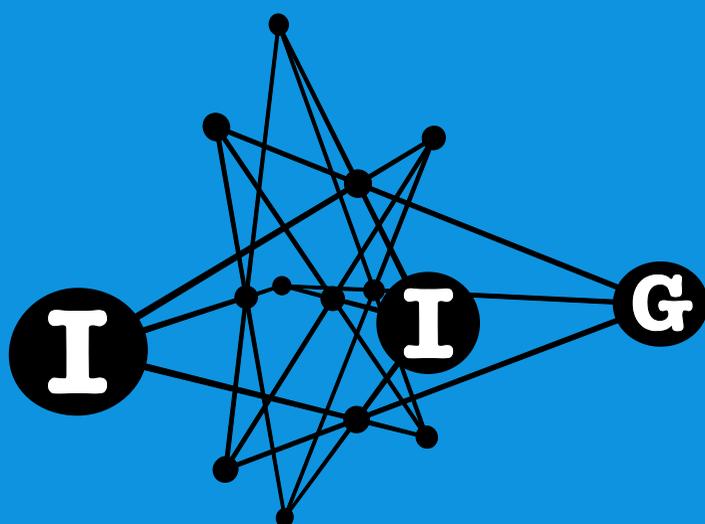


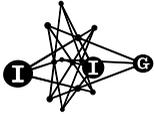
Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial



A note on locally elliptic actions on cube complexes

Nils Leder and Olga Varghese



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We deduce from Sageev's results that whenever a group acts locally elliptically on a finite-dimensional $\text{CAT}(0)$ cube complex, then it must fix a point. As an application, we partially prove a conjecture by Marquis concerning actions on buildings and we give an example of a group G such that G does not have property (T), but G and all its finitely generated subgroups can not act without a fixed point on a finite-dimensional $\text{CAT}(0)$ cube complex, answering a question by Barnhill and Chatterji.

1. Introduction

The questions we investigate in this note are concerned with fixed points on $\text{CAT}(0)$ cube complexes. Roughly speaking, a cube complex is a union of cubes of any dimension which are glued together along isometric faces. Let \mathcal{C} be a class of finite-dimensional $\text{CAT}(0)$ cube complexes. A group G is said to have property FC if any simplicial action of G on any member of \mathcal{C} has a fixed point. For a subclass \mathcal{A} consisting of simplicial trees the study of property FA was initiated by Serre [1980].

Bass [1976] introduced a weaker property FA' for groups. A group has property FA' if any simplicial action of G on any member of \mathcal{A} is locally elliptic, i.e. each $g \in G$ fixes some point on a tree. We define a generalization of property FA' . A group G has property FC' if any simplicial action of G on any member of \mathcal{C} is locally elliptic, i.e. each $g \in G$ fixes some point on a $\text{CAT}(0)$ cube complex.

A finitely generated group which is acting locally elliptically on a simplicial tree has a global fixed point; see [Serre 1980, §6.5, Corollary 2]. The following result of Sageev is well known to the experts. It follows from the proof of Theorem 5.1 in [Sageev 1995].

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Theorem A. *Let G be a finitely generated group acting by simplicial isometries on a finite-dimensional CAT(0) cube complex. If the G -action is locally elliptic, then G has a global fixed point.*

In particular, a finitely generated group G has property FC' if and only if G has property FC.

The result of Theorem A was also observed by Caprace and Lytchak in [Chatterji et al. 2016, Proposition B.8] and was proven for median spaces in [Fioravanti 2018, Theorem 3.1].

Before we state the corollaries of Theorem A, we observe that the result in Theorem A is not true for infinite-dimensional CAT(0) cube complexes. Let G be a finitely generated torsion group. Then, by the Bruhat–Tits fixed point theorem [Bridson and Haefliger 1999, Corollary II 2.8] follows, that G has property FC' and thus by Theorem A the group G has property FC. Free Burnside groups are finitely generated torsion groups and thus these groups have always property FC, but many of these groups act without a fixed point on infinite-dimensional CAT(0) cube complexes; see [Osajda 2018, Theorem 1].

The next corollary follows from Theorem A and is known in the case of trees by a result of Tits [1970, Proposition 3.4].

Corollary B. *Let G be a group acting by simplicial isometries on a finite-dimensional CAT(0) cube complex X . If the G -action is locally elliptic, then G has a global fixed point in $X \cup \partial X$, where ∂X denotes the visual boundary of X .*

Proof. For the proof we need the following result by Caprace [2010, Theorem 1.1]:

Let X be a finite-dimensional CAT(0) cube complex and $\{X_\alpha\}_{\alpha \in A}$ be a filtering family of closed convex nonempty subsets. Then either the intersection $\bigcap_{\alpha \in A} X_\alpha$ is nonempty or the intersection of the visual boundaries $\bigcap_{\alpha \in A} \partial X_\alpha$ is a nonempty subset of ∂X .

Recall that a family \mathcal{F} of subsets of a given set is called *filtering* if for all E, F in \mathcal{F} there exists $D \in \mathcal{F}$ such that $D \subseteq E \cap F$.

Let X be a finite-dimensional CAT(0) cube complex and Φ a simplicial action of G on X . For $S \subseteq G$ we define the set $\text{Fix}(S) = \{x \in X \mid \Phi(s)(x) = x \text{ for all } s \in S\}$. It is closed and convex. If S is a finite set, it follows by Theorem A that $\text{Fix}(S)$ is nonempty. Further, we define $\text{Fix}(G)^\partial = \{\xi \in \partial X \mid \Phi(g)(\xi) = \xi \text{ for all } g \in G\}$.

Now we consider the following family $\mathcal{F} = \{\text{Fix}(S) \mid S \subseteq G \text{ and } \#S < \infty\}$. If $S, T \subseteq G$ are finite subsets, we have $\text{Fix}(S \cup T) \subseteq \text{Fix}(S) \cap \text{Fix}(T)$ and thus \mathcal{F} is a filtering. The result of Caprace stated above implies that

$$\bigcap \mathcal{F} = \text{Fix}(G) \text{ is nonempty}$$

or

$$\bigcap \{\partial \text{Fix}(S) \mid S \subseteq G \text{ and } \#S < \infty\} \subseteq \text{Fix}(G)^\partial \text{ is nonempty.}$$

□

Since the Davis realization of a right-angled building carries the structure of a finite-dimensional CAT(0) cube complex, we can apply Corollary B to confirm the following conjecture by Marquis [2015, Conjecture 2] in the special case of right-angled buildings.

Conjecture. *Let G be a group acting by type-preserving simplicial isometries on a building Δ . If the G -action on the Davis realization X of Δ is locally elliptic, then G has a global fixed point in $X \cup \partial X$.*

Another fixed point property of interest is Kazhdan's property (T). Niblo and Reeves [1997, Theorem B] proved in that if a group G has Kazhdan's property (T), then G also has property FC. Barnhill and Chatterji raised the following question [2008, Question 5.3]:

Question. *Is FC equivalent to (T), or does there exist a group G such that G does not have property (T), but G and all its finite-index subgroups have property FC?*

With the next result we can answer this question in the negative.

Corollary C. *Let G be the first Grigorchuk group. Then G and all its finitely generated subgroups have property FC, but G doesn't have property (T). In particular, all finite-index subgroups of G also have property FC.*

Proof. The first Grigorchuk group G is a finitely generated infinite torsion group (see [Grigorchuk 1980]) and thus G and all its finitely generated subgroups have property FC. But G does not have property (T) since G is amenable, see [Grigorchuk 1984]. □

Further, many free Burnside groups have property FC, but don't have property (T), see [Osajda 2018, Theorem 1]. Other examples of groups with property FC and without property (T) were given by Cornulier in [Cornulier 2015] and by Genevois in [Genevois 2019].

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2. Proof of Theorem A

In this section we give the proof of Theorem A, which is hidden in the proof of Theorem 5.1 in [Sageev 1995] by Sageev. For definitions and properties of CAT(0) cube complexes see [Sageev 1995].

We first need the following result.

Proposition. *Let X be a d -dimensional CAT(0) cube complex and \mathcal{S} be a finite set of hyperplanes in X . If $\#\mathcal{S} \geq d + d \cdot (d + 1)$, then there exist three hyperplanes in \mathcal{S} that do not intersect pairwise.*

Proof. Let $\mathcal{T} = \{J_1, \dots, J_k\} \subseteq \mathcal{S}$ be a maximal set of pairwise intersecting hyperplanes. Then by Helly's Theorem for CAT(0) cube complexes or [Sageev 1995, Theorem 4.14] follows that $\bigcap \mathcal{T}$ is not empty. Further, since the dimension of X is d we have: $k \leq d$. By maximality of \mathcal{T} , for each hyperplane $J \in \mathcal{S} - \mathcal{T}$ there exists $i = 1, \dots, k$ such that $J \cap J_i = \emptyset$. This yields a well-defined map

$$q : \mathcal{S} - \mathcal{T} \rightarrow \{1, \dots, k\}, J \mapsto \min\{i \mid J \cap J_i = \emptyset\}.$$

Let B_i denote the preimage $q^{-1}(i)$ for $i = 1, \dots, k$. Since $\#\mathcal{S} \geq d + d \cdot (d + 1)$ and $k \leq d$, we have $\#(\mathcal{S} - \mathcal{T}) \geq d \cdot (d + 1)$. Thus, by the pigeon-hole principle there exists $j \in \{1, \dots, k\}$ such that $\#B_j \geq d + 1$. By maximality of \mathcal{T} , not all hyperplanes of B_j intersect pairwise, i.e there are $H_1, H_2 \in B_j$ such that $H_1 \cap H_2 = \emptyset$. Then, J_j, H_1, H_2 are three hyperplanes that do not intersect each other. \square

Proof of Theorem A. Let G be a finitely generated group with a symmetric generating set $Y = \{g_1, \dots, g_n\}$. Let X be a d -dimensional CAT(0) cube complex, $v \in X$ be a vertex and $G \rightarrow \text{Isom}(X)$ be a simplicial locally elliptic action.

For $i = 1, \dots, n$ we choose a combinatorial geodesic λ_i from v to $g_i(v)$. Further, we denote by \mathcal{S}_i the set of hyperplanes crossed by λ_i . We have $\#\mathcal{S}_i = D(v, g_i(v))$, where we denote by D the metric on the 1-skeleton of X . Hence the union $\mathcal{S} := \bigcup_{i=1}^n \mathcal{S}_i$ is a finite set.

Let us assume that the action has no global fixed point. Then the Bruhat–Tits fixed point theorem implies that the orbit of v is unbounded. Thus, there exists $g \in G$ such that

$$N := D(v, g(v)) \geq \#\mathcal{S} \cdot (d + d(d + 1)).$$

Since Y generates G , we can write $g = g_{i_1} \dots g_{i_l}$ with $g_{i_j} \in Y$ for $i = 1, \dots, l$. We define

$$v_j := g_{i_1} \dots g_{i_j}(v) \text{ and } \gamma_j := g_{i_1} \dots g_{i_j}(\lambda_{i_{j+1}}).$$

The map γ_j is a combinatorial geodesic from v_j to v_{j+1} . Hence $\alpha := \gamma_1 \dots \gamma_l \lambda_{g_{i_1}}$ is a combinatorial path from v to $g(v)$. Since $D(v, g(v)) = N$, there exists a set of hyperplanes $\mathcal{T} = \{K_1, \dots, K_N\}$ such that α crosses each hyperplane in \mathcal{T} .

By construction, for each K_i in \mathcal{T} there exists $J \in \mathcal{S}$ such that $K_i = hJ$ for some $h \in G$. By pigeon-hole principle there exists a hyperplane $J \in \mathcal{S}$ such that

$$\#\{K \in \mathcal{T} \mid \exists h \in G : K = hJ\} \geq d + d(d + 1).$$

By the Proposition there exist three hyperplanes h_1J, h_2J and h_3J in

$$\{K \in \mathcal{T} \mid \exists h \in G : K = hJ\}$$

whose pairwise intersection is empty. But each of these hyperplanes is crossed precisely once by a combinatorial geodesic from v to $g(v)$. Therefore one of these hyperplanes separates the other two.

It is not difficult to verify the following: If there exist a hyperplane $J \subseteq X$ and $g, h \in G$ such that J, gJ, hJ do not intersect pairwise and gJ separates J and hJ , then g, h or hg^{-1} is hyperbolic.

This completes the proof. \square

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