follows that $\widetilde{G}_{\tilde{\xi}}$ is the direct product $\widetilde{G}_{\tilde{\xi}} = \widetilde{K}_{\tilde{\xi}} \times X$ for a subgroup $X \cong \mathrm{SO}(3) \cong \mathrm{PSU}(2)$ of $\mathrm{GL}(3, \mathbb{R})$. In short, $\widetilde{G}_{\tilde{\xi}} = \mathrm{SU}(2) \times \mathrm{PSU}(2)$. However $\widetilde{G}_{\tilde{\xi}} \cong G_\xi = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\langle(-\iota, -\iota)\rangle$, which is not a direct product of $\mathrm{SU}(2)$ and $\mathrm{PSU}(2)$. Eventually, we have reached a contradiction.

Therefore $\widetilde{\Gamma} = \Gamma$ in this case too. The proof of Theorem 1.5 is complete.

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Received 27 Nov 2018. Revised 12 Mar 2019.

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Innovation in Incidence Geometry

Vol. 17 No. 3 2019