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a journal of mathematics

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mathematical sciences publishers

2008 Vol. 1, No. 1

Difference inequalities, comparison tests, and some consequences

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(Communicated by Gerry Ladas)

We study the behavior of nonnegative sequences which satisfy certain difference inequalities. Several comparison tests involving difference inequalities are developed for nonnegative sequences. Using the aforementioned comparison tests, it is possible to determine the global stability and boundedness character for nonnegative solutions of particular rational difference equations in a range of their parameters.

1. Introduction

There has been a significant amount of work done at the University of Rhode Island pertaining to the boundedness character of rational difference equations. Recently a general boundedness result has appeared in the literature. This result proves the boundedness of solutions for many special cases of the k-th order rational difference equation [Camouzis et al. 2006, Theorem 6]. In this paper we intend to generalize this result.

Rather than working with solutions of difference equations, we intend to work with sequences which satisfy recursive inequalities, which we call difference inequalities. This approach bears relevance to the field of difference equations, as every solution to a difference equation satisfies several difference inequalities. The use of difference inequalities provides a general and efficient way to obtain bounds, attracting intervals, and convergence results for a variety of difference equations. These four theorems presented below provide the theoretical groundwork needed.

The first theorem demonstrates that the previously mentioned boundedness result extends to the framework of difference inequalities. In fact Theorem 1 demonstrates a much stronger result. Theorem 1 acts as a comparison test between difference inequalities, showing that any sequence of nonnegative real numbers which satisfies one of the assumed difference inequalities also satisfies a Riccati inequality.

MSC2000: 39A10.

Keywords: difference equation, boundedness, global stability, difference inequality.

A Riccati inequality is a difference inequality of the form

$$x_n \leq \frac{\alpha + \beta \max_{i=1,\dots,k}(x_{n-i})}{A + B \max_{i=1,\dots,k}(x_{n-i})}, \qquad n \geq J,$$

where J is a nonnegative integer. It is easy to see that with $A, B > 0$ and $\alpha, \beta \geq 0$ any nonnegative sequence which satisfies a Riccati inequality is bounded. Theorem 2 will demonstrate something stronger, however, namely a comparison between any nonnegative sequence which satisfies a Riccati inequality, and a solution to a particular associated Riccati equation.

Combining Theorem 1 and Theorem 2 a strong comparison is made between the solutions of certain rational difference equations and the solutions of associated Riccati equations. Using this comparison it is possible to prove global convergence results for certain rational difference equations in a range of their parameters. This global convergence result is given in Theorem 4.

2. Boundedness by iteration

Here the general theorem which proves boundedness through the method of iteration [Camouzis et al. 2006, Theorem 6] is extended to the framework of difference inequalities. Nonnegative sequences which satisfy certain difference inequalities are shown to satisfy a Riccati inequality. A direct result of this is that every solution of every rational difference equation which is bounded through the method of iteration satisfies a Ricatti inequality.

Theorem 1. *Suppose that we have a sequence of nonnegative real numbers* $\{x_n\}_{n=1}^{\infty}$ *which satisfies the inequality*

$$x_n \leq \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{i=1}^{k} B_i x_{n-i}}, \qquad n \geq J, \tag{1}$$

with nonnegative parameters.

Let us define the sets of indices

$$I_\beta = \{i \in \{1, 2, \dots, k\} : \beta_i > 0\} \quad \text{and} \quad I_B = \{i \in \{1, 2, \dots, k\} : B_i > 0\}.$$

Suppose that the following conditions hold true:

(1) $A > 0$.

(2) *There exists a positive integer* η, *such that for every sequence* $\{c_m\}_{m=1}^{\infty}$ *with* $c_m \in I_\beta$, *for* $m = 1, 2, \dots$, *there exist positive integers*, $N_1, N_2 \leq \eta$, *such that* $\sum_{m=N_1}^{N_2} c_m \in I_B$.

Then $\{x_n\}_{n=1}^{\infty}$ *satisfies a Riccati inequality for* $n \geq J + k\eta$.

In particular, if $A \geq \sum_{i=1}^{k} \beta_i$, then, for $n \geq J + k\eta$,

$$x_n \leq \frac{\alpha\eta}{A} + \frac{\left(\sum_{i=1}^{k} \beta_i\right) \max_{i=1,\ldots,k\eta} (x_{n-i})}{A + \min_{i \in I_B} (B_i) \max_{i=1,\ldots,k\eta} (x_{n-i})}, \tag{2}$$

and if $A < \sum_{i=1}^{k} \beta_i$, then for $n \geq J + k\eta$,

$$x_n \leq \left(\frac{\alpha\eta}{A} + \frac{\left(\sum_{i=1}^{k} \beta_i\right) \max_{i=1,\ldots,k\eta} (x_{n-i})}{A + \min_{i \in I_B} (B_i) \max_{i=1,\ldots,k\eta} (x_{n-i})}\right) \left(\frac{\sum_{i=1}^{k} \beta_i}{A}\right)^{\eta-1}. \tag{3}$$

Proof. Let us consider a particular term x_N in $\{x_n\}_{n=1}^{\infty}$. Now for x_N, with $N \geq \max(J, k+1)$, let us define a finite sequence $\{c_m\}_{m=1}^{\tau}$ recursively based on x_N, $\{x_n\}_{n=1}^{\infty}$, and I_β. We will define this sequence by letting

$$c_1 = \min\left(i : x_{N-i} = \max_{\rho \in I_\beta} \left(x_{N-\rho}\right)\right), \tag{4}$$

and supposing that $c_1, \ldots, c_{t-1}$ exist, and $N - \sum_{m=1}^{t-1} c_m \geq \max(J, k+1)$, and then letting

$$c_t = \min\left(i : x_{N-i-\sum_{m=1}^{t-1} c_m} = \max_{\rho \in I_\beta} \left(x_{N-\rho-\sum_{m=1}^{t-1} c_m}\right)\right).$$

Notice that this is a finite sequence, and that τ is the first integer such that $N - \sum_{m=1}^{\tau} c_m < \max(J, k+1)$. This finite sequence $\{c_m\}_{m=1}^{\tau}$ has two noteworthy properties. First it is a finite sequence $\{c_m\}_{m=1}^{\tau}$ with $c_m \in I_\beta$ for $m = 1, \ldots, \tau$; second,

$$\max_{i \in I_\beta} \left(x_{N-i-\sum_{m=1}^{t-1} c_m}\right) = x_{N-\sum_{m=1}^{t} c_m}. \tag{5}$$

We will use these properties to establish bounds for the term x_N.

For the sake of notation let us define $c_0 = 0$. Now we will show by induction that when $N \geq \max(J, k+1)$, for all t such that $1 \leq t \leq \tau$, we have

$$x_N \leq \left(\frac{\alpha}{A}\right)\left(\sum_{D=0}^{t-1}\left(\frac{\sum_{i=1}^{k} \beta_i}{A}\right)^{D}\right) + \frac{\left(\sum_{i=1}^{k} \beta_i\right)^{t} \left(x_{N-\sum_{m=1}^{t} c_m}\right)}{\prod_{L=0}^{t-1}\left(A + \sum_{i=1}^{k} B_i x_{N-i-\sum_{m=0}^{L} c_m}\right)}. \tag{6}$$

First we will establish the base case

$$x_N \leq \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{N-i}}{A + \sum_{i=1}^{k} B_i x_{N-i}} \leq \frac{\alpha}{A} + \frac{\left(\sum_{i=1}^{k} \beta_i\right) \max_{i \in I_\beta} (x_{N-i})}{A + \sum_{i=1}^{k} B_i x_{N-i}}.$$

Now using Equation (4) we get that

$$x_N \leq \frac{\alpha}{A} + \frac{\left(\sum_{i=1}^{k} \beta_i\right) x_{N-c_1}}{A + \sum_{i=1}^{k} B_i x_{N-i}}.$$

This is since $\max_{i \in I_\beta}(x_{N-i}) = x_{N-c_1}$, by (4). Thus (6) holds for $t = 1$. Now suppose (6) holds for $t < \tau$, we must show that it holds for $t + 1$.

$$x_N \leq \left(\frac{\alpha}{A}\right)\left(\sum_{D=0}^{t-1}\left(\frac{\sum_{i=1}^{k}\beta_i}{A}\right)^D\right) + \frac{\left(\sum_{i=1}^{k}\beta_i\right)^t \left(x_{N-\sum_{m=1}^{t} c_m}\right)}{\prod_{L=0}^{t-1}\left(A + \sum_{i=1}^{k} B_i x_{N-i-\sum_{m=0}^{L} c_m}\right)} \tag{7a}$$

$$\leq \left(\frac{\alpha}{A}\right)\left(\sum_{D=0}^{t-1}\left(\frac{\sum_{i=1}^{k}\beta_i}{A}\right)^D\right) + \frac{\left(\sum_{i=1}^{k}\beta_i\right)^t \left(\alpha + \sum_{i=1}^{k}\beta_i x_{N-i-\sum_{m=1}^{t} c_m}\right)}{\prod_{L=0}^{t}\left(A + \sum_{i=1}^{k} B_i x_{N-i-\sum_{m=0}^{L} c_m}\right)} \tag{7b}$$

$$\leq \left(\frac{\alpha}{A}\right)\left(\sum_{D=0}^{t}\left(\frac{\sum_{i=1}^{k}\beta_i}{A}\right)^D\right) + \frac{\left(\sum_{i=1}^{k}\beta_i\right)^{t+1} \left(\max_{i \in I_\beta}\left(x_{N-i-\sum_{m=1}^{t} c_m}\right)\right)}{\prod_{L=0}^{t}\left(A + \sum_{i=1}^{k} B_i x_{N-i-\sum_{m=0}^{L} c_m}\right)} \tag{7c}$$

$$\leq \left(\frac{\alpha}{A}\right)\left(\sum_{D=0}^{t}\left(\frac{\sum_{i=1}^{k}\beta_i}{A}\right)^D\right) + \frac{\left(\sum_{i=1}^{k}\beta_i\right)^{t+1} \left(x_{N-\sum_{m=1}^{t+1} c_m}\right)}{\prod_{L=0}^{t}\left(A + \sum_{i=1}^{k} B_i x_{N-i-\sum_{m=0}^{L} c_m}\right)}. \tag{7d}$$

Our induction assumption is (7a). We get (7b) from (7a) using our original inequality (1). We get (7c) from (7b), since $A > 0$ and our parameters are nonnegative. We get (7d) from (7c) since

$$\max_{i \in I_\beta}\left(x_{N-i-\sum_{m=1}^{t} c_m}\right) = x_{N-\sum_{m=1}^{t+1} c_m},$$

from (5). Thus we have shown that (6) holds for all t such that $1 \leq t \leq \tau$.

Since τ is the first integer such that $N - \sum_{m=1}^{\tau} c_m < \max(J, k+1)$, then

$$N - \max(J, k+1) < \sum_{m=1}^{\tau} c_m < k\tau.$$

Thus $\tau > (N - \max(J, k+1))/k$. So if we choose $N \geq J + k\eta$, where η is the integer η defined in Condition (2), then $\tau \geq \eta$. We know there exist positive integers $N_1, N_2 \leq \eta$, so that $\sum_{m=N_1}^{N_2} c_m \in I_B$; this is from Condition (2) in our original assumptions. Thus

$$\sum_{m=1}^{N_2} c_m = \sum_{m=0}^{N_1-1} c_m + i,$$

for some $i \in I_B$. Since $N_2 \le \eta \le \tau$, by Equation (6)

$$x_N \le \left(\frac{\alpha}{A}\right)\left(\sum_{D=0}^{N_2-1}\left(\frac{\sum_{i=1}^k \beta_i}{A}\right)^D\right) + \frac{\left(\sum_{i=1}^k \beta_i\right)^{N_2}\left(x_{N-\sum_{m=1}^{N_2} c_m}\right)}{\prod_{L=0}^{N_2-1}\left(A+\sum_{i=1}^k B_i x_{N-i-\sum_{m=0}^{L} c_m}\right)} \tag{8a}$$

$$\le \left(\frac{\alpha}{A}\right)\left(\sum_{D=0}^{N_2-1}\left(\frac{\sum_{i=1}^k \beta_i}{A}\right)^D\right) + \frac{\left(\sum_{i=1}^k \beta_i\right)^{N_2}\left(x_{N-\sum_{m=1}^{N_2} c_m}\right)}{A^{N_2-1}\left(A+\sum_{i=1}^k B_i x_{N-i-\sum_{m=0}^{N_1-1} c_m}\right)} \tag{8b}$$

$$\le \left(\frac{\alpha}{A}\right)\left(\sum_{D=0}^{N_2-1}\left(\frac{\sum_{i=1}^k \beta_i}{A}\right)^D\right) + \frac{\left(\sum_{i=1}^k \beta_i\right)^{N_2}\left(x_{N-\sum_{m=1}^{N_2} c_m}\right)}{A^{N_2-1}\left(A+\left(\min_{i\in I_B}(B_i)\right)\left(x_{N-\sum_{m=1}^{N_2} c_m}\right)\right)}. \tag{8c}$$

We get (8a) directly from (6) with $t = N_2$. We get (8b) from (8a), since $A > 0$ and our parameters are nonnegative. This expression is obtained by reducing all of the terms of the product in the denominator of this fraction, except for the term where $L = N_1 - 1$, which is kept as it is needed to establish a bound. We get (8c) from (8b) since

$$\sum_{m=1}^{N_2} c_m = \sum_{m=0}^{N_1-1} c_m + i,$$

for some $i \in I_B$. Now we will consider two cases, namely

$$A \ge \sum_{i=1}^k \beta_i \quad \text{and} \quad A < \sum_{i=1}^k \beta_i.$$

Considering the former case, since $1 \le N_2 \le \eta$, we have that,

$$x_N \le \left(\frac{\alpha}{A}\right)\left(\sum_{D=0}^{N_2-1}\left(\frac{\sum_{i=1}^k \beta_i}{A}\right)^D\right) + \frac{\left(\sum_{i=1}^k \beta_i\right)^{N_2}\left(x_{N-\sum_{m=1}^{N_2} c_m}\right)}{A^{N_2-1}\left(A+\left(\min_{i\in I_B}(B_i)\right)\left(x_{N-\sum_{m=1}^{N_2} c_m}\right)\right)} \tag{9a}$$

$$\le \frac{\alpha\eta}{A} + \frac{\left(\sum_{i=1}^k \beta_i\right) x_{N-\sum_{m=1}^{N_2} c_m}}{A+\left(\min_{i\in I_B}(B_i)\right) x_{N-\sum_{m=1}^{N_2} c_m}}. \tag{9b}$$

Notice that our bound in (9b) is increasing with respect to $x_{N-\sum_{m=1}^{N_2} c_m}$, and that $1 \le \sum_{m=1}^{N_2} c_m \le k\eta$; thus, by (9b),

$$x_N \le \frac{\alpha\eta}{A} + \frac{\left(\sum_{i=1}^k \beta_i\right)\max_{i=1,\ldots,k\eta}(x_{N-i})}{A+\left(\min_{i\in I_B}(B_i)\right)\max_{i=1,\ldots,k\eta}(x_{N-i})},$$

for all $N \ge J + k\eta$. Thus we have shown the inequality (2).

If $A < \sum_{i=1}^{k} \beta_i$, then, since $1 \leq N_2 \leq \eta$, we have,

$$x_N \leq \left(\frac{\alpha}{A}\right)\left(\sum_{D=0}^{N_2-1}\left(\frac{\sum_{i=1}^{k}\beta_i}{A}\right)^{D}\right) + \frac{\left(\sum_{i=1}^{k}\beta_i\right)^{N_2}\left(x_{N-\sum_{m=1}^{N_2} c_m}\right)}{A^{N_2-1}\left(A+\left(\min_{i\in I_B}(B_i)\right)\left(x_{N-\sum_{m=1}^{N_2} c_m}\right)\right)} \tag{10a}$$

$$\leq \left(\frac{\alpha\eta}{A} + \frac{\left(\sum_{i=1}^{k}\beta_i\right) x_{N-\sum_{m=1}^{N_2} c_m}}{A + \left(\min_{i\in I_B}(B_i)\right) x_{N-\sum_{m=1}^{N_2} c_m}}\right)\left(\frac{\sum_{i=1}^{k}\beta_i}{A}\right)^{\eta-1}. \tag{10b}$$

Notice that our bound in Equation (10b) is increasing with respect to $x_{N-\sum_{m=1}^{N_2} c_m}$, and that $1 \leq \sum_{m=1}^{N_2} c_m \leq k\eta$. Thus, by (10b),

$$x_n \leq \left(\frac{\alpha\eta}{A} + \frac{\left(\sum_{i=1}^{k}\beta_i\right)\max_{i=1,\dots,k\eta}(x_{n-i})}{A + \left(\min_{i\in I_B}(B_i)\right)\max_{i=1,\dots,k\eta}(x_{n-i})}\right)\left(\frac{\sum_{i=1}^{k}\beta_i}{A}\right)^{\eta-1},$$

for all $N \geq J + k\eta$. Thus we have shown the inequality (3) and the theorem is proved. □

Theorem 1 immediately establishes the boundedness character for a number of special cases of the k-th order rational difference equation. These boundedness results were completely established in [Camouzis et al. 2006]. For related works, see [Kocić and Ladas 1993; Kulenović and Ladas 2002; Camouzis et al. 2004a; 2004b; 2005a; 2005b; 2006; Ladas 2004; Camouzis and Ladas 2005; Grove and Ladas 2005; Camouzis 2006].

Since Theorem 1 only assumes that the inequality (1) eventually holds for our sequence $\{x_n\}_{n=1}^{\infty}$, it is also possible to quickly establish the boundedness character for several nonautonomous rational difference equations.

3. Comparison tests of the maximum and minimum

The following two theorems deal with comparison tests involving the maximum and minimum. One important consequence of these tests is that when combined with Theorem 1 they allow for the comparison between solutions of certain special cases of the k-th order rational difference equation and solutions of a Riccati type difference equation.

Theorem 2. *Let $g : [0, \infty) \to [0, \infty)$ be defined and increasing for all $x \in [0, \infty)$.*

Suppose that we have a sequence of nonnegative real numbers $\{x_n\}_{n=1}^{\infty}$ which satisfies the inequality, $x_n \leq g(\max(x_{n-1}, \dots, x_{n-k}))$, with $n \geq N$. Let $\{y_n\}_{n=0}^{\infty}$ be a solution of the difference equation $y_n = g(y_{n-1})$, given $n = 1, 2, \dots,$ and with $y_0 = \max(x_{N-1}, \dots, x_{N-k})$, then, for all $n \geq N$,

$$\max(x_{n-1}, \dots, x_{n-k}) \leq \max(y_{\lfloor \frac{n-N}{k} \rfloor}, \dots, y_{n-N}). \tag{11}$$

Proof. This result follows by strong induction. From our assumptions we have that $\max(x_{N-1}, \dots, x_{N-k}) = y_0$. This establishes the base case for $n = N$. Now suppose that

$$\max\left(x_{n-1}, \dots, x_{n-k}\right) \leq \max\left(y_{\lfloor \frac{n-N}{k} \rfloor}, \dots, y_{n-N}\right),$$

for all $N \leq n < J$. Then, for all $N \leq n < J$,

$$x_n \leq g\left(\max\left(x_{n-1}, \dots, x_{n-k}\right)\right) \leq g\left(\max\left(y_{\lfloor \frac{n-N}{k} \rfloor}, \dots, y_{n-N}\right)\right).$$

Since g is defined and increasing for all $x \in [0, \infty)$,

$$x_n \leq \max\left(g\left(y_{\lfloor \frac{n-N}{k} \rfloor}\right), \dots, g\left(y_{n-N}\right)\right) = \max\left(y_{1+\lfloor \frac{n-N}{k} \rfloor}, \dots, y_{n+1-N}\right).$$

From this it follows that,

$$\max\left(x_{J-1}, \dots, x_{J-k}\right) \leq \max_{n=J-1,\dots,J-k}\left(\max\left(y_{1+\lfloor \frac{n-N}{k} \rfloor}, \dots, y_{n+1-N}\right)\right).$$

Thus,

$$\max\left(x_{J-1}, \dots, x_{J-k}\right) \leq \max\left(y_{1+\lfloor \frac{J-k-N}{k} \rfloor}, \dots, y_{J-N}\right) = \max\left(y_{\lfloor \frac{J-N}{k} \rfloor}, \dots, y_{J-N}\right).$$

This proves that Equation (11) holds for J, and completes the proof by induction. □

Theorem 3. *Let $g : [0, \infty) \to [0, \infty)$ be defined and increasing for all $x \in [0, \infty)$.*

Suppose that we have a sequence of nonnegative real numbers $\{x_n\}_{n=1}^{\infty}$ which satisfies the inequality, $x_n \geq g(\min(x_{n-1}, \dots, x_{n-k}))$ with $n \geq N$. Let $\{y_n\}_{n=0}^{\infty}$ be a solution of the difference equation $y_n = g(y_{n-1})$, with $n = 1, 2, \dots,$ and with $y_0 = \min(x_{N-1}, \dots, x_{N-k})$, then for all $n \geq N$,

$$\min\left(x_{n-1}, \dots, x_{n-k}\right) \geq \min\left(y_{\lfloor \frac{n-N}{k} \rfloor}, \dots, y_{n-N}\right). \tag{12}$$

Proof. This result follows by strong induction. From our assumptions we have that $\min(x_{N-1}, \dots, x_{N-k}) = y_0$. This establishes the base case for $n = N$. Now suppose that

$$\min\left(x_{n-1}, \dots, x_{n-k}\right) \geq \min\left(y_{\lfloor \frac{n-N}{k} \rfloor}, \dots, y_{n-N}\right),$$

for all $N \leq n < J$. Then, for all $N \leq n < J$,

$$x_n \geq g\left(\min\left(x_{n-1}, \dots, x_{n-k}\right)\right) \geq g\left(\min\left(y_{\lfloor \frac{n-N}{k} \rfloor}, \dots, y_{n-N}\right)\right).$$

Since g is defined and increasing for all $x \in [0, \infty)$,

$$x_n \geq \min\left(g\left(y_{\lfloor \frac{n-N}{k} \rfloor}\right), \dots, g\left(y_{n-N}\right)\right) = \min\left(y_{1+\lfloor \frac{n-N}{k} \rfloor}, \dots, y_{n+1-N}\right).$$

From this it follows that

$$\min(x_{J-1}, \ldots, x_{J-k}) \geq \min_{n=J-1,\ldots,J-k}\left(\min\left(y_{1+\lfloor \frac{n-N}{k} \rfloor}, \ldots, y_{n+1-N}\right)\right).$$

Thus,

$$\min(x_{J-1}, \ldots, x_{J-k}) \geq \min\left(y_{1+\lfloor \frac{J-k-N}{k} \rfloor}, \ldots, y_{J-N}\right) = \min\left(y_{\lfloor \frac{J-N}{k} \rfloor}, \ldots, y_{J-N}\right).$$

This proves that Equation (12) holds for J, and completes the proof by induction. □

Theorem 2 and its dual Theorem 3 provide a general and useful method for obtaining tighter bounds on both the solutions of difference equations and sequences which satisfy difference inequalities. Indeed using Theorem 2 it is sometimes possible to obtain upper bounds for the solutions of certain difference equations which are arbitrarily close to an equilibrium. The discovery of such bounds coupled with a thorough understanding of semicycle analysis may yield some interesting convergence results. We will leave this idea for future investigation.

4. A convergence result for difference inequalities

Here we will give one example which demonstrates convergence even in the framework of difference inequalities. The convergence result here also settles an open problem in rational difference equations in the case $A = \sum_{i=1}^{k} \beta_i$.

Theorem 4. *Suppose that we have a sequence of nonnegative real numbers* $\{x_n\}_{n=1}^{\infty}$ *which satisfies the inequality*,

$$x_n \leq \frac{\sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{i=1}^{k} B_i x_{n-i}}, \qquad n \geq J,$$

with nonnegative parameters.

Let us define the sets of indices

$$I_\beta = \{i \in \{1, 2, \ldots, k\} : \beta_i > 0\} \quad \text{and} \quad I_B = \{i \in \{1, 2, \ldots, k\} : B_i > 0\}.$$

Suppose that the following conditions hold true:

(1) $A \geq \sum_{i=1}^{k} \beta_i$.

(2) *There exists a positive integer* η, *such that for every sequence* $\{c_m\}_{m=1}^{\infty}$ *with* $c_m \in I_\beta$ *for* $m = 1, 2, \ldots$ *there exists positive integers*, $N_1, N_2 \leq \eta$, *such that* $\sum_{m=N_1}^{N_2} c_m \in I_B$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges to* 0.

Proof. By Theorem 1, for $n \geq J + k\eta$,

$$x_n \leq \frac{\left(\sum_{i=1}^{k} \beta_i\right) \max_{i=1,\dots,k\eta} (x_{n-i})}{A + \left(\min_{i \in I_B} (B_i)\right) \max_{i=1,\dots,k\eta} (x_{n-i})}.$$

Dividing the numerator and denominator by $\sum_{i=1}^{k} \beta_i$, we may rewrite the inequality in the form,

$$x_n \leq \frac{\max_{i=1,\dots,k\eta} (x_{n-i})}{\rho + C \max_{i=1,\dots,k\eta} (x_{n-i})},$$

where $\rho \geq 1$ and $C > 0$. Applying Theorem 2 we get that for $\{y_n\}_{n=0}^{\infty}$, a solution of the difference equation,

$$y_n = \frac{y_{n-1}}{\rho + C y_{n-1}}, \qquad n = 1, 2, \dots, \tag{13}$$

with $y_0 = \max\left(x_{J+k\eta-1}, \dots, x_{J+k\eta-k}\right)$, then for all $n \geq J + k\eta$,

$$\max\left(x_{n-1}, \dots, x_{n-k}\right) \leq \max\left(y_{\lfloor \frac{n-J-k\eta}{k} \rfloor}, \dots, y_{n-J-k\eta}\right).$$

Since $\{y_n\}_{n=0}^{\infty}$ is decreasing and bounded below by zero, $\{y_n\}_{n=0}^{\infty}$ converges. Since the only equilibrium of equation Equation (13) is zero, $\{y_n\}_{n=0}^{\infty}$ converges to zero.

Since $\{y_n\}_{n=0}^{\infty}$ converges to zero, given $\epsilon > 0$, there exists a natural number N sufficiently large so that $y_n < \epsilon$ for all $n \geq N$. Choose D to be a natural number so that $N = \lfloor (D - J - k\eta) / k \rfloor$. Then, for $n \geq D$,

$$x_{n-1} \leq \max(x_{n-1}, \dots, x_{n-k}) \leq \max\left(y_{\lfloor \frac{n-J-k\eta}{k} \rfloor}, \dots, y_{n-J-k\eta}\right) < \max(\epsilon, \dots, \epsilon) = \epsilon.$$

Thus, given $\epsilon > 0$, there exists a natural number D sufficiently large so that $x_n < \epsilon$ for all $n \geq D$. Therefore $\{x_n\}_{n=1}^{\infty}$ converges to 0. □

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Received: 2007-10-28 Accepted: 2007-10-28

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