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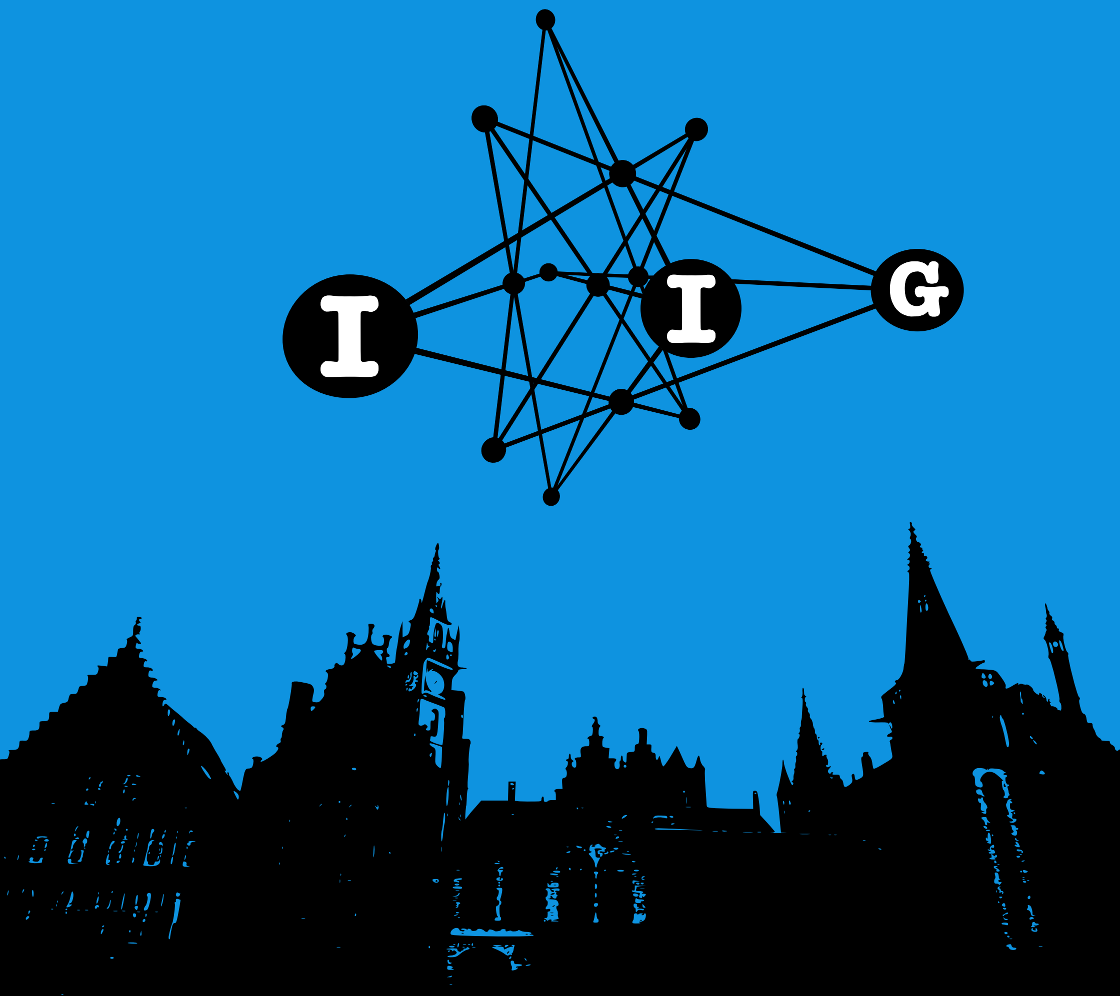


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# A note on locally elliptic actions on cube complexes

Nils Leder and Olga Varghese

We deduce from Sageev's results that whenever a group acts locally elliptically on a finite-dimensional CAT(0) cube complex, then it must fix a point. As an application, we partially prove a conjecture by Marquis concerning actions on buildings and we give an example of a group $G$ such that $G$ does not have property (T), but $G$ and all its finitely generated subgroups can not act without a fixed point on a finite-dimensional CAT(0) cube complex, answering a question by Barnhill and Chatterji.

## 1. Introduction

The questions we investigate in this note are concerned with fixed points on CAT(0) cube complexes. Roughly speaking, a cube complex is a union of cubes of any dimension which are glued together along isometric faces. Let $\mathcal{C}$ be a class of finite-dimensional CAT(0) cube complexes. A group $G$ is said to have property F$\mathcal{C}$ if any simplicial action of $G$ on any member of $\mathcal{C}$ has a fixed point. For a subclass $\mathcal{A}$ consisting of simplicial trees the study of property F$\mathcal{A}$ was initiated by Serre [1980].

Bass [1976] introduced a weaker property F$\mathcal{A}'$ for groups. A group has property F$\mathcal{A}'$ if any simplicial action of $G$ on any member of $\mathcal{A}$ is locally elliptic, i.e. each $g \in G$ fixes some point on a tree. We define a generalization of property F$\mathcal{A}'$. A group $G$ has property F$\mathcal{C}'$ if any simplicial action of $G$ on any member of $\mathcal{C}$ is locally elliptic, i.e. each $g \in G$ fixes some point on a CAT(0) cube complex.

A finitely generated group which is acting locally elliptically on a simplicial tree has a global fixed point; see [Serre 1980, §6.5, Corollary 2]. The following result of Sageev is well known to the experts. It follows from the proof of Theorem 5.1 in [Sageev 1995].

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**Theorem A.** *Let $G$ be a finitely generated group acting by simplicial isometries on a finite-dimensional* CAT(0) *cube complex. If the $G$-action is locally elliptic, then $G$ has a global fixed point.*

*In particular, a finitely generated group $G$ has property* $\mathrm{F}\mathcal{C}'$ *if and only if $G$ has property* $\mathrm{F}\mathcal{C}$.

The result of Theorem A was also observed by Caprace and Lytchak in [Chatterji et al. 2016, Proposition B.8] and was proven for median spaces in [Fioravanti 2018, Theorem 3.1].

Before we state the corollaries of Theorem A, we observe that the result in Theorem A is not true for infinite-dimensional CAT(0) cube complexes. Let $G$ be a finitely generated torsion group. Then, by the Bruhat–Tits fixed point theorem [Bridson and Haefliger 1999, Corollary II 2.8] follows, that $G$ has property $\mathrm{F}\mathcal{C}'$ and thus by Theorem A the group $G$ has property $\mathrm{F}\mathcal{C}$. Free Burnside groups are finitely generated torsion groups and thus these groups have always property $\mathrm{F}\mathcal{C}$, but many of these groups act without a fixed point on infinite-dimensional CAT(0) cube complexes; see [Osajda 2018, Theorem 1].

The next corollary follows from Theorem A and is known in the case of trees by a result of Tits [1970, Proposition 3.4].

**Corollary B.** *Let $G$ be a group acting by simplicial isometries on a finite-dimensional* CAT(0) *cube complex $X$. If the $G$-action is locally elliptic, then $G$ has a global fixed point in $X \cup \partial X$, where $\partial X$ denotes the visual boundary of $X$.*

*Proof.* For the proof we need the following result by Caprace [2010, Theorem 1.1]:

Let $X$ be a finite-dimensional CAT(0) cube complex and $\{X_\alpha\}_{\alpha\in A}$ be a filtering family of closed convex nonempty subsets. Then either the intersection $\bigcap_{\alpha\in A} X_\alpha$ is nonempty or the intersection of the visual boundaries $\bigcap_{\alpha\in A} \partial X_\alpha$ is a nonempty subset of $\partial X$.

Recall that a family $\mathcal{F}$ of subsets of a given set is called *filtering* if for all $E$, $F$ in $\mathcal{F}$ there exists $D \in \mathcal{F}$ such that $D \subseteq E \cap F$.

Let $X$ be a finite-dimensional CAT(0) cube complex and $\Phi$ a simplicial action of $G$ on $X$. For $S \subseteq G$ we define the set $\mathrm{Fix}(S) = \{x \in X \mid \Phi(s)(x) = x \text{ for all } s \in S\}$. It is closed and convex. If $S$ is a finite set, it follows by Theorem A that $\mathrm{Fix}(S)$ is nonempty. Further, we define $\mathrm{Fix}(G)^\partial = \{\xi \in \partial X \mid \Phi(g)(\xi) = \xi \text{ for all } g \in G\}$.

Now we consider the following family $\mathcal{F} = \{\mathrm{Fix}(S) \mid S \subseteq G \text{ and } \#S < \infty\}$. If $S, T \subseteq G$ are finite subsets, we have $\mathrm{Fix}(S \cup T) \subseteq \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ and thus $\mathcal{F}$ is a filtering. The result of Caprace stated above implies that

$$\bigcap \mathcal{F} = \mathrm{Fix}(G) \text{ is nonempty}$$

or

$$\bigcap \{\partial\mathrm{Fix}(S) \mid S \subseteq G \text{ and } \#S < \infty\} \subseteq \mathrm{Fix}(G)^\partial \text{ is nonempty.}$$

□

Since the Davis realization of a right-angled building carries the structure of a finite-dimensional CAT(0) cube complex, we can apply Corollary B to confirm the following conjecture by Marquis [2015, Conjecture 2] in the special case of right-angled buildings.

**Conjecture.** *Let $G$ be a group acting by type-preserving simplicial isometries on a building $\Delta$. If the $G$-action on the Davis realization $X$ of $\Delta$ is locally elliptic, then $G$ has a global fixed point in $X \cup \partial X$.*

Another fixed point property of interest is Kazhdan's property (T). Niblo and Reeves [1997, Theorem B] proved in that if a group $G$ has Kazhdan's property (T), then $G$ also has property F$\mathcal{C}$. Barnhill and Chatterji raised the following question [2008, Question 5.3]:

**Question.** *Is* F$\mathcal{C}$ *equivalent to* (T), *or does there exist a group $G$ such that $G$ does not have property* (T), *but $G$ and all its finite-index subgroups have property* F$\mathcal{C}$?

With the next result we can answer this question in the negative.

**Corollary C.** *Let $G$ be the first Grigorchuk group. Then $G$ and all its finitely generated subgroups have property* F$\mathcal{C}$, *but $G$ doesn't have property* (T). *In particular, all finite-index subgroups of $G$ also have property* F$\mathcal{C}$.

*Proof.* The first Grigorchuk group $G$ is a finitely generated infinite torsion group (see [Grigorchuk 1980]) and thus $G$ and all its finitely generated subgroups have property F$\mathcal{C}$. But $G$ does not have property (T) since $G$ is amenable, see [Grigorchuk 1984]. □

Further, many free Burnside groups have property F$\mathcal{C}$, but don't have property (T), see [Osajda 2018, Theorem 1]. Other examples of groups with property F$\mathcal{C}$ and without property (T) were given by Cornulier in [Cornulier 2015] and by Genevois in [Genevois 2019].

***Acknowledgement.*** We would like to thank Rémi Coulon for pointing us on Theorem 5.1 in [Sageev 1995]. Further, we want to thank Elia Fioravanti and Anthony Genevois for making us aware of important references.

## 2. Proof of Theorem A

In this section we give the proof of Theorem A, which is hidden in the proof of Theorem 5.1 in [Sageev 1995] by Sageev. For definitions and properties of CAT(0) cube complexes see [Sageev 1995].

We first need the following result.

**Proposition.** *Let $X$ be a $d$-dimensional* CAT(0) *cube complex and $\mathcal{S}$ be a finite set of hyperplanes in $X$. If $\#\mathcal{S} \geq d + d \cdot (d+1)$, then there exist three hyperplanes in $\mathcal{S}$ that do not intersect pairwise.*

*Proof.* Let $\mathcal{T} = \{J_1, \dots, J_k\} \subseteq S$ be a maximal set of pairwise intersecting hyperplanes. Then by Helly's Theorem for CAT(0) cube complexes or [Sageev 1995, Theorem 4.14] follows that $\bigcap \mathcal{T}$ is not empty. Further, since the dimension of $X$ is $d$ we have: $k \leq d$. By maximality of $\mathcal{T}$, for each hyperplane $J \in \mathcal{S} - \mathcal{T}$ there exists $i = 1, \dots, k$ such that $J \cap J_i = \varnothing$. This yields a well-defined map

$$q : \mathcal{S} - \mathcal{T} \to \{1, \dots, k\},\ J \mapsto \min\{i \mid J \cap J_i = \varnothing\}.$$

Let $B_i$ denote the preimage $q^{-1}(i)$ for $i = 1, \dots, k$. Since $\#\mathcal{S} \geq d + d \cdot (d+1)$ and $k \leq d$, we have $\#(\mathcal{S} - \mathcal{T}) \geq d \cdot (d+1)$. Thus, by the pigeon-hole principle there exists $j \in \{1, \dots, k\}$ such that $\#B_j \geq d+1$. By maximality of $\mathcal{T}$, not all hyperplanes of $B_j$ intersect pairwise, i.e there are $H_1, H_2 \in B_j$ such that $H_1 \cap H_2 = \varnothing$. Then, $J_j, H_1, H_2$ are three hyperplanes that do not intersect each other. □

*Proof of Theorem A.* Let $G$ be a finitely generated group with a symmetric generating set $Y = \{g_1, \dots, g_n\}$. Let $X$ be a $d$-dimensional CAT(0) cube complex, $v \in X$ be a vertex and $G \to \mathrm{Isom}(X)$ be a simplicial locally elliptic action.

For $i = 1, \dots, n$ we choose a combinatorial geodesic $\lambda_i$ from $v$ to $g_i(v)$ . Further, we denote by $\mathcal{S}_i$ the set of hyperplanes crossed by $\lambda_i$. We have $\#\mathcal{S}_i = D(v, g_i(v))$, where we denote by $D$ the metric on the 1-skeleton of $X$. Hence the union $\mathcal{S} := \bigcup_{i=1}^n \mathcal{S}_i$ is a finite set.

Let us assume that the action has no global fixed point. Then the Bruhat–Tits fixed point theorem implies that the orbit of $v$ is unbounded. Thus, there exists $g \in G$ such that

$$N := D(v, g(v)) \geq \#\mathcal{S} \cdot (d + d(d+1)).$$

Since $Y$ generates $G$, we can write $g = g_{i_1} \dots g_{i_l}$ with $g_{i_j} \in Y$ for $i = 1, \dots, l$. We define

$$v_j := g_{i_1} \dots g_{i_j}(v) \text{ and } \gamma_j := g_{i_1} \dots g_{i_j}(\lambda_{i_{j+1}}).$$

The map $\gamma_j$ is a combinatorial geodesic from $v_j$ to $v_{j+1}$. Hence $\alpha := \gamma_l \dots \gamma_1 \lambda_{g_{i_1}}$ is a combinatorial path from $v$ to $g(v)$. Since $D(v, g(v)) = N$, there exists a set of hyperplanes $\mathcal{T} = \{K_1, \dots, K_N\}$ such that $\alpha$ crosses each hyperplane in $T$.

By construction, for each $K_i$ in $\mathcal{T}$ there exists $J \in \mathcal{S}$ such that $K_i = hJ$ for some $h \in G$. By pigeon-hole principle there exists a hyperplane $J \in \mathcal{S}$ such that

$$\#\{K \in \mathcal{T} \mid \exists h \in G : K = hJ\} \geq d + d(d+1).$$

By the Proposition there exist three hyperplanes $h_1 J$, $h_2 J$ and $h_3 J$ in

$$\{K \in \mathcal{T} \mid \exists h \in G : K = hJ\}$$

whose pairwise intersection is empty. But each of these hyperplanes is crossed precisely once by a combinatorial geodesic from $v$ to $g(v)$. Therefore one of these hyperplanes separates the other two.

It is not difficult to verify the following: If there exist a hyperplane $J \subseteq X$ and $g, h \in G$ such that $J$, $gJ$, $hJ$ do not intersect pairwise and $gJ$ separates $J$ and $hJ$, then $g$, $h$ or $hg^{-1}$ is hyperbolic.

This completes the proof. □

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# Tits arrangements on cubic curves

Michael Cuntz and David Geis

We classify affine rank three Tits arrangements whose roots are contained in the locus of a homogeneous cubic polynomial. We find that there exist irreducible affine Tits arrangements which are not locally spherical.

## 1. Introduction

Given a real reflection group, its set of reflecting hyperplanes defines a possibly infinite arrangement of hyperplanes with the property that every chamber is an open simplicial cone. Thus geometrically, a reflection group may be viewed as a so-called *simplicial arrangement*.

Of course, very few simplicial arrangements come from reflection groups. Another source, for example, are the *Weyl groupoids* (see [Heckenberger and Welker 2011; Cuntz 2011a]). As Weyl groups are invariants of different types of algebras in Lie theory, Weyl groupoids are invariants of (in a certain sense) more general quantum groups, the so-called Nichols algebras (see for example [Heckenberger 2006]). Not much is known about the infinite dimensional Nichols algebras which produce infinite Weyl groupoids. To go beyond the theory of finite dimensional Nichols algebras, it turns out that one needs an appropriate notion of infinite simplicial arrangement, which is the main contribution of [Cuntz et al. 2017], where these are called *Tits arrangements*. Thus a deep understanding of Tits arrangements would be beneficial for many reasons.

However, even the case of finite simplicial arrangements is poorly understood. Simplicial arrangements are quite rare; it is a highly non-trivial problem to classify simplicial arrangements, even finding further examples is very difficult. Among the known (irreducible) simplicial arrangements of rank three (see [Grünbaum 2009; Cuntz 2012]), one observes that the projective root vectors of almost all of them

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are contained in a cubic curve since this holds for the known infinite families of irreducible arrangements $\mathcal{R}(1)$ and $\mathcal{R}(2)$; see [Cuntz 2011b]. Thus one could hope to obtain many, maybe even almost all of the infinite Tits arrangements of rank three by concentrating on those for which the root vectors lie on a cubic curve. Also, there is no infinite family of infinite Tits arrangements known yet, even in the affine case, i.e. when the Tits cone is a half space.

There is yet another reason why cubic curves are interesting in this context: simplicial arrangements of rank three are combinatorially extremal in the sense that they have very few double points; one of the keys for the main result in [Green and Tao 2013] is the fact that arrangements with few double points are close to having the property that their root vectors lie on a cubic curve.

In this paper, we give a classification of *affine rank three Tits arrangements* whose corresponding projective root vectors are contained in the locus of a homogeneous cubic polynomial. Our strategy for the classification builds upon the results obtained in [Cuntz et al. 2017] and on elementary tools from the geometry of the (real) projective plane, like Bézout's theorem and the fact that the conic in $\mathbb{P}^2(\mathbb{R})$ is a selfdual curve.

We find that there are only two classes of irreducible affine Tits arrangements satisfying the above property: namely the arrangement of type $\tilde{A}_2$ whose corresponding projective root vectors are contained in the union of three projective lines, and a new class of arrangements which we call $\tilde{A}_2^0$ (see Figure 1). The projective root vectors of $\tilde{A}_2^0$ are contained in the union of a projective conic $\sigma$ and a projective line $\mathfrak{l}$ touching $\sigma$. It turns out that the arrangement $\tilde{A}_2^0$ is an example of an irreducible affine Tits arrangement which is not locally spherical. More precisely, we have the following main theorem (precise definitions are given in Section 2):

**Theorem.** *Let the pair $(\mathcal{A}, T)$ be an affine rank three Tits arrangement and assume that the projective root vectors of $\mathcal{A}$ are contained in the locus of a homogeneous cubic polynomial. Then up to projectivities, $\mathcal{A}$ is either a near pencil, an arrangement of type $\tilde{A}_2$, or it is an arrangement of type $\tilde{A}_2^0$.*

This result is established by proving Theorem 2 in Section 3. The necessary definitions and notations are collected in Section 2. In Section 4 we discuss some related open questions.

## 2. Definitions and notation

We start with the notion of a Tits arrangement in $\mathbb{R}^r$ (see [Cuntz et al. 2017]).

**Definition 1.** Let $\mathcal{A}$ be a (possibly infinite) set of linear hyperplanes in $V := \mathbb{R}^r$ and let $T$ be an open convex cone in $V$. We say that $\mathcal{A}$ *is locally finite in $T$* if for every $x \in T$ there exists a neighborhood $U_x \subset T$ of $x$, such that the set

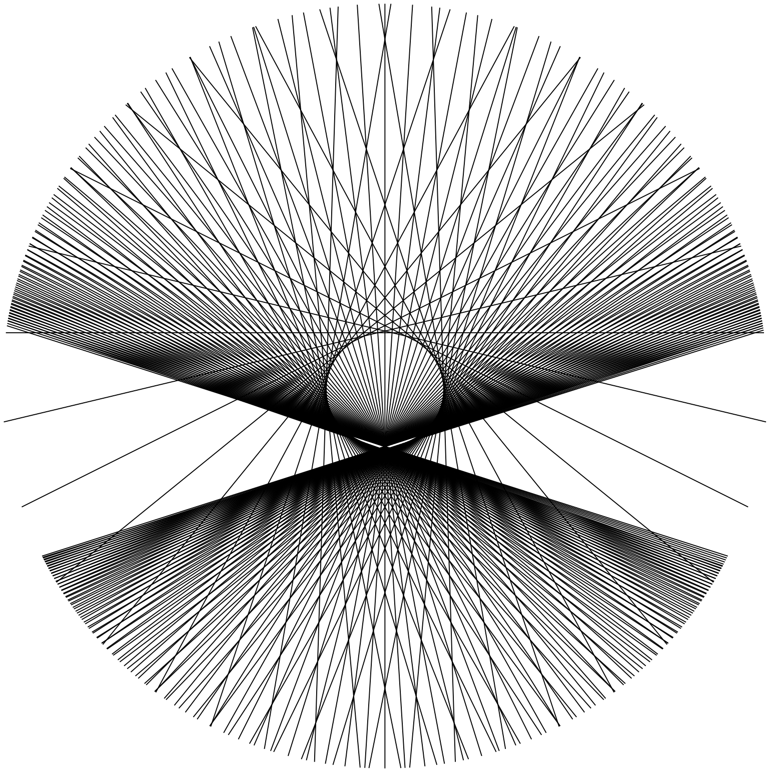

**Figure 1.** A subset of the arrangement of type $\tilde{A}_2^0$.

$\{H \in \mathcal{A} \mid H \cap U_x \neq \varnothing\}$ is finite. A *hyperplane arrangement (of rank r)* is a pair $(\mathcal{A}, T)$, where $T$ is a convex open cone in $V$, and $\mathcal{A}$ is a set of linear hyperplanes such that the following holds:

- $H \cap T \neq \varnothing$ for all $H \in \mathcal{A}$.
- $\mathcal{A}$ is locally finite in $T$.

Denote by $\overline{T}$ the topological closure of $T$ with respect to the standard topology of $V$. If $X \subset \overline{T}$ then the *localization at X (in $\mathcal{A}$)* is defined as

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}.$$

If $X = \{x\}$ we write $\mathcal{A}_x$ instead of $A_{\{x\}}$ and call $(\mathcal{A}_x, T)$ the *parabolic subarrangement at x*. The connected components of $T \setminus \bigcup_{H \in \mathcal{A}} H$ are called *chambers* or *cells*. If $K$ is a chamber then its *walls* are given by the hyperplanes contained in the set

$$W^K := \{H \leq V \mid \dim(H) = r - 1, \langle H \cap \overline{K} \rangle_{\mathbb{R}} = H, H \cap K = \varnothing\}.$$

The arrangement $(\mathcal{A}, T)$ is called *thin* if $W^K \subset \mathcal{A}$ for each chamber $K$. A *simplicial hyperplane arrangement (of rank r)* is an arrangement $(\mathcal{A}, T)$ such that each chamber $K$ is an open simplicial cone. $T$ is called the *Tits cone* of the arrangement. Finally, a simplicial arrangement is called a *Tits arrangement* if it is also thin.

**Remark 1.** i) If the pair $(\mathcal{A}, T)$ is a Tits arrangement, we usually omit the reference to $T$, since it should always be clear from the context.

ii) For a simplicial arrangement $(\mathcal{A}, T)$, the closure of $T$ can be reconstructed from the chambers of $\mathcal{A}$: we have $\overline{T} = \overline{\bigcup_{K \in \mathcal{K}(\mathcal{A})} K}$. In particular, the Tits cone $T$ is determined by $\mathcal{A}$. For details on this, see [Cuntz et al. 2017, Lemma 3.24].

**Definition 2.** Let the pair $(\mathcal{A}, T)$ be a Tits arrangement and denote the set of chambers by $\mathcal{K}$. Then we have the following thin chamber complex

$$\mathcal{S}(\mathcal{A}, T) := \Big\{\overline{K} \cap \bigcap_{H \in X} H \mid K \in \mathcal{K}, X \subset W^K\Big\},$$

whose poset-structure is given by set-wise inclusion. If $U \leq V$ has dimension 1 such that $v := \overline{K} \cap U \in \mathcal{S}(\mathcal{A}, T)$, then $v$ is called a *vertex*. Similarly, if $U' \leq V$ has dimension 2 such that $e := \overline{K} \cap U' \in \mathcal{S}(\mathcal{A}, T)$, then $e$ is called an *edge* or *segment*. A Tits arrangement $(\mathcal{A}, T)$ is called *locally spherical* if all vertices meet $T$. Finally, if $v$ is a vertex then we define its *weight* to be $w(v) := |\mathcal{A}_v|$.

**Remark 2.** Let $(\mathcal{A}, T)$ be a Tits arrangement and let $v$ be a vertex. If $v$ meets $T$, then clearly $w(v) < \infty$ because $\mathcal{A}$ is locally finite in $T$. However, if $v$ is contained in $\partial T$, then we necessarily have $w(v) = \infty$ because $\mathcal{A}$ is thin. Altogether, it follows that $w(v) = \infty$ if and only if $v \in \partial T$.

**Definition 3.** i) Let $(\mathcal{A}, T)$ be a Tits arrangement in $V := \mathbb{R}^r$. If there is a linear form $0 \neq \alpha \in V^*$ such that $T = \alpha^{-1}(\mathbb{R}_{>0})$ is a half-space, then we say that $(\mathcal{A}, T)$ is an *affine Tits arrangement*.

ii) Let $(\mathcal{A}, T)$ be a Tits arrangement in $\mathbb{R}^r$. If $T = \mathbb{R}^r$, then $\mathcal{A}$ is called a *spherical Tits arrangement*.

iii) Let $(\mathcal{A}, T)$ be a Tits arrangement in $\mathbb{R}^3$. Assume that there is $H_0 \in \mathcal{A}$ and a single vertex $v$ such that $v$ is contained in every $H \in \mathcal{A} \setminus \{H_0\}$ while $H_0$ does not contain $v$. Then $\mathcal{A}$ is called a *near pencil (arrangement)*. A Tits arrangement $(\mathcal{A}, T)$ in $\mathbb{R}^3$ which is not a near pencil arrangement is said to be *irreducible*.

**Remark 3.** i) If $(\mathcal{A}, T)$ is affine with corresponding linear form $\alpha$, the boundary $\partial T$ of the Tits cone is given by the hyperplane $\ker(\alpha)$.

ii) If $(\mathcal{A}, T)$ is spherical, then we have $|\mathcal{A}| < \infty$. Indeed, for such an arrangement we have $0 \in T$ and $0 \in H$ for every $H \in \mathcal{A}$; as by definition $\mathcal{A}$ is locally finite in $T$, this proves the claim. Moreover, we note that a spherical Tits arrangement in $\mathbb{R}^r$ induces a simplicial cell decomposition of the unit sphere $\mathbb{S}_{r-1}$.

iii) Near pencil arrangements are usually considered trivial.

In the following sections we will be concerned with the case of an affine Tits arrangement $(\mathcal{A}, T)$ of rank three. Then we may view $\mathcal{A}$ as set of projective lines in $\mathbb{P}^2(\mathbb{R})$ and the boundary $\partial T$ of $T$ is again a projective line. Further, in $\mathbb{P}^2(\mathbb{R})$ we have a duality between projective lines and projective points, for which we require the following notation.

**Notation.** Let $(\mathcal{A}, T)$ be a Tits arrangement of rank three. By abuse of notation we denote the set of projective lines $\{\mathfrak{g} \mid \exists H \in \mathcal{A} : \mathfrak{g} = \pi(H)\}$ by $\mathcal{A}$ as well; here $\pi : \{U \leq \mathbb{R}^3 \mid \dim U \geq 1\} \to \{U \leq \mathbb{P}^2(\mathbb{R})\}$ is the natural projection. If $p \in (\mathbb{P}^2(\mathbb{R}))^*$ then we denote the corresponding dual line by $p^* \subset \mathbb{P}^2(\mathbb{R})$. Likewise, if $\mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$ is a projective line, then its corresponding dual point is denoted by $\mathfrak{l}^* \in \mathbb{P}^2(\mathbb{R})$. Similarly, if $\mathcal{A}$ is a set of projective lines in $\mathbb{P}^2(\mathbb{R})$, we write $\mathcal{A}^* \subset (\mathbb{P}^2(\mathbb{R}))^*$ for the corresponding set of dual projective points (and vice versa). For a set of projective lines $\mathcal{A}$, we call $\mathcal{A}^*$ its corresponding *dual point set*.

Finally, we introduce the following notion of isomorphism of Tits arrangements.

**Definition 4.** Set $V := \mathbb{R}^3$ and let $(\mathcal{A}, T)$, $(\mathcal{A}', T')$ be two affine Tits arrangements in $V$. Then these are called *(projectively) isomorphic* if there exists $\phi \in \mathrm{PGL}(V^*)$ such that both $\phi(\mathcal{A}^*) = (\mathcal{A}')^*$ and $\phi(\partial T^*) = (\partial T')^*$.

## 3. Results and proofs

Now we are ready to prove our main theorem. The main strategy can be summarized as follows: according to the possible factorizations of a homogeneous cubic polynomial $P$, there are naturally three cases to consider. Namely, $P$ may factor as a product of three linear polynomials, or it may factor as a product of an irreducible quadratic polynomial and a linear polynomial, or $P$ may be irreducible. We examine all three cases and collect all (up to projectivity) affine Tits arrangements $\mathcal{A}$ such that $\mathcal{A}^* \subset V(P)$.

We start with the following lemma which will be used extensively to rule out the possibility of existence of certain Tits arrangements. It basically says that near pencils are the only rank three Tits arrangements containing a segment bounded by two vertices of weight two.

**Lemma 1.** *Let $\mathcal{A}$ be a Tits arrangement of rank three. Suppose there is a line $\mathfrak{g} \in \mathcal{A}$ containing two vertices $v_1, v_2$ of weight two such that there is no other vertex contained in the bounded segment between $v_1$ and $v_2$ on $\mathfrak{g}$. Then $\mathcal{A}$ is a near pencil.*

*Proof.* Denote by $\mathfrak{g}_1, \mathfrak{g}_2$ the two lines meeting $\mathfrak{g}$ in $v_1$ respectively $v_2$ and set $v := \mathfrak{g}_1 \cap \mathfrak{g}_2$. Since $w(v_1) = w(v_2) = 2$, the vertices $v_1, v_2$ are in the interior of the Tits cone by Remark 2, and it follows that there are two chambers with vertices

$v_1, v_2, v$. Every other line $\mathfrak{g}' \in \mathcal{A} \setminus \{\mathfrak{g}\}$ has to avoid these two chambers and hence needs to pass through $v$. □

We state some more lemmata, which will turn out to be useful and may be interesting in their own right.

**Lemma 2.** *Let $\mathcal{A}$ be an affine Tits arrangement of rank three. Then there is at most one vertex of $\mathcal{A}$ contained in $\partial T$.*

*Proof.* By Remark 2, we have $|\mathcal{A}_v| = \infty$ for every vertex $v \in \partial T$. Now suppose there were two different vertices $v, w \in \partial T$. There is a chamber $K$ having $v$ as a vertex. As $\mathcal{A}$ is thin and locally finite, it follows that $K$ has to be contained in the cone $C$ generated by two neighboring lines passing through $v$. Let $\epsilon > 0$ and consider the ball $B_\epsilon(v)$ centered at $v$ with radius $\epsilon$. Write $C_\epsilon := B_\epsilon(v) \cap C$ and note that there are infinitely many lines passing through $w$ which accumulate at $\partial T$. From this, it follows that there is a line passing through $w$ and intersecting $C_\epsilon$ for every $\epsilon > 0$. Hence, $\mathcal{A}$ contains a line intersecting $K$, a contradiction. □

**Lemma 3.** *Let $\mathcal{A}$ be a Tits arrangement of rank three. Assume that $\mathcal{A}^* \subset V(P)$ for some homogeneous polynomial $P \in \mathbb{R}[x, y, z]$ of degree $d$. Write $P = Q \cdot \prod_{1 \leq i \leq s} l_i$, where the $l_1, \ldots, l_s \in \mathbb{R}[x, y, z]$ are (not necessarily distinct) linear forms and $Q$ has no linear factors. Then $\mathcal{A}$ determines at most $s$ vertices of weight exceeding $d$.*

*Proof.* A vertex $v$ determined by $\mathcal{A}$ is a point in $\mathbb{P}^2(\mathbb{R})$. Therefore, the dual $\mathfrak{l} := v^*$ is a line in $(\mathbb{P}^2(\mathbb{R}))^*$. By Bézout's theorem we know that $|\mathcal{A}^* \cap \mathfrak{l}| \leq |V(P) \cap \mathfrak{l}| \leq d$, unless $\mathfrak{l}$ is a component of $V(P)$. As by assumption $V(P)$ contains at most $s$ linear components, this proves the claim. □

**Lemma 4.** *Let $\mathcal{A}$ be a Tits arrangement of rank three. Suppose there is a vertex $v$ of weight two which is surrounded by vertices $v_1, v_2, v_3, v_4$ of weight three. Then $\mathcal{A}$ is spherical and $|\mathcal{A}| \in \{6, 7\}$.*

*Proof.* Notice first that all the vertices $v, v_1, v_2, v_3, v_4$ are in the interior of the Tits cone by Remark 2. We denote the lines intersecting in $v$ by $\mathfrak{l}_1, \mathfrak{l}_2$ and we agree that $v_1, v_3 \in \mathfrak{l}_1$ while $v_2, v_4 \in \mathfrak{l}_2$. It is clear that there are no further vertices lying in the segment between $v_1$ and $v_4$ and the same is true for the segments between $v_1$ and $v_2$, $v_2$ and $v_3$, $v_3$ and $v_4$. Denote the line passing through $v_i$ and $v_j$ by $\mathfrak{l}_{i,j}$ and observe that the spherical arrangement $\mathcal{B} \subset \mathcal{A}$ defined by $\mathcal{B} := \{\mathfrak{l}_{i,j} \mid 1 \leq i < j \leq 4\}$ is simplicial. Moreover, there are chambers $K_{i,j}$ of $\mathcal{A}$ which do not contain $v$ but which contain $v_i, v_j$ for $\{i, j\} \in \{\{1, 4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}\}$. As $\mathcal{A}$ is simplicial, these chambers are simplicial cones.

Now suppose there was a line $\mathfrak{g}$ supporting an edge of a chamber $K_{i,j}$ such that $g \in \mathcal{A} \setminus \mathcal{B}$. Then $\mathfrak{g}$ needs to pass through either $v_i$ or $v_j$. But then the weight of either $v_i$ or $v_j$ needs to be strictly greater than three, contradicting our assumption. This shows that there is precisely one line which one may add to $\mathcal{B}$ in such a way that the obtained arrangement is simplicial with $w(v_i) = 3$ for $1 \leq i \leq 4$. Namely, the line passing through the points $\mathfrak{l}_{1,2} \cap \mathfrak{l}_{3,4}$ and $\mathfrak{l}_{1,4} \cap \mathfrak{l}_{2,3}$. □

The next proposition and theorem will simplify the proof of Proposition 2.

**Proposition 1.** *Let $\mathcal{A}$ be a Tits arrangement of rank three. Assume that $\mathcal{A}^*$ is contained in the union of two lines. Then $\mathcal{A}$ is a near pencil.*

*Proof.* Clearly, we may assume that $|\mathcal{A}| > 4$. So suppose that $\mathcal{A}^* \subset \mathfrak{l}_1 \cup \mathfrak{l}_2$. Then after dualizing the lines $\mathfrak{l}_1$ and $\mathfrak{l}_2$ become two points $v_1, v_2 \in \mathbb{P}^2(\mathbb{R})$ and we have $\mathcal{A}_{v_1} \cup \mathcal{A}_{v_2} = \mathcal{A}$. Without loss of generality we may assume that $w(v_1) = |\mathcal{A}_{v_1}| > 2$. Moreover, as $\mathcal{A}$ is a Tits arrangement there must exist a line $\mathfrak{l} \in \mathcal{A}$ such that $v_1 \notin \mathfrak{l}$. But $\mathfrak{l}$ and every other line of $\mathcal{A}$ which does not pass through $v_1$ has to pass through $v_2$. It follows from Lemma 3 that $v_2$ is the only point on $\mathfrak{l}$ which could potentially be a vertex of weight greater than two. But as $|\mathcal{A}_{v_1}| > 2$, the line $\mathfrak{l}$ contains at least one segment bounded by two vertices of weight two. Hence by Lemma 1 it follows that $\mathcal{A}$ is a near pencil (and a posteriori, there are at most two lines passing through $v_2$). □

**Remark 4.** In the situation of Proposition 1 there is a unique $\alpha \in \mathcal{A}^*$ such that $\mathcal{A}^* \setminus \{\alpha\}$ is contained in one of the two lines $\mathfrak{l}_1, \mathfrak{l}_2$ while $\alpha$ lies in the other one.

**Theorem 1.** *Near pencils are the only Tits arrangements of rank three whose dual point sets lie on a conic.*

*Proof.* Let $P \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of degree two and set $\sigma := V(P)$. Suppose that $\mathcal{A}^* \subset \sigma$ for some rank three Tits arrangement $\mathcal{A}$. First, assume that $P$ factors in two distinct linear polynomials. Then by Proposition 1 the only Tits arrangements lying on $\sigma$ are near pencils. If $P$ factors as a square of a linear polynomial, then every $\alpha \in \mathcal{A}^*$ lies on a single line which means that all lines of $\mathcal{A}$ pass through a single point. Hence $\mathcal{A}$ is not simplicial. Now, finally suppose that $P$ is irreducible. By Lemma 3, the weight of any vertex of $\mathcal{A}$ is bounded by two. But this implies that $\mathcal{A}$ is a near pencil consisting of three lines. □

The next proposition is a first step towards our main theorem. Before stating it, we need to introduce the affine reflection arrangement of type $\tilde{A}_2$.

**Definition 5.** Let $V := \mathbb{R}^3$ with standard basis $e_1, e_2, e_3$ and denote the corresponding dual basis vectors by $\alpha_1, \alpha_2, \alpha_3 \in V^*$. Consider the matrix

$$C := \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

For $1 \leq i, j \leq 3$ we define reflections $\sigma_i : V^* \to V^*$ via the formulae

$$\sigma_i(\alpha_j) := \alpha_j - C_{i,j}\alpha_i.$$

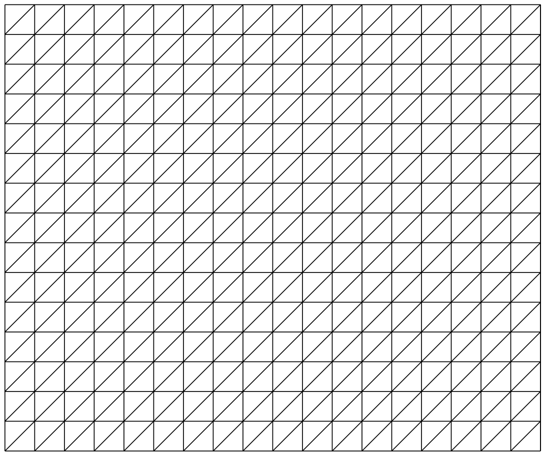

**Figure 2.** A subset of an arrangement of type $\tilde{A}_2$.

The (infinite) subgroup of $\mathrm{GL}(V^*)$ generated by $\sigma_1, \sigma_2, \sigma_3$ is called $\tilde{A}_2$. Let $\mathcal{O}$ be the union of the orbits of $\alpha_1, \alpha_2, \alpha_3$ under $\tilde{A}_2$. Define the arrangement $\mathcal{A} := \{\alpha^\perp \mid \alpha \in \mathcal{O}\}$. It is well known that $\mathcal{A}$ defines an affine rank three Tits arrangement. Any arrangement which is projectively isomorphic to $\mathcal{A}$ is then said *to be of type* $\tilde{A}_2$. (See Figure 2 for a visualization of such an arrangement.)

**Remark 5.** i) The matrices which give rise to affine Tits arrangement in the manner of Definition 5 are called *generalized Cartan matrices of affine type*. A complete classification as well as explicit descriptions of the corresponding root systems can be found for instance in [Kac 1990]. We observe that all arrangements obtainable in this way are locally spherical. Thus, one might believe that this is true for *every* affine Tits arrangement. However, this is not the case (see the proof of Corollary 1).

ii) We observe that for any arrangement $(\mathcal{A}, T)$ of type $\tilde{A}_2$, there are three points $v_1, v_2, v_3 \in \partial T$ such that $\mathcal{A} = \mathcal{A}_{v_1} \cup \mathcal{A}_{v_2} \cup \mathcal{A}_{v_3}$. However, none of these forms a vertex of $\mathcal{A}$. In Figure 2, the points $v_1, v_2, v_3$ are the intersection points at infinity of classes of mutually parallel lines. Moreover, by the above we see that the dual point set $\mathcal{A}^*$ is contained in the union of three lines (dual to $v_1, v_2, v_3$ respectively).

**Proposition 2.** *Let $\mathcal{A}$ be an affine Tits arrangement of rank three. Suppose that $\mathcal{A}^*$ is contained in the union of at most three lines and assume that $\mathcal{A}$ is not a near pencil. Then $\mathcal{A}$ is an arrangement of type $\tilde{A}_2$.*

*Proof.* Taking into account Theorem 1 it is enough to consider the case where $\mathcal{A}^*$ is contained in the union of exactly three lines: $\mathcal{A}^* \subset \mathfrak{l}_1 \cup \mathfrak{l}_2 \cup \mathfrak{l}_3$. We define $v_1 := \mathfrak{l}_1^*, v_2 := \mathfrak{l}_2^*, v_3 := \mathfrak{l}_3^*$ so that $\mathcal{A} = \mathcal{A}_{v_1} \cup \mathcal{A}_{v_2} \cup \mathcal{A}_{v_3}$. Now we consider three cases:

a) Suppose that there is exactly one $i \in \{1, 2, 3\}$ such that $|\mathcal{A}_{v_i}| = \infty$. We may assume that $i = 1$. Choose a line $\mathfrak{l} \in \mathcal{A}$ such that $\mathfrak{l}$ passes through $v_2$ but not through $v_1$. As $|\mathcal{A}_{v_3}| < \infty$, we conclude that $\mathfrak{l}$ contains infinitely many vertices of weight two but only finitely many vertices of weight possibly greater than two. Thus, we find that $\mathfrak{l}$ contains a segment bounded by vertices of weight two. Lemma 1 now shows that $\mathcal{A}$ must be a near pencil, contradicting our assumption.

b) Suppose that there are exactly two indices $i, j \in \{1, 2, 3\}$ such that $|\mathcal{A}_{v_i}| = |\mathcal{A}_{v_j}| = \infty$. We may assume that $\{i, j\} = \{1, 2\}$. Again, we choose a line $\mathfrak{l}$ which passes through $v_2$ but not through $v_1$. Because $|\mathcal{A}_{v_3}| < \infty$, we may again conclude that $\mathfrak{l}$ contains infinitely many vertices of weight two but only finitely many vertices of weight possibly greater than two. As in part a), Lemma 1 tells that $\mathcal{A}$ must be a near pencil, which is impossible.

c) Suppose that $|\mathcal{A}_{v_1}| = |\mathcal{A}_{v_2}| = |\mathcal{A}_{v_3}| = \infty$. Then the corresponding points $v_1, v_2, v_3$ all lie on the line $\partial T$. In the affine space $\mathbb{E} := \mathbb{P}^2(\mathbb{R}) \setminus \partial T$, the lines through $v_1, v_2, v_3$ are given by three respective classes of mutually parallel lines. This shows that $\mathcal{A}$ must be of type $\tilde{A}_2$ (compare Figure 2). □

**Remark 6.** i) While case a) in the above proof is possible only for near pencil arrangements, we note that there is no Tits arrangement at all satisfying the conditions of case b).

ii) If we drop the condition on $\mathcal{A}$ to be affine, then we find some more possible (spherical) arrangements such that $\mathcal{A}^*$ is contained in the union of three lines: for instance the arrangement of type A(10,3) (as denoted in [Grünbaum 2009]) and some of its subarrangements.

Our next goal is to show that there is no affine Tits arrangement $\mathcal{A}$ such that $\mathcal{A}^*$ is contained in the locus of an irreducible homogeneous cubic polynomial. This is established in the following result, which uses the well known fact that an irreducible singular cubic curve in $\mathbb{P}^2(\mathbb{R})$ has precisely one singular point, which is either an isolated point, a cusp or a double point.

**Lemma 5.** *There is no affine Tits arrangement of rank three whose dual point set is contained in the locus of an irreducible homogeneous cubic polynomial.*

*Proof.* Assume that $\mathcal{A}^* \subset C := V(P)$ for some homogeneous irreducible cubic polynomial $P$. By Lemma 3, we see that $w(v) \leq 3$ for every vertex $v$. In particular, the arrangement $\mathcal{A}$ cannot be a near pencil. Assume that there exists a vertex $v$ of weight two. Then the above together with Lemma 1 implies that every neighbor of $v$ has weight three. But then by Lemma 4, we conclude that $\mathcal{A}$ is spherical, a contradiction. Thus, every vertex of $\mathcal{A}$ has weight three. In particular, every $\alpha \in \mathcal{A}^*$ is a smooth point of $C$: every line through a pair of different points $\alpha, \beta \in \mathcal{A}^*$ meets $C$ in a unique third point denoted $\alpha \oplus \beta \in \mathcal{A}^*$. Moreover, as $\mathcal{A}$ is assumed to be affine, the set $\mathcal{A}^*$ is infinite with unique accumulation point $(\partial T)^*$, in particular $(\partial T)^* \in C$. Now we consider two cases.

Case a): Assume that $(\partial T)^*$ is a smooth point of $C$. Then we can find $\alpha \in \mathcal{A}^*$ such that the line through $\alpha$ and $(\partial T)^*$ meets $C$ in a third point $\gamma$. Now choose a sequence $(\alpha_n)_{n\in\mathbb{N}}$ from $\mathcal{A}^* \setminus \{\alpha\}$ converging towards $(\partial T)^*$. Then by the above, we see that the sequence $(\alpha \oplus \alpha_n)_{n\in\mathbb{N}}$ consists of elements from $\mathcal{A}^*$ and converges towards $\gamma$. As by construction $(\partial T)^* \neq \gamma$, this is a contradiction to $\mathcal{A}$ having a unique accumulation point.

Case b): Assume that $(\partial T)^*$ is a singular point of $C$. As $(\partial T)^*$ cannot be an isolated point, we may assume that $C$ has either a double point or a cusp at $(\partial T)^*$.

i) Assume first that $(\partial T)^*$ is a cusp of $C$. By the Weierstrass normal form for cubic polynomials we may assume that $P := y^2z - x^3$, in particular $(\partial T)^* = (0 : 0 : 1)$. Then in the affine $z = 1$ part of $\mathbb{P}^2(\mathbb{R})$, the curve $C$ consists of two branches $C_1, C_2$ meeting in $(\partial T)^*$ and given explicitly by

$$C_1 := \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid y = x^{3/2}\},$$
$$C_2 := \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid y = -x^{3/2}\}.$$

As $(\partial T)^*$ is the unique accumulation point of $\mathcal{A}^*$, there is a point $\alpha_1 \in \mathcal{A}^* \cap (C_1 \cup C_2)$ whose $x$-coordinate is maximal (when the $z$-coordinate is normalized to 1). In particular, we have $(0 : 1 : 0) \notin \mathcal{A}^* \cap C$. Without loss of generality, we may assume that $\alpha_1 \in C_2$. Let $\alpha_2 \in C_2 \setminus \{\alpha_1\}$ be the point on $C_2$ whose $x$-coordinate is exceeded only by $\alpha_1$. Similarly, let $\alpha_3 \in C_2 \setminus \{\alpha_1, \alpha_2\}$ be the point on $C_2$ whose $x$-coordinate is exceeded only by $\alpha_1, \alpha_2$. One checks that the point of $\mathcal{A}^* \cap C_1$ with maximal $x$-coordinate is then given by $\beta := \alpha_1 \oplus \alpha_2$ (remember that $\alpha \oplus \alpha' \in \mathcal{A}^*$ for $\alpha \neq \alpha' \in \mathcal{A}^*$). In particular, if $\gamma$ denotes the second point of $C$ on the tangent to $C$ at $\beta$, then we see that $\alpha_3$ necessarily has $x$-coordinate strictly smaller than $\gamma$. But then the point $((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \oplus \alpha_1 \in \mathcal{A}^* \cap C_2$ is different from $\alpha_2$ and has $x$-coordinate strictly greater than $\alpha_3$ but less than $\alpha_1$, contradicting the choice of $\alpha_3$.

ii) Now assume that $C$ has a double point at $(\partial T)^*$. We may assume that $C := V(P)$ for $P = y^2z - x^2(x+z)$. Then in the affine $z = 1$ part of $\mathbb{P}^2(\mathbb{R})$, the curve $C$ is given by the union of the following three sets $C_1, C_2, C_3$ given explicitly by

$$C_1 := \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid x > 0,\ y = x(x+1)^{1/2}\},$$
$$C_2 := \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid x > 0,\ y = -x(x+1)^{1/2}\},$$
$$C_3 := \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid -1 \leq x \leq 0,\ y = \pm x(x+1)^{1/2}\}.$$

Using the fact that $(\partial T)^* = (0 : 0 : 1)$ is an accumulation point of $\mathcal{A}^*$, we see that both $C_1 \cap \mathcal{A}^*$, $C_2 \cap \mathcal{A}^*$ are infinite. Focusing on these two sets, one can use the same techniques as in i) to obtain another contradiction. □

**Remark 7.** If one drops the assumption that $A$ is affine, then there are candidates for (spherical) Tits arrangements $\mathcal{A}$ such that $\mathcal{A}^* \subset V(P)$ for an irreducible cubic polynomial $P$: namely all spherical arrangements having only vertices of weight two or three. Since these are precisely the arrangements $A(6, 1)$, $A(7, 1)$ and the near pencils with at most four lines, we will not elaborate on this further.

It remains to consider the possibility that $\mathcal{A}^*$ is contained in the locus of a cubic homogeneous polynomial having an irreducible quadratic factor. As preparation, we formulate the following remark and definition which will be useful later.

**Definition 6.** a) Let $\sigma$ be an irreducible conic in $\mathbb{P}^2(\mathbb{R})$ and consider a subset $M \subset \sigma$. There exists a projectivity $\Psi$ such that $\Psi(\sigma)$ is given by the polynomial $P := x^2 + y^2 - z^2$ and is thus contained entirely in the affine $z = 1$ part of $\mathbb{P}^2(\mathbb{R})$. We say that $p_1, \dots, p_k \in M$ are *consecutive with respect to* $\Psi$, if for any $1 \le i \le k-1$ it is true that one of the segments on $\Psi(\sigma)$ bounded by $\Psi(p_i)$, $\Psi(p_{i+1})$ contains no other point of $\Psi(M)$.

b) Consider the map $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ sending $v_1, v_2 \in \mathbb{R}^3$ to their vector product $v_1 \times v_2$. This induces a map $\psi : \big(\mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})\big) \setminus \Delta \to (\mathbb{P}^2(\mathbb{R}))^*$, where $\Delta := \{(x, x) \mid x \in \mathbb{P}^2(\mathbb{R})\}$; $(\psi(\langle v_1\rangle, \langle v_2\rangle)$ is the line dual to the projective point $\langle v_1 \times v_2\rangle)$. By a slight abuse of notation, we write $\psi(v_1, v_2) = v_1 \times v_2 \in (\mathbb{P}^2(\mathbb{R}))^*$ for two different projective points $v_1, v_2 \in \mathbb{P}^2(\mathbb{R})$. Observe that for $p, q \in (\mathbb{P}^2(\mathbb{R}))^*$ the vector product $p \times q$ gives the vertex in $\mathbb{P}^2(\mathbb{R})$ obtained as the intersection of the dual lines $p^*, q^*$. Similarly, if $v, v'$ are two points in $\mathbb{P}^2(\mathbb{R})$, then the vector product $v \times v'$ gives the point in $(\mathbb{P}^2(\mathbb{R}))^*$ which is dual to the line passing through $v$ and $v'$.

Now we can prove the following statement. (Compare Theorem 3.6 in [Cuntz 2011b], where case c) of the following proposition is examined for spherical Tits arrangements.)

**Proposition 3.** *Suppose that $\mathcal{A}$ is an affine rank three Tits arrangement and assume that $\mathcal{A}^* \subset \sigma \cup \mathfrak{l}$ for some irreducible conic $\sigma \subset (\mathbb{P}^2(\mathbb{R}))^*$ and an arbitrary line $\mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$. Then the following statements hold*:

a) $|\mathcal{A}^* \cap \sigma| = \infty$, *unless $\mathcal{A}$ is a near pencil.*

b) $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$ *and* $(\partial T)^* \in \mathfrak{l}$.

c) *If* $|\sigma \cap \mathfrak{l}| = 0$ *then $\mathcal{A}$ is a near pencil.*

d) *If* $|\sigma \cap \mathfrak{l}| = 1$ *then* $\sigma \cap \mathfrak{l} = (\partial T)^*$, *unless $\mathcal{A}$ is a near pencil.*

e) *If* $|\sigma \cap \mathfrak{l}| = 2$ *then $\mathcal{A}$ is a near pencil.*

*Proof.* a) Define $\mathcal{B}^* := \mathcal{A}^* \cap \sigma$ and suppose that $|\mathcal{B}^*| < \infty$. Since $\mathcal{A}$ is affine and hence necessarily infinite, the set $L^* := \mathcal{A}^* \cap \mathfrak{l}$ is infinite. We have $\mathcal{A} = \mathcal{B} \cup L$ and it is easy to see that we find a line in $\mathcal{B}$ containing a segment bounded by two vertices of weight two: pick an arbitrary line $\mathfrak{g} \in \mathcal{B} \setminus L$. Then each of the infinitely many lines in $L$ produces a unique vertex on $\mathfrak{g}$. On the other hand, there are only finitely many lines in $\mathcal{B}$, which could turn some of the above vertices into triple points. By Lemma 1 we conclude that $\mathcal{A}$ is a near pencil.

b) If $\mathcal{A}$ is a near pencil then both statements are easily seen to be true: in this case, all lines except one pass through the point $\mathfrak{l}^*$, thus $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$. Using the fact that $\mathcal{A}$ is locally finite in $T$, we conclude that $\mathfrak{l}^* \in \partial T$, that is $(\partial T)^* \in \mathfrak{l}$. So we may assume that $\mathcal{A}$ is not a near pencil. We show that the second statement is a consequence of the first. So suppose that $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$ and assume that $(\partial T)^* \notin \mathfrak{l}$. Dualizing we obtain that the point $\mathfrak{l}^*$ does not lie on the line $\partial T$. Hence $\mathfrak{l}^*$ lies in $T$ and there are infinitely many lines of $\mathcal{A}$ passing through $\mathfrak{l}^*$. But since $\mathcal{A}$ is locally finite in $T$ this is impossible. So it suffices to prove that $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$. We show that $|\mathcal{A}^* \cap \mathfrak{l}| < \infty$ gives a contradiction: fix some $q \in \mathcal{A}^* \cap \sigma$ and for $q \neq p \in \sigma \cap \mathcal{A}^*$ consider the corresponding dual lines $p^*, q^*$ in $\mathbb{P}^2(\mathbb{R})$. Different choices of $p$ will yield different vertices on $q^*$ and part a) implies that there are infinitely many such vertices. If $|\mathcal{A}^* \cap \mathfrak{l}| < \infty$, then only finitely many of these vertices can be turned into triple points. Thus, $q^*$ must contain a segment bounded by two double points, which by Lemma 1 implies that $\mathcal{A}$ is a near pencil. This is the desired contradiction.

For the proof of c) and d) it suffices to note that $\mathcal{A}$ must be a near pencil if $(\partial T)^* \notin \sigma$. Indeed, if this is the case, then we necessarily have $|\mathcal{A}^* \cap \sigma| < \infty$, since points of $\mathcal{A}^*$ may accumulate only in a neighborhood of $(\partial T)^*$ (because $\mathcal{A}$ is locally finite in $T$). But then part a) implies that $\mathcal{A}$ is a near pencil.

e) After applying a projectivity as in part a) of Definition 6, we may assume that $\sigma = V(P)$ where $P := x^2 + y^2 - z^2$. So $\sigma$ is contained entirely in the affine $z = 1$ part of $(\mathbb{P}^2(\mathbb{R}))^*$. We write $\sigma'$ for the conic in $\mathbb{P}^2(\mathbb{R})$ defined by the same polynomial.

Suppose that $\mathcal{A}$ is not a near pencil. As points of $\mathcal{A}^*$ may accumulate only in a neighborhood of $(\partial T)^*$, we have $(\partial T)^* \in \sigma \cap \mathfrak{l}$. Observe that for $p = (a : b : 1) \in \sigma \cap \mathcal{A}^* \subset (\mathbb{P}^2(\mathbb{R}))^*$ the corresponding dual line $p^*$ is the tangent to $\sigma'$ at the point $(-a : -b : 1) \in \mathbb{P}^2(\mathbb{R})$. In particular, if $(\partial T)^* = (x : y : 1)$, this implies that there is a sequence of tangent lines to $\sigma'$ converging towards the tangent line at the point $(-x : -y : 1)$, and this tangent line is precisely $\partial T$. It remains to identify the dual lines $q^*$ corresponding to $q \in \mathfrak{l} \cap \mathcal{A}^*$. We may assume without loss of generality that in the $z = 1$ part of $(\mathbb{P}^2(\mathbb{R}))^*$ the line $\mathfrak{l}$ is given by the equation $y = \lambda$ for some $0 \leq \lambda < 1$. Hence any $q \in \mathfrak{l}$ will have homogeneous coordinates $q = (x_0 : \lambda : 1)$. So if $\lambda > 0$, the equation of the dual line $q^*$ in the $z = 1$ part of $\mathbb{P}^2(\mathbb{R})$ will be $y = -\frac{x_0 \cdot x}{\lambda} - \frac{1}{\lambda}$; if on the other hand $\lambda = 0$, then the equation of $q^*$ will be $x = -\frac{1}{x_0}$. Hence if $\lambda > 0$, then all lines pass through the point $(0 : -\frac{1}{\lambda} : 1)$ which implies that $\mathfrak{l}^* = (0 : -\frac{1}{\lambda} : 1)$; if $\lambda = 0$, then all lines pass through $\mathfrak{l}^* = (0 : 1 : 0)$. This shows that $\mathfrak{l}^* \notin \sigma'$. Since $(\partial T)^* \in \mathfrak{l}$ we conclude that $\mathfrak{l}^* \in \partial T$. Now we take $\partial T$ as line at infinity. Doing so, we obtain $\mathcal{A}$ as union of tangent lines to a parabola together with infinitely many parallel lines each of which being non-parallel to the symmetry axis of the parabola. But then $\mathcal{A}$ is not simplicial. □

The following lemma will be the key to proving the main theorem.

**Lemma 6.** *Let $\sigma$ be an irreducible conic together with a projectivity $\Psi$ as in part a) of Definition 6. Assume that $\mathfrak{l}$ is a line touching $\sigma$ and let $\mathcal{A}$ be an irreducible affine rank three Tits arrangement. If $\mathcal{A}^* \subset \sigma \cup \mathfrak{l}$, then $\mathcal{A}$ is determined by specifying four points on $\sigma$ which are consecutive with respect to $\Psi$. More precisely, if $p_{-1}, p_0, p_1, p_2, p_3, p_4 \in \mathcal{A}^* \cap \sigma$ are six consecutive points (with respect to $\Psi$), then we have the following formulae for $p_{-1}$ and $p_4$ in terms of $p_0, \dots, p_3$:*

$$p_4 = \left(p_0 \times \left(\mathfrak{l}^* \times (p_1 \times p_3)\right)\right) \times \left(p_1 \times \left(\mathfrak{l}^* \times (p_2 \times p_3)\right)\right), \tag{1}$$

$$p_{-1} = \left(p_2 \times \left(\mathfrak{l}^* \times (p_0 \times p_1)\right)\right) \times \left(p_3 \times \left(\mathfrak{l}^* \times (p_0 \times p_2)\right)\right). \tag{2}$$

*Moreover, if $v$ is a vertex of weight two of $\mathcal{A}$ with lines $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{A}$ passing through $v$, then $\mathfrak{l}^* \in \mathfrak{g}_1$ or $\mathfrak{l}^* \in \mathfrak{g}_2$.*

*Proof.* Denote by $L_1, L_2 \subset \mathcal{A}$ the set of lines corresponding to elements in $\mathcal{A}^* \cap \sigma$, $\mathcal{A}^* \cap \mathfrak{l}$ respectively. Observe that every $\mathfrak{h} \in L_2$ passes through the point $\mathfrak{l}^*$ while no line belonging to $L_1$ passes through $\mathfrak{l}^*$: if $\mathfrak{l}^* \in \mathfrak{g}$ and $\mathfrak{g}^* \in \sigma$ for some $\mathfrak{g}$, then $\mathfrak{g}^* = \mathfrak{l} \cap \sigma = (\partial T)^*$, by part d) of Proposition 3. As $\mathcal{A}$ is thin by definition, we conclude that $\mathfrak{g} \notin \mathcal{A}$.

Note also that every vertex of weight two of $\mathcal{A}$ must lie on a line belonging to $L_2$. Indeed, assume there was a vertex $v$ of weight two such that $v = \mathfrak{g} \cap \mathfrak{g}'$ for some $\mathfrak{g}, \mathfrak{g}' \in L_1$. As $\mathcal{A}^* \subset \sigma \cup \mathfrak{l}$ and because no line belonging to $L_1$ passes through $\mathfrak{l}^*$, we may use Lemma 3 to conclude that every neighbor of $v$ has weight bounded by three. But then by Lemma 1 every neighbor of $v$ has weight precisely three, because by assumption $\mathcal{A}$ is not a near pencil. By Lemma 4 we obtain that $\mathcal{A}$ is spherical, a contradiction. In particular, it follows that for every vertex $v'$ obtained as intersection of elements in $L_1$ there is a line $\mathfrak{h} \in L_2$ passing through $v'$. Also, every vertex of weight two is a neighbor of $\mathfrak{l}^*$, proving the last claim of the lemma.

These conditions already suffice to prove the claim. Let $p_0, p_1, p_2, p_3 \in \mathcal{A}^* \cap \sigma$ be four consecutive points (with respect to $\Psi$). We need to construct the points $p_{-1}, p_4 \in \mathcal{A}^* \cap \sigma$ such that both $p_{-1}, p_0, p_1, p_2$ and $p_1, p_2, p_3, p_4$ are consecutive (with respect to $\Psi$). By symmetry, it suffices to construct $p_4$. For this, denote the line corresponding to $p_i$ by $\mathfrak{g}_i$ and let $\mathfrak{h}$ be the line passing through the vertices $\mathfrak{l}^*, \mathfrak{g}_1 \cap \mathfrak{g}_3$. Similarly, denote by $\mathfrak{h}'$ the line passing through the vertices $\mathfrak{l}^*, \mathfrak{g}_2 \cap \mathfrak{g}_3$. Then $\mathfrak{g}_4$ is the line passing through the vertices $\mathfrak{g}_0 \cap \mathfrak{h}, \mathfrak{g}_1 \cap \mathfrak{h}'$. From this, one reads off that (1) holds. This completes the proof. □

**Remark 8.** Let $P \in \mathbb{R}[x, y, z]$ be a homogeneous cubic polynomial having an irreducible quadratic factor and let $\mathcal{A}$ be a spherical Tits arrangement such that $\mathcal{A}^* \subset V(P)$. If $\mathcal{A}$ is not a near pencil, then one may use Lemma 3 to conclude that there are two possibilities for $\mathcal{A}$: either $\mathcal{A}$ is the arrangement $A(7, 1)$ or $\mathcal{A}$ belongs to the infinite family $\mathcal{R}(1)$ (see [Grünbaum 2009]).

Now we can construct the arrangement of type $\tilde{A}_2^0$ and prove that up to projective isomorphism, it is the only non-trivial affine rank three Tits arrangement whose dual point set is contained in the locus of a cubic polynomial having an irreducible quadratic factor:

**Proposition 4.** *Up to projectivity, there is only one irreducible affine rank three Tits arrangement $\mathcal{A}$ such that $\mathcal{A}^*$ is contained in the locus of a cubic polynomial $P$ having an irreducible quadratic factor. The arrangement $\mathcal{A}$ may be defined by the following set of dual points*:

$$\mathcal{A}^* = \left\{ \left(k : \tfrac{k(k-1)}{2} : 1\right), \left(1 : \tfrac{k}{2} : 0\right) \mid k \in \mathbb{Z} \right\}.$$

*Proof.* Let $\mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$ be the line corresponding to the linear factor of $P$ and let $\sigma \subset (\mathbb{P}^2(\mathbb{R}))^*$ be the irreducible conic corresponding to the quadratic factor of $P$. We then have $\mathcal{A}^* \subset \sigma \cup \mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$ and by Proposition 3 we may assume that $\mathfrak{l}$ touches $\sigma$ at the point $(\partial T)^*$.

Let $p_1, p_2, p_3, p_4 \in \mathcal{A}^* \cap \sigma$ be four consecutive points (with respect to some projectivity $\Psi$). After a change of coordinates we may assume that

$$(\partial T)^* = (0 : 1 : 0)\,,\ p_2 = (1 : 0 : 1)\,,$$
$$p_3 = (2 : 1 : 1)\,,\ p_4 = (3 : 3 : 1)\,.$$

We then have $p_1 = (x : y : z)$ for some $x, y, z \in \mathbb{R}$. Now consider the vertices $v := p_2 \times p_3$, $v' := p_1 \times p_4 \in \mathbb{P}^2(\mathbb{R})$ and let $\mathfrak{g} \subset \mathbb{P}^2(\mathbb{R})$ be the line passing through $v$ and $v'$. Then the last claim of Lemma 6 implies that $\mathfrak{g} \in \mathcal{A}$ and that $\mathfrak{g}$ passes through the vertex $\mathfrak{l}^*$. As $\mathfrak{l}^* \in \partial T$, we may write $\mathfrak{l}^* = (a : 0 : b)$ for certain $a, b \in \mathbb{R}$. In order to prove the statement we will distinguish four cases.

Case 1. Assume that $x = y = 0$. This implies that $p_1 = (0 : 0 : 1)$. We claim that $\mathfrak{l}^* = (0 : 0 : 1)$. To see this write $\mathfrak{l}^* = (a : 0 : b)$ for some $a, b \in \mathbb{R}$ as above. The fact that $\mathfrak{g}$ passes through $\mathfrak{l}^*$ implies that $a = 0$ and therefore we have $\mathfrak{l}^* = (0 : 0 : 1)$.

Now consider the projectivity $\Phi : (\mathbb{P}^2(\mathbb{R}))^* \to (\mathbb{P}^2(\mathbb{R}))^*$ taking the point $p_i$ to $p_{i+1}$ for $1 \le i \le 4$. We obtain

$$\mathcal{A}^* \cap \sigma = \left\{ \Phi^k(p_1) \mid k \in \mathbb{Z} \right\} = \left\{ \left(k : \tfrac{k(k-1)}{2} : 1\right) \mid k \in \mathbb{Z} \right\},$$

using Lemma 6 and induction. Observe that the lines of $\mathcal{A}$ corresponding to points in $\mathcal{A}^* \cap \mathfrak{l}$ are exactly the lines passing through $\mathfrak{l}^*$ and a vertex of the form $p \times p'$

for $p, p' \in \mathcal{A}^* \cap \sigma$ (see the proof of Lemma 6). We conclude that $\mathcal{A}^* \cap \mathfrak{l} = \{(1 : \frac{k}{2} : 0) \mid k \in \mathbb{Z}\}$. It is now easy to check that $\mathcal{A}^* = \{(k : \frac{k(k-1)}{2} : 1), (1 : \frac{k}{2} : 0) \mid k \in \mathbb{Z}\}$ defines an irreducible affine Tits arrangement.

Case 2. Assume that $x \neq 0$ and $y = 0$. Then we may assume that $p_1 = (1 : 0 : z)$. Write $\mathfrak{l}^* = (a : 0 : b)$ for $a, b \in \mathbb{R}$. The fact that $\mathfrak{g}$ passes through $\mathfrak{l}^*$ implies that $a \neq 0$. Thus, we may assume that $\mathfrak{l}^* = (1 : 0 : b)$. It follows that $z = \frac{b+4}{3}$ and therefore $p_1 = (1 : 0 : \frac{b+4}{3})$. Observe that the five given points $(\partial T)^*, p_1, p_2, p_3, p_4$ on $\sigma$ determine its equation. Using this together with Lemma 6, the condition $p_5 \in \sigma$ implies that $b \in \{-1, -\frac{3}{2}, -\frac{7}{3}, -3\}$. As $p_0, p_5 \neq p_i$ for $1 \le i \le 4$, we conclude that $b \in \{-1, -\frac{3}{2}, -3\}$ is impossible. In the remaining case $b = -\frac{7}{3}$, we observe that the conic $\sigma$ may be defined by the polynomial $f = -\frac{10}{3}X^2 + 2XY + \frac{28}{3}XZ - \frac{10}{3}YZ - 6Z^2$. By assumption, we know that the line $\mathfrak{l}$ touches $\sigma$ at the point $(\partial T)^*$. Thus, as $\mathfrak{l}^* = (1 : 0 : -\frac{7}{3})$, there exists $0 \neq \lambda \in \mathbb{R}$ such that the following equations are satisfied:

$$1 = \lambda \, \frac{\partial f}{\partial X}\Big|_{(\partial T^*)}, \quad 0 = \lambda \, \frac{\partial f}{\partial Y}\Big|_{(\partial T^*)}, \quad -\frac{7}{3} = \lambda \, \frac{\partial f}{\partial Z}\Big|_{(\partial T^*)}.$$

The first equation gives $\lambda = \frac{1}{2}$. But then the third equation reads $-\frac{7}{3} = -\frac{5}{3}$. This contradiction shows that Case 2 cannot occur.

Case 3. Assume that $x = 0$ and $y \neq 0$. Then without loss of generality, we may assume that $p_1 = (0 : 1 : z)$. Again, we write $\mathfrak{l}^* = (a : 0 : b)$ for suitable $a, b \in \mathbb{R}$ and as $\mathfrak{g}$ passes through $\mathfrak{l}^*$, we obtain $a \neq 0$. Thus, we may assume that $\mathfrak{l}^* = (1 : 0 : b)$, leading to $z = -\frac{b+3}{3}$. We conclude that $p_1 = (0 : 1 : -\frac{b+3}{3})$. The relation $p_5 \in \sigma$ gives $b \in \{-3, -1\}$. As $p_5 \neq p_i$ for $1 \le i \le 4$, we conclude that this is impossible.

Case 4. Assume that both $x \neq 0$ and $y \neq 0$. Then we may suppose that $p_1 = (1 : y : z)$. Write $\mathfrak{l}^* = (a : 0 : b)$ for suitable $a, b \in \mathbb{R}$. As before, by considering the line $\mathfrak{g}$, we conclude that $-3za - 3ay - by + 4a + b = 0$. Suppose that $a = 0$. Then without loss of generality $b = 1$ and we have $y = 1$, in particular $p_1 = (1 : 1 : z)$. As $p_5 \in \sigma$, we conclude that $z \in \{\frac{1}{3}, \frac{1}{2}\}$. Again, this is not possible because $p_0, p_5 \neq p_i$ for $1 \le i \le 4$.

Hence, we may assume that $a = 1$. In particular, we have $z = \frac{4}{3} - \frac{b(y-1)}{3} - y$ and $p_1 = (1 : y : \frac{4}{3} - \frac{b(y-1)}{3} - y)$.

Suppose that $b \neq -3$. Using the condition $p_5 \in \sigma$, we compute that $y$ is one of $1, \frac{-3b^2-10b-7}{2(b+3)}, \frac{2b^2+5b+3}{2(b^2+3b+3)}$. As $p_1 \neq p_4$, we can exclude the case $y = 1$.

Likewise, if $y$ is the last of the three numbers, we obtain $p_1 = p_5$, a contradiction. So we must have $y = \frac{-3b^2-10b-7}{2(b+3)}$. This implies that $p_1 = (1 : \frac{-3b^2-10b-7}{2(b+3)} : \frac{b^2+4b+5}{2})$. Therefore, the conic $\sigma$ may be defined by the polynomial

$$f := (b-1)X^2 + 2XY - (b-7)XZ - 2(b+4)YZ - 6Z^2.$$

To see this, one only has to check that $f(p_i) = 0$ for $1 \leq i \leq 5$. The line $\mathfrak{l}$ touches $\sigma$ at the point $(\partial T)^* = (0:1:0)$. Therefore, as $\mathfrak{l}^* = (1:0:b)$, we know that there exists $0 \neq \lambda \in \mathbb{R}$ such that the following equations hold:

$$1 = \lambda \, \frac{\partial f}{\partial_X}\Big|_{(\partial T^*)}, \quad 0 = \lambda \, \frac{\partial f}{\partial_Y}\Big|_{(\partial T^*)}, \quad b = \lambda \, \frac{\partial f}{\partial_Z}\Big|_{(\partial T^*)}.$$

The first equation gives $\lambda = \frac{1}{2}$. Thus, the third equation yields $b = -2$ and we obtain $p_1 = \left(1 : \frac{1}{2} : \frac{1}{2}\right) = (2:1:1) = p_3$, a contradiction.

It remains to consider the case $b = -3$. Then we have $\mathfrak{l}^* = (1:0-3)$ and $p_1 = \left(1 : y : \frac{1}{3}\right)$. Clearly, we have $y \neq 1$ because $p_1 \neq p_4$. Then Lemma 6 yields $p_5 = (3:3:1) = p_4$, another contradiction. □

**Corollary 1.** *There are irreducible affine rank three Tits arrangements which are not locally spherical.*

*Proof.* This follows from Proposition 4. The arrangement constructed there is such an example: the vertex $\mathfrak{l}^*$ is incident with infinitely many lines of $\mathcal{A}$. □

Finally, using Proposition 2, Lemma 5, Proposition 3, and Proposition 4, we obtain the promised main theorem:

**Theorem 2.** *Let $\mathcal{A}$ be an affine rank three Tits arrangement such that $\mathcal{A}^*$ is contained in the locus of a homogeneous polynomial of degree three. Then up to projectivity, $\mathcal{A}$ is either a near pencil, an arrangement of type $\tilde{A}_2$, or it is an arrangement of type $\tilde{A}_2^0$.*

## 4. Open questions and related problems

In this section we point out some possibly interesting related problems. First, we ask if there exists an affine rank three Tits arrangement $\mathcal{A}$ (viewed as arrangement of lines in the real projective plane) such that $\mathcal{A}^*$ is contained entirely in the locus of an irreducible homogeneous polynomial.

**Problem 1.** *Is there an irreducible homogeneous polynomial $P \in \mathbb{R}[x,y,z]$ such that $\mathcal{A}^* \subset V(P)$ for a suitable irreducible affine rank three Tits arrangement $\mathcal{A}$?*

Observe that given a Tits arrangement $\mathcal{A}$ and an irreducible homogeneous polynomial $P$ of degree $d$ such that $\mathcal{A}^* \subset V(P)$, it follows immediately that $\mathcal{A}$ is locally spherical. Indeed, suppose there was a vertex $v$ of $\mathcal{A}$ such that infinitely many lines of $\mathcal{A}$ pass through $v$. Then after dualizing it follows that infinitely many points of $\mathcal{A}^*$ lie on the line $v^*$. But since by assumption $\mathcal{A}^* \subset V(P)$, it follows that infinitely many points lie on the intersection $V(P) \cap v^*$. But Bézout's theorem tells that $|V(P) \cap v^*| \leq d \cdot 1 = d < \infty$, because $P$ was assumed to be irreducible and hence $v^*$ cannot be a component of $V(P)$. This contradiction shows that $\mathcal{A}$ must be locally spherical.

This leads to the next problem. Are there other examples of irreducible affine rank three Tits arrangements which are not locally spherical?

**Problem 2.** *Classify* (*up to projectivities*) *all irreducible affine rank three Tits arrangements $\mathcal{A}$ which are not locally spherical.*

Observe that if $\mathcal{A}$ is not locally spherical, then by Lemma 2 there is precisely one vertex $v$ on the boundary of the Tits cone $T$. In particular, it follows that for every line $\mathfrak{l} \neq v^*$ we have $|\mathcal{A}^* \cap \mathfrak{l}| < \infty$. If in addition we know that $\mathcal{A}^* \subset V(P)$ for some homogeneous polynomial $P$ of degree $d$, then by Bézout's theorem the last inequality can be strengthened to

$$|\mathcal{A}^* \cap \mathfrak{l}| \leq |V(P) \cap \mathfrak{l}| \leq d$$

for every line $\mathfrak{l} \neq v^*$ which is not a component of $V(P)$.

We close this section by proposing the following final problem which is probably the most difficult:

**Problem 3.** *Classify* (*up to projectivities*) *all affine rank three Tits arrangements $\mathcal{A}$ such that $\mathcal{A}^* \subset V(P)$ for some homogeneous polynomial $P \in \mathbb{R}[x, y, z]$.*

A solution to the last problem seems to be an important step towards a classification of all affine rank three Tits arrangements. Indeed, if $\mathcal{A}$ is such an arrangement and if $\mathcal{A} = \bigcup_{i \in I} L_i$ for some finite index set $I$ and sets of mutually parallel lines $L_i$, $i \in I$, then $\mathcal{A}^*$ is contained in the locus of a polynomial $P$ of degree $|I|$: the polynomial $P$ is a product of linear factors corresponding to the sets $L_i$, $i \in I$. For example, affine Tits arrangements coming from Nichols algebras of diagonal type are always of this type.

Even if we enlarge $\mathcal{A}$ by finitely many countable subsets of tangent lines to certain conics, we still find a polynomial $P'$ such that the enlarged arrangement is contained in the locus of $P'$. The polynomial $P'$ may be taken as the product of $P$ together with the irreducible quadratic polynomials defining the (dual) conics in question. This gives the impression that the class of rank three affine Tits arrangements lying on the locus of some polynomial is rather large, as demonstrated by the fact that only usage of at most quadratic polynomials already leads to nontrivial considerations. It may even be conjectured that for every irreducible rank three affine Tits arrangement $\mathcal{B}$ there is a certain polynomial $Q$ such that $\mathcal{B}^* \subset V(Q)$. If this is true, then clearly a solution to Problem 3 amounts to a complete classification of affine rank three Tits arrangements.

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# Chamber graphs of minimal parabolic sporadic geometries

Veronica Kelsey and Peter Rowley

We explore the minimal characteristic two parabolic geometries for the finite sporadic simple groups, as introduced by Ronan and Stroth. The chamber graphs of the geometries are studied, with the aid of Magma, focusing on their disc structure and geodesic closures. For the larger sporadic geometries which are beyond computational reach we give bounds on the diameter of their chamber graphs.

## 1. Introduction

In this paper, with the aid of computer programs [Kelsey and Rowley 2019], we investigate the chamber graphs of the characteristic two minimal parabolic geometries for the finite sporadic simple groups which are listed in [Ronan and Stroth 1984]. The motivation for the Ronan and Stroth catalogue was to obtain geometries which captured certain features seen in the buildings associated with the finite groups of Lie type.

The common thread of these geometries is a generalization of the idea of a minimal parabolic subgroup of a group of Lie type. We briefly review minimal parabolic subgroups, following Ronan and Stroth. Suppose $G$ is a finite group, $p$ a prime and $S \in Syl_p(G)$. Set $B = N_G(S)$. A subgroup $P$ of $G$ which properly contains $B$ with $O_p(P) \neq 1$ and for which $B$ is contained in a unique maximal subgroup of $P$ is called a *minimal parabolic subgroup* of $G$ with respect to $B$.

Let $P_1, \dots, P_n$ be minimal parabolic subgroups of $G$ with respect to $B$. Put $I = \{1, \dots, n\}$. If $\langle P_i \mid i \in I\rangle = G$ and $\langle P_j \mid j \in J\rangle \neq G$ for all proper subsets $J$ of $I$, we call $\{P_i \mid i \in I\}$ a *characteristic $p$ minimal parabolic system of $G$ of rank $n$*.

From now on we suppose $\{P_i \mid i \in I\}$ is a rank $n$ minimal parabolic system. For nonempty $J \subseteq I$, we set $P_J = \langle P_j \mid j \in J\rangle$ and for $J = \varnothing$, $P_J = B$. If for all

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subsets $J, K \subseteq I$ we have

$$P_j \cap P_k = P_{J\cap K}$$

the minimal parabolic system $\{P_i \mid i \in I\}$ is called a *geometric system*.

We shall concentrate here on the case $p = 2$ with systems that are geometric. In fact, it is the chamber graph of these geometries we focus on. Chamber graphs were employed by Tits to give an alternative approach to buildings; see [Ronan 2009; Tits 1981]. They have proved to be a fruitful way of viewing buildings and so it is natural to study the chamber graphs of related geometries.

We recollect the salient features of chamber systems and chamber graphs that we need. Let $\Gamma$ be the geometry associated with $\{P_i \mid i \in I\}$. In the group theory context, the chambers of the chamber system are $\{Bg \mid g \in G\}$. The chambers are the vertices of the chamber graph $\mathcal{C}(\Gamma)$.

Two (distinct) chambers $Bg$ and $Bh$ of $\mathcal{C}(\Gamma)$ are $i$-adjacent if $gh^{-1} \in P_i$, and two chambers are adjacent in the chamber graph, $\mathcal{C}(\Gamma)$, if they are $i$-adjacent for some $i \in I$. Since $B$ is self-normalizing in $G$, $\mathcal{C}(\Gamma)$ may also be described as having $\{B^g \mid g \in G\}$ as its vertex set with $B^g$ and $B^h$ $i$-adjacent if $gh^{-1} \in P_i$.

All the chamber systems we consider here will be flag transitive. See [Buekenhout 1995, Chapter 3] for further background on group geometries.

In [Ronan and Stroth 1984] a dictionary of rank 2 subdiagrams is given, resulting in diagrams for these geometries analogous to the Dynkin diagrams of buildings. Usually these diagrams for the sporadic geometries have just one rank 2-subdiagram which is not associated with a crystallographic root system. So in this sense they look very close to buildings. This raises the question as to how chamber graphs of buildings and chamber graphs of the sporadic geometries compare. We recall that all essential properties of a building are encoded in its chamber graph (see [Tits 1981], for example) and so we cannot expect them to be too similar.

For $\gamma$ a chamber of $\mathcal{C}(\Gamma)$ and $i \in \mathbb{N}$,

$$\Delta_i(\gamma) = \{\gamma' \in \mathcal{C}(\Gamma) \mid d(\gamma, \gamma') = i\},$$

where $d(\ ,\ )$ is the usual distance metric on the chamber graph $\mathcal{C}(\Gamma)$. We refer to $\Delta_i(\gamma)$ as the *$i$-th disc of $\gamma$*. For $\gamma, \gamma' \in \mathcal{C}(\Gamma)$ any path of shortest distance between them in $\mathcal{C}(\Gamma)$ is called a geodesic. The geodesic closure of a set of chambers $X$ is defined to be the set $\overline{X}$ of all chambers lying on some geodesic of $\gamma, \gamma'$, for any pair $\gamma, \gamma' \in X$. The graph theoretic structure and size of $\Delta_i(\gamma)$ tells us much about $\mathcal{C}(\Gamma)$. Suppose $d = \operatorname{Diam} \mathcal{C}(\Gamma)$, the diameter of $\mathcal{C}(\Gamma)$, then we call $\Delta_d(\gamma)$ the *last disc of $\gamma$*.

Assume that $\gamma \in \mathcal{C}(\Gamma)$ is such that $\operatorname{Stab}_G(\gamma) = B$. If $G$ is a Lie type group and $\Gamma$ its associated building, then the last disc of $\gamma$ displays a number of interesting facets of $\Gamma$. Firstly, $S$ acts simply transitively on the chambers in the last disc of $\gamma$ (and so the size of this disc is $|S|$). More importantly if we choose any $\gamma'$ in the last disc of $\gamma$, then the geodesic closure of $\gamma$ and $\gamma'$ gives the chambers of an apartment of $\Gamma$.

Accordingly, for the minimal parabolic sporadic geometries we investigate here we shall be looking for those with a small number of $B$-orbits in the last disc, and for these we shall also probe their geodesic closures. The minimal parabolic geometries of $M_{12}$, $M_{24}$, $J_2$, $J_3$, $He$, $McL$ and $Ru$ fall into this category.

## 2. Statement of results

Our first result concerns the diameter of $\mathcal{C}(\Gamma)$.

**Theorem 2.1.** *The diameter, or bounds for the diameter, of the chamber graphs of the minimal parabolic sporadic geometries are as shown in Table 1.*

In the table, the second column gives the set $\{P_i/O_2(P_i) \mid i \in I\}$, which we refer to as the set of *induced panel residues* of $\Gamma$. The third column gives the diameter of $\mathcal{C}(\Gamma)$, and the last gives the number $n_{\mathrm{orbits}}$ of $B$ orbits of $\Delta_d(\gamma_0)$.The use of $-$ indicates we have no information.

In Theorem 2.1, $M_{23}$ has two different minimal parabolic geometries whose induced panel residues are the same. They differ in the choice of $2^4 : L_3(2)$ ( $=\langle P_1, P_3\rangle$ or $\langle P_3, P_4\rangle$ in [Ronan and Stroth 1984]) in $H = 2^4 : Alt(7)$. One choice leaves a 1-space of $O_2(H)$ invariant and the other a 3-space of $O_2(H)$ invariant. The former is called the 1-geometry and the latter the 3-geometry. Also in Theorem 2.1, to distinguish two of the $McL$ geometries we use the same notation for minimal parabolic subgroups as in [Ronan and Stroth 1984].

Surveying the last column of Theorem 2.1 we see a number of geometries for which the last disc consists of relatively few $B$-orbits. These geometries certainly warrant further attention — indeed, those of $M_{24}$ and $He$ have been dissected in [Carr and Rowley 2018].

There has been considerable effort expended in collecting geometries, just as in [Ronan and Stroth 1984], which share properties similar to those in buildings. See [Buekenhout 1979a; 1979b; 1995; Kantor 1981; Ronan and Smith 1980; Tits 1980] for an overview of these. The, so-called, GABs which stands for geometries that are almost buildings are among this collection. Perversely, from the point of view of the number of $B$-orbits in the last disc these geometries are very different from buildings; see [Kelsey and Rowley 2019]. In this sense some of the sporadic geometries in Theorem 2.1 are more like buildings.

| group | induced panel residues | $d = \mathrm{Diam}\,\mathcal{C}(\Gamma)$ | $n_{\mathrm{orbits}}$ |
|---|---|---|---|
| $M_{12}$ | $\{L_2(2),\ L_2(2)\}$ | 12 | 1 |
| $M_{22}$ | $\{L_2(2),\ Sym(5)\}$ | 5 | 12 |
| $M_{23}$ | $\{L_2(2),\ L_2(2),\ Sym(5)\}$ 1-geometry | 7 | 228 |
| | $\{L_2(2),\ L_2(2),\ Sym(5)\}$ 3-geometry | 7 | 224 |
| $M_{24}$ | $\{L_2(2),\ L_2(2),\ L_2(2)\}$ | 17 | 2 |
| $J_2$ | $\{L_2(2),\ L_2(4)\}$ | 8 | 2 |
| $J_3$ | $\{L_2(2),\ L_2(4)\}$ | 14 | 1 |
| $J_4$ | $\{L_2(2),\ L_2(2),\ Sym(5)\}$ | $12 \leq d \leq 75$ | – |
| $Co_3$ | $\{L_2(2),\ L_2(2),\ L_2(2)\}$ | $13 \leq d$ | – |
| $Co_2$ | $\{L_2(2),\ L_2(2),\ Sym(5)\}$ | 15 | 86 |
| $Co_1$ | $\{L_2(2),\ L_2(2),\ L_2(2),\ L_2(2)\}$ | $15 \leq d \leq 48$ | – |
| $HS$ | $\{L_2(2),\ Sym(5)\}$ | 8 | 39 |
| $He$ | $\{L_2(2),\ L_2(2),\ L_2(2)\}$ | 21 | 1 |
| $Ly$ | $\{L_2(2),\ Sym(9)\}$ | $5 \leq d$ | – |
| | $\{L_2(2),\ Sym(5)\}$ | $15 \leq d$ | – |
| $McL$ | $\{L_2(2),\ L_2(2),\ L_2(2)\}$ | 20 | 4 |
| | $\{L_2(2),\ L_2(2),\ Sym(5)\}$ $\{P_1,\ P_1^{\sigma},\ P_5\}$ | 11 | 1596 |
| | $\{L_2(2),\ L_2(2),\ Sym(5)\}$ $\{P_1^{\sigma},\ P_2^{\sigma},\ P_5\}$ | 10 | 2042 |
| | $\{L_2(2),\ L_2(2),\ L_2(2),\ L_2(2)\}$ | 14 | 881 |
| $O'N$ | $\{L_2(2),\ L_3(4).2\}$ | $5 \leq d$ | – |
| $Ru$ | $\{L_2(2),\ Sym(5)\}$ | 12 | 3 |
| $Sz$ | $\{L_2(2),\ L_2(2),\ L_2(4)\}$ | 16 | 57 |
| $Fi_{22}$ | $\{L_2(2),\ L_2(2),\ Sym(5)\}$ | $8 \leq d \leq 18$ | – |
| $Fi_{23}$ | $\{L_2(2),\ L_2(2),\ L_2(2),\ Sym(5)\}$ | $11 \leq d \leq 32$ | – |
| $Fi_{24}'$ | $\{L_2(2),\ L_2(2),\ L_2(2),\ L_2(2)\}$ | $21 \leq d \leq 90$ | – |
| $Th$ | $\{L_2(2),\ Alt(9)\}$ | $9 \leq d \leq 11$ | – |
| $HN$ | $\{L_2(2),\ Alt(5) \wr \mathbb{Z}_2\}$ | $9 \leq d \leq 11$ | – |
| $\mathbb{B}$ | $\{L_2(2),\ L_2(2),\ L_2(2),\ Sym(5)\}$ | $17 \leq d \leq 64$ | – |
| $\mathbb{M}$ | $\{L_2(2),\ L_2(2),\ L_2(2),\ L_2(2),\ L_2(2)\}$ | $42 \leq d \leq 344$ | – |

**Table 1.** Information on the the diameter of the chamber graphs of the minimal parabolic sporadic geometries. The second column gives the set $\{P_i/O_2(P_i) \mid i \in I\}$, the third gives the diameter of $\mathcal{C}(\Gamma)$, and the last gives the number $n_{\mathrm{orbits}}$ of $B$ orbits of $\Delta_d(\gamma_0)$.The use of – indicates we have no information.

Our second result describes the disc structure of some of the minimal parabolic sporadic geometries.

**Theorem 2.2.** *Let $G$ denote one of the sporadic simple groups $M_{12}$, $M_{22}$, $M_{23}$, $J_2$, $J_3$, $Co_2$, $HS$, $McL$ and $Ru$. Let $\Gamma$ denote a minimal parabolic geometry associated to one of these groups. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let $\gamma_0$ be a fixed chamber of $\mathcal{C}$. Put $B = \mathrm{Stab}_G(\gamma_0)$ and let $n_{\text{orbits}}$ be the number of $B$ orbits of $\Delta_d(\gamma_0)$.*

(i) *If $G \cong M_{12}$ and $\Gamma$ has induced panel residues $\{L_2(2), L_2(2)\}$, then $\mathcal{C}$ has* 1485 *chambers*, 44 *$B$-orbits, diameter* 12 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\Delta_i(\gamma_0)\rvert$ | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 384 | 320 | 192 | 64 | 16 |
| $n_{\text{orbits}}$ | 2 | 2 | 2 | 2 | 3 | 4 | 6 | 6 | 6 | 6 | 3 | 1 |

(ii) *If $G \cong M_{22}$ and $\Gamma$ has induced panel residues $\{L_2(2), L_2(2)\}$, then $\mathcal{C}$ has* 3465 *chambers*, 60 *$B$-orbits, diameter* 5 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\lvert\Delta_i(\gamma_0)\rvert$ | 16 | 56 | 432 | 1040 | 1920 |
| $n_{\text{orbits}}$ | 4 | 6 | 15 | 17 | 17 |

(iii) *If $G \cong M_{23}$ and $\Gamma$ has induced panel residues $\{L_2(2), L_2(2), Sym(5)\}$, the* 1*-geometry, then $\mathcal{C}$ has* 79,695 *chambers*, 835 *$B$-orbits, diameter* 7 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\lvert\Delta_i(\gamma_0)\rvert$ | 18 | 92 | 664 | 3104 | 10,728 | 36,032 | 29,056 |
| $n_{\text{orbits}}$ | 5 | 13 | 32 | 81 | 157 | 318 | 228 |

(iv) *If $G \cong M_{23}$ and $\Gamma$ has induced panel residues $\{L_2(2), L_2(2), Sym(5)\}$, the* 3*-geometry, then $\mathcal{C}$ has* 79,695 *chambers*, 835 *$B$-orbits, diameter* 7 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\lvert\Delta_i(\gamma_0)\rvert$ | 18 | 92 | 664 | 3104 | 10,728 | 36,544 | 28,544 |
| $n_{\text{orbits}}$ | 5 | 13 | 32 | 81 | 157 | 322 | 224 |

(v) *If $G \cong J_2$ and $\Gamma$ has induced panel residues $\{L_2(2), L_2(4)\}$, then $\mathcal{C}$ has* 1575 *chambers*, 20 *$B$-orbits, diameter* 8 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\lvert\Delta_i(\gamma_0)\rvert$ | 6 | 16 | 48 | 128 | 384 | 640 | 288 | 64 |
| $n_{\text{orbits}}$ | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 2 |

(vi) *If* $G \cong J_3$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), L_2(4)\}$, *then* $\mathcal{C}$ *has* 130,815 *chambers*, 370 $B$*-orbits*, *diameter* 14 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 6 | 16 | 48 | 128 | 384 | 1024 | 3072 | 7936 | 20,736 | 42,240 | 42,432 | 10,944 | 1656 | 192 |
| $n_{\text{orbits}}$ | 2 | 2 | 2 | 2 | 3 | 4 | 10 | 22 | 55 | 114 | 115 | 30 | 7 | 1 |

(vii) *If* $G \cong Co_3$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2).L_2(2), L_2(2)\}$, *then* $\mathcal{C}$ *has* 484,147,125 *chambers*, 484,680 $B$*-orbits and this disc structure as far as* $i = 14$ (note incomplete data here):

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 6 | 24 | 84 | 258 | 792 | 2344 | 6976 | 19,552 | 53,728 |
| $n_{\text{orbits}}$ | 3 | 6 | 12 | 20 | 34 | 56 | 100 | 162 | 281 |

| $i$-th disc | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 144,960 | 382,464 | 1,006,720 | 2,567,232 | 6,494,720 |
| $n_{\text{orbits}}$ | 512 | 999 | 1991 | 3963 | 8133 |

(viii) *If* $G \cong Co_2$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), L_2(2), Sym(5)\}$, *then* $\mathcal{C}$ *has* 161,382,375 *chambers*, 2791 $B$*-orbits*, *diameter* 15 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 18 | 92 | 664 | 3104 | 11,264 | 46,912 | 159,360 | 5,501,44 | 1,597,952 |
| $n_{\text{orbits}}$ | 5 | 11 | 28 | 53 | 83 | 139 | 187 | 265 | 303 |

| $i$-th disc | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 4,143,104 | 11,051,008 | 27,033,600 | 47,185,920 | 47,054,848 | 22,544,384 |
| $n_{\text{orbits}}$ | 338 | 377 | 365 | 347 | 203 | 86 |

(ix) *If* $G \cong HS$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), Sym(5)\}$, *then* $\mathcal{C}$ *has* 86,625 *chambers*, 270 $B$*-orbits*, *diameter* 8 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 16 | 56 | 440 | 1312 | 7872 | 17,664 | 40,448 | 18816 |
| $n_{\text{orbits}}$ | 4 | 6 | 15 | 19 | 47 | 50 | 89 | 39 |

(x) *If* $G \cong McL$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), L_2(2), L_2(2)\}$, *then* $\mathcal{C}$ *has* 7,016,625 *chambers*, 57,866 $B$*-orbits*, *diameter* 20 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 6 | 20 | 56 | 144 | 376 | 936 | 2210 | 5124 | 11,656 | 26,640 | 60,544 | 136,032 |
| $n_{\text{orbits}}$ | 3 | 5 | 8 | 13 | 24 | 45 | 82 | 135 | 216 | 383 | 714 | 1408 |

| $i$-th disc | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 284,880 | 588,800 | 1,162,272 | 1,934,416 | 2,019,280 | 745,408 | 37,568 | 256 |
| $n_{\text{orbits}}$ | 2638 | 5033 | 9432 | 15,379 | 16,026 | 6002 | 315 | 4 |

(xi) *If* $G \cong McL$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), L_2(2), Sym(5)\}$, $\{P_1, P_1^\sigma, P_5\}$, *then* $\mathcal{C}$ *has* 7,016,625 *chambers*, 57,866 $B$*-orbits*, *diameter* 11 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 18 | 112 | 770 | 3964 | 17400 | 71440 | 294760 | 1078784 | 2789696 | 2555840 | 203840 |
| $n_{\text{orbits}}$ | 5 | 16 | 52 | 138 | 358 | 998 | 3037 | 9182 | 22326 | 20157 | 1596 |

(xii) *If* $G \cong McL$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), L_2(2), Sym(5)\}$, $\{P_1^\sigma, P_2^\sigma, P_5\}$, *then* $\mathcal{C}$ *has* 7,016,625 *chambers*, 57,866 $B$*-orbits*, *diameter* 10 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 18 | 116 | 880 | 5288 | 28,062 | 154,772 | 711,008 | 2,560,688 | 3,296,208 | 259,584 |
| $n_{\text{orbits}}$ | 5 | 16 | 53 | 162 | 518 | 1814 | 6418 | 20769 | 26068 | 2042 |

(xiii) *If* $G \cong McL$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), L_2(2), L_2(2), L_2(2)\}$, *then* $\mathcal{C}$ *has* 7,016,625 *chambers*, 57,866 $B$*-orbits*, *diameter* 14 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 8 | 40 | 176 | 704 | 2384 | 7936 | 26,048 | 79,616 | 238,720 |
| $n_{\text{orbits}}$ | 4 | 11 | 26 | 66 | 134 | 253 | 560 | 1228 | 2651 |

| $i$-th disc | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 661,632 | 1,581,184 | 2,658,560 | 1,646,848 | 112768 |
| $n_{\text{orbits}}$ | 5844 | 12,564 | 20,777 | 12,866 | 881 |

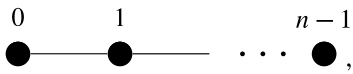
(xiv) *If* $G \cong Ru$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), Sym(5)\}$, *then* $\mathcal{C}$ *has* 8,906,625 *chambers*, 847 $B$*-orbits*, *diameter* 12 *and this disc structure*:

| $i$-th disc | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\|\Delta_i(\gamma_0)\|$ | 16 | 56 | 440 | 1344 | 10560 | 32000 | 231936 | 647168 | 3588096 | 3997696 | 385024 | 12288 |
| $n_{\text{orbits}}$ | 4 | 6 | 11 | 12 | 27 | 33 | 65 | 94 | 304 | 250 | 37 | 3 |

## 3. Diameters and geodesic closures

We first give three results concerning the diameter of chamber graphs. For $\Gamma$ a geometry and $x \in \Gamma$, the residue of $x$, denoted $\Gamma_x$, is the subgeometry consisting of all $y \in \Gamma$ incident with $x$.

**Lemma 3.1.** *Suppose that* $\Gamma$ *is a string geometry with diagram*




*where the type* 0 *and type* 1 *objects are, respectively, the points and lines of* $\Gamma$. *Let* $\mathcal{G}(\Gamma)$ *be the point-line collinearilty graph of* $\Gamma$. *Assume that*

(i) $G = Aut(\Gamma)$ *acts flag transitively on* $\Gamma$;

(ii) *for $x$ a point of* $\Gamma$, *the chamber graph* $\mathcal{C}(\Gamma_x)$ *is connected with* $\operatorname{Diam}\mathcal{C}(\Gamma_x) = e$; *and*,

(iii) $\mathcal{G}(\Gamma)$ *is connected with* $\operatorname{Diam}\mathcal{G}(\Gamma) = f$.

*Then*

$$\operatorname{Diam}\mathcal{C}(\Gamma) \leq f(1+e).$$

*Proof.* Let $\gamma_1 = \{x_1, x_2, \ldots, x_n\}$ be a chamber of $\Gamma$ with $x = x_1$, a point and $\ell = x_2$ a line. Note that $x$ and $\ell$ are incident. Let $y$ be a point incident with $\ell$ and $y \neq x$. Since $\Gamma$ is a string geometry $\gamma_2 = \{y, \ell, x_3, \ldots, x_n\}$ is a chamber of $\Gamma$. Moreover, in $\mathcal{C}(\Gamma)$, $d(\gamma_1, \gamma_2) = 1$. Also $\{\ell, x_3, \ldots x_n\}$ is a chamber in $\Gamma_y$. Hence for any chamber $\gamma$ of $\Gamma$ which contains $y$, we have $d(\gamma_1, \gamma) \leq 1 + e$. Let $\gamma_0$ be a chamber of $\Gamma$. Because, by assumption, $\mathcal{G}(\Gamma)$ is connected, a straight forward induction argument shows $d(\gamma_0, \gamma) \leq f(1+e)$ for any chamber $\gamma$ of $\Gamma$. Hence, as $G$ is flag transitive on $\Gamma$, we deduce that $\operatorname{Diam}\mathcal{C}(\Gamma) \leq f(1+e)$. □

**Lemma 3.2.** *Suppose* $\Gamma = \{P_1, \ldots, P_n\}$ *is a minimal parabolic geometry*, *and set* $a_i = [P_i : B]$, *for* $i = 1, \ldots, n$. *Let*

$$a = \sum_{i=1}^{n} (a_i - 1) \quad \text{and} \quad b = \sum_{i=1}^{n} \big((a_i - 1)(a - (a_i - 1))\big).$$

*Then*

$$\operatorname{Diam}\mathcal{C}(\Gamma) \geq \left\lceil \log_{a-1}\left(\frac{a-2}{b}\big(|\mathcal{C}(\Gamma)| - (1+a)\big) + 1\right)\right\rceil + 1.$$

*Proof.* Let $\gamma$ be a type $i$ neighbour of $\gamma_0$, then $\gamma$ is $i$-adjacent to all other type $i$ neighbours of $\gamma_0$. And so $\gamma$ is joined to at least $a_i - 1$ chambers in $\Delta_1(\gamma_0) \cup \{\gamma_0\}$. Hence $\gamma$ has at most $a - (a_i - 1)$ neighbours in $\Delta_2(\gamma_0)$. There are $(a_i - 1)$ chambers of type $i$ in $\Delta_1(\gamma_0)$, and so there are at most $\sum_{i=1}^{n} \big((a_i - 1)[a - (a_i - 1)]\big)$ chambers in the second disc.

For $i \geq 2$, each chamber in $\Delta_i(\gamma_0)$ has at most $a-1$ neighbours in $\Delta_{i+1}(\gamma_0)$. Consequently the number of chambers in $\Delta_{i+1}(\gamma_0)$ is at most $(a-1)|\Delta_i(\gamma_0)|$. Hence summing across the discs up to and including $\Delta_{k+2}(\gamma_0)$, there are at most $1+a+b+b(a-1)+\cdots+b(a-1)^k$ chambers. Set $d = \operatorname{Diam} \mathcal{C}(\Gamma)$. Then

$$|\mathcal{C}(\Gamma)| \leq 1+a+b+b(a-1)+\cdots+b(a-1)^{d-2} = 1+a+\frac{b\big((a-1)^{d-1}-1\big)}{a-2}$$

and hence

$$(a-1)^{d-1} \geq \frac{a-2}{b}\big(|\mathcal{C}(\Gamma)|-(1+a)\big)+1.$$

Taking log base $a-1$ gives the inequality in the lemma. □

**Lemma 3.3.** *Suppose $\Gamma$ is a rank 2 geometry with point-line collinearity graph $\mathcal{G}(\Gamma)$. If $\operatorname{Diam} \mathcal{G}(\Gamma) = f$, then $2f-1 \leq \operatorname{Diam} \mathcal{C}(\Gamma) \leq 2f+1$.*

*Proof.* Given a path $\{x_0, x_1, \ldots, x_\ell\}$ with lines $l_{i+1}$ joining $x_i$ to $x_{i+1}$ for $0 \leq i \leq \ell-1$ in $\mathcal{G}(\Gamma)$, there is a corresponding path in $\mathcal{C}(\Gamma)$ given by

$$\{(x_0, l_1), (x_1, l_2), (x_1, l_2), \ldots, (x_\ell, l_\ell)\}.$$

If the path in $\mathcal{G}(\Gamma)$ is a geodesic then so is the corresponding path in $\mathcal{C}(\Gamma)$, as any shorter path in $\mathcal{C}(\Gamma)$ results in a shorter path in $\mathcal{G}(\Gamma)$.

Hence the longest geodesic in $\mathcal{G}(\Gamma)$ of length $f$ gives rise to a geodesic of length $2f-1$ in $\mathcal{C}(\Gamma)$. If there is a vertex $x_{-1}$ joined to $x_0$ by $l_0$ such that $d(x_0, x_f) = d(x_{-1}, x_f)$ then prepending $(x_0, l_0)$ to the induced path in $\mathcal{C}(\Gamma)$ creates a geodesic of length $2f$. The same situation occurring at $x_f$ can result in a geodesic of length $2f+1$. □

*Proof of Theorem 1.2.* The combined efforts of Magma [Cannon and Playoust 1997], and the code used in [Carr and Rowley 2018] or [Kelsey and Rowley 2019] yield the data on disc structure given in Theorem 2.2. □

*Proof of Theorem 1.1.* The diameters for the geometries associated with $M_{12}$, $M_{22}$, $M_{23}$, $J_2$, $J_3$, $Co_2$, $HS$, $McL$ and $Ru$ follow from Theorem 2.2. For the geometries associated with $M_{24}$ and $He$ see [Carr and Rowley 2018] and for $Suz$ see [Kelsey and Rowley 2019]. The bounds for the $Th$ and $HN$ geometries follow from [Rowley and Taylor 2011] and Lemma 3.3. Now let $\Gamma$ be the characteristic two minimal parabolic geometry for one of the groups $J_4$, $Co_1$, $Fi_{22}$, $Fi_{23}$, $Fi_{24}'$, $\mathbb{B}$ and $\mathbb{M}$ given in [Ronan and Stroth 1984]. These are all string geometries. Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for $\Gamma$, where we will nominate in each case which objects play the role of points. Set $f = \operatorname{Diam} \mathcal{G}(\Gamma)$ and for $x$ a point of $\Gamma$ let $e$ denote the diameter of $\mathcal{C}(\Gamma_x)$. We aim to determine, or obtain bounds for, $e$ and $f$, first looking at $\Gamma$ for $J_4$. Call those objects whose stabilizer in $J_4$ has shape $2^{1+12}3M_{22}2$ and $2^{3+12+2}(Sym(3) \times Sym(5))$ points and lines respectively. Now subgroups $H$

of $J_4$ of shape $2^{2+12}2M_{22}2$ have $|Z(H)| = 2$ and are self normalizing ($H$ is in fact a maximal subgroup, see [Conway et al. 1985]). Thus we may identify the points of $\Gamma$ with the $2A$ conjugacy class of $J_4$. Let $x$ be a point of $\Gamma$ and $l$ a line incident with $x$. Now $l$ is incident with seven points and under this identification they correspond to the seven involutions in the minimal normal subgroup of the stabilizer of $l$ of order $2^3$. Since the stabilizer of $x$ is transitive on the lines incident with $x$ and the first disc of the commuting involution graph of $2A$ has size 194106, we conclude that $\mathcal{G}(\Gamma)$ is the same as the commuting involution graph for $2A$. Therefore, by [Bates et al. 2007, Theorem 1.1] $\mathcal{G}(\Gamma)$ has diameter 3. From [Rowley 2010] the diameter of the chamber graph of the $3^{\cdot}M_{22}.2$ geometry is 24. Thus $f = 3$ and $e = 24$ for $J_4$. Now using [Segev 1988], [Rowley and Walker 1996, 2011; 2012b; 2012a; 2016; 2004a; 2004b] and [Rowley 2019] we have the values for $f$ in the table below. (For $Co_1$, $Fi_{23}$, $Fi'_{24}$ and $\mathbb{M}$ we note the given reference deals with the point-line collinearity graph for their maximal parabolic geometries which is the same as that for its minimal parabolic geometries.) The values given for $e$ are obtained from Theorem 2.2 except for $\mathbb{M}$, where $e \le 3(17+1) = 48$ follows from Lemma 3.1, using the data for $Co_1$.

| Group | $e$ | $f$ | point-stabilizer |
|---|---|---|---|
| $J_4$ | 24 | 3 | $2^{1+12}.3^{\cdot}M_{22}.2$ |
| $Co_1$ | 17 | 3 | $2^{11}.M_{24}$ |
| $Fi_{22}$ | 5 | 3 | $2^{10}.M_{22}$ |
| $Fi_{23}$ | 7 | 4 | $2^{11}.M_{23}$ |
| $Fi'_{24}$ | 17 | 5 | $2^{11}.M_{24}$ |
| $\mathbb{B}$ | 15 | 4 | $2^{1+22}.Co_2$ |
| $\mathbb{M}$ | $\le 48$ | $\le 6$ | $2^{1+24}.Co_1$ |

Applying Lemma 3.1 yields the bounds for $\mathcal{C}(\Gamma)$ as stated in Theorem 2.1. The given lower bounds for $\operatorname{Diam} \mathcal{C}(\Gamma)$ may be obtained using Lemma 3.2. □

We single out for special attention those chamber graphs having few $B$-orbits in the last disc.

**Theorem 3.4.** *Let $\gamma_i$ be $B$-orbit representatives for the chambers in the disc $\gamma_0$. The geodesic closure of $B$-orbit representatives of the last disc are given below.*

(i) *If $G \cong M_{12}$ and $\Gamma$ has induced panel residues $\{L_2(2), L_2(2)\}$, then $\mathcal{C}$ has the following geodesic closure*:

| disc $i$ of $\mathcal{C}(\Gamma)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\{\overline{\gamma_0, \gamma_1}\} \cap \Delta_i(\gamma_0)\rvert$ | 1 | 4 | 8 | 12 | 16 | 16 | 16 | 16 | 16 | 12 | 8 | 4 | 1 |

(ii) *If* $G \cong J_2$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), L_2(4)\}$, *then for* $i = 1, 2$, *the two* $B$*-orbits have the following geodesic closure data*:

| disc $i$ of $\mathcal{C}(\Gamma)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\lvert\{\overline{\gamma_0, \gamma_i}\} \cap \Delta_i(\gamma_0)\rvert$ | 1 | 5 | 8 | 8 | 8 | 8 | 8 | 5 | 1 |

(iii) *If* $G \cong J_3$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), L_2(4)\}$, *then* $\mathcal{C}$ *has the following geodesic closure*:

| disc $i$ of $\mathcal{C}(\Gamma)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\{\overline{\gamma_0, \gamma_1}\} \cap \Delta_i(\gamma_0)\rvert$ | 1 | 6 | 16 | 40 | 52 | 56 | 56 | 56 | 52 | 48 | 40 | 16 | 6 | 1 |

(iv) *If* $G \cong McL$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), L_2(2), L_2(2)\}$, *then, for* $i = 1, 2$, *the four* $B$*-orbits have the following geodesic closure data*:

| *disc* $i$ *of* $\mathcal{C}(\Gamma)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\{\overline{\gamma_0, \gamma_i}\} \cap \Delta_i(\gamma_0)\rvert$ | 1 | 5 | 14 | 28 | 32 | 38 | 44 | 46 | 52 | 46 | 48 |
| $\lvert\{\overline{\gamma_0, \gamma_3}\} \cap \Delta_i(\gamma_0)\rvert$ | 1 | 5 | 15 | 28 | 34 | 32 | 30 | 32 | 36 | 36 | 32 |
| $\lvert\{\overline{\gamma_0, \gamma_4}\} \cap \Delta_i(\gamma_0)\rvert$ | 1 | 5 | 15 | 28 | 32 | 32 | 36 | 38 | 36 | 34 | 32 |

| *disc* $i$ *of* $\mathcal{C}(\Gamma)$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\{\overline{\gamma_0, \gamma_i}\} \cap \Delta_i(\gamma_0)\rvert$ | 46 | 52 | 46 | 44 | 38 | 32 | 28 | 14 | 5 | 1 |
| $\lvert\{\overline{\gamma_0, \gamma_3}\} \cap \Delta_i(\gamma_0)\rvert$ | 34 | 36 | 38 | 36 | 32 | 32 | 28 | 15 | 5 | 1 |
| $\lvert\{\overline{\gamma_0, \gamma_4}\} \cap \Delta_i(\gamma_0)\rvert$ | 36 | 36 | 32 | 30 | 32 | 34 | 28 | 15 | 5 | 1 |

(v) *If* $G \cong Ru$ *and* $\Gamma$ *has induced panel residues* $\{L_2(2), Sym(5)\}$, *then for* $i = 1, 2, 3$, *the three* $B$*-orbits have the following geodesic closure data*:

| disc $i$ of $\mathcal{C}(\Gamma)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\{\overline{\gamma_0, \gamma_i}\} \cap \Delta_i(\gamma_0)\rvert$ | 1 | 14 | 40 | 40 | 40 | 40 | 40 | 40 | 40 | 40 | 40 | 14 | 1 |

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# Maximal cocliques in the Kneser graph on plane-solid flags in PG(6, $q$)

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For $q \geq 27$ we determine the independence number $\alpha(\Gamma)$ of the Kneser graph $\Gamma$ on plane-solid flags in PG(6, $q$). More precisely we describe all maximal independent sets of size at least $q^{11}$ and show that every other maximal example has cardinality at most a constant times $q^{10}$.

## 1. Introduction

For integers $n \geq 2$ and prime powers $q$ we denote by $\mathrm{PG}(n, q)$ the $n$-dimensional projective space over the finite field $\mathbb{F}_q$. A *flag* $F$ of $\mathrm{PG}(n, q)$ is a set of nontrivial subspaces of $\mathrm{PG}(n, q)$ such that $U \subseteq U'$ or $U' \subseteq U$ for all $U, U' \in F$. Here nontrivial means different from $\varnothing$ and $\mathrm{PG}(n, q)$. The set $\{\dim(U) \mid U \in F\}$ is called the *type* of the flag $F$. Two flags $F_1$ and $F_2$ of $\mathrm{PG}(n, q)$ are said to be in *general position*, if for all subspaces $U_1 \in F_1$ and $U_2 \in F_2$ we have $U_1 \cap U_2 = \varnothing$ or $\langle U_1, U_2\rangle = \mathrm{PG}(n, q)$.

If $S$ is a nonempty subset of $\{0, 1, \dots, n-1\}$, then the *Kneser graph* of flags of type $S$ is the simple graph whose vertices are the flags of type $S$ of $\mathrm{PG}(n, q)$ with two flags $F$ and $F'$ adjacent if and only if they are in general position. Note that this graph, among other generalizations of Kneser graphs, has already been defined in [Güven 2012].

For $|S| = 1$ the Kneser graph of type $S$ is also known simply as $q$-Kneser graph and the size of maximal cocliques therein was determined in [Frankl and Wilson 1986]. Furthermore, for $|S| > 1$ maximal cocliques in this graph were studied in [Blokhuis and Brouwer 2017] for $S = \{1, 2\}$ and $n = 4$, in [Blokhuis, Brouwer and Güven 2014] for $S = \{0, n-1\}$ and $n \geq 2$, and in [Blokhuis, Brouwer and Szőnyi 2014] for $S = \{0, 2\}$ and $n = 4$. The result given in the second of these works was already conjectured in [Mussche 2009] and is also given in [Güven 2012].

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In this paper we determine the independence number $\alpha(\Gamma)$ for the Kneser graph $\Gamma$ of type $\{2, 3\}$ in $\mathrm{PG}(6, q)$ for $q \geq 27$. We point out that a flag of type $\{2, 3\}$ of $\mathrm{PG}(6, q)$ is a self-dual object, hence any independent set of $\Gamma$ can also be seen as an independent set of the Kneser graph of the same type in the dual space of $\mathrm{PG}(6, q)$. To simplify notation, we also denote a flag $\{E, S\}$ of type $\{2, 3\}$ by $(E, S)$ where $E$ is a plane and $S$ is a solid. We first provide some examples.

**Example 1.1.** For a hyperplane $H$ of $\mathrm{PG}(6, q)$ and a maximal set $\mathcal{E}$ of mutually intersecting planes of $H$, we denote by $\Lambda(H, \mathcal{E})$ the set of all flags $(E, S)$ of type $\{2, 3\}$ of $\mathrm{PG}(6, q)$ that satisfy $S \subseteq H$ or $E \in \mathcal{E}$. Dually, for a point $P$ of $\mathrm{PG}(6, q)$ and a maximal set $\mathcal{S}$ of 3-dimensional subspaces on $P$ any two of which share at least a line, denote by $\Lambda(P, \mathcal{S})$, the set of all flags $(E, S)$ of type $\{2, 3\}$ of $\mathrm{PG}(6, q)$ that satisfy $P \in E$ or $S \in \mathcal{S}$.

Indeed, the following special case of this example was already covered in a more general setting in [Blokhuis and Brouwer 2017].

**Example 1.2.** For an incident point-hyperplane pair $(P, H)$ of $\mathrm{PG}(6, q)$, denote by $\Lambda(P, H)$ the set of all flags $(E, S)$ of type $\{2, 3\}$ that satisfy $P \in E$ or $P \in S \subseteq H$ and let $\Lambda(H, P)$ be the set of all flags $(E, S)$ of type $\{2, 3\}$ that satisfy $S \subseteq H$ or $P \in E \subseteq H$.

For an incident point-line pair $(P, l)$ of $\mathrm{PG}(6, q)$, let $\Lambda(P, l)$ be the set of all flags $(E, S)$ of type $\{2, 3\}$ that satisfy $P \in E$ or $l \subseteq S$.

For an incident pair $(U, H)$ of a 4-dimensional space $U$ and a hyperplane $H$ of $\mathrm{PG}(6, q)$, let $\Lambda(H, U)$ be the set of all flags $(E, S)$ of type $\{2, 3\}$ that satisfy $S \subseteq H$ or $E \subseteq U$.

We shall show in Proposition 3.2 that the sets described in Example 1.1 are maximal independent sets in the Kneser graph of flags of type $\{2, 3\}$ in $\mathrm{PG}(6, q)$. Notice that the condition on $\mathcal{E}$ means that $\mathcal{E}$ is an independent set of the Kneser graph of planes of $H \simeq \mathrm{PG}(5, q)$ and the condition on $\mathcal{S}$ means that $\mathcal{S}$ is an independent set of the Kneser graph of planes of the quotient space $\mathbb{P}/P \simeq \mathrm{PG}(5, q)$.

The sets constructed in Example 1.2 are special cases of the ones in Example 1.1 and hence also maximal independent sets. Here we use independent sets $\mathcal{E}$ and $\mathcal{S}$ of maximal size. Notice that the first and second examples as well as the third and forth examples in Example 1.2 are dual to each other.

In order to state our first theorem, we need the Gaussian coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, which are defined (for integer $n$ and $k$) by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \prod_{i=1}^{k} \frac{q^{n+1-i} - 1}{q^i - 1} \quad \text{if } 0 \leq k \leq n$$

and $\begin{bmatrix} n \\ k \end{bmatrix}_q := 0$ otherwise.

**Theorem 1.3.** *For $q \geq 27$, the independence number of the Kneser graph of flags of type $\{2, 3\}$ of* PG$(6, q)$ *is*

$$\begin{bmatrix}6\\4\end{bmatrix}_q \cdot \begin{bmatrix}4\\3\end{bmatrix}_q + \begin{bmatrix}5\\3\end{bmatrix}_q \cdot q^3$$

*and the independent sets attaining this bound are precisely the four examples described in Example 1.2.*

**Remarks. 1.** The independence number is of order $q^{11}$. As our proof of this theorem is geometric it also provides a stability result for independence sets. Essentially it says that, for large values of prime powers $q$, Example 1.1 describes all maximal independent sets with at least $27q^{10}$ elements. A precise formulation is given in Theorem 6.5.

**2.** Since we essentially show that any large independent set on the Kneser graph of plane-solid flags in PG$(6, q)$ is given by Example 1.1, any Hilton-Milner type result for the Kneser graph of type $\{2\}$ in PG$(5, q)$ translates to a Hilton-Milner type result for the Kneser graph of plane-solid flags in PG$(6, q)$. In particular, in the main theorem of Section 6 of [Blokhuis, Brouwer and Szőnyi 2012], a Hilton-Milner type result for the Kneser graph of planes in PG$(5, q)$ is given (the three largest examples are determined) and thus the second largest maximal EKR-set of plane-solid flags in PG$(6, q)$ has size

$$\begin{bmatrix}6\\4\end{bmatrix}_q \cdot \begin{bmatrix}4\\3\end{bmatrix}_q + \left(\begin{bmatrix}5\\3\end{bmatrix}_q - (q^6 - q^3)\right) q^3.$$

and its structure can be derived from Example 1.1 and said Hilton-Milner result. However, note hat the flags provided by the sets $\mathcal{E}$ and $\mathcal{S}$ in Example 1.1 contribute only a small amount of flags (order $q^9$) to the total size (order $q^{11}$) of the maximal examples.

**3.** Every upper bound $b$ for the independence number of a graph with $n$ vertices leads to the lower bound $\chi \geq n/b$ for its chromatic number $\chi$. In our situation this shows that the chromatic number of the Kneser graph of plane-solid flags of PG$(6, q)$, $q \geq 27$, has chromatic number at least $q^4 - q^2 + 2q + 1$. On the other hand, if $U$ is a 4-space, then the sets $\Lambda(P, \varnothing)$ with $P \in U$ are independent sets whose union covers every vertex, so the chromatic number is at most $q^4 + q^3 + q^2 + q + 1$. Using independent sets of the form $\Lambda(P, l)$ a simple construction given in Section 7 shows that this trivial upper bound can be slightly improved.

**4.** We keep all estimations in this paper as easy as possible and as such prove Theorem 1.3 only for $q > 27$. Only a more detailed approach, especially in Lemma 4.2, shows that Theorem 1.3 holds for $q = 27$. This will appear in the Ph.D. thesis of the second author.

## 2. Preliminaries

Let $q$ be a prime power and $\mathbb{F}_q$ the finite field of order $q$. For integer $n, d \geq 0$, the number of $d$-dimensional subspaces of an $\mathbb{F}_q$-vector space of dimension $n$ is given by the Gaussian coefficient $\left[{n \atop d}\right]_q$ (see bottom of page 40 for the definition). If $0 \leq d \leq n$, and if $D$ is a $d$-dimensional subspace of an $n$-dimensional $\mathbb{F}_q$ vector space $V$ then $D$ has exactly $q^{d(n-d)}$ complements in $V$. These two facts can be found in Section 3.1 of [Hirschfeld 1998]. We define

$$s_q(l, k, d, n) := q^{(l+1)(d-k)} \cdot \begin{bmatrix} n-k-l-1 \\ d-k \end{bmatrix}_q.$$

We also set $s_q(k, d, n) := s_q(-1, k, d, n)$, $s_q(d, n) := s_q(-1, d, n)$ and $s_q(n) := s_q(0, n)$ and omit the subscript $q$ in the following.

**Lemma 2.1.** *Given two skew subspaces in* $\mathrm{PG}(n, q)$ *of dimensions $k$ and $l$ respectively and any integer $d$ the number of $d$-subspaces of* $\mathrm{PG}(n, q)$ *that contain the $k$-subspace and are skew to the $l$-subspace is $s(l, k, d, n)$.*

*Proof.* We prove this for the underlying $\mathbb{F}_q$-vector space $V$ of dimension $n+1$ and two skew subspaces $K$ and $L$ of dimension $k+1$ and $l+1$ respectively, where we have to count the number of subspaces $D$ of dimension $d+1$ that contain $K$ and are skew to $L$. Every such subspace $D$ gives rise to a subspace $D+L$ of dimension $d+l+2$ of $V$. Going to the factor space $V/(K+L)$, we see that $V$ has $\left[{n-k-l-1 \atop d-k}\right]_q$ subspaces $U$ of dimension $d+l+2$ that contain $K+L$. For such a subspace $U$ we see in the quotient space $U/K$ that $U$ has $q^{(d-k)(l+1)}$ subspaces $D$ of dimension $d+1$ with $U = L+D$. $\square$

**Lemma 2.2.** *If $n \geq 5$ and if $\mathcal{E}$ is a set of planes of* $\mathrm{PG}(n, q)$ *such that any two distinct planes of $\mathcal{E}$ meet in a line, then $|\mathcal{E}| \leq s(n-2)$.*

*Proof.* If there exists a line contained in all planes of $\mathcal{E}$, then $|\mathcal{E}| \leq s(1, 2, n) = s(n-2)$. Otherwise there exist planes $E_1$, $E_2$, $E_3 \in \mathcal{E}$ such that $E_1 \cap E_3$ and $E_2 \cap E_3$ are distinct lines, which implies that $E_3$ is contained in the 3-space $U := \langle E_1, E_2 \rangle$. In this case, for every further plane $E$ of $\mathcal{E}$ at least two of the lines $E \cap E_1$, $E \cap E_2$ and $E \cap E_3$ are distinct, so $E$ is contained in $U$. Thus, in this case, every plane of $\mathcal{E}$ is one of the $s(2, 3)$ planes of $U$. $\square$

The following result has been proven in Theorem 1.4 of [Blokhuis et al. 2010], where it was formulated in its dual version.

**Result 2.3.** *For $q \geq 3$ the independence number $\alpha(\Gamma)$ of the Kneser graph $\Gamma$ of type $\{3\}$ in* $\mathrm{PG}(6, q)$ *is given by*

$$\alpha(\Gamma) = s(3, 5) = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1.$$

*For each hyperplane $H$ of* PG(6, $q$) *the set consisting of all solids of $H$ is an independent set of* $\Gamma$ *with* $\alpha(\Gamma)$ *vertices. Every other maximal independent set has cardinality at most* $q^6+2q^5+3q^4+3q^3+2q^2+q+1$.

We shall conclude this section with the following result, which is one specific case of the main Theorem of [Frankl and Wilson 1986], which solves the Erdős–Ko–Rado problem for vector spaces in general.

**Result 2.4.** *If $\mathcal{E}$ is an independent set of the Kneser graph of type* $\{2\}$ *in* PG(5, $q$), *then* $|\mathcal{E}| \le s(1, 4)$ *and equality holds if and only if $\mathcal{E}$ is the set of all planes on a point or the set of all planes in a hyperplane of* PG(5, $q$).

## 3. Sets of flags of type {2, 3}

In this section we study sets of flags of type $\{2, 3\}$ of PG(6, $q$). Recall that we also denote a flag $\{E, S\}$ of type $\{2, 3\}$ as the ordered pair $(E, S)$ where $E$ is the plane and $S$ the solid of the flag. Note that two distinct such flags $(E, S)$ and $(E', S')$ are adjacent in $\Gamma$ if and only if $E\cap S' = \varnothing = E'\cap S$. Let $\pi_2$ and $\pi_3$ be the maps from the set of all flags of type $\{2, 3\}$ to the set of subspaces of PG(6, $q$) with $\pi_2(f) := E$ and $\pi_3(f) := S$ for all flags $f = (E, S)$ of type $\{2, 3\}$. For any set $C$ of such flags, we define $\pi_i(C) := \{\pi_i(f) : f \in C\}$, $i = 2, 3$.

**Lemma 3.1.** *Let $\Gamma$ be the Kneser graph of flags of type* $\{2, 3\}$ *of* PG(6, $q$), *let $C$ be an independent set of $\Gamma$, let $H$ be a hyperplane and let $P$ be a point of* PG(6, $q$).

(i) *Let $\mathcal{E}$ be the set whose elements are the planes $E$ of $H$ for which there exists a solid $S$ with $(E, S) \in C$ and $E = H \cap S$. Then $E \cap E' \ne \varnothing$ for all $E, E' \in \mathcal{E}$, that is, $\mathcal{E}$ is an independent set of the Kneser graph of planes of $H$. Hence* $|\mathcal{E}| \le s(1, 4)$.

(ii) *Let $\mathcal{S}$ be the set whose elements are the solids $S$ for which there exists a flag $(E, S) \in C$ with $P \in S \setminus E$. Then* $|\mathcal{S}| \le s(1, 4)$.

*Proof.* (i) For $E, E' \in \mathcal{E}$ let $(E, S)$ and $(E', S')$ be flags of $C$ with $S \cap H = E$ and $S' \cap H = E'$. Then $S' \cap E = E' \cap E$ and $S \cap E' = E' \cap E$. Since $C$ is independent, it follows that $E \cap E' \ne \varnothing$. Thus $\mathcal{E}$ is an independent set of the Kneser graph of planes of $H$ and Result 2.4 shows $|\mathcal{E}| \le s(1, 4)$.

(ii) This is a special case of the dual statement of part (i). □

In the following proposition we investigate the sets constructed in Example 1.1 up to duality.

**Proposition 3.2.** *Let $H$ be a hyperplane of* PG(6, $q$) *and let $\Gamma$ be the Kneser graph of flags of type* $\{2, 3\}$ *of* PG(6, $q$).

(i) $\Lambda(H, \varnothing)$ *is an independent set of* $\Gamma$.

(ii) *The maximal independent sets of* $\Gamma$ *that contain* $\Lambda(H, \varnothing)$ *are the sets* $\Lambda(H, \mathcal{E})$ *for maximal independent sets* $\mathcal{E}$ *of planes of* $H$.

(iii) *For every maximal independent set* $\mathcal{E}$ *of the Kneser graph of planes of* $H$ *we have*

$$|\Lambda(H, \mathcal{E})| = s(3,5) \cdot s(3) + |\mathcal{E}| \cdot q^3$$
$$\leq q^{11} + 2q^{10} + 5q^9 + 7q^8 + 10q^7 + 11q^6 + 11q^5 + 9q^4 + 7q^3 + 4q^2 + 2q + 1.$$

*If equality holds, that is, if* $|\mathcal{E}| = s(1,4)$, *then either there exists a point* $P$ *in* $H$ *such that* $\mathcal{E}$ *consists of all planes of* $H$ *that contain* $P$, *or there exists a* 4*-dimensional subspace of* $H$ *such that* $\mathcal{E}$ *consists of all planes of this* 4*-dimensional subspace.*

*Proof.* (i) This follows from the fact that every solid of $H$ meets every plane of $H$.

(ii) If $\mathcal{E}$ is an independent set of planes in the Kneser graph of planes of $H$, then every solid of $H$ meets every plane of $\mathcal{E}$ nontrivially and every two planes of $\mathcal{E}$ meet nontrivially. Therefore $\Lambda(H, \mathcal{E})$ is an independent set of $\Gamma$. In order to prove the assertion, it therefore suffices to consider an independent set $C$ of $\Gamma$ with $\Lambda(H, \varnothing) \subseteq C$ and to show that $C$ is contained in $\Lambda(H, \mathcal{E})$ for a set $\mathcal{E}$ of mutually intersecting planes of $H$.

Let $C$ be an independent set with $\Lambda(H, \varnothing) \subseteq C$. Let $\mathcal{E}$ be the set of all planes $E$ of $H$ such that $C$ contains a flag $(E, S)$ with $E = S \cap H$. Lemma 3.1 shows that the planes of $\mathcal{E}$ are mutually intersecting. It remains to show that $C \subseteq \Lambda(H, \mathcal{E})$. Suppose on the contrary that there exists a flag $(E, S) \in C$ with $S \not\subseteq H$ and $H \cap S \neq E$. Then $S \cap H$ is a plane and $E \cap H$ is a line of this plane and $H$ contains a solid $S'$ that is skew to the line $E \cap H$. This implies that $S'$ meets the plane $S \cap H$ in a point and therefore $S'$ contains a plane $E'$ with $E' \cap S \cap H = \varnothing$. Then $(E', S') \in \Lambda(H, \varnothing) \subseteq C$ with $S' \cap E = \varnothing = S \cap E'$ and since $C$ is independent this is a contradiction.

(iii) Since $H$ contains $s(3,5)$ solids all of which contain $s(2,3) = s(3)$ planes, we have $|\Lambda(H, \varnothing)| = s(3,5) \cdot s(3)$. Every plane $E$ of $H$ lies on $s(2,3,6) - s(2,3,5) = q^3$ solids $S$ with $S \cap H = E$. Hence $|\Lambda(H, \mathcal{E})| = |\Lambda(H, \varnothing)| + |\mathcal{E}| \cdot q^3$. Result 2.4 shows $|\mathcal{E}| \leq s(1,4)$ with equality if and only if all planes of $\mathcal{E}$ contain a common point of $H$ or lie in a common 4-subspace of $H$. □

**Lemma 3.3.** *Let* $\mathcal{C}$ *be an independent set of the Kneser graph of type* $\{2,3\}$ *in* PG$(6,q)$ *and let* $\xi \in \mathbb{N}$ *be such that every solid of* PG$(6,q)$ *occurs in at most* $\xi$ *flags of* $C$. *Let* $(E, S)$ *be an element of* $C$. *Then there are at most*

$$s(2) \cdot s(1,4) \cdot \xi = (q^8 + 2q^7 + 4q^6 + 5q^5 + 6q^4 + 5q^3 + 4q^2 + 2q + 1) \cdot \xi$$

*flags* $(E', S') \in \mathcal{C}$ *with* $E' \cap E = \varnothing$ *and* $S' \cap E \neq \varnothing$.

*Proof.* Since $C$ is independent, every flag $(E', S') \in \mathcal{C}$ with $E' \cap E = \varnothing$ and $S' \cap E \neq \varnothing$ has the property that $S' \cap E$ is a point $P$ with $P \notin E'$. Hence for every such flag there exists a point $P$ in $E$ with $P \in S' \setminus E'$. Since $E$ has $s(2)$ points and since every solid occurs in at most $\xi$ flags of $C$ Lemma 3.1(ii) proves the statement. □

We now proceed to prove our theorem in three steps, where we consider two special cases in the first two steps: In the first step we only consider independent sets $C$ in which no plane or solid occurs in more than $s(1)$ flags of $C$ and in the second step we consider independent sets $C$ in which no plane or solid occurs in more than $s(2)$ flags of $C$.

## 4. The first special case

In this section we consider an independent set $C$ of the Kneser graph of type $\{2, 3\}$ in PG$(6, q)$ that has the property that every plane and every solid of PG$(6, q)$ occurs in at most $q + 1$ flags of $C$. Our aim is to prove an upper bound for $|C|$. For every point $P$ we denote the set of all flags $(E, S) \in C$ with $P \in E$ by $\Delta_P(C)$.

**Lemma 4.1.** *Let $P_1$, $P_2$ and $P_3$ be noncollinear points of* PG$(6, q)$.

(i) *If*

$$|\Delta_{P_1}(C)| > (q + 1)(6q^6 + 10q^5 + 17q^4 + 15q^3 + 15q^2 + 9q + 5), \qquad (1)$$

*then there are flags $f_i = (E_i, S_i) \in C$ for $i \in \{1, 2, 3\}$ with* $\dim(\langle E_1, E_2, E_3\rangle) \geq 5$, *$P_2, P_3 \notin S_1, S_2, S_3$ as well as $E_i \cap E_j = P_1$ and $P_2, P_3 \notin \langle E_i, E_j\rangle$ for all distinct $i, j \in \{1, 2, 3\}$.*

(ii) *If there are flags $f_1$, $f_2$ and $f_3$ with the properties stated in* (i) *and if*

$$|\Delta_{P_2}(C)| > (q + 1)(6q^6 + 10q^5 + 17q^4 + 18q^3 + 15q^2 + 9q + 5),$$

*then there are flags $f'_i = (E'_i, S'_i) \in C$ for $i \in \{1, 2, 3\}$ with* $\dim(\langle E'_1, E'_2, E'_3\rangle) \geq 5$, *$P_1, P_3 \notin S'_1, S'_2, S'_3$,* $\dim(S_i \cap S'_j) \leq 1$ *for all $i, j \in \{1, 2, 3\}$ as well as $E'_i \cap E'_j = P_2$ and $P_1, P_3 \notin \langle E'_i, E'_j\rangle$ for all distinct $i, j \in \{1, 2, 3\}$.*

*Proof.* (i) We frequently make use of the fact that every plane and every solid occurs in at most $q + 1$ flags of $C$. We also make use of the following properties:

(Q1) There are $2 \cdot s(1, 3, 6) - s(2, 3, 6) = 2 \cdot s(1, 4) - s(3)$ solids that contain $P_1$ and a point of $\{P_2, P_3\}$.

(Q2) If $E$ is a plane on $P_1$ and $P$ is a point not contained in $E$, then every plane $E'$ on $P_1$ with $E' \cap E \neq P_1$ or $P \in \langle E, E'\rangle$ meets the solid $\langle P, E\rangle$ in at least a line and hence there are at most $s(0, 1, 3) \cdot s(1, 2, 6) = s(2) \cdot s(4)$ such planes $E'$.

(Q3) If $E_1$ and $E_2$ are planes with $E_1 \cap E_2 = P_1$, then there exist less than $s(1, 0, 2, 4) = q^4$ planes in $\langle E_1, E_2\rangle$ with $E \cap E_1 = E \cap E_2 = P_1$.

From (Q1) and the bound in (1) we see that there exists a flag $(E_1, S_1)$ in $C$ with $P_1 \in E_1$ and $P_2, P_3 \notin S_1$. According to (Q1) and (Q2) the number of flags $(E, S) \in \Delta_{P_1}(C)$ for which $E_1 \cap E \neq P_1$ or for which $\langle E_1, E\rangle$ or $S$ contains a point of $\{P_2, P_3\}$ is at most

$$(q+1)(2s(1,4)-s(3)+2s(2)s(4)) = (q+1)(4q^6+6q^5+10q^4+9q^3+9q^2+5q+3)$$

which is smaller than the right-hand side of (1). Therefore, we find a flag $(E_2, S_2) \in \Delta_{P_1}(C)$ such that $E_1 \cap E_2 = P_1$ and neither of the spaces $\langle E_1, E_2\rangle$ or $S_2$ contains one of the points $P_2$ and $P_3$. Notice that $\dim(\langle E_1, E_2\rangle) = 4$, so for the remaining flag $(E_3, S_3)$ we need that $E_3$ is not contained in $\langle E_1, E_2\rangle$. Using (Q1), (Q2) and (Q3) a similar argument shows that at most

$$(q+1)(2\cdot s(1,4) - s(3) + 4\cdot s(2)\cdot s(4) + q^4) \\ = (q+1)(6q^6 + 10q^5 + 17q^4 + 15q^3 + 15q^2 + 9q + 5) \quad (2)$$

flags of $\Delta_{P_1}(C)$ do not satisfy all of the properties we want for the final flag $(E_3, S_3)$. Since this is the right-hand side of (1) and thus smaller than $|\Delta_{P_1}(C)|$ we find a flag $(E_3, S_3)$ with the desired properties.

(ii) We can argue analogously to the proof of (i). However, when choosing the flags $(E'_i, S'_i)$ for $i \in \{1, 2, 3\}$ we additionally have to avoid all flags $(E, S) \in \Delta_{P_2}(C)$ for which $S$ meets one of the solids $S_1$, $S_2$ and $S_3$ in a plane $\pi$ with $P_1 \notin \pi$. For $j \in \{1, 2, 3\}$ each $S_j$ has $q^3$ planes that do not contain $P_1$, so in total there are at most $3q^3$ solids $S$ that must not appear in any of our desired flags $(E'_i, S'_i)$ for $i \in \{1, 2, 3\}$ and were not considered before. Therefore, it is sufficient to check that the sum of the number in (2) and the number $3q^3(q+1)$ is the right-hand side of (1) and thus smaller than $|\Delta_{P_2}(C)|$, which is obviously true. □

**Lemma 4.2.** *Let $P_1$ and $P_2$ be two distinct points of* PG$(6, q)$ *and let $E_1$, $E_2$ and $E_3$ be planes such that $E_i \cap E_j = P_1$ and $P_2 \notin \langle E_i, E_j\rangle$ for all distinct $i, j \in \{1, 2, 3\}$. Furthermore, let $\mathcal{S}$ be the set of all solids of* PG$(6, q)$ *with $P_2 \in S$ and $S \cap E_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$. Then we have $|\mathcal{S}| \le 3q^6 + 6q^5 + 7q^4 + 4q^3 + 2q^2 + q + 1$.*

*Proof.* Let $\mathcal{E}$ be the set of all planes that contain $P_2$ but not $P_1$ and that meet all the planes $E_1$, $E_2$ and $E_3$. There are $s(1, 3, 6)$ solids on $P_2$ that contain $P_1$. If a solid on $P_2$ does not contain $P_1$ nor any plane of $\mathcal{E}$, then it meets all the planes $E_1$, $E_2$ and $E_3$ in unique points (different from $P_1$) and these three intersection points together with $P_2$ span the solid. Hence, there are at most $(q^2 + q)^3$ such solids. Finally, each plane of $\mathcal{E}$ lies in at most $s(0, 2, 3, 6)$ solids that do not contain $P_1$, which shows that the number of solids on $P_2$ that meet $E_1$, $E_2$ and $E_3$ is at most

$$s(1,3,6)+(q^2+q)^3+|\mathcal{E}|\cdot s(0,2,3,6). \tag{3}$$

It remains to determine an upper bound on $|\mathcal{E}|$. We put $U := \langle E_1, E_2\rangle$ with $P_2 \notin U$ and if $E \in \mathcal{E}$ we know from $P_2 \in E$ and $P_1 \notin E$ that $E \cap U$ is a line. We show that every point of $E_3 \setminus \{P_1\}$ lies on at most $q$ planes of $\mathcal{E}$. To see this, let $Q$ be a point of $E_3 \setminus \{P_1\}$ and suppose that $Q$ lies on at least one plane $E$ of $\mathcal{E}$. Since the lines $\langle P_2, Q\rangle$ and $E \cap U$ of $E$ are distinct, they meet in a unique point $R$, and $E \cap U$ is a line on $R$. Since $P_2$ is not contained in $\langle E_1, E_3\rangle$ nor in $\langle E_2, E_3\rangle$ we have $R \notin E_1$ and $R \notin E_2$. This implies that $R$ lies on exactly $q$ lines of $U$ that meet $E_1$ and $E_2$ but do not contain $P_1$. Since every plane of $\mathcal{E}$ on $Q$ is generated by $P_2$ and such a line, we see that $Q$ lies on at most $q$ planes of $\mathcal{E}$. As there are $q^2+q$ choices for $Q$, we find $|\mathcal{E}| \le (q^2+q)q$. Using this upper bound for $|\mathcal{E}|$, the statement follows from (3). □

**Lemma 4.3.** *Let $P$ be a point and suppose that there are flags $(E_i, S_i) \in \Delta_P(C)$, for $i \in \{1,2,3\}$, such that $E_i \cap E_j = P$ for distinct $i, j \in \{1,2,3\}$. Then every point $Q$ with $Q \notin \langle E_i, E_j\rangle$ and $Q \notin S_i$ for all $i, j \in \{1,2,3\}$ satisfies*

$$|\Delta_Q(C)| \le 3q^8+12q^7+21q^6+28q^5+26q^4+18q^3+12q^2+8q+4.$$

*Proof.* For $i \in \{1,2,3\}$ exactly $n := s(0,2,6)-s(3,0,2,6)$ planes on $Q$ meet $S_i$. Since every plane lies in at most $q+1$ flags of $C$, it follows that there exists at most $3n(q+1)$ flags $(E,S) \in \Delta_Q(C)$ such that $E$ has nonempty intersection with at least one of the solids $S_1$, $S_2$ or $S_3$.

Every other flag $f = (E,S) \in \Delta_Q(C)$ has the property that its solid $S$ meets $E_1$, $E_2$ and $E_3$. Lemma 4.2 shows that there at most $n' := 3q^6+6q^5+7q^4+4q^3+2q^2+q+1$ such solids. Since each solid lies in at most $q+1$ flags of $C$, there are at most $n'(q+1)$ such flags. Therefore $|\Delta_Q(C)| \le (3n+n')(q+1)$ proving the desired bound. □

**Lemma 4.4.** *Let $S_1$ and $S_2$ be solids of* PG$(6,q)$ *with* $\dim(S_1 \cap S_2) \le 1$ *and let $P$ be a point that is not contained in $S_1 \cup S_2$. Then the number of planes that contain $P$ and meet $S_1$ and $S_2$ nontrivially is at most* $2q^6+2q^5+3q^4+2q^3+2q^2+q+1$.

*Proof.* We have $d := \dim(S_1 \cap S_2) \in \{0,1\}$. A line through $P$ meets $S_1$ and $S_2$ if and only if it meets one and hence both of the subspaces $U_1 := \langle P, S_2\rangle \cap S_1$ and $U_2 := \langle P, S_1\rangle \cap S_2$, that is, if the line is contained in the subspace $V := \langle U_1, P\rangle$. The subspaces $U_1$ and $U_2$ have the same dimension $u$ where $u = 1$ if $d = 0$ and $u \in \{1,2\}$ when $d = 1$. We have $\dim(V) = u+1$.

A plane on $P$ that meets $V$ only in $P$ is spanned by $P$, a point of $S_1 \setminus U_1$ and a point of $S_2 \setminus U_2$, so there are $(s(3)-s(u))^2$ such planes. The number of planes on $P$ that meet $V$ in a line is equal to the number $s(0,1,u+1)$ of lines of $V$ on

$P$ multiplied with the number $s(1, 2, 6) - s(1, 2, u+1)$ of planes that meet $V$ in a given line. Finally there are $s(0, 2, u+1)$ planes on $P$ that are contained in $V$. Hence, the total number of planes on $P$ that meet $S_1$ and $S_2$ nontrivially is

$$(s(3) - s(u))^2 + s(0, 1, u+1)(s(1, 2, 6) - s(1, 2, u+1)) + s(0, 2, u+1)$$

The larger value occurs for $u = 2$ and gives the bound in the lemma. □

**Lemma 4.5.** *Let $P_1$, $P_2$ and $P_3$ be noncollinear points of* $\mathrm{PG}(6, q)$. *Suppose that for $i \in \{1, 2\}$ and $r \in \{1, 2, 3\}$ there exist flags $f_{i,r} = (E_{i,r}, S_{i,r}) \in \Delta_{P_i}(C)$ such that*

- $\forall r, s \in \{1, 2, 3\} : \dim(S_{1,r} \cap S_{2,s}) \leq 1$ *and*
- $\forall i \in \{1, 2\}, \forall \{r, s, t\} = \{1, 2, 3\} : P_{3-i}, P_3 \notin \langle E_{i,r}, E_{i,s}\rangle \cup S_{i,r}$ *and* $E_{i,r} \cap E_{i,s} = P_i$.

*Then* $|\Delta_{P_3}(C)| \leq 24q^7 + 54q^6 + 71q^5 + 67q^4 + 48q^3 + 33q^2 + 22q + 11$.

*Proof.* Because $C$ is independent we know that for every $(E, S) \in C$ and every $i \in \{1, 2\}$ we have $S \cap E_{i,r} \neq \varnothing$ for all $r \in \{1, 2, 3\}$ or $E \cap S_{i,r} \neq \varnothing$ for at least one $r \in \{1, 2, 3\}$. For $i \in \{1, 2\}$ Lemma 4.2 shows that the number of solids of $\mathrm{PG}(6, q)$ that contain $P_3$ and meet $E_{i,1}$, $E_{i,2}$ and $E_{i,3}$ is at most

$$m := 3q^6 + 6q^5 + 7q^4 + 4q^3 + 2q^2 + q + 1. \tag{4}$$

For every flag $(E, S) \in \Delta_{P_3}(C)$ for which $S$ is not such a solid we know that $E$ is a plane that meets $S_{1,r}$ and $S_{2,s}$ for some $r, s \in \{1, 2, 3\}$. For any choice of $r, s \in \{1, 2, 3\}$, Lemma 4.4 shows that there exist at most

$$n := 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

planes on $P_3$ that meet $S_{1,r}$ and $S_{2,s}$. Since every plane and every solid occurs in at most $q+1$ flags of $C$, it follows that $|\Delta_{P_3}(C)| \leq (2m + 9n)(q+1)$, as claimed. □

**Proposition 4.6.** *Let $C$ be an independent set of the Kneser graph of type* $\{2, 3\}$ *in* $\mathrm{PG}(6, q)$ *with $q \geq 7$ that has the property that every plane and every solid of* $\mathrm{PG}(6, q)$ *is contained in at most $s(1) = q+1$ flags of $C$. Then*

$$|C| \leq 24q^{10} + 79q^9 + 155q^8 + 210q^7 + 216q^6 + 187q^5 + 140q^4 + 93q^3 + 51q^2 + 22q + 5.$$

*Proof.* Let $P_1$ and $P_2$ be distinct points of $\mathrm{PG}(6, q)$ such that $|\Delta_{P_1}(C)|, |\Delta_{P_2}(C)| \geq |\Delta_P(C)|$ for all points $P \neq P_1$. If every flag $(E, S) \in C$ satisfies $\langle P_1, P_2\rangle \cap E \neq \varnothing$, then

$$\begin{aligned} |C| &\leq (s(2, 6) - s(1, -1, 2, 6)) \cdot s(1) \\ &= q^{10} + 3q^9 + 5q^8 + 7q^7 + 8q^6 + 8q^5 + 7q^4 + 5q^3 + 3q^2 + 2q + 1 \end{aligned}$$

since there are $s(2, 6) - s(1, -1, 2, 6)$ planes that meet the line $\langle P_1, P_2\rangle$ and since every plane lies in at most $q+1$ flags of $C$. Therefore, we may assume that $C$ contains a flag $f = (E, S)$ with $\langle P_1, P_2\rangle \cap E = \varnothing$ and thus $\dim(S \cap \langle P_1, P_2\rangle) \leq 0$.

Every flag $(E', S') \in C$ either satisfies $E' \cap S \neq \varnothing$ or $E' \cap S = \varnothing \neq S' \cap E$. Lemma 3.3 shows that at most

$$(q^8 + 2q^7 + 4q^6 + 5q^5 + 6q^4 + 5q^3 + 4q^2 + 2q + 1) \cdot s(1) \tag{5}$$

flags $(E', S')$ of $C$ satisfy $E' \cap S = \varnothing \neq S' \cap E$. Before we count all flags $f' = (E', S')$ with $E' \cap S \neq \varnothing$ we note that we either have

$$|\Delta_P(C)| \leq |\Delta_{P_2}(C)| \leq 6q^7 + 16q^6 + 27q^5 + 35q^4 + 33q^3 + 24q^2 + 14q + 5 \tag{6}$$

for all $P \in \mathrm{PG}(6, q) \setminus \langle P_1, P_2 \rangle$ or

$$|\Delta_{P_1}(C)| \geq |\Delta_{P_2}(C)| > 6q^7 + 16q^6 + 27q^5 + 35q^4 + 33q^3 + 24q^2 + 14q + 5.$$

If the second situation occurs, then Lemma 4.1 provides flags $f_{i,j} \in C$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ required to apply Lemma 4.5 proving

$$|\Delta_P(C)| \leq 24q^7 + 54q^6 + 71q^5 + 67q^4 + 48q^3 + 33q^2 + 22q + 11 \tag{7}$$

for all $P \in \mathrm{PG}(6, q) \setminus \langle P_1, P_2 \rangle$. Since the bound in (7) is weaker than the bound given in (6) we know that it also holds in the first case. In particular, (7) holds for all $P \in S \setminus (S \cap \langle P_1, P_2 \rangle)$. Note that we chose $f$ such that $S \cap \langle P_1, P_2 \rangle$ is at most a point. Now, if $\hat{P} := S \cap \langle P_1, P_2 \rangle \neq \varnothing$, then, since $P_1$ and $P_2$ are distinct, there is an index $i \in \{1, 2\}$ such that $\hat{P} \neq P_i$ and, using the flags $f_{i,1}$, $f_{i,2}$ and $f_{i,3}$, we may apply Lemma 4.3 to see that

$$|\Delta_{\hat{P}}(C)| \leq 3q^8 + 12q^7 + 21q^6 + 28q^5 + 26q^4 + 18q^3 + 12q^2 + 8q + 4,$$

which is weaker than the bound in (7) for $q \geq 7$. Therefore, the number of all flags $(E', S')$ of $C$ with $E' \cap S \neq \varnothing$ is at most

$$\begin{aligned}
&(s(3) - 1) \cdot (24q^7 + 54q^6 + 71q^5 + 67q^4 + 48q^3 + 33q^2 + 22q + 11) \\
&\quad + 3q^8 + 12q^7 + 21q^6 + 28q^5 + 26q^4 + 18q^3 + 12q^2 + 8q + 4 \\
&\quad = 24q^{10} + 78q^9 + 152q^8 + 204q^7 + 207q^6 + 176q^5 + 129q^4 + 84q^3 + 45q^2 + 19q + 4.
\end{aligned}$$

Together with the upper bound in (5) for the remaining flags of $C$, this provides the claimed upper bound on the cardinality of $C$. □

## 5. The second special case

In this section we generalize the results of Section 4 to Kneser graphs of type {2, 3} in PG(6, $q$) with the property that every plane and every solid of PG(6, $q$) occurs in at most $q^2 + q + 1$ flags of $C$. Let Γ be a Kneser graph with that property.

**Lemma 5.1.** *Let $E$ be a plane and suppose that the solids $S$ with $(E, S) \in C$ span a subspace $H$ of dimension at least* 5*. Suppose also that every plane of* PG(6, $q$) *occurs in at most $s(2) = q^2 + q + 1$ flags of $C$. Then the number of flags $(E', S') \in C$ with $E' \cap E = \varnothing$ is at most $s(1, 4) \cdot s(2) \cdot (s(2) + 1)$.*

*Proof.* Let $M$ be the set consisting of all flags $(E', S')$ of $C$ such that $S' \cap E = \varnothing$ and let $N$ be the set consisting of all flags $(E', S')$ of $C$ such that $S' \cap E \neq \varnothing$ and $E' \cap E = \varnothing$. Every flag $(E', S') \in C$ with $E' \cap E = \varnothing$ lies in $M \cup N$. Lemma 3.3 applied with $\xi = s(2)$ shows that $|N| \leq s(1, 4) \cdot s(2)^2$. For an upper bound on the number of flags in $M$, we let $\mathcal{E}$ denote the set of all planes that occur in a flag of $M$. The hypothesis of this lemma shows that $|M| \leq |\mathcal{E}| \cdot s(2)$. In order to prove the statement, it remains to show that $|\mathcal{E}| \leq s(1, 4)$.

Consider $E' \in \mathcal{E}$. Let $S'$ be a solid with $(E', S') \in M$. Then $S' \cap E = E' \cap E = \varnothing$. Since $C$ is independent and since $S' \cap E = \varnothing$ the plane $E'$ meets every solid $S$ for which $(E, S) \in C$. Then every such solid $S$ is spanned by $E$ and a point of $E'$, so $H \subseteq \langle E, E' \rangle$. Since $H$ has dimension at least 5, it follows that $H$ has dimension 5 and that $H = \langle E, E' \rangle$ for all $E' \in \mathcal{E}$. Lemma 3.1 shows that $|\mathcal{E}| \leq s(1, 4)$. □

**Proposition 5.2.** *Let $C$ be an independent set of the Kneser graph of type* {2, 3} *in* PG(6, $q$) *with $q \geq 8$ that has the property that every plane and every solid of* PG(6, $q$) *occurs in at most $s(2) = q^2 + q + 1$ flags of $C$. Then*

$$|C| \leq 24q^{10} + 79q^9 + 155q^8 + 210q^7 + 218q^6 + 189q^5 + 142q^4 + 95q^3 + 53q^2 + 22q + 5.$$

*Proof.* Let $\mathcal{E}$ be the set consisting of all planes of PG(6, $q$) that lie in at least $q + 2$ flags of $C$, and let $\mathcal{S}$ be the set consisting of all solids of PG(6, $q$) that lie in at least $q + 2$ flags of $C$. We distinguish three cases.

Case 1. We assume that $|\mathcal{E}| \leq s(4)$ and $|\mathcal{S}| \leq s(4)$. In this case we choose a subset $C'$ of $C$ such that every plane and every solid of $C'$ lies in at most $q + 1$ flags of $C'$. Since every plane and solid lies in at most $q^2 + q + 1$ flags of $C$, we can find such a subset with $|C'| \geq |C| - (|\mathcal{E}| + |\mathcal{S}|)q^2$ and then $|C| \leq |C'| + 2 \cdot s(4) \cdot q^2$. Now the statement follows by applying Proposition 4.6 to $C'$.

Case 2. We assume that $|\mathcal{E}| > s(4)$. Lemma 2.2 proves the existence of planes $E_1, E_2 \in \mathcal{E}$ satisfying $\dim(E_1 \cap E_2) \leq 0$. From Lemma 5.1 we know that at most

$$2 \cdot s(1, 4) \cdot s(2) \cdot (s(2) + 1) \tag{8}$$

flags $(E, S) \in C$ satisfy $E \cap E_1 = \varnothing$ or $E \cap E_2 = \varnothing$. It remains to find an upper bound on the number of flags in $C$ whose planes meet both $E_1$ and $E_2$. Therefore, we count the number of planes of PG(6, $q$) that meet $E_1$ and $E_2$. First consider the case that $E_1 \cap E_2$ is a point $Q$. In this case there are $s(0, 2, 6)$ planes on $Q$, there are $(s(2) - 1)^2(s(1, 2, 6) - (2 \cdot s(0, 1, 2) - 1))$ planes that do not contain $Q$ and meet both $E_1$ and $E_2$ in exactly one point and there are $2 \cdot s(0, -1, 1, 2)(s(2) - 1)$

planes that do not contain $Q$ and meet $E_1$ or $E_2$ in a line and the other plane in a point. Thus, in this case the number of planes that meet $E_1$ and $E_2$ is equal to

$$n := 2q^8 + 4q^7 + 6q^6 + 4q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1.$$

If $E_1$ and $E_2$ are skew than a similar calculation shows that there are even less than $n$ planes that meet $E_1$ and $E_2$, so that $n$ is an upper bound for the number of planes that meet $E_1$ and $E_2$ in both situations. Since every plane lies in at most $s(2)$ flags of $C$, it follows that there are at most $n \cdot s(2)$ flags $(E, S) \in C$ such that $E$ meets $E_1$ and $E_2$. Together with the count in (8) we find $|C| \le n \cdot s(2) + 2 \cdot s(1, 4) \cdot s(2) \cdot (s(2) + 1)$ and this bound is better than the one in the statement.

Case 3. We assume that $|\mathcal{S}| > s(4)$. This is dual to Case 2. □

## 6. Proof of the theorem

In this section, $\Gamma$ denotes the Kneser graph of plane-solid flags in PG(6, $q$) and $C$ denotes a maximal independent set of $\Gamma$.

**Lemma 6.1.** (i) *Every solid $S$ of* PG(6, $q$) *has a subspace $U$ with the following property*: *For every plane $E$ of $S$ we have $(E, S) \in C$ if and only if $U \subseteq E$.*

(ii) *For every plane $E$ of* PG(6, $q$) *there exists a subspace $U$ containing $E$ with the following property*: *For every solid $S$ on $E$ we have $(E, S) \in C$ if and only if $S \subseteq U$.*

*Proof.* Since the two statements are dual to each other, it suffices to prove the first statement. Thus consider a plane $E$ and let $\mathcal{S}$ be the set of solids $S$ satisfying $E \subseteq S$ and $(E, S) \in C$. In the quotient space PG(6, $q$)$/E$ the set $\{S/E \mid S \in \mathcal{S}\}$ is a set of points and we have to show that this set is a subspace of PG(6, $q$)$/E$. In that regard, it is sufficient to show for any two distinct solids $S_1, S_2 \in \mathcal{S}$ and every solid $S$ with $E \subseteq S \subseteq \langle S_1, S_2\rangle$ we have $S \in \mathcal{S}$. Let $S$ be such a solid. If $(E', S')$ is any flag of $C$ then either $S' \cap E \neq \varnothing$ or $E'$ meets $S_1 \setminus E$ and $S_2 \setminus E$. In the second case $E'$ meets $\langle S_1, S_2\rangle$ in a line and hence $E'$ meets $S$. Thus for every $(E', S') \in C$ we have $E \cap S' \neq \varnothing$ or $E' \cap S \neq \varnothing$. This shows that $C \cup \{(E, S)\}$ is an independent set of $\Gamma$ and since $C$ is a maximal independent set we have $(E, S) \in C$, that is, $S \in \mathcal{S}$. □

**Definition 6.2.** A plane $E$ will be called *saturated* (for $C$) if $(E, S) \in C$ for all solids $S$ of PG(6, $q$) that contain $E$. Dually, a solid $S$ will be called *saturated* (for $C$), if $(E, S) \in C$ for all planes $E$ of $S$.

**Lemma 6.3.** (i) *For every saturated solid $S$ and every flag $(E', S') \in C$ we have $E' \cap S \neq \varnothing$.*

(ii) *If $S$ is a solid with $S \cap E' \neq \varnothing$ for all flags $(E', C')$ of $C$, then $S$ is saturated.*

(iii) *If $S$ and $S'$ are saturated solids, then* $\dim(S \cap S') \geq 1$.

(iv) *Let $H$ be a hyperplane of* $\mathrm{PG}(6, q)$ *and suppose that $E \subseteq H$ for all flags $(E, S) \in C$. Then every solid of $H$ is saturated.*

*Proof.* (i) Suppose that there is a flag $(E', S') \in C$ with $E' \cap S = \varnothing$. Since $\mathrm{PG}(6, q)$ has dimension 6, it follows $S' \cap S$ is a point $P$ with $P \notin E'$. Let $E$ be a plane of $S$ with $P \notin E$. Then $E \cap S' = \varnothing$. As $S$ is a saturated solid we have $(E, S) \in C$. But then $(E, S)$ and $(E', S')$ are flags of the independent set $C$ with $E \cap S' = \varnothing$ and $E' \cap S = \varnothing$, a contradiction.

(ii) Let $E$ be a plane of $S$. We have to show that $(E, S) \in C$. Since $S \cap E' \neq \varnothing$ for every flag $(E', S)$ of $C$, the set $C \cup \{(E, S)\}$ is independent. Maximality of $C$ implies $(E, S) \in C$.

(iii) Assume to the contrary that $S$ and $S'$ only meet in a point $P$. Choose planes $E$ of $S$ and $E'$ of $S'$ with $P \notin E, E'$. Then $S \cap E' = \varnothing = S' \cap E$. Hence $(E, S)$ and $(E', S')$ are adjacent elements of the Kneser graph $\Gamma$. As $C$ is independent, this is a contradiction.

(iv) Let $S$ be a solid of $H$. The dimension formula shows that $S \cap E \neq \varnothing$ for all planes $E$ of $H$. Therefore part (ii) shows that $S$ is saturated. □

**Lemma 6.4.** *Let $C$ be a maximal independent set of* $\Gamma$. *If there are more than $c := q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$ saturated solids for $C$, then $C = \Lambda(H, \mathcal{E})$ for some hyperplane $H$ and some maximal independent set $\mathcal{E}$ of the Kneser graph of planes of $H$ (cf. Example 1.1).*

*Proof.* Let $\mathcal{S}$ be the set of saturated solids in $\Pi_3(C)$. We have $c > q^6 + 2q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1$ and according to Lemma 6.3(iii) we have $\dim(S_1 \cap S_2) \geq 1$ for all $S_1, S_2 \in \mathcal{S}$. Result 2.3 shows that there exists a hyperplane $H$ containing all saturated solids. If there would be a flag $(E, S) \in C$ such that $E \not\subseteq H$, then according to Lemma 6.3(i) all solids of $\mathcal{S}$ would have nonempty intersection with the line $E \cap H$ and thus

$$|\mathcal{S}| \leq s(3, 5) - s(1, -1, 3, 5) = q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1,$$

which is a contradiction. Hence $E \subseteq H$ for all planes $E \in \Pi_2(C)$. Lemma 6.3(iv) shows that all solids of $H$ are saturated. This means that $\Lambda(H, \varnothing) \subseteq C$. Proposition 3.2 now proves the statement. □

**Theorem 6.5.** *Suppose that $q \geq 8$ and that $C$ is a maximal independent set in* $\Gamma$ *with*

$$|C| > 26q^{10} + 83q^9 + 159q^8 + 216q^7 + 222q^6 + 193q^5 + 144q^4 + 97q^3 + 53q^2 + 22q + 5.$$

*Then $C = \Lambda(H, \mathcal{E})$ for a hyperplane $H$ and a maximal set $\mathcal{E}$ of mutually intersecting planes of $H$, or $C = \Lambda(P, \mathcal{S})$ for a point $P$ and a maximal set $\mathcal{S}$ of solids on $P$ any two of which share at least a line.*

*Proof.* The class of examples described in Example 1.1 is closed under duality. In view of Lemma 6.4 we may assume that there exists at most $c := q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$ saturated solids, and, dually, that there are at most $c$ saturated planes. For every saturated plane $E$ choose one hyperplane $H_E$ on $E$, and for every saturated solid $S$ choose a point $P_S$ of $S$. Let $C'$ be the subset of $C$ that is obtained from $C$ by removing all flags $(E, S)$ such that $E$ is saturated and $S$ is not contained in $H_E$ and by removing all flags $(E, S)$ such that $S$ is saturated and $E$ does not contain $P_S$. Then $|C'| \geq |C| - 2cq^3$, that is

$$|C| \leq 2(q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1) \cdot q^3 + |C'|. \tag{9}$$

Lemma 6.1 shows that every plane $E$ which is not saturated for $C$ has the property that the solids $S$ with $(E, S) \in C$ span a proper subspace of PG(6, $q$). Therefore the construction of $C'$ implies that every plane $E$ has the property that the solids $S$ with $(E, S) \in C'$ span a proper subspace of PG(6, $q$). Consequently every plane of PG(6, $q$) lies in at most $q^2 + q + 1$ flags of $C'$. Dually, every solid $S$ of PG(6, $q$) lies in at most $q^2 + q + 1$ flags of $C'$. Therefore Proposition 5.2 proves an upper bound for $|C'|$. Now (9) proves the bound for $|C|$ that is given in the statement. □

**Corollary 6.6.** *For $q > 27$ the maximal independent set in the Kneser graph of flags of type $\{2, 3\}$ in PG(6, $q$) with $|C| \geq q^{11} + 2q^{10}$ are the independent sets described in Example 1.1.*

Theorem 1.3 follows from this corollary and Proposition 3.2(ii) for $q > 27$ and for $q = 27$ consider Remark 4 on page 41.

## 7. Bounds on the chromatic number of Γ

Let $\Gamma$ be the Kneser graph of flags of type $\{2, 3\}$ in PG(6, $q$). The chromatic number of $\Gamma$ is the smallest number $\chi$ such that the vertex set can be represented as the union of $\chi$ independent sets. Using the upper bound $\alpha$ for the size of such an independent set this immediately gives the bound $\chi \geq \frac{n}{\alpha}$. With the upper bound from Theorem 1.3 we find

**Proposition 7.1.** *For $q \geq 27$, the chromatic number of $\Gamma$ is at least $q^4 - q^2 + 2q + 1$.*

On the other hand, if $V$ is a subspace of dimension 4 of PG(6, $q$), then the independent sets $\Lambda(P, \varnothing)$ with $P \in V$ comprise all vertices of $\Gamma$, so we have the trivial upper bound $\chi \leq s(4) = q^4 + q^3 + q^2 + q + 1$. We can slightly improve this bound using the following construction.

**Proposition 7.2.** *The chromatic number $\chi$ of $\Gamma$ satisfies $\chi \leq q^4 + q^3 + q^2 + 1$.*

*Proof.* Consider a point $P$, a line $l$, a plane $E$ and a 4-space $V$ that are mutually incident. Let $Q$ be a point of $V$ that is not in $E$. Let $l_1, \dots, l_q$ be the lines of plane $\langle l, Q\rangle$ with $P \in l_i$ and $Q \notin l_i$, let $E_1, \dots, E_q$ be the planes of $\langle E, Q\rangle$ with $l \subseteq E_i$ and $Q \notin E_i$, and let $S_1, \dots, S_q$ be the solids of $V$ with $E \subseteq S_i$ and $Q \notin S_i$. For $i \in \{1, \dots, q\}$ put $M_i := l_i \cup (E_i \setminus l) \cup (S_i \setminus E)$. Then $|M_i| = q^3 + q^2 + q + 1$ with $M_i \cap M_j = P$ for distinct $i, j \in \{1, \dots, q\}$ and the union of the sets $M_1, \dots, M_q$ is $\{P\} \cup V \setminus \langle P, Q\rangle$. Let $\{Q_1, \dots, Q_q\} = \langle P, Q\rangle \setminus \{P\}$ and consider the independent set $\Lambda(X, \langle X, Q_i\rangle)$ for $X \in M_i$ and $i \in \{1, \dots, q\}$. Then for $i \in \{1, \dots, q\}$ all lines of $V$ on $Q_i$ occur in one of these sets and every solid that contains $Q_i$ contains a line $\langle X, Q_i\rangle$ with $X \in M_i$. Therefore the union of the sets $\Lambda(X, \langle X, Q_i\rangle)$ for $i \in \{1, \dots, q\}$ covers all vertices of $\Gamma$. □

In some situations, having a Hilton-Milner result for the size of the independent sets used (here these are Erdős–Ko–Rado sets in PG$(6, q)$) is a good tool to determine the chromatic number of a graph exactly and with little effort. However, we are convinced that this is not the case in this situation. The reason is, that the second largest independent sets are still almost as large as the largest independent sets, as we have stated in Remark 2 on page 41.

However, one could use the fact that every independent set which is essentially different from the largest examples (that is, different from those given in Example 1.1) is much smaller. Indeed, we have given this some thought, but are convinced that this is not quite simple and would go far beyond the scope of this work.

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