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On the asymptotic behavior of unions of sets of lengths in atomic monoids

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Let M be a commutative cancellative atomic monoid. We use unions of sets of lengths in M to construct the $\mathscr{V}$-Delta set of M. We first derive some basic properties of $\mathscr{V}$-Delta sets and then show how they offer a method to investigate the asymptotic behavior of the sizes of unions of sets of lengths.

A central focus of number theory is the study of number theoretic functions and their asymptotic behavior. This has led to similar investigations concerning nonunique factorizations in integral domains and moniods. Suppose that M is a commutative cancellative monoid in which each nonunit can be factored into a product of irreducible elements (such a monoid is known as *atomic*). For a nonunit x in M, let $L(x)$ represent the maximum length of a factorization of x into irreducibles and $l(x)$ the minimum such length. The functions

$$\overline{L}(x) = \lim_{k\to\infty} \frac{L(x^n)}{n} \qquad \text{and} \qquad \overline{l}(x) = \lim_{k\to\infty} \frac{l(x^n)}{n}$$

have been studied in the literature by Anderson and Pruis [1991] and Geroldinger and Halter-Koch [1992]. Chapman and Smith [1998] defined the notion of a *generalized set of lengths*, and showed [Chapman and Smith 1993b] that the size of a generalized set of lengths (denoted $\Phi(n)$) satisfies

$$\overline{\Phi}(R) = \lim_{n\to\infty} \frac{\Phi(n)}{n} = \frac{D(G)^2 - 4}{2D(G)}, \tag{1}$$

for a ring of algebraic integers R where $D(G)$ represents Davenport's constant of the ideal class group G of R (the Davenport constant is defined in [Geroldinger and Halter-Koch 2006, Section 3.4]). Since a generalized set of lengths is actually

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a union of certain length sets, we will refer to these sets with the more descriptive term *unions of sets of lengths*. The value $\overline{\Phi}(R)$ has also been explored for various semigroup rings over fields [Anderson et al. 1993, Theorem 3.3]. In this note, we examine the limit $\overline{\Phi}(R)$ in greater detail. By generalizing the well known notion of the Delta set of a monoid M [Geroldinger and Halter-Koch 2006, Section 1.4], we find new bounds for the value $\overline{\Phi}(M)$ which allows us to determine exact calculations in several instances recently addressed in the literature (see Examples 3 and 4). We will begin with a review of the necessary definitions and notations from the theory of nonunique factorizations. The reader is directed to the monograph [Geroldinger and Halter-Koch 2006] for a complete survey of recent results in this area.

Throughout our work, we assume that M is an atomic commutative cancellative monoid with sets $\mathcal{I}(M)$ of irreducible elements and $M^{\bullet}$ of nonunits. The set of lengths of $x \in M^{\bullet}$ is $\mathcal{L}(x) = \{n \mid x = x_1 \cdots x_n \text{ with each } x_i \in \mathcal{I}(M)\}$. Also, define $L(x) = \max \mathcal{L}(x)$ and $l(x) = \min \mathcal{L}(x)$. The quotient $L(x)/l(x)$ is called the *elasticity* of x and the constant $\rho(M) = \sup\left\{\frac{L(x)}{l(x)} \mid x \in M^{\bullet}\right\}$ is known as the *elasticity* of M. A survey of the results in the literature concerning elasticity can be found in [Anderson 1997]. If $\mathcal{L}(x) = \{n_1, \ldots, n_t\}$ with the n_i's listed in increasing order, then the Delta set of x is $\Delta(x) = \{n_i - n_{i-1} \mid 2 \le i \le t\}$. The Delta set of M is then defined as $\Delta(M) = \cup_{x \in M^{\bullet}} \Delta(x)$. If $d = \gcd \Delta(M)$, Geroldinger [1988, Proposition 4] has shown that $d \in \Delta(M)$. Hence, it follows that

$$\{d, qd\} \subseteq \Delta(M) \subseteq \{d, 2d, \ldots, qd\}, \tag{2}$$

for some positive integer q. While the concept of the Delta set of a monoid M has been widely studied, there are few exact computations of specific Delta sets in the literature. If $\mathcal{B}(\mathbb{Z}_n)$ represents the block monoid ([Geroldinger and Halter-Koch 2006] or Example 2) on the cyclic group of order n, then

$$\Delta(\mathcal{B}(\mathbb{Z}_n)) = \{1, 2, \ldots, n-2\}$$

[Geroldinger and Halter-Koch 2006, Theorem 6.7.1]. The Delta sets of several numerical monoids [Bowles et al. 2006] and several congruence monoids [Baginski et al. 2008] have been computed under restricted conditions. In particular, an example is constructed in [Bowles et al. 2006, Proposition 4.9] where both containments in Equation (2) are strict.

The notion of a set of lengths was generalized in [Chapman and Smith 1998] as follows: With M as above, for each $n \in \mathbb{N}$ set $\mathcal{W}(n) = \{m \in M \mid n \in \mathcal{L}(m)\}$ and

$$\mathcal{V}(n) = \bigcup_{m \in \mathcal{W}(n)} \mathcal{L}(m).$$

We refer to the set $\mathcal{V}(n)$ as a *union of sets of lengths*. In [Chapman and Smith 1998], the basic properties of these sets are determined. Moreover, for block monoids $\mathcal{B}(G)$ where G is a finite abelian group, the authors argue that the sequence $\{\mathcal{V}(n)\}_{n=1}^{\infty}$ does not uniquely characterize G. We will often need to refer to the maximum and minimum values in $\mathcal{V}(n)$. Hence for each $n \in \mathbb{N}$ we set

$$\lambda_n(M) = \min \mathcal{V}(n) \quad \text{and} \quad \rho_n(M) = \sup \mathcal{V}(n).$$

When the monoid M is understood, we will merely use the notation λ_n and ρ_n. The sequence $\{\rho_n\}_{n=1}^{\infty}$ has been an object of study in its own right [Geroldinger and Halter-Koch 2006, Section 1.4] and [Geroldinger and Hassler $\geq$ 2008] and it is shown in [Geroldinger and Halter-Koch 2006, Proposition 1.4.2] that

$$\rho(M) = \lim_{n\to\infty} \frac{\rho_n(M)}{n}.$$

Finally, for each $n \in \mathbb{N}$, set $\Phi(n) = |\mathcal{V}(n)|$. Some basic properties of the Φ-function are explored in [Chapman and Smith 1990, Section 2] and several additional computations of the limit

$$\overline{\Phi}(M) = \lim_{n\to\infty} \frac{\Phi(n)}{n}$$

can be found in the literature [Chapman and Smith 1993a, Theorem 2.7 and Theorem 2.10].

For our purposes, we extend the notion of the Delta set to unions of sets of lengths as follows: For a fixed monoid M, suppose for each $n \in \mathbb{N}$ that

$$\mathcal{V}(n) = \{v_{1,n}, \ldots, v_{t,n}\},$$

where $v_{i,n} < v_{i+1,n}$ for $1 \leq i < t$. Define the $\mathcal{V}(n)$-Delta set of M to be

$$\Delta\mathcal{V}(n) = \{v_{i,n} - v_{i-1,n} \mid 2 \leq i \leq t\}$$

and the $\mathcal{V}$-Delta set of M to be

$$\Delta_{\mathcal{V}}(M) = \bigcup_{n\in\mathbb{N}} \Delta\big(\mathcal{V}(n)\big).$$

In addition, set $\mathcal{V}^*(M) = \sup \Delta_{\mathcal{V}}(M)$ and $\mathcal{V}_*(M) = \min \Delta_{\mathcal{V}}(M)$. Clearly,

$$\Delta\mathcal{V}(1) = \varnothing.$$

Example 1. Let $\mathbb{N}_0$ represent the nonnegative integers. Consider the additive submonoid $M = \{(x_1, x_2, x_3) \mid x_1 + 3x_2 = 4x_3 \text{ with each } x_i \in \mathbb{N}_0\}$ of $\mathbb{N}_0^3$. Such a monoid is known as a *Diophantine monoid* [Chapman et al. 2002]. A characterization of Diophantine monoids can be found in [Geroldinger and Halter-Koch 2006, Theorem 2.7.14]. It follows from [Chapman et al. 2000, Proposition 4.8], that $\Delta(M) = \{2\}$. Using elementary number theory, it follows that the irreducible

	λ_n	ρ_n
$n \equiv 0 \pmod 4$	$2\lfloor \frac{n}{4} \rfloor$	2n
$n \equiv 1 \pmod 4$	$2\lfloor \frac{n-1}{4} \rfloor + 1$	2n-1
$n \equiv 2 \pmod 4$	$2\lfloor \frac{n}{4} \rfloor + 2$	2n
$n \equiv 3 \pmod 4$	$2\lfloor \frac{n-1}{4} \rfloor + 3$	2n-1

Table 1. Example 1: values for λ_n and ρ_n for $n = 0, 1, 2, 3$.

elements of M are $v_1 = (4, 0, 1)$, $v_2 = (0, 4, 3)$ and $v_3 = (1, 1, 1)$. The following two facts will be key in determining $\Delta_{\mathcal{V}}(M)$:

- using the relation $v_1 + v_2 = 4v_3$, it is clear that an irreducible factorization in M which contains both v_1 and v_2 can be increased in length by 2;
- by [Chapman and Smith 1993a, Lemma 2.8], if a and b are in $\mathcal{V}(n)$, then $a \equiv b \pmod 2$.

By observing that λ_n is obtained by factoring nv_3 and ρ_n by factoring $2nv_3$, if n is even or $(2n-1)v_3$, if n is odd, we obtain the values given in Table 1. We list the first few values of $\mathcal{V}(n)$ below:

$$\begin{aligned} &\mathcal{V}(1) = \{1\}, && \mathcal{V}(5) = \{3, 5, 7, 9\}, \\ &\mathcal{V}(2) = \{2, 4\}, && \mathcal{V}(6) = \{4, 6, 8, 10, 12\}, \\ &\mathcal{V}(3) = \{3, 5\}, && \mathcal{V}(7) = \{5, 7, 9, 11, 13\}, \\ &\mathcal{V}(4) = \{2, 4, 6, 8\}, && \mathcal{V}(8) = \{4, 6, 8, 10, 12, 14, 16\}. \end{aligned}$$

We have that $\Delta\big(\mathcal{V}(n)\big) = \{2\}$ for all n and hence $\Delta_{\mathcal{V}}(M) = \{2\}$. Notice here that $\Delta_{\mathcal{V}}(M) = \Delta(M)$. $\square$

Example 2. Let G be an abelian group and $\mathcal{F}(G)$ represent the free abelian monoid on G. Set

$$\mathcal{B}(G) = \left\{ \prod_{g_i \in G} g_i^{n_i} \mid \sum_{g_i \in G} n_i g_i = 0 \right\}.$$

$\mathcal{B}(G)$ is a submonoid of $\mathcal{F}(G)$ known as the *block monoid* on G. Its irreducible elements are known as *minimal zero-sequences*. Using the results of [Chapman and Smith 1998], we can write out the unions of sets of lengths, and in turn the $\mathcal{V}(n)$-Delta sets of block monoids on relatively simple groups. For instance, if

$G = \mathbb{Z}_5$, then [Chapman and Smith 1998, Example 5.4] yields:

$$\begin{aligned} \rho_n &= \lfloor \frac{5n}{2} \rfloor && \text{for } n \geq 2, \\ \lambda_1 &= 1,\ \lambda_k = 2 && \text{for } k = 2, 3, 4, 5, \\ \lambda_k &= \lambda_{(k-5)} + 2 && \text{for } k \geq 6, \end{aligned}$$

for all $n \geq 1$, $\mathcal{V}(n) = [\lambda_n, \rho_n] \cap \mathbb{Z}$ Hence, $\Delta\mathcal{V}(n) = \{1\}$ for each $n > 1$ in $\mathbb{N}$ and thus $\Delta_{\mathcal{V}}(\mathcal{B}(\mathbb{Z}_5)) = \{1\}$. Notice that our previous remark yields that $\Delta(\mathcal{B}(\mathbb{Z}_5)) = \{1, 2, 3\}$. □

We consider some basic properties of the $\mathcal{V}$-Delta set of M in the following lemma.

Lemma 1. *Let M be an atomic monoid with* $\min \Delta(M) = d$ *and* $\max \Delta(M) = qd$ *for $q \geq 1$.*

(1) $\mathcal{V}_*(M) = d$.

(2) $\mathcal{V}^*(M) \leq qd$.

(3) $\{d\} \subseteq \Delta_{\mathcal{V}}(M) \subseteq \{d, 2d, \ldots, qd\}$.

Proof. Choose $n \in \mathbb{N}$ and let $v_{i+1,n},\ v_{i,n}$ be in $\mathcal{V}(n)$. We may choose x_1 and x_2 in $M^\bullet$ such that $\{n, v_{i+1,n}\} \subseteq \mathcal{L}(x_1)$ and $\{n, v_{i,n}\} \subseteq \mathcal{L}(x_2)$. By Equation (2), $\mathcal{L}(x_1)$ is a subset of $n + d\mathbb{Z}$ which contains n and whose consecutive elements are at most qd apart. The same statement holds for $\mathcal{L}(x_2)$, therefore the union, $\mathcal{L}(x_1) \cup \mathcal{L}(x_2)$, also possesses all these properties. Note that the union is a subset of $\mathcal{V}(n)$, so since $v_{i+1,n}$ and $v_{i,n}$ are consecutive elements of $\mathcal{V}(n)$, they in particular must be consecutive elements of $\mathcal{L}(x_1) \cup \mathcal{L}(x_2)$. Therefore $v_{i+1,n} - v_{i,n} = td$ for some $1 \leq t \leq q$. This shows that $\Delta\mathcal{V}(n) \subseteq \{d, 2d, \ldots, qd\}$, which in turn implies (2) and (3). It also determines that $\mathcal{V}_*(M) \geq d$, so we are left with just showing $d \in \Delta_{\mathcal{V}}(M)$.

Since $d \in \Delta(M)$, there is an $x \in M$ and $l_1, l_2 \in \mathcal{L}(x)$ with $l_2 - l_1 = d$. Consider $\mathcal{V}(l_1)$, to which both l_1 and l_2 belong. They must be consecutive elements of $\mathcal{V}(l_1)$ since we have just shown that consecutive elements are at least d apart. Hence $d \in \Delta(\mathcal{V}(l_1)) \subset \Delta_{\mathcal{V}}(M)$. □

Note that Example 2 indicates that the inequality in Lemma 1 regarding $\mathcal{V}^*(M)$ may be strict. The next corollary will later be useful and follows immediately from Lemma 1.

Corollary 1. *If $\Delta(M) = \{d\}$, then $\Delta_{\mathcal{V}}(M) = \{d\}$.*

We apply the $\mathcal{V}$-Delta set to limits of the form Equation (1). Unlike the $\overline{L}(x)$ and $\overline{l}(x)$ functions, there is no known argument that $\overline{\Phi}(M)$ exists for a general atomic monoid M. Hence, our analysis of Equation (1) will involve the use of lim inf and lim sup. Moreover, we must assume that $\Phi(n)$ is finite for all n, since this is necessary for $\limsup_{n\to\infty}$ to be finite. Indeed, if $\Phi(n)$ were infinite for some n, then so would be $\Phi(kn)$ for all k: if x has a factorization of length n and of length m, then x^k has factorizations of lengths kn and km. In [Chapman and Smith 1990], an atomic monoid which statisfies $\Phi(n) < \infty$ for all nonnegative n is called *Φ-finite*.

Our main theorem will use the stronger hypothesis that M has finite elasticity. The following proposition shows this is a necessary condition for

$$\limsup_{n\to\infty} \frac{\Phi(n)}{n}$$

to be finite, and the main theorem shows that it is sufficient as well.

Proposition 1. *Let M be an atomic Φ-finite monoid. If $\rho(M) = \infty$, then*

$$\limsup_{n\to\infty} \frac{\Phi(n)}{n} = \infty.$$

Proof.

Since $\rho(M) = \infty$, there are x_t such that $a_t = L(x_t)$ and $b_t = l(x_t)$ satisfying

$$\lim_{t\to\infty} \frac{a_t}{b_t} = \infty.$$

But all the $\mathcal{V}(n)$ are finite and $a_t \in \mathcal{V}(b_t)$, implying that for every $M > 0$ there is an $N > 0$ such that for all $t > N$, $b_t > M$. Therefore we may assume that the sequence is chosen such that the b_t are strictly increasing.

Since $\Phi(n)$ is finite for each n, $\mathcal{V}^*(b_t)$ exists and $\mathcal{V}^*(b_t) \geq a_t$. Pruning the sequence if necessary, we may assume that the b_t are chosen such that

$$\lim_{t\to\infty} \frac{\mathcal{V}^*(b_t)}{b_t} = \infty.$$

We may estimate

$$\Phi(b_t) \geq \frac{\mathcal{V}^*(b_t) - \mathcal{V}_*(b_t) + 1}{qd}.$$

Since $\mathcal{V}_*(b_t) \leq b_t$, we find that

$$\frac{\Phi(b_t)}{b_t} \geq \frac{\mathcal{V}^*(b_t)}{b_t qd} - \frac{1}{qd} + \frac{1}{b_t qd}.$$

Taking $\liminf$ of both sides, we see that

$$\liminf_{t\to\infty} \frac{\Phi(b_t)}{b_t} \geq \infty,$$

since the b_t are strictly increasing. Therefore

$$\limsup_{n\to\infty} \frac{\Phi(n)}{n} = \infty. \qquad \square$$

Now our main theorem:

Theorem 1. *Let M be an atomic monoid with $\rho(M) < \infty$. Then M is Φ-finite and moreover*

$$\frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}^*(M)} \leq \liminf_{n\to\infty} \frac{\Phi(n)}{n} \leq \limsup_{n\to\infty} \frac{\Phi(n)}{n} \leq \frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}_*(M)}. \tag{3}$$

Proof. Let $n \in \mathbb{N}$ and suppose that $m \in \mathcal{V}(n)$. It follows that

$$\frac{1}{\rho(M)} \leq \frac{m}{n} \leq \rho(M)$$

and hence

$$\frac{n}{\rho(M)} \leq m \leq n\rho(M),$$

which shows that M is Φ-finite. We further obtain that

$$\frac{\left(\rho(M) - \frac{1}{\rho(M)}\right)n + 1}{\mathcal{V}^*(M)} \leq \Phi(n) \leq \frac{\left(\rho(M) - \frac{1}{\rho(M)}\right)n + 1}{\mathcal{V}_*(M)}.$$

Thus,

$$\left(\frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}^*(M)}\right)n + \frac{1}{\mathcal{V}^*(M)} \leq \Phi(n) \leq \left(\frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}_*(M)}\right)n + \frac{1}{\mathcal{V}_*(M)}.$$

After dividing by n and taking the respective $\liminf$ and $\limsup$, we get that

$$\frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}^*(M)} \leq \liminf_{n\to\infty} \frac{\Phi(n)}{n} \leq \limsup_{n\to\infty} \frac{\Phi(n)}{n} \leq \frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}_*(M)}. \qquad \square$$

If $\Delta(M) = \{d\}$, then Corollary 1 implies that $\mathcal{V}^*(M) = \mathcal{V}_*(M) = d$ and Theorem 1 reduces to the following.

Corollary 2. *Let M be an atomic monoid with $\rho(M) < \infty$. If $\Delta(M) = \{d\}$, then*

$$\overline{\Phi}(M) = \frac{\rho(M)^2 - 1}{\rho(M)d}. \tag{4}$$

Corollary 2 immediately has some nice applications.

Example 3. A numerical monoid is an additive submonoid of the nonnegative integers. Every numerical monoid S has a unique minimal set of generators, and we will use the notation $S = \langle a_1, a_2, \ldots, a_t\rangle$ to represent the minimal generating set (which we assume is written in linear order). S is *primitive* if

$$1 = \gcd\{s \mid s \in S\}.$$

Every numerical monoid S is isomorphic to a unique primitive numerical monoid, so when working with numerical monoids, we can always assume that S is a primitive numerical monoid. By [Bowles et al. 2006], there exists a method for calculating $\max \Delta(S)$ in finite time and

$$\min \Delta(S) = \gcd\{a_i - a_{i-1} \mid i \in \{2, 3, \ldots, t\}\} = d.$$

By [Chapman et al. 2006, Theorem 2.1], $\rho(S) = a_t/a_1$. Hence for a numerical monoid, Equation (3) reduces to

$$\frac{a_t^2 - a_1^2}{\mathcal{V}^* a_1 a_t} \le \liminf_{n\to\infty} \frac{\Phi(n)}{n} \le \limsup_{n\to\infty} \frac{\Phi(n)}{n} \le \frac{a_t^2 - a_1^2}{\mathcal{V}_* a_1 a_t}.$$

If we know further that the generators of S form an arithmetic sequence (that is, $S = \langle a, a+d, a+2d, \ldots, a+kd\rangle$ for some positive integers d and k), then [Bowles et al. 2006, Theorem 3.9] indicates that $\Delta(S) = \{d\}$. In this case we obtain an exact calculation of $\overline{\Phi}(S)$ as

$$\overline{\Phi}(S) = \frac{k(2a+kd)}{a(a+kd)} = k\left(\frac{1}{a} + \frac{1}{a+kd}\right). \qquad \square$$

Example 4. Let a and b be positive integers with $a \le b$ and $a^2 \equiv a \pmod b$. The set of numbers $M(a, b) = \{x \mid x \in \mathbb{N} \text{ and } x \equiv a \pmod b\} \cup \{1\}$ forms a multiplicative monoid known as an *arithmetical congruence monoid* (ACM). ACMs have been the focus of three recent papers in the literature [Banister et al. 2007a, 2007b, Baginski et al. 2008]. An ACM is called *local* if $\gcd(a, b) = p^\alpha$ for some prime number p and positive integer α. It follows from elementary number theory that a local ACM $M(a, b)$ has a minimal index, which we denote by β, for which $p^\beta \in M(a, b)$. There are two relevant known results for a local ACM $M(a, b)$:

- $\rho(M(a, b)) = \frac{\alpha+\beta-1}{\alpha}$ [Banister et al. 2007b, Theorem 2.4]
- if $\alpha = \beta > 1$, then $\Delta(M(a, b)) = \{1\}$ [Baginski et al. 2008, Theorem 3.1].

Hence, for an ACM as above where $\alpha = \beta > 1$ (for instance, $M(4, 12)$), Equation (4) reduces to

$$\overline{\Phi}(M(a, b)) = \frac{(2\alpha-1)^2 - \alpha^2}{\alpha(2\alpha-1)}. \qquad \square$$

We close with a few comments:

- The proof in [Chapman and Smith 1993b] of Equation (1) relies on a different technique than that used above. The proof relies on knowing the exact structure of the sets in an infinite subsequence of the sequence $\mathcal{V}(1), \mathcal{V}(2), \ldots$.
- By a recent result of [Freeze and Geroldinger ≥ 2008],

$$\mathcal{V}^*(\mathcal{B}(G)) = \mathcal{V}_*(\mathcal{B}(G)) = 1$$

  for all abelian groups G. Combined with Theorem 1, this yields a simpler proof of Equation (1) than the original proof in [Chapman and Smith 1993b].

- Connected to the last remark is a question posed in [Chapman and Smith 1998, Section 5]: for $\mathcal{B}(\mathbb{Z}_n)$, does $\rho_3 = \max \mathcal{V}(3) = n + 1$? This question has been answered in the affirmative by [Gao and Geroldinger $\geq$ 2008].

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