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Johnny Henderson, John Ehrke and Curtis Kunkel

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Five-point boundary value problems for n-th order differential equations by solution matching

Johnny Henderson, John Ehrke and Curtis Kunkel

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For the ordinary differential equation

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}), \qquad n \geq 3,$$

solutions of three-point boundary value problems on $[a, b]$ are matched with solutions of three-point boundary value problems on $[b, c]$ to obtain solutions satisfying five-point boundary conditions on $[a, c]$.

1. Introduction

We are concerned with the existence and uniqueness of solutions of boundary value problems on an interval $[a, c]$ for the n-th order ordinary differential equation

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}), \tag{1}$$

satisfying the five-point boundary conditions

$$\begin{aligned} y(a) - y(x_1) = y_1, \qquad y^{(i-1)}(b) = y_{i+1}, \quad 1 \leq i \leq n-2, \\ y(x_2) - y(c) = y_n, \end{aligned} \tag{2}$$

where $a < x_1 < b < x_2 < c$ and $y_1, \dots, y_n \in \mathbb{R}$.

It is assumed throughout that $f : [a, c] \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that solutions of initial value problems for (1) are unique and exist on all of $[a, c]$. Moreover, the points $a < x_1 < b < x_2 < c$ are fixed throughout.

Nonlocal boundary value problems, for which the number of boundary points is possibly greater than the order of the ordinary differential equation, have received considerable interest. For a small sample of such works, we refer the reader to the papers by Bai and Fang [2003], Gupta [1997], Gupta and Trofimchuk [1998], Infante [2005], Ma [1997; 2002] and Webb [2005].

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Monotonicity conditions will be imposed on f. Sufficient conditions will be given such that, if $y_1(x)$ is a solution of a three-point boundary value problem on $[a, b]$, and if $y_2(x)$ is a solution of another three-point boundary value problem on $[b, c]$, then $y(x)$ defined by

$$y(x) = \begin{cases} y_1(x), & a \le x \le b, \\ y_2(x), & b \le x \le c, \end{cases}$$

will be a desired unique solution of (1), (2). In particular, a monotonicity condition is imposed on $f(x, r_1, \ldots, r_n)$ insuring that each three-point boundary value problem for (1) satisfying any one of the following conditions:

$$y(a) - y(x_1) = y_1, \qquad y^{(i-1)}(b) = y_{i+1}, \quad 1 \le i \le n-2, \\ y^{(n-2)}(b) = m, \quad m \in \mathbb{R}, \tag{3}$$

$$y(a) - y(x_1) = y_1, \qquad y^{(i-1)}(b) = y_{i+1}, \quad 1 \le i \le n-2, \\ y^{(n-1)}(b) = m, \quad m \in \mathbb{R}, \tag{4}$$

$$y^{(i-1)}(b) = y_{i+1}, \qquad 1 \le i \le n-2, \quad y^{(n-2)}(b) = m, \\ y(x_2) - y(c) = y_n, \quad m \in \mathbb{R} \tag{5}$$

or

$$y^{(i-1)}(b) = y_{i+1}, \qquad 1 \le i \le n-2, \quad y^{(n-1)}(b) = m, \\ y(x_2) - y(c) = y_n, \quad m \in \mathbb{R} \tag{6}$$

has at most one solution.

We will impose an additional hypothesis that solutions for (1) satisfying any of (3), (4), (5) or (6) exist. Then we will construct a unique solution of (1), (2).

Solution matching techniques were first used by Bailey et al. [1968]. They considered solutions of two-point boundary value problems for the second order equation $y''(x) = f(x, y(x), y'(x))$ by matching solution of initial value problems. Since then, there have been numerous papers in which solutions of two-point boundary value problems on $[a, b]$ were matched with solutions of two-point boundary value problems on $[b, c]$ to obtain solutions of three-point boundary value problems on $[a, c]$. See, for example [Barr and Miletta 1974; Das and Lalli 1981; Henderson 1983; Moorti and Garner 1978; Rao et al. 1981]. In 1973, Barr and Sherman [1973] used solution matching techniques to obtain solutions of three-point boundary value problems for third order differential equations from solutions of two-point problems. They also generalized their results to equations of arbitrary order by obtaining solutions of n-th equations. More recently, Henderson and Prasad [2001] and Eggensperger et al. [2004] used matching methods for solutions of multipoint boundary value problems on time scales. Finally, Henderson and

Tisdale [2005] adapted the matching methods to obtain solutions of five-point problems for third order equations. The present work extends the results of Henderson and Tisdale [2005] to n-th order five-point boundary value problems (1), (2) on $[a, c]$.

The monotonicity hypothesis on f which will play a fundamental role in uniqueness of solutions (and later existence as well), is given by:

(A) For all $w \in \mathbb{R}$,

$$f(x, v_1, \dots, v_{n-2}, v_{n-1}, w) > f(x, u_1, \dots, u_{n-2}, u_{n-1}, w),$$

- (a) when $x \in (a, b]$, $(-1)^{n-i} u_i \geq (-1)^{n-i} v_i$, $1 \leq i \leq n-2$, and $v_{n-1} > u_{n-1}$, or
- (b) when $x \in [b, c)$, $v_i \geq u_i$, $1 \leq i \leq n-2$, and $v_{n-1} > u_{n-1}$.

2. Uniqueness of solutions

In this section, we establish that under condition (A) solutions of the three-point boundary value problems, as well as the five-point problem are unique when they exist.

Theorem 2.1. *Let $y_1, \dots, y_n \in \mathbb{R}$ be given and assume condition* (A) *is satisfied. Then, given $m \in \mathbb{R}$, each of the boundary value problems for* (1) *satisfying any of conditions* (3), (4), (5) *or* (6) *has at most one solution.*

Proof. We will establish the result only for (1), (3). Arguments for the other boundary value problems are very similar.

In order to reach a contradiction, we assume that for some $m \in \mathbb{R}$, there are distinct solutions, α and β, of (1), (3), and set $w = \alpha - \beta$. Then

$$w(a) - w(x_1) = w^{(i-1)}(b) = 0, \qquad 1 \leq i \leq n-1.$$

By the uniqueness of solutions of initial value problems for (1), we may assume with no loss of generality that $w^{(n-1)}(b) < 0$. It follows from the boundary conditions satisfied by w that there exists $a < r < b$ such that

$$w^{(n-1)}(r) = 0 \quad \text{and} \quad w^{(n-1)}(x) < 0 \text{ on } (r, b].$$

Since $w^{(i-1)}(b) = 0,\ 1 \leq i \leq n-1$, it follows in turn that

$$(-1)^{n-j} w^{(j)}(x) > 0, \quad 0 \leq j \leq n-2, \text{ on } [r, b).$$

This leads to

$$w^{(n)}(r) = \lim_{x \to r^+} \frac{w^{(n-1)}(x)}{x - r} \leq 0.$$

However, from condition (A), we have

$$\begin{aligned} w^{(n)}(r) &= \alpha^{(n)}(r) - \beta^{(n)}(r) \\ &= f(r, \alpha(r), \alpha'(r), \dots, \alpha^{(n-2)}(r), \alpha^{(n-1)}(r)) \\ &\qquad - f(r, \beta(r), \beta'(r), \dots, \beta^{(n-2)}(r), \beta^{(n-1)}(r)) \\ &= f(r, \alpha(r), \alpha'(r), \dots, \alpha^{(n-2)}(r), \alpha^{(n-1)}(r)) \\ &\qquad - f(r, \beta(r), \beta'(r), \dots, \beta^{(n-2)}(r), \alpha^{(n-1)}(r)) \\ &> 0, \end{aligned}$$

which is a contradiction. Thus, (1), (3) has at most one solution. The proof is complete. □

Theorem 2.2. *Let $y_1, \dots, y_n \in \mathbb{R}$ be given. Assume condition* (A) *is satisfied. Then, the boundary value problem* (1), (2) *has at most one solution.*

Proof. Again, we argue by contradiction. Assume for some values $y_1, \dots, y_n \in \mathbb{R}$, there exist distinct solutions α and β of (1) and (2). Also, let $w = \alpha - \beta$. Then

$$w(a) - w(x_1) = w^{(i-1)}(b) = w(x_2) - w(c) = 0, \quad 1 \le i \le n-2.$$

By Theorem 2.1, $w^{(n-2)}(b) \neq 0$ and $w^{(n-1)}(b) \neq 0$. We assume with no loss of generality that $w^{(n-2)}(b) > 0$. Then, from the boundary conditions satisfied by w, there are points $a < r_1 < b < r_2 < c$ so that

$$w^{(n-2)}(r_1) = w^{(n-2)}(r_2) = 0, \quad \text{and} \quad w^{(n-2)}(x) > 0 \text{ on } (r_1, r_2).$$

There are two cases to analyze, that is, $w^{(n-1)}(b) > 0$ and $w^{(n-1)}(b) < 0$. The arguments for the two cases are completely analagous, therefore we will treat only the first case $w^{(n-1)}(b) > 0$. In view of the fact that $w^{(n-2)}(b) > 0$ and $w^{(n-2)}(r_2) = 0$, there exists $b < r < r_2$ so that

$$w^{(n-1)}(r) = 0, \quad \text{and} \quad w^{(n-1)}(x) > 0 \text{ on } [b, r).$$

Then

$$w^{(j)}(x) > 0, \quad 0 \le j \le n-2, \text{ on } (b, r].$$

This leads to

$$w^{(n)}(r) = \lim_{x \to r^-} \frac{w^{(n-1)}(x)}{x - r} \le 0.$$

However, again from condition (A), we have

$$\begin{aligned} w^{(n)}(r) &= \alpha^{(n)}(r) - \beta^{(n)}(r) \\ &= f(r, \alpha(r), \alpha'(r), \ldots, \alpha^{(n-2)}(r), \alpha^{(n-1)}(r)) \\ &\qquad - f(r, \beta(r), \beta'(r), \ldots, \beta^{(n-2)}(r), \beta^{(n-1)}(r)) \\ &= f(r, \alpha(r), \alpha'(r), \ldots, \alpha^{(n-2)}(r), \alpha^{(n-1)}(r)) \\ &\qquad - f(r, \beta(r), \beta'(r), \ldots, \beta^{(n-2)}(r), \alpha^{(n-1)}(r)) \\ &> 0, \end{aligned}$$

which contradicts the initial assumption. Thus, (1), (2) has at most one solution, and the proof is complete. □

3. Existence of solutions

In this section, we show that solutions of (1) satisfying each of (3), (4), (5) and (6) are monotone functions of m. Then, we use these monotonicity properties to obtain solutions of (1), (2).

For notation purposes, given $m \in \mathbb{R}$, let $\alpha(x,m)$, $u(x,m)$, $\beta(x,m)$ and $v(x,m)$ denote the solutions, when they exist, of the boundary value problems for (1) satisfying, respectively, (3), (4), (5) and (6).

Theorem 3.1. *Suppose that the monotonicity hypothesis* (A) *is satisfied and that, for each $m \in \mathbb{R}$, there exist solutions of* (1) *satisfying each of the conditions* (3), (4), (5) *and* (6). *Then, $\alpha^{(n-1)}(b,m)$ and $\beta^{(n-1)}(b,m)$ are, respectively, strictly increasing and decreasing functions of m with ranges all of $\mathbb{R}$.*

Proof. The "strictness" of the conclusion arises from Theorem 2.1. Let $m_1 > m_2$ and let $w(x) = \alpha(x,m_1) - \alpha(x,m_2)$. Then,

$$w(x_1) - w(a) = w^{(i-1)}(b) = 0,\ 1 \le i \le n-2,\ w^{(n-2)}(b) = m_1 - m_2 > 0$$

and $w^{(n-1)}(b) \neq 0$.

Contrary to the conclusion, assume $w^{(n-1)}(b) < 0$. Since there exists $a < r_1 < b$ so that $w^{(n-2)}(r_1) = 0$ and $w^{(n-2)}(x) > 0$ on $(r_1, b]$, it follows that there exists $r_1 < r_2 < b$ such that

$$w^{(n-1)}(r_2) = 0 \quad \text{and} \quad w^{(n-1)}(x) < 0 \text{ on } (r_2, b].$$

We also have

$$(-1)^{n-j} w^{(j)}(x) > 0, \quad 0 \le j \le n-2 \text{ on } [r_2, b).$$

As in the other proofs above, we arrive at the same contradiction, that is, $w^{(n)}(r_2) \le 0$ and $w^{(n)}(r_2) > 0$. Thus, $w^{(n-1)}(b) > 0$ and, as a consequence, $\alpha^{(n-1)}(b,m)$ is a strictly increasing function of m.

We next argue that $\{\alpha^{(n-1)}(b, m) \mid m \in \mathbb{R}\} = \mathbb{R}$. Let $k \in \mathbb{R}$ and consider the solution $u(x, k)$ of (1), (4) with u as defined above. Consider also the solution $\alpha(x, u^{(n-2)}(b, k))$ of (1), (3). Then $\alpha(x, u^{(n-2)}(b, k))$ and $u(x, k)$ are solutions of the same type boundary value problems (1), (3). Hence by Theorem 2.1, the functions are identical. Therefore,

$$\alpha^{(n-1)}(b, u^{(n-2)}(b, k)) = u^{(n-1)}(b, k) = k,$$

and the range of $\alpha^{(n-1)}(b, m)$, as a function of m, is the set of real numbers.

The argument for $\beta^{(n-1)}(b, m)$ is quite similar. This completes the proof. □

In a similar way, we also have a monotonicity result on $(n-2)$-derivatives of $u(x, m)$ and $v(x, m)$.

Theorem 3.2. *Assume the hypotheses of Theorem 3.1. Then, $u^{(n-2)}(b, m)$ and $v^{(n-2)}(b, m)$ are, respectively, strictly increasing and decreasing functions of m with ranges all of $\mathbb{R}$.*

We now provide our existence result.

Theorem 3.3. *Assume the hypotheses of Theorem 3.1. Then (1), (2) has a unique solution.*

Proof. The existence is immediate from either Theorem 3.1 or Theorem 3.2. Making use of Theorem 3.2, there exists a unique $m_0 \in \mathbb{R}$ such that $u^{(n-2)}(b, m_0) = v^{(n-2)}(b, m_0)$. Then

$$y(x) = \begin{cases} u(x, m_0), & a \le x \le b, \\ v(x, m_0), & b \le x \le c, \end{cases}$$

is a solution of (1), (2). By Theorem 2.2, $y(x)$ is the unique solution. The proof is complete. □

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Johnny_Henderson@baylor.edu
Department of Mathematics, Baylor University, Waco, TX 76798-7328, United States
http://www.baylor.edu/math/index.php?id=22228

john.ehrke@acu.edu
Department of Mathematics, Abilene Christian University, Abilene, TX 79699-8012, United States

ckunkel@utm.edu
Department of Mathematics and Statistics, 424 Humanities Building, University of Tennessee at Martin, Martin, TN 38238, United States